EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 17, No. 4, 2024, 3973-3993
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

An Innovative Perspective on Bipolar Fuzzy Fantastic Ideals in BCK/BCI-Algebras

M. Balamurugan[1], Khalil H. Hakami[2,*], Moin A. Ansari[2,*], K. Loganathan[3]

[1] *Department of Mathematics, Vel Tech Rangarajan Dr. Sagunthala R&D Institute of Science and Technology, Chennai 600062, Tamil Nadu, India*
[2] *Department of Mathematics, College of Science, Jazan University, P.O. Box. 114, Jazan 45142, Kingdom of Saudi Arabia*
[3] *Department of Mathematics and Statistics, Manipal University Jaipur, Jaipur-303007, India*

Abstract. In this paper, we propose the concept of $(\in, \in \vee(\varphi^{*}, \check{q_\varphi}))$-bipolar fuzzy ideals in BCK/BCI-algebras. We show that an $(\in, \in \vee\check{q_\varphi})$-bipolar fuzzy ideal is an $(\in, \in \vee(\varphi^{*}, \check{q_\varphi}))$-bipolar fuzzy ideal. For a BCK/BCI-algebra, it has been shown that an $(\in, \in \vee(\varphi^{*}, \check{q_\varphi})$-bipolar fuzzy ideal is an $(\in, \in \vee\check{q})$-bipolar fuzzy ideal of $\check{\aleph}$, but not conversely, and then an example is given. We introduce the concept of $(\in, \in \vee(\varphi^{*}, \check{q_\varphi}))$-bipolar fuzzy fantastic ideals in BCK/BCI-algebras. It has been shown that an $(\in, \in \vee(\varphi^{*}, \check{q_\varphi}))$-bipolar fuzzy fantastic ideal is an $(\in, \in)$-bipolar fuzzy ideal in BCK/BCI-algebras. Furthermore, the connection between $(\in, \in \vee(\varphi^{*}, \check{q_\varphi}))$-bipolar fuzzy fantastic ideals and fantastic ideals are established.

2020 Mathematics Subject Classifications: 06F35, 03G25, 03B52

Key Words and Phrases: BCK/BCI-algebra, fuzzy logic, bipolar fuzzy ideal ($\mathcal{BFI}$), bipolar fuzzy fantastic ideal ($\mathcal{BFFI}$), $(\in, \in \vee(\varphi^{*}, \check{q_\varphi}))$-$\mathcal{BFI}$, $(\in, \in \vee(\varphi^{*}, \check{q_\varphi}))$-$\mathcal{BFFI}$

---

1. Introduction

Zadeh [45] introduced the concept of fuzzy set theory in 1965 as a mathematical framework for handling uncertainty and vagueness. Rather than being merely inside or outside a set, it broadens the classical thought of set theory to allow for varying degrees of membership among its elements, from 0 to 1. This method is especially suitable for areas with non-binary information or where there is imprecision, such as artificial intelligence, control systems, and decision-making. Axiomatic systems of propositional calculi are formal systems that use axioms and inference rules to come up with theorems in propositional logic.

---

*Corresponding author.
*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v17i4.5527

Email addresses: drbalamuruganm@veltech.edu.in (M. Balamurugan), khakami@jazanu.edu.sa (Khalil H. Hakami), maansari@jazanu.edu.sa (Moin A. Ansari), loganathankaruppusamy304@gmail.com (K. Loganathan)

This has been first described by Imai et al. [25, 26]. These systems are foundational in the study of logic and are crucial for understanding the formal properties and limitations of logical systems.

A bipolar fuzzy set ($\mathcal{BFS}$) was introduced by Zhang [47] to extend the classical fuzzy set theory by incorporating both positive and negative membership degrees. This method can better reflect real-life situations where an element can show different levels of belonging and not belonging to a set at the same time. It also has a more complex semantic interpretation compared to traditional fuzzy sets. Bipolar logic and bipolar fuzzy logic are conceptual frameworks introduced by Zhang [46]. These frameworks extend classical binary logic and traditional fuzzy logic by incorporating bipolarity, which means that they handle positive and negative information separately. This allows for a more nuanced representation of reality, reflecting the inherent dualities found in many real-world situations.

Bipolar fuzzy subalgebras and bipolar fuzzy ideals are concepts introduced by Lee [31–33] to extend the theory of BCK/BCI-algebras into the realm of fuzzy logic. Saied et al. [44] studied bipolar-valued fuzzy BCK/BCI-algebras, which is a niche but fascinating area within mathematical logic and algebra. Rosenfeld [43] extended the classical concept of a group in algebra to accommodate the notion of fuzziness, which is characterized by elements having degrees of membership rather than crisp membership. Aslam [17] presented the idea of bipolar fuzzy ideals in LA-semigroups, which gives the study of algebraic structures using fuzzy set theory a big new dimension. Akram et al. [2, 3] introduced the concept of m-polar fuzzy Lie ideals of Lie algebras.

Al-Masarwah et al. (See [5, 10–14]) extended the concept of bipolar/m-polar fuzzy subalgebras and ideals to BCK/BCI-algebras. The concept of $(\in, \in, \vee q)$-fuzzy subgroup was introduced by Bhakat [20] as a generalization in the context of fuzzy group theory. Chen et al. [21] worked on m-polar fuzzy sets as an extension of bipolar fuzzy sets, which is a significant contribution to the field of fuzzy set theory. Farooq et al. [22] presented a topic related to m-polar fuzzy groups. Ibrar [24] studied ordered semigroups and their structures using (α, β)-bipolar fuzzy generalized bi-ideals. Jana [28] studied $(\in, \in \vee q)$-bipolar fuzzy BCK-algebras, which belong to a specialized branch of mathematics that deals with algebraic structures and fuzzy logic. Bipolar fuzzy UP-algebras are a concept in mathematics introduced by Kawila et al. [29].

Balamurugan et al. [18, 42] investigated $(\acute{\in}, \acute{\in} \vee \acute{q}_{\check{k}})$-uni-intuitionistic fuzzy soft h-Ideals in subtraction BG-algebras and $(\in, \in \vee \check{q})$-bipolar fuzzy-ideals of BCK/BCI-algebras. Al-Maswarwah et al. [6–9] developed multipolar fuzzy ideals of BCK/BCI-algebras. Balamurugan [19]presented the concept of complex fuzzy ideals in BCK/BCI-algebras. Iampan et al. [23] discussed anti-intuitionistic fuzzy soft b-ideals in BCK/BCI algebras. Moin [15] introduced roughness in JU-algebras. Mohseni et al. [35] studied the concept of multipolar fuzzy p-ideals of BCI-algebras. Muhiuddin et al. (see [36–41]) applied the concept to multipolar fuzzy ideals in BCK/BCI algebras. Al-Kadi et al. [4] looked into a group of BCI-algebras called bipolar fuzzy BCI-implicative ideals. These ideals probably share some properties with both fuzzy logic and BCI-algebras. Abuhijileh et al. [1] developed the concept of complex fuzzy group based on Rosenfeld's approach. Mahmood [34] studied bipolar complex fuzzy soft sets and their applications in decision-making. Jaleel et al. [27]

demonstrated interval-valued bipolar complex fuzzy soft set as a generalization of fuzzy set, interval-valued fuzzy set, bipolar fuzzy set, complex fuzzy set, and soft set. Ali et al. [30] extended KU-algebras and investigated properties based on them whereas Moin et al. [16] introduced intersectional soft ideals and their quotients on KU-algebras,

The article is organized as follows: Section 2 proceeds with a recapitulation of all required definitions and properties. In Section 3, we present $(\in, \in \vee(\varphi^{*}, \check{q_{\varphi}}))$-bipolar fuzzy ideals in BCK/BCI-algebras. In Section 4, $(\in, \in \vee(\varphi^{*}, \check{q_{\varphi}}))$-bipolar fuzzy fantastic ideals in BCK/BCI-algebras are proposed and their properties are discussed in detail. Finally, in Section 5, the conclusions and scope of future research are given.

2. Preliminaries

BCK-algebras and BCI-algebras are types of algebraic structures used in the study of non-classical logics, particularly in the context of certain types of implication algebras. These algebras generalize certain aspects of set theory, logic and have applications in some areas, such as theoretical computer science and mathematical logic.

Definition 1. *[25, 26] A BCK-algebra is a structure* $(\check{\aleph}; \between, 0)$ *consisting of a non-empty set* $\check{\aleph}$*, a binary operation* $\between$ *on* $\check{\aleph}$*, and a constant* $0 \in \check{\aleph}$*, satisfying the following axioms:*
(C_1) $((\check{\varrho}_0 \between \check{\varrho}_1) \between (\check{\varrho}_0 \between \check{\varrho}_2)) \between (\check{\varrho}_2 \between \check{\varrho}_1) = 0,$
(C_2) $(\check{\varrho}_0 \between (\check{\varrho}_0 \between \check{\varrho}_1)) \between \check{\varrho}_1 = 0,$
(C_3) $\check{\varrho}_0 \between \check{\varrho}_0 = 0,$
(C_4) $0 \between \check{\varrho}_0 = 0,$
(C_5) $\check{\varrho}_0 \between \check{\varrho}_1 = 0$ *and* $\check{\varrho}_1 \between \check{\varrho}_0 = 0 \Rightarrow \check{\varrho}_0 = \check{\varrho}_1,$
$\forall \check{\varrho}_0, \check{\varrho}_1, \check{\varrho}_2 \in \check{\aleph}$

A non-void subset $\check{A}$ is an ideal of $\check{\aleph}$ if (I_1) $0 \in \check{A}$, (I_2) $\forall \check{\varrho}_0, \check{\varrho}_1 \in \check{\aleph}, \check{\varrho}_0 \between \check{\varrho}_1 \in \tilde{A}, \check{\varrho}_1 \in \tilde{A} \Rightarrow \check{\varrho}_0 \in \check{A}$.

A non-void subset $\check{A}$ is a fantastic ideal of $\check{\aleph}$ if (I_1) and (I_3) $\forall \check{\varrho}_0, \check{\varrho}_1, \check{\varrho}_2 \in \check{\aleph}, (\check{\varrho}_0 \between \check{\varrho}_1) \between \check{\varrho}_2 \in \check{A}, \check{\varrho}_2 \in \check{A} \Rightarrow \check{\varrho}_0 \between (\check{\varrho}_1 \between (\check{\varrho}_1 \between \check{\varrho}_0)) \in \check{A}$.

A bipolar fuzzy set ($\mathcal{BFS}$) is denoted by $\bar{\zeta} = (\bar{\zeta}_-, \bar{\zeta}_+)$, where $\bar{\zeta}_- : \check{\aleph} \to [-1, 0]$ and $\bar{\zeta}_+ : \check{\aleph} \to [0, 1]$.

Definition 2. *[42] A* $\mathcal{BFS}$ $\bar{\zeta}$ *is a* $\mathcal{BFI}$ *of* $\check{\aleph}$ *if it meets the ensuing assertions:*
(i) $\bar{\zeta}_-(0) \leq \bar{\zeta}_-(\check{\varrho}_0)$, $\bar{\zeta}_+(0) \geq \bar{\zeta}_+(\check{\varrho}_0)$.
(ii) $\bar{\zeta}_-(\check{\varrho}_0) \leq \bar{\zeta}_-(\check{\varrho}_0 \between \check{\varrho}_1) \vee \bar{\zeta}_-(\check{\varrho}_1)$,
(iii) $\bar{\zeta}_+(\check{\varrho}_0) \geq \bar{\zeta}_+(\check{\varrho}_0 \between \check{\varrho}_1) \wedge \bar{\zeta}_+(\check{\varrho}_1), \forall \check{\varrho}_0, \check{\varrho}_1 \in \check{\aleph}$.

Definition 3. *[42] A* $\mathcal{BFS}$ $\bar{\zeta}$ *is a* $(\in, \in \vee \check{q})$*-*$\mathcal{BFI}$ *of* $\check{\aleph}$ *if it meets the ensuing assertions:*
(i) $\bar{\zeta}_-(0) \leq \bar{\zeta}_-(\check{\varrho}_0)$, $\bar{\zeta}_+(0) \geq \bar{\zeta}_+(\check{\varrho}_0)$.
(ii) $\bar{\zeta}_-(\check{\varrho}_0) \leq \bar{\zeta}_-(\check{\varrho}_0 \between \check{\varrho}_1) \vee \bar{\zeta}_-(\check{\varrho}_1) \vee -\frac{1}{2}$,
(iii) $\bar{\zeta}_+(\check{\varrho}_0) \geq \bar{\zeta}_+(\check{\varrho}_0 \between \check{\varrho}_1) \wedge \bar{\zeta}_+(\check{\varrho}_1) \wedge \frac{1}{2}, \forall \check{\varrho}_0, \check{\varrho}_1 \in \check{\aleph}$.

3. $(\in, \in \vee(\varphi^*, \check{q_\varphi}))$-bipolar fuzzy ideals

In this section, we investigate an $(\in, \in \vee(\varphi^*, \check{q_\varphi}))$-bipolar fuzzy ideal of BCK/BCI-algebras.

Definition 4. *A $\mathcal{BFS}$ $\bar{\zeta}$ is an $(\in, \in \vee\check{q_\varphi})$-$\mathcal{BFI}$ of $\check{\aleph}$ if it meets the ensuing assertions:*
(i) $\bar{\zeta}_-(0) \leq \bar{\zeta}_-(\check{\varrho}_0)$, $\bar{\zeta}_+(0) \geq \bar{\zeta}_+(\check{\varrho}_0)$.
(ii) $\bar{\zeta}_-(\check{\varrho}_0) \leq \bar{\zeta}_-(\check{\varrho}_0 \between \check{\varrho}_1) \vee \bar{\zeta}_-(\check{\varrho}_1) \vee (\frac{\varphi}{2} - \frac{1}{2})$,
(iii) $\bar{\zeta}_+(\check{\varrho}_0) \geq \bar{\zeta}_+(\check{\varrho}_0 \between \check{\varrho}_1) \wedge \bar{\zeta}_+(\check{\varrho}_1) \wedge (-\frac{\varphi}{2} + \frac{1}{2}), \forall \check{\varrho}_0, \check{\varrho}_1 \in \check{\aleph}$.

Definition 5. *A $\mathcal{BFS}$ $\bar{\zeta}$ is an $(\in, \in \vee(\varphi^*, \check{q_\varphi}))$-$\mathcal{BFI}$ of $\check{\aleph}$ if it meets the ensuing assertions:*
(i) $\bar{\zeta}_-(0) \leq \bar{\zeta}_-(\check{\varrho}_0)$, $\bar{\zeta}_+(0) \geq \bar{\zeta}_+(\check{\varrho}_0)$.
(ii) $\bar{\zeta}_-(\check{\varrho}_0) \leq \bar{\zeta}_-(\check{\varrho}_0 \between \check{\varrho}_1) \vee \bar{\zeta}_-(\check{\varrho}_1) \vee (\frac{\varphi}{2} - \frac{\varphi^*}{2})$,
(iii) $\bar{\zeta}_+(\check{\varrho}_0) \geq \bar{\zeta}_+(\check{\varrho}_0 \between \check{\varrho}_1) \wedge \bar{\zeta}_+(\check{\varrho}_1) \wedge (-\frac{\varphi}{2} + \frac{\varphi^*}{2}), \forall \check{\varrho}_0, \check{\varrho}_1 \in \check{\aleph}$.

Definition 6. *A $\mathcal{BFS}$ $\bar{\zeta}$ is an $(\in, \in \vee(\varphi^*, \check{q_\varphi}))$-$\mathcal{BFI}$ of $\check{\aleph}$ if it fulfills the ensuing assertions:*
(i) $(\check{\varrho}_0 \between \check{\varrho}_1, \check{s}) \in \bar{\zeta}_-, (\check{\varrho}_1, \check{t}) \in \bar{\zeta}_- \Rightarrow (\check{\varrho}_0, \check{s} \vee \check{t}) \in \vee(\varphi^*, \check{q_\varphi})\bar{\zeta}_-$,
(ii) $(\check{\varrho}_0 \between \check{\varrho}_1, \check{u}) \in \bar{\zeta}_+, (\check{\varrho}_1, \check{v}) \in \bar{\zeta}_+ \Rightarrow (\check{\varrho}_0, \check{u} \wedge \check{v}) \in \vee(\varphi^*, \check{q_\varphi})\bar{\zeta}_+$,
for all $\check{\varrho}_0, \check{\varrho}_1 \in \check{\aleph}$, $\check{s}, \check{t} \in [-1, 0)$ *and* $\check{u}, \check{v} \in (0, 1]$.

Example 1. *Take a BCK/BCI-algebra $\check{\aleph} = \{0, \check{u}, \check{\varrho}_0, \check{\varrho}_1, \check{\varrho}_2\}$ with the subsequent Cayley table:*

$\between$	0	$\check{u}$	$\check{\varrho}_0$	$\check{\varrho}_1$	$\check{\varrho}_2$
0	0	0	0	0	0
$\check{u}$	$\check{u}$	0	$\check{u}$	0	$\check{u}$
$\check{\varrho}_0$	$\check{\varrho}_0$	$\check{\varrho}_0$	0	0	0
$\check{\varrho}_1$	$\check{\varrho}_1$	$\check{u}$	$\check{\varrho}_1$	0	$\check{\varrho}_1$
$\check{\varrho}_2$	$\check{\varrho}_2$	$\check{\varrho}_2$	$\check{\varrho}_2$	$\check{\varrho}_2$	0

Define a $\mathcal{BFS}$ $\bar{\zeta}$ of $\check{\aleph}$ as follows:

$$\bar{\zeta}(\check{u}) = \begin{cases} (-0.72, 0.52), & \check{u} = 0; \\ (-0.42, 0.22), & \check{u} = \check{u}; \\ (-0.22, 0.06), & \check{u} = \check{\varrho}_0; \\ (-0.52, 0.12), & \check{u} = \check{\varrho}_1; \\ (-0.12, 0.22), & \check{u} = \check{\varrho}_2. \end{cases}$$

It is easy to show that $\bar{\zeta}$ is an $(\in, \in \vee(\varphi^*, \check{q_\varphi}))$-$\mathcal{BFI}$ of $\check{\aleph}$.

Theorem 1. *A $\mathcal{BFS}$ $\bar{\zeta}$ is an $(\in, \in \vee(\varphi^*, \check{q_\varphi}))$-$\mathcal{BFI}$ of $\check{\aleph}$ if and only if satisfies*

$$\bar{\zeta}_-(\check{\varrho}_0) \leq \bar{\zeta}_-(\check{\varrho}_0 \between \check{\varrho}_1) \vee \bar{\zeta}_-(\check{\varrho}_1) \vee (\tfrac{\varphi}{2} - \tfrac{\varphi^*}{2})$$

and

$$\bar{\zeta}_+(\breve{\varrho}_0) \geq \bar{\zeta}_+(\breve{\varrho}_0 \between \breve{\varrho}_1) \wedge \bar{\zeta}_+(\breve{\varrho}_1) \wedge (-\tfrac{\varphi}{2} + \tfrac{\varphi^*}{2})$$

for all $\breve{\varrho}_0, \breve{\varrho}_1 \in \check{\aleph}$.

Proof. Let $\bar{\zeta}$ be an $(\in, \in \vee(\varphi^*, \check{q_\varphi}))$-$\mathcal{BFI}$ of $\check{\aleph}$. If $\bar{\zeta}_-(\breve{\varrho}_0 \between \breve{\varrho}_1) \vee \bar{\zeta}_-(\breve{\varrho}_1) > \frac{\varphi}{2} - \frac{\varphi^*}{2}$ and $\bar{\zeta}_+(\breve{\varrho}_0 \between \breve{\varrho}_1) \wedge \bar{\zeta}_+(\breve{\varrho}_1) < -\frac{\varphi}{2} + \frac{\varphi^*}{2}$, then

$$\bar{\zeta}_-(\breve{\varrho}_0) \leq \bar{\zeta}_-(\breve{\varrho}_0 \between \breve{\varrho}_1) \vee \bar{\zeta}_-(\breve{\varrho}_1)$$

and

$$\bar{\zeta}_+(\breve{\varrho}_0) \geq \bar{\zeta}_+(\breve{\varrho}_0 \between \breve{\varrho}_1) \wedge \bar{\zeta}_+(\breve{\varrho}_1).$$

Assume that $\bar{\zeta}_-(\breve{\varrho}_0) > \bar{\zeta}_-(\breve{\varrho}_0 \between \breve{\varrho}_1) \vee \bar{\zeta}_-(\breve{\varrho}_1)$ and $\bar{\zeta}_+(\breve{\varrho}_0) < \bar{\zeta}_+(\breve{\varrho}_0 \between \breve{\varrho}_1) \wedge \bar{\zeta}_+(\breve{\varrho}_1)$. Let us take $\check{s} \in \neg\bar{\zeta}$ and $\breve{u} \in \bar{\zeta}$ such that

$$\bar{\zeta}_-(\breve{\varrho}_0) > \check{s} > \bar{\zeta}_-(\breve{\varrho}_0 \between \breve{\varrho}_1) \vee \bar{\zeta}_-(\breve{\varrho}_1)$$

and

$$\bar{\zeta}_+(\breve{\varrho}_0) < \breve{u} < \bar{\zeta}_+(\breve{\varrho}_0 \between \breve{\varrho}_1) \wedge \bar{\zeta}_+(\breve{\varrho}_1).$$

Then $(\breve{\varrho}_0 \between \breve{\varrho}_1, \check{s}) \in \bar{\zeta}_-, (\breve{\varrho}_1, \check{s}) \in \bar{\zeta}_-$ and $(\breve{\varrho}_0 \between \breve{\varrho}_1, \breve{u}) \in \bar{\zeta}_+, (\breve{\varrho}_1, \breve{u}) \in \bar{\zeta}_+$ but $(\breve{\varrho}_0, \check{s} \vee \check{s}) = (\breve{\varrho}_0, \check{s}) \overline{\in \vee(\varphi^*, \check{q_\varphi})\bar{\zeta}_-}$ and $(\breve{\varrho}_0, \breve{u} \wedge \breve{u}) = (\breve{\varrho}_0, \breve{u}) \overline{\in \vee(\varphi^*, \check{q_\varphi})\bar{\zeta}_+}$, a contradiction.

Hence, $\bar{\zeta}_-(\breve{\varrho}_0) \leq \bar{\zeta}_-(\breve{\varrho}_0 \between \breve{\varrho}_1) \vee \bar{\zeta}_-(\breve{\varrho}_1)$ whenever $\bar{\zeta}_-(\breve{\varrho}_0 \between \breve{\varrho}_1) \vee \bar{\zeta}_-(\breve{\varrho}_1) > \frac{\varphi}{2} - \frac{\varphi^*}{2}$ and $\bar{\zeta}_+(\breve{\varrho}_0) \geq \bar{\zeta}_+(\breve{\varrho}_0 \between \breve{\varrho}_1) \wedge \bar{\zeta}_+(\breve{y})$ whenever $\bar{\zeta}_+(\breve{\varrho}_0 \between \breve{\varrho}_1) \wedge \bar{\zeta}_+(\breve{\varrho}_1) < -\frac{\varphi}{2} + \frac{\varphi^*}{2}$.

Suppose that $\bar{\zeta}_-(\breve{\varrho}_0 \between \breve{\varrho}_1) \vee \bar{\zeta}_-(\breve{\varrho}_1) \leq \frac{\varphi}{2} - \frac{\varphi^*}{2}$ and $\bar{\zeta}_+(\breve{\varrho}_0 \between \breve{\varrho}_1) \wedge \bar{\zeta}_+(\breve{\varrho}_1) \geq -\frac{\varphi}{2} + \frac{\varphi^*}{2}$. Then, $(\breve{\varrho}_0 \between \breve{\varrho}_1, \frac{\varphi}{2} - \frac{\varphi^*}{2}) \in \bar{\zeta}_-$, $(\breve{\varrho}_1, \frac{\varphi}{2} - \frac{\varphi^*}{2}) \in \bar{\zeta}_-$ and $(\breve{\varrho}_0 \between \breve{\varrho}_1, -\frac{\varphi}{2} + \frac{\varphi^*}{2}) \in \bar{\zeta}_+$, $(\breve{\varrho}_1, -\frac{\varphi}{2} + \frac{\varphi^*}{2}) \in \bar{\zeta}_+$, which imply that

$$(\breve{\varrho}_0, (\tfrac{\varphi}{2} - \tfrac{\varphi^*}{2}) \wedge (\tfrac{\varphi}{2} - \tfrac{\varphi^*}{2})) = (\breve{\varrho}_0, (\tfrac{\varphi}{2} - \tfrac{\varphi^*}{2})) \in \vee(\varphi^*, \check{q_\varphi})\bar{\zeta}_-$$
$$(\breve{\varrho}_0, (-\tfrac{\varphi}{2} + \tfrac{\varphi^*}{2}) \wedge (-\tfrac{\varphi}{2} + \tfrac{\varphi^*}{2})) = (\breve{\varrho}_0, (-\tfrac{\varphi}{2} + \tfrac{\varphi^*}{2})) \in \vee(\varphi^*, \check{q_\varphi})\bar{\zeta}_+.$$

Thus, $\bar{\zeta}_-(\breve{\varrho}_0) \leq \frac{\varphi}{2} - \frac{\varphi^*}{2}$ and $\bar{\zeta}_+(\breve{\varrho}_0) \geq -\frac{\varphi}{2} + \frac{\varphi^*}{2}$. Otherwise, $\bar{\zeta}_-(\breve{\varrho}_0) + \frac{\varphi}{2} - \frac{\varphi^*}{2} > \frac{\varphi}{2} - \frac{\varphi^*}{2} + \frac{\varphi}{2} - \frac{\varphi^*}{2} = \varphi - \varphi^*$ and $\bar{\zeta}_+(\breve{\varrho}_0) - \frac{\varphi}{2} + \frac{\varphi^*}{2} < -\frac{\varphi}{2} + \frac{\varphi^*}{2} - \frac{\varphi}{2} + \frac{\varphi^*}{2} = -\varphi + \varphi^*$, a contradiction. And so,

$$\bar{\zeta}_-(\breve{\varrho}_0) \leq \bar{\zeta}_-(\breve{\varrho}_0 \between \breve{\varrho}_1) \vee \bar{\zeta}_-(\breve{\varrho}_1) \vee (\tfrac{\varphi}{2} - \tfrac{\varphi^*}{2})$$

and

$$\bar{\zeta}_+(\breve{\varrho}_0) \geq \bar{\zeta}_+(\breve{\varrho}_0 \between \breve{\varrho}_1) \wedge \bar{\zeta}_+(\breve{\varrho}_1) \wedge (-\tfrac{\varphi}{2} + \tfrac{\varphi^*}{2})$$

for all $\breve{\varrho}_0, \breve{\varrho}_1 \in \check{\aleph}$.

On the contrary, let us assume that an $(\in, \in \vee(\varphi^*, \check{q_\varphi}))$-$\mathcal{BFI}$ of $\check{\aleph}$ holds. Let $\breve{\varrho}_0, \breve{\varrho}_1, \breve{\varrho}_2 \in \check{\aleph}$ and $\breve{u}, \breve{v} \in (0, 1]$ and $\check{s}, \check{t} \in \neg\bar{\zeta}$ such that $(\breve{\varrho}_0 \between \breve{\varrho}_1, \check{s}) \in \bar{\zeta}_-, (\breve{\varrho}_1, \check{t}) \in \bar{\zeta}_-$ and $(\breve{\varrho}_0 \between \breve{\varrho}_1, \breve{u}) \in \bar{\zeta}_+, (\breve{\varrho}_1, \breve{v}) \in \bar{\zeta}_+$. Then, $\bar{\zeta}_-(\breve{\varrho}_0 \between \breve{\varrho}_1) \leq \check{s}, \bar{\zeta}_-(\breve{\varrho}_0 \between \breve{\varrho}_1) \leq \check{t}$ and $\bar{\zeta}_+(\breve{\varrho}_0 \between \breve{\varrho}_1) \geq \breve{u}, \bar{\zeta}_+(\breve{\varrho}_1) \leq \breve{v}$. If $\bar{\zeta}_-(\breve{\varrho}_0) > \check{s} \vee \check{t}$ and $\bar{\zeta}_+(\breve{\varrho}_0) < \breve{u} \vee \breve{v}$, then $\bar{\zeta}_-(\breve{\varrho}_0 \between \breve{\varrho}_1) \vee \bar{\zeta}_-(\breve{\varrho}_1) \leq \frac{\varphi}{2} - \frac{\varphi^*}{2}$ and $\bar{\zeta}_+(\breve{\varrho}_0 \between \breve{\varrho}_1) \wedge \bar{\zeta}_+(\breve{\varrho}_1) \geq -\frac{\varphi}{2} + \frac{\varphi^*}{2}$.
Otherwise, we get

$$\bar{\zeta}_-(\breve{\varrho}_0) \leq \bar{\zeta}_-(\breve{\varrho}_0 \between \breve{\varrho}_1) \vee \bar{\zeta}_-(\breve{\varrho}_1) \vee (\tfrac{\varphi}{2} - \tfrac{\varphi^*}{2}) \leq \bar{\zeta}_-(\breve{\varrho}_0 \between \breve{\varrho}_1) \vee \bar{\zeta}_-(\breve{\varrho}_1) \leq \check{s} \vee \check{t}$$

and

$$\bar{\zeta}_+(\breve{\varrho}_0) \geq \bar{\zeta}_+(\breve{\varrho}_0 \between \breve{\varrho}_1) \wedge \bar{\zeta}_+(\breve{\varrho}_1) \wedge (-\tfrac{\varphi}{2} + \tfrac{\varphi^*}{2}) \geq \bar{\zeta}_+(\breve{\varrho}_0 \between \breve{\varrho}_1) \wedge \bar{\zeta}_+(\breve{\varrho}_1) \geq \breve{u} \wedge \breve{v},$$

a contradiction. In that case

$$\bar{\zeta}_-(\breve{\varrho}_0) + \check{s} \vee \check{t} < 2\bar{\zeta}_-(\breve{\varrho}_0) \leq 2(\bar{\zeta}_-(\breve{\varrho}_0 \between \breve{\varrho}_1) \vee \bar{\zeta}_-(\breve{\varrho}_1) \vee (\tfrac{\varphi}{2} - \tfrac{\varphi^*}{2})) = \varphi - \varphi^*,$$

and

$$\bar{\zeta}_-(\breve{\varrho}_0) + \breve{v} \wedge \breve{u} > 2\bar{\zeta}_+(\breve{\varrho}_0) \geq 2(\bar{\zeta}_+(\breve{\varrho}_0 \between \breve{\varrho}_1) \wedge \bar{\zeta}_+(\breve{\varrho}_1) \wedge (-\tfrac{\varphi}{2} + \tfrac{\varphi^*}{2})) = -\varphi + \varphi^*.$$

Hence, $(\breve{\varrho}_0, \check{s} \vee \check{t}) \in \vee(\varphi^*, \breve{q_\varphi})\bar{\zeta}_-$ and $(\breve{\varrho}_0, \breve{u} \wedge \breve{v}) \in \vee(\varphi^*, \breve{q_\varphi})\bar{\zeta}_+$.
Therefore, $\bar{\zeta}$ is an $(\in, \in \vee(\varphi^*, \breve{q_\varphi}))$-$\mathcal{BFI}$ of $\check{\aleph}$.

Lemma 1. *Every $(\in, \in \vee\breve{q_\varphi})$-$\mathcal{BFI}$ is an $(\in, \in \vee(\varphi^*, \breve{q_\varphi}))$-$\mathcal{BFI}$ of $\check{\aleph}$, but the converse may not be true in general.*

Proof. Straightforward.

Example 2. *Take a BCI-algebra $\check{\aleph} = \{0, \breve{u}, \breve{\varrho}_0, \breve{\varrho}_1\}$ with Cayley table:*

$\between$	0	$\breve{u}$	$\breve{\varrho}_0$	$\breve{\varrho}_1$
0	0	$\breve{u}$	$\breve{\varrho}_0$	$\breve{\varrho}_1$
$\breve{u}$	$\breve{u}$	0	$\breve{\varrho}_1$	$\breve{\varrho}_0$
$\breve{\varrho}_0$	$\breve{\varrho}_0$	$\breve{\varrho}_1$	0	$\breve{u}$
$\breve{\varrho}_1$	$\breve{\varrho}_1$	$\breve{\varrho}_0$	$\breve{u}$	0

Define a $\mathcal{BFS}$ $\bar{\zeta}$ of $\check{\aleph}$ as follows:

$$\bar{\zeta}(\breve{u}) = \begin{cases} (-0.67, 0.67), & \breve{u} = 0; \\ (-0.17, 0.57), & \breve{u} = \breve{u}; \\ (-0.57, 0.57), & \breve{u} = \breve{\varrho}_0; \\ (-0.17, 0.47), & \breve{u} = \breve{\varrho}_1. \end{cases}$$

Hence, $(\in, \in \vee(\varphi^*, \breve{q_\varphi}))$-$\mathcal{BFI}$ of $\check{\aleph}$, but is not $\mathcal{BFI}$ of $\check{\aleph}$ because $\bar{\zeta}_+(\breve{\varrho}_1) = 0.47 \ngeq 0.57 = \bar{\zeta}_+(\breve{\varrho}_1 \between \breve{u}) \wedge \bar{\zeta}_+(\breve{u})$.

4. $(\in, \in \vee(\varphi^*, \breve{q_\varphi}))$-bipolar fuzzy fantastic ideals

This section investigates $(\in, \in \vee(\varphi^*, \breve{q_\varphi}))$-bipolar fuzzy fantastic ideals of BCK/BCI-algebras.

Definition 7. *A $\mathcal{BFS}$ $\bar{\zeta}$ is a $\mathcal{BFFI}$ of $\check{\aleph}$ if it fulfills the Definition 5(i) and the resulting assertions:*
(i) $\bar{\zeta}_-(\breve{\varrho}_0 \between (\breve{\varrho}_1 \between (\breve{\varrho}_1 \between \breve{\varrho}_0))) \leq \bar{\zeta}_-((\breve{\varrho}_0 \between \breve{\varrho}_1) \between \breve{\varrho}_2)) \vee \bar{\zeta}_-(\breve{\varrho}_2)$,
(ii) $\bar{\zeta}_+(\breve{\varrho}_0 \between (\breve{\varrho}_1 \between (\breve{\varrho}_1 \between \breve{\varrho}_0))) \geq \bar{\zeta}_+((\breve{\varrho}_0 \between \breve{\varrho}_1) \between \breve{\varrho}_2)) \wedge \bar{\zeta}_+(\breve{\varrho}_2), \forall \breve{\varrho}_0, \breve{\varrho}_1, \breve{\varrho}_2 \in \check{\aleph}$.

Definition 8. *A $\mathcal{BFS}$ $\bar{\zeta}$ is an $(\in, \in \vee(\varphi^*, \check{q_\varphi}))$-$\mathcal{BFFI}$ of $\check{\aleph}$ if it fulfills the ensuing assertions:*
(i) $((\breve{\varrho}_0 \between \breve{\varrho}_1) \between \breve{\varrho}_2), \check{s}) \in \bar{\zeta}_-, (\breve{\varrho}_2, \check{t}) \in \bar{\zeta}_- \Rightarrow (\breve{\varrho}_0 \between (\breve{\varrho}_1 \between (\breve{\varrho}_1 \between \breve{\varrho}_0)), \check{s} \vee \check{t}) \in \vee(\varphi^*, \check{q_\varphi})\bar{\zeta}_-$,
(ii) $((\breve{\varrho}_0 \between \breve{\varrho}_1) \between \breve{\varrho}_2), \check{u}) \in \bar{\zeta}_-, (\breve{\varrho}_2, \check{v}) \in \bar{\zeta}_- \Rightarrow (\breve{\varrho}_0 \between (\breve{\varrho}_1 \between (\breve{\varrho}_1 \between \breve{\varrho}_0)), \check{u} \wedge \check{v}) \in \vee(\varphi^*, \check{q_\varphi})\bar{\zeta}_+$,
$\forall \breve{\varrho}_0, \breve{\varrho}_1, \breve{\varrho}_2 \in \check{\aleph}$, $\check{s}, \check{t} \in [-1, 0)$ *and* $\check{u}, \check{v} \in (0, 1]$.

Example 3. *Consider $\check{\aleph} = \{0, \check{u}, \breve{\varrho}_0, \breve{\varrho}_1, \breve{\varrho}_2\}$ be a BCK-algebra in Example 1, and now we define a $\mathcal{BFS}$ $\bar{\zeta}$ of $\check{\aleph}$ as*

$$\bar{\zeta}(\check{u}) = \begin{cases} (-0.76, 0.53), & \check{u} = 0; \\ (-0.46, 0.23), & \check{u} = \check{u}; \\ (-0.26, 0.32), & \check{u} = \breve{\varrho}_0; \\ (-0.56, 0.13), & \check{u} = \breve{\varrho}_1; \\ (-0.16, 0.03), & \check{u} = \breve{\varrho}_2. \end{cases}$$

It is easy to show that $\bar{\zeta}$ is an $(\in, \in \vee(\varphi^*, \check{q_\varphi}))$-$\mathcal{BFFI}$ of $\check{\aleph}$ and $\mathcal{BFFI}$ of $\check{\aleph}$.

Theorem 2. *A $\mathcal{BFS}$ $\bar{\zeta}$ is a $\mathcal{BFFI}$ of $\check{\aleph}$ $\Leftrightarrow$ the succeeding assertions are holds:*
(i) $(\breve{\varrho}_0, \check{s}) \in \bar{\zeta}_- \Rightarrow (0, \check{s}) \in \bar{\zeta}_-$ *and* $(\breve{\varrho}_0, \check{u}) \in \bar{\zeta}_+ \Rightarrow (0, \check{u}) \in \bar{\zeta}_+$,
(ii) $((\breve{\varrho}_0 \between \breve{\varrho}_1) \between \breve{\varrho}_2), \check{s}) \in \bar{\zeta}_-, (\breve{\varrho}_2, \check{t}) \in \bar{\zeta}_- \Rightarrow (\breve{\varrho}_0 \between (\breve{\varrho}_1 \between (\breve{\varrho}_1 \between \breve{\varrho}_0)), \check{s} \vee \check{t}) \in \bar{\zeta}_-$,
(iii) $((\breve{\varrho}_0 \between \breve{\varrho}_1) \between \breve{\varrho}_2), \check{u}) \in \bar{\zeta}_+, (\breve{\varrho}_2, \check{v}) \in \bar{\zeta}_+ \Rightarrow (\breve{\varrho}_0 \between (\breve{\varrho}_1 \between (\breve{\varrho}_1 \between \breve{\varrho}_0)), \check{u} \wedge \check{v}) \in \bar{\zeta}_+$, *for all* $\breve{\varrho}_0, \check{s}, \check{t} \in [-1, 0)\breve{\varrho}_1, \breve{\varrho}_2 \in \check{\aleph}$ *and* $\check{u}, \check{v} \in (0, 1]$.

Proof. Suppose that Definition 5 (i) is hold and $\breve{\varrho}_0 \in \check{\aleph}$, $\check{u} \in (0, 1]$, $\check{s} \in [-1, 0)$ such that $(\breve{\varrho}_0, \check{s}) \in \bar{\zeta}_-$ and $(\breve{\varrho}_0, \check{u}) \in \bar{\zeta}_+$. Then

$$\bar{\zeta}_-(0) \leq \bar{\zeta}_-(\breve{\varrho}_0) \leq \check{s} \quad \text{and} \quad \bar{\zeta}_+(0) \geq \bar{\zeta}_+(\breve{\varrho}_0) \geq \check{u},$$

and so

$$(0, \check{s}) \in \bar{\zeta}_- and (0, \check{u}) \in \bar{\zeta}_+.$$

Since $(\breve{\varrho}_0, \bar{\zeta}(\breve{\varrho}_0)) \in \bar{\zeta}_-$ and $(\breve{\varrho}_0, \bar{\zeta}(\breve{\varrho}_0)) \in \bar{\zeta}_+$ for all $\breve{\varrho}_0 \in \check{\aleph}$, it follows from (i) that $(0, \bar{\zeta}(\breve{\varrho}_0)) \in \bar{\zeta}_-$ and $(0, \bar{\zeta}(\breve{\varrho}_0)) \in \bar{\zeta}_+$ so that $\bar{\zeta}_-(0) \leq \bar{\zeta}_-(\breve{\varrho}_0)$ and $\bar{\zeta}_+(0) \geq \bar{\zeta}_+(\breve{\varrho}_0)$ for all $\breve{\varrho}_0 \in \check{\aleph}$. Assume that Definition 7 holds.

Let $\breve{\varrho}_0, \breve{\varrho}_1, \breve{\varrho}_2 \in \check{\aleph}$, and $\check{s}, \check{t} \in [-1, 0)$, $\check{u}, \check{v} \in (0, 1]$ be such that $((\breve{\varrho}_0 \between \breve{\varrho}_1) \between \breve{\varrho}_2, \check{s}) \in \bar{\zeta}_-$, $(\breve{\varrho}_2, \check{t}) \in \bar{\zeta}_-$, and $((\breve{\varrho}_0 \between \breve{\varrho}_1) \between \breve{\varrho}_2, \check{u}) \in \bar{\zeta}_+, (\breve{\varrho}_2, \check{v}) \in \bar{\zeta}_+$. Then $\bar{\zeta}_-((\breve{\varrho}_0 \between \breve{\varrho}_1) \between \breve{\varrho}_2) \leq \check{s}, \bar{\zeta}_-(\breve{\varrho}_2) \leq \check{t}$ and $\bar{\zeta}_+((\breve{\varrho}_0 \between \breve{\varrho}_1) \between \breve{\varrho}_2) \geq \check{u}, \bar{\zeta}_+(\breve{\varrho}_2) \geq \check{v}$. It follows from Definition 7,

$$\bar{\zeta}_-(\breve{\varrho}_0 \between (\breve{\varrho}_1 \between (\breve{\varrho}_1 \between \breve{\varrho}_0))) \leq \bar{\zeta}_-((\breve{\varrho}_0 \between \breve{\varrho}_1) \between \breve{\varrho}_2)) \vee \bar{\zeta}_-(\breve{\varrho}_2) \leq \check{s} \vee \check{t}$$

and

$$\bar{\zeta}_+(\breve{\varrho}_0 \between (\breve{\varrho}_1 \between (\breve{\varrho}_1 \between \breve{\varrho}_0))) \geq \bar{\zeta}_+((\breve{\varrho}_0 \between \breve{\varrho}_1) \between \breve{\varrho}_2)) \wedge \bar{\zeta}_+(\breve{\varrho}_2) \geq \breve{u} \wedge \breve{v}.$$

So, that $(\breve{\varrho}_0 \between (\breve{\varrho}_1 \between (\breve{\varrho}_1 \between \breve{\varrho}_0)), \breve{s} \vee \breve{t}) \in \bar{\zeta}_-$ and $(\breve{\varrho}_0 \between (\breve{\varrho}_1 \between (\breve{\varrho}_1 \between \breve{\varrho}_0)), \breve{u} \wedge \breve{v}) \in \bar{\zeta}_+$.

Next, suppose that (ii) and (iii) are holds. For every $\breve{\varrho}_0, \breve{\varrho}_1, \breve{\varrho}_2 \in \breve{\aleph}$, $((\breve{\varrho}_0 \between \breve{\varrho}_1) \between \breve{\varrho}_2, \bar{\zeta}_-((\breve{\varrho}_0 \between \breve{\varrho}_1) \between \breve{\varrho}_2)) \in \bar{\zeta}_-, (\breve{\varrho}_2, \bar{\zeta}_-(\breve{\varrho}_2)) \in \bar{\zeta}_-$ and $((\breve{\varrho}_0 \between \breve{\varrho}_1) \between \breve{\varrho}_2, \bar{\zeta}_+((\breve{\varrho}_0 \between \breve{\varrho}_1) \between \breve{\varrho}_2)) \in \bar{\zeta}_+, (\breve{\varrho}_2, \bar{\zeta}_+(\breve{\varrho}_2))$. Hence, $(\breve{\varrho}_0 \between (\breve{\varrho}_1 \between (\breve{\varrho}_1 \between \breve{\varrho}_0)), \bar{\zeta}_-((\breve{\varrho}_0 \between \breve{\varrho}_1) \between \breve{\varrho}_2) \vee \bar{\zeta}_-(\breve{\varrho}_2)) \in \bar{\zeta}_-$ and $(\breve{\varrho}_0 \between (\breve{\varrho}_1 \between (\breve{\varrho}_1 \between \breve{\varrho}_0)), \bar{\zeta}_+((\breve{\varrho}_0 \between \breve{\varrho}_1) \between \breve{\varrho}_2) \wedge \bar{\zeta}_+(\breve{\varrho}_2)) \in \bar{\zeta}_+$ by (ii), and (iii), respectively and thus,

$$\bar{\zeta}_-(\breve{\varrho}_0 \between (\breve{\varrho}_1 \between (\breve{\varrho}_1 \between \breve{\varrho}_0))) \leq \bar{\zeta}_-((\breve{\varrho}_0 \between \breve{\varrho}_1) \between \breve{\varrho}_2) \vee \bar{\zeta}_-(\breve{\varrho}_2),$$

and

$$\bar{\zeta}_+(\breve{\varrho}_0 \between (\breve{\varrho}_1 \between (\breve{\varrho}_1 \between \breve{\varrho}_0))) \geq \bar{\zeta}_+((\breve{\varrho}_0 \between \breve{\varrho}_1) \between \breve{\varrho}_2) \wedge \bar{\zeta}_+(\breve{\varrho}_2).$$

Theorem 3. *A $\mathcal{BFS}$ $\bar{\zeta}$ is an $(\in, \in \vee(\varphi^*, \breve{q_\varphi}))$-$\mathcal{BFFI}$ of $\breve{\aleph}$ $\Leftrightarrow$ satisfies the succeeding assertions:*
(i) $\bar{\zeta}_-(0) \leq \bar{\zeta}_-(\breve{\varrho}_0) \vee (\frac{\varphi}{2} - \frac{\varphi^*}{2})$ and $\bar{\zeta}_+(0) \geq \bar{\zeta}_+(\breve{\varrho}_0) \wedge (-\frac{\varphi}{2} + \frac{\varphi^*}{2})$,
(ii) $\bar{\zeta}_-(\breve{\varrho}_0 \between (\breve{\varrho}_1 \between (\breve{\varrho}_1 \between \breve{\varrho}_0))) \leq \bar{\zeta}_-((\breve{\varrho}_0 \between \breve{\varrho}_1) \between \breve{\varrho}_2) \vee \bar{\zeta}_-(\breve{\varrho}_2) \vee (\frac{\varphi}{2} - \frac{\varphi^*}{2})$,
(iii) $\bar{\zeta}_+(\breve{\varrho}_0 \between (\breve{\varrho}_1 \between (\breve{\varrho}_1 \between \breve{\varrho}_0))) \geq \bar{\zeta}_+((\breve{\varrho}_0 \between \breve{\varrho}_1) \between \breve{\varrho}_2) \wedge \bar{\zeta}_+(\breve{\varrho}_2) \wedge (-\frac{\varphi}{2} + \frac{\varphi^*}{2})$
for all $\breve{\varrho}_0, \breve{\varrho}_1, \breve{\varrho}_2 \in \breve{\aleph}$.

Proof. Suppose $\bar{\zeta}$ be an $(\in, \in \vee(\varphi^*, \breve{q_\varphi}))$-$\mathcal{BFFI}$ of $\breve{\aleph}$. Let $\breve{\varrho}_0 \in \breve{\aleph}$ be such that $\bar{\zeta}_-(\breve{\varrho}_0) > (\frac{\varphi}{2} - \frac{\varphi^*}{2})$ and $\bar{\zeta}_+(\breve{\varrho}_0) < (-\frac{\varphi}{2} + \frac{\varphi^*}{2})$. If $\bar{\zeta}_-(0) > \bar{\zeta}_-(\breve{\varrho}_0)$ and $\bar{\zeta}_+(0) < \bar{\zeta}_+(\breve{\varrho}_0)$, $\bar{\zeta}_-(0) > \breve{s} > \bar{\zeta}_-(\breve{\varrho}_0)$ and $\bar{\zeta}_+(0) < \breve{u} < \bar{\zeta}_+(\breve{\varrho}_0)$ for every $\breve{s} \in (\frac{\varphi}{2} - \frac{\varphi^*}{2}, 0)$ and $\breve{u} \in (0, -\frac{\varphi}{2} + \frac{\varphi^*}{2})$, so we get $(\breve{\varrho}_0, \breve{s}) \in \bar{\zeta}_-$, $(0, \breve{s})\overline{\in}\bar{\zeta}_-$ and $(\breve{\varrho}_0, \breve{u}) \in \bar{\zeta}_+$, $(0, \breve{v})\overline{\in}\bar{\zeta}_+$.

Since $\bar{\zeta}_-(0) + \breve{s} > \varphi - \varphi^*$ and $\bar{\zeta}_+(0) + \breve{u} < -\varphi + \varphi^*$, so we have $(0, \breve{s})\overline{\breve{q_\varphi}}\bar{\zeta}_-$ and $(0, \breve{u})\overline{\breve{q_\varphi}}\bar{\zeta}_+$. It follows that $(0, \breve{s})\overline{\in \vee \breve{q_\varphi}}\bar{\zeta}_-$ and $(0, \breve{u})\overline{\in \vee \breve{q_\varphi}}\bar{\zeta}_+$, a contradiction. Hence, $\bar{\zeta}_-(0) \leq \bar{\zeta}_-(\breve{\varrho}_0)$ and $\bar{\zeta}_+(0) \geq \bar{\zeta}_+(\breve{\varrho}_0)$. Now if $\bar{\zeta}_-(\breve{\varrho}_0) \leq (\frac{\varphi}{2} - \frac{\varphi^*}{2})$ and $\bar{\zeta}_+(\breve{\varrho}_0) \geq (-\frac{\varphi}{2} + \frac{\varphi^*}{2})$, then $(\breve{\varrho}_0, (\frac{\varphi}{2} - \frac{\varphi^*}{2})) \in \bar{\zeta}_-$ and $(\breve{\varrho}_0, (-\frac{\varphi}{2} + \frac{\varphi^*}{2})) \in \bar{\zeta}_+$. Thus, $(0, \frac{\varphi}{2} - \frac{\varphi^*}{2})) \in \vee\breve{q_\varphi}\bar{\zeta}_-$ and $(0, -\frac{\varphi}{2} + \frac{\varphi^*}{2}) \in \vee\breve{q_\varphi}\bar{\zeta}_+$. Thus, $\bar{\zeta}_-(0) \leq \frac{\varphi}{2} - \frac{\varphi^*}{2}$ and $\bar{\zeta}_+(0) \geq -\frac{\varphi}{2} + \frac{\varphi^*}{2}$.

Otherwise, $\bar{\zeta}_-(\breve{\varrho}_0) + \frac{\varphi}{2} - \frac{\varphi^*}{2} > \frac{\varphi}{2} - \frac{\varphi^*}{2} + \frac{\varphi}{2} - \frac{\varphi^*}{2} = \varphi - \varphi^*$ and $\bar{\zeta}_+(\breve{\varrho}_0) + -\frac{\varphi}{2} + \frac{\varphi^*}{2} < -\frac{\varphi}{2} + \frac{\varphi^*}{2} + -\frac{\varphi}{2} + \frac{\varphi^*}{2} = -\varphi + \varphi^*$, a contradiction. Consequently, $\bar{\zeta}_-(0) \leq \{\bar{\zeta}_-(\breve{\varrho}_0), \frac{\varphi}{2} - \frac{\varphi^*}{2}\}$ and $\bar{\zeta}_+(0) \geq \{\bar{\zeta}_+(\breve{\varrho}_0), -\frac{\varphi}{2} + \frac{\varphi^*}{2}\}$, for all $\breve{\varrho}_0 \in \breve{\aleph}$.

Let $\breve{\varrho}_0, \breve{\varrho}_1, \breve{\varrho}_2 \in \breve{\aleph}$. Suppose that $\bar{\zeta}_-((\breve{\varrho}_0 \between \breve{\varrho}_1) \between \breve{\varrho}_2) \vee \bar{\zeta}_-(\breve{\varrho}_2) > \frac{\varphi}{2} - \frac{\varphi^*}{2}$ and $\bar{\zeta}_+((\breve{\varrho}_0 \between \breve{\varrho}_1) \between \breve{\varrho}_2) \wedge \bar{\zeta}_+(\breve{\varrho}_2) < -\frac{\varphi}{2} + \frac{\varphi^*}{2}$. Then $\bar{\zeta}_-(\breve{\varrho}_0 \between (\breve{\varrho}_1 \between (\breve{\varrho}_1 \between \breve{\varrho}_0))) \leq \bar{\zeta}_-((\breve{\varrho}_0 \between \breve{\varrho}_1) \between \breve{\varrho}_2) \vee \bar{\zeta}_-(\breve{\varrho}_2)$ and $\bar{\zeta}_+(\breve{\varrho}_0 \between (\breve{\varrho}_1 \between (\breve{\varrho}_1 \between \breve{\varrho}_0))) \geq \bar{\zeta}_+((\breve{\varrho}_0 \between \breve{\varrho}_1) \between \breve{\varrho}_2) \wedge \bar{\zeta}_+(\breve{\varrho}_2)$.
If not, then $\bar{\zeta}_-(\breve{\varrho}_0 \between (\breve{\varrho}_1 \between (\breve{\varrho}_1 \between \breve{\varrho}_0))) > \breve{s} > \bar{\zeta}_-((\breve{\varrho}_0 \between \breve{\varrho}_1) \between \breve{\varrho}_2) \vee \bar{\zeta}_-(\breve{\varrho}_2)$ and $\bar{\zeta}_+(\breve{\varrho}_0 \between (\breve{\varrho}_1 \between (\breve{\varrho}_1 \between \breve{\varrho}_0))) < \breve{u} < \bar{\zeta}_+((\breve{\varrho}_0 \between \breve{\varrho}_1) \between \breve{\varrho}_2) \wedge \bar{\zeta}_+(\breve{\varrho}_2)$, for some $\breve{s} \in (\frac{\varphi}{2} - \frac{\varphi^*}{2}, 0), \breve{u} \in (0, -\frac{\varphi}{2} + \frac{\varphi^*}{2})$.

It follows that $((\breve{\varrho}_0 \between \breve{\varrho}_1) \between \breve{\varrho}_2, \breve{s}) \in \bar{\zeta}_-$ and $(\breve{\varrho}_2, \breve{s}) \in \bar{\zeta}_-$ but $(\breve{\varrho}_0 \between (\breve{\varrho}_1 \between (\breve{\varrho}_1 \between \breve{\varrho}_0)), \breve{s} \vee \breve{s}) = (\breve{\varrho}_0 \between (\breve{\varrho}_1 \between (\breve{\varrho}_1 \between \breve{\varrho}_0)), \breve{s}) \in \vee \breve{q}_\varphi \bar{\zeta}_-$ and $((\breve{\varrho}_0 \between \breve{\varrho}_1) \between \breve{\varrho}_2, \breve{u}) \in \bar{\zeta}_+$ and $(\breve{\varrho}_2, \breve{u}) \in \bar{\zeta}_+$ but $(\breve{\varrho}_0 \between (\breve{\varrho}_1 \between (\breve{\varrho}_1 \between \breve{\varrho}_0)), \breve{u} \vee \breve{u}) = (\breve{\varrho}_0 \between (\breve{\varrho}_1 \between (\breve{\varrho}_1 \between \breve{\varrho}_0)), \breve{u}) \in \vee \breve{q}_\varphi \bar{\zeta}_+$ which is a contradiction.

Hence, $\bar{\zeta}_-(\breve{\varrho}_0 \between (\breve{\varrho}_1 \between (\breve{\varrho}_1 \between \breve{\varrho}_0))) \leq \bar{\zeta}_-((\breve{\varrho}_0 \between \breve{\varrho}_1) \between \breve{\varrho}_2) \vee \bar{\zeta}_-(\breve{\varrho}_2)$ whenever $\bar{\zeta}_-((\breve{\varrho}_0 \between \breve{\varrho}_1) \between \breve{\varrho}_2) \vee \bar{\zeta}_-(\breve{\varrho}_2) > \frac{\varphi}{2} - \frac{\varphi^*}{2}$ and $\bar{\zeta}_+(\breve{\varrho}_0 \between (\breve{\varrho}_1 \between (\breve{\varrho}_1 \between \breve{\varrho}_0))) \geq \bar{\zeta}_+((\breve{\varrho}_0 \between \breve{\varrho}_1) \between \breve{\varrho}_2) \wedge \bar{\zeta}_+(\breve{\varrho}_2)$ whenever $\bar{\zeta}_+((\breve{\varrho}_0 \between \breve{\varrho}_1) \between \breve{\varrho}_2) \wedge \bar{\zeta}_+(\breve{\varrho}_2) < -\frac{\varphi}{2} + \frac{\varphi^*}{2}$.

If $\bar{\zeta}_-((\breve{\varrho}_0 \between \breve{\varrho}_1) \between \breve{\varrho}_2) \vee \bar{\zeta}_-(\breve{\varrho}_2) \leq \frac{\varphi}{2} - \frac{\varphi^*}{2}$, then $((\breve{\varrho}_0 \between \breve{\varrho}_1) \between \breve{\varrho}_2, \frac{\varphi}{2} - \frac{\varphi^*}{2}) \in \bar{\zeta}_-$ and $(\breve{\varrho}_2, \frac{\varphi}{2} - \frac{\varphi^*}{2}) \in \bar{\zeta}_-$, which imply that $(\breve{\varrho}_0 \between (\breve{\varrho}_1 \between (\breve{\varrho}_1 \between \breve{\varrho}_0)), \frac{\varphi}{2} - \frac{\varphi^*}{2}) = (\breve{\varrho}_0 \between (\breve{\varrho}_1 \between (\breve{\varrho}_1 \between \breve{\varrho}_0)), \frac{\varphi}{2} - \frac{\varphi^*}{2}) \vee \frac{\varphi}{2} - \frac{\varphi^*}{2}) \in \vee \breve{q}_\varphi \bar{\zeta}_-$ and if $\bar{\zeta}_+((\breve{\varrho}_0 \between \breve{\varrho}_1) \between \breve{\varrho}_2) \wedge \bar{\zeta}_+(\breve{\varrho}_2) \geq \frac{\varphi}{2} - \frac{\varphi^*}{2}$, then $((\breve{\varrho}_0 \between \breve{\varrho}_1) \between \breve{\varrho}_2, \frac{\varphi}{2} - \frac{\varphi^*}{2}) \in \bar{\zeta}_+$ and $(\breve{\varrho}_2, -\frac{\varphi}{2} + \frac{\varphi^*}{2}) \in \bar{\zeta}_+$, which imply that $(\breve{\varrho}_0 \between (\breve{\varrho}_1 \between (\breve{\varrho}_1 \between \breve{\varrho}_0)), -\frac{\varphi}{2} + \frac{\varphi^*}{2}) = (\breve{\varrho}_0 \between (\breve{\varrho}_1 \between (\breve{\varrho}_1 \between \breve{\varrho}_0)), -\frac{\varphi}{2} + \frac{\varphi^*}{2}) \wedge (-\frac{\varphi}{2} + \frac{\varphi^*}{2}) \in \vee \breve{q}_\varphi \bar{\zeta}_+$.

Therefore, $\bar{\zeta}_-(\breve{\varrho}_0 \between (\breve{\varrho}_1 \between (\breve{\varrho}_1 \between \breve{\varrho}_0))) \leq \frac{\varphi}{2} - \frac{\varphi^*}{2}$ and $\bar{\zeta}_+(\breve{\varrho}_0 \between (\breve{\varrho}_1 \between (\breve{\varrho}_1 \between \breve{\varrho}_0))) \geq -\frac{\varphi}{2} + \frac{\varphi^*}{2}$, because if $\bar{\zeta}_-(\breve{\varrho}_0 \between (\breve{\varrho}_1 \between (\breve{\varrho}_1 \between \breve{\varrho}_0))) > \frac{\varphi}{2} - \frac{\varphi^*}{2}$ and $\bar{\zeta}_+(\breve{\varrho}_0 \between (\breve{\varrho}_1 \between (\breve{\varrho}_1 \between \breve{\varrho}_0))) < -\frac{\varphi}{2} + \frac{\varphi^*}{2}$, then $\bar{\zeta}_-(\breve{\varrho}_0 \between (\breve{\varrho}_1 \between (\breve{\varrho}_1 \between \breve{\varrho}_0))) + \frac{\varphi}{2} - \frac{\varphi^*}{2} > \frac{\varphi}{2} - \frac{\varphi^*}{2} + \frac{\varphi}{2} - \frac{\varphi^*}{2} = \varphi - \varphi^*$ and $\bar{\zeta}_+(\breve{\varrho}_0 \between (\breve{\varrho}_1 \between (\breve{\varrho}_1 \between \breve{\varrho}_0))) - \frac{\varphi}{2} + \frac{\varphi^*}{2} < -\frac{\varphi}{2} + \frac{\varphi^*}{2} - \frac{\varphi}{2} + \frac{\varphi^*}{2} = -\varphi + \varphi^*$, which is a contradiction. Hence,

$$\bar{\zeta}_-(\breve{\varrho}_0 \between (\breve{\varrho}_1 \between (\breve{\varrho}_1 \between \breve{\varrho}_0))) \leq \bar{\zeta}_-((\breve{\varrho}_0 \between \breve{\varrho}_1) \between \breve{\varrho}_2) \vee \bar{\zeta}_-(\breve{\varrho}_2) \vee (\tfrac{\varphi}{2} - \tfrac{\varphi^*}{2}),$$

and

$$\bar{\zeta}_+(\breve{\varrho}_0 \between (\breve{\varrho}_1 \between (\breve{\varrho}_1 \between \breve{\varrho}_0))) \geq \bar{\zeta}_+((\breve{\varrho}_0 \between \breve{\varrho}_1) \between \breve{\varrho}_2) \wedge \bar{\zeta}_+(\breve{\varrho}_2) \wedge (-\tfrac{\varphi}{2} + \tfrac{\varphi^*}{2}),$$

for all $\breve{\varrho}_0, \breve{\varrho}_1, \breve{\varrho}_2 \in \breve{\aleph}$.

Conversely, assume that $\bar{\zeta}$ satisfies the conditions of (i), (ii), and (iii). Let $\breve{\varrho}_0 \in \breve{\aleph}$ and $\breve{u} \in (0, 1]$ and $\breve{s} \in [-1, 0)$ be such that $(\breve{\varrho}_0, \breve{s}) \in \bar{\zeta}_-$ and $(\breve{\varrho}_0, \breve{u}) \in \bar{\zeta}_+$. Then, $\bar{\zeta}_-(\breve{\varrho}_0) \leq \breve{s}$ and $\bar{\zeta}_+(\breve{\varrho}_0) \geq \breve{u}$.

Suppose that $\bar{\zeta}_-(0) \geq \breve{s}$ and $\bar{\zeta}_+(0) \leq \breve{u}$. If $\bar{\zeta}_-(\breve{\varrho}_0) > \frac{\varphi}{2} - \frac{\varphi^*}{2}$ and $\bar{\zeta}_+(\breve{\varrho}_0) < -\frac{\varphi}{2} + \frac{\varphi^*}{2}$, then $\bar{\zeta}_-(0) \leq \bar{\zeta}_-(\breve{\varrho}_0) \vee (\frac{\varphi}{2} - \frac{\varphi^*}{2}) = \bar{\zeta}_-(\breve{\varrho}_0) \leq \breve{s}$ and $\bar{\zeta}_+(0) \geq \bar{\zeta}_+(\breve{\varrho}_0) \wedge (-\frac{\varphi}{2} + \frac{\varphi^*}{2}) = \bar{\zeta}_+(\breve{\varrho}_0) \geq \breve{u}$, a contradiction. Hence, we know that $\bar{\zeta}_-(\breve{\varrho}_0) \leq \frac{\varphi}{2} - \frac{\varphi^*}{2}$ and $\bar{\zeta}_+(\breve{\varrho}_0) \geq -\frac{\varphi}{2} + \frac{\varphi^*}{2}$ and so we get

$$\bar{\zeta}_-(0) + \breve{s} < 2\bar{\zeta}_-(0) \leq \bar{\zeta}_-(\breve{\varrho}_0) \vee (\tfrac{\varphi}{2} - \tfrac{\varphi^*}{2}) = \varphi - \varphi^*$$

and

$$\bar{\zeta}_+(0) + \breve{\varrho}_2 > 2\bar{\zeta}_-(0) \geq \bar{\zeta}_+(\breve{\varrho}_0) \wedge (-\tfrac{\varphi}{2} + \tfrac{\varphi^*}{2}) = -\varphi + \varphi^*.$$

Thus, $(0, \breve{s}) \in \vee \bar{\zeta}_-$ and $(0, \breve{u}) \in \vee \bar{\zeta}_+$.

Let $\breve{\varrho}_0, \breve{\varrho}_1, \breve{\varrho}_2 \in \breve{\aleph}$, $\breve{u}, \breve{v} \in (0, 1]$ and $\breve{s}, \breve{t} \in [1, 0)$ be such that $((\breve{\varrho}_0 \between \breve{\varrho}_1) \between \breve{\varrho}_2, \breve{s}) \in \bar{\zeta}_-$, $(\breve{\varrho}_2, \breve{t}) \in \bar{\zeta}_-$ and $((\breve{\varrho}_0 \between \breve{\varrho}_1) \between \breve{\varrho}_2, \breve{u}) \in \bar{\zeta}_+$, $(\breve{\varrho}_1, \breve{v}) \in \bar{\zeta}_+$. Then $\bar{\zeta}_-((\breve{\varrho}_0 \between \breve{\varrho}_1) \between \breve{\varrho}_2) \leq \breve{s}$, $\bar{\zeta}_-(\breve{\varrho}_2) \leq \breve{t}$ and $\bar{\zeta}_+((\breve{\varrho}_0 \between \breve{\varrho}_1) \between \breve{\varrho}_2) \geq \breve{u}$, $\bar{\zeta}_+(\breve{\varrho}_1) \geq \breve{v}$.

Suppose that $\bar{\zeta}_-(\breve{\varrho}_0 \between (\breve{\varrho}_1 \between (\breve{\varrho}_1 \between \breve{\varrho}_0))) > \breve{s} \vee \breve{t}$ and $\bar{\zeta}_+(\breve{\varrho}_0 \between (\breve{\varrho}_1 \between (\breve{\varrho}_1 \between \breve{\varrho}_0))) < \breve{u} \wedge \breve{v}$. If $\bar{\zeta}_-((\breve{\varrho}_0 \between \breve{\varrho}_1) \between \breve{\varrho}_2) \vee \bar{\zeta}_-(\breve{\varrho}_2) \geq \frac{\varphi}{2} - \frac{\varphi^*}{2}$ and $\bar{\zeta}_+((\breve{\varrho}_0 \between \breve{\varrho}_1) \between \breve{\varrho}_2) \wedge \bar{\zeta}_+(\breve{\varrho}_1) \leq -\frac{\varphi}{2} + \frac{\varphi^*}{2}$. Then

$$\begin{aligned}
\bar{\zeta}_-(\breve{\varrho}_0 \between (\breve{\varrho}_1 \between (\breve{\varrho}_1 \between \breve{\varrho}_0))) &\leq \bar{\zeta}_-((\breve{\varrho}_0 \between \breve{\varrho}_1) \between \breve{\varrho}_2) \vee \bar{\zeta}_-(\breve{\varrho}_2) \vee (\frac{\varphi}{2} - \frac{\varphi^*}{2}) \\
&\leq \bar{\zeta}_-((\breve{\varrho}_0 \between \breve{\varrho}_1) \between \breve{\varrho}_2) \vee \bar{\zeta}_-(\breve{\varrho}_2) \\
&\leq \breve{s} \vee \breve{t}
\end{aligned}$$

and

$$\begin{aligned}
\bar{\zeta}_+(\breve{\varrho}_0 \between (\breve{\varrho}_1 \between (\breve{\varrho}_1 \between \breve{\varrho}_0))) &\geq \bar{\zeta}_+((\breve{\varrho}_0 \between \breve{\varrho}_1) \between \breve{\varrho}_2) \wedge \bar{\zeta}_+(\breve{\varrho}_2) \wedge (-\frac{\varphi}{2} + \frac{\varphi^*}{2}) \\
&\geq \bar{\zeta}_+((\breve{\varrho}_0 \between \breve{\varrho}_1) \between \breve{\varrho}_2) \wedge \bar{\zeta}_+(\breve{\varrho}_2) \\
&\geq \breve{u} \wedge \breve{v},
\end{aligned}$$

a contradiction. Thus, $\bar{\zeta}_-((\breve{\varrho}_0 \between \breve{\varrho}_1) \between \breve{\varrho}_2) \vee \bar{\zeta}_-(\breve{\varrho}_2) \leq \frac{\varphi}{2} - \frac{\varphi^*}{2}$ and $\bar{\zeta}_+((\breve{\varrho}_0 \between \breve{\varrho}_1) \between \breve{\varrho}_2) \wedge \bar{\zeta}_+(\breve{\varrho}_2) \geq -\frac{\varphi}{2} + \frac{\varphi^*}{2}$. In that case

$$\begin{aligned}
\bar{\zeta}_-(\breve{\varrho}_0 \between (\breve{\varrho}_1 \between (\breve{\varrho}_1 \between \breve{\varrho}_0))) + \breve{s} \vee \breve{t} &< 2\bar{\zeta}_-(\breve{\varrho}_0 \between (\breve{\varrho}_1 \between (\breve{\varrho}_1 \between \breve{\varrho}_0))) \\
&\leq 2((\bar{\zeta}_-(\breve{\varrho}_0 \between \breve{\varrho}_1) \between \breve{\varrho}_2) \vee \bar{\zeta}_-(\breve{\varrho}_2) \vee \frac{\varphi}{2} - \frac{\varphi^*}{2}) \\
&= \varphi - \varphi^*,
\end{aligned}$$

and

$$\begin{aligned}
\bar{\zeta}_-(\breve{\varrho}_0 \between (\breve{\varrho}_1 \between (\breve{\varrho}_1 \between \breve{\varrho}_0))) + \breve{u} \wedge \breve{v} &> 2\bar{\zeta}_+(\breve{\varrho}_0 \between (\breve{\varrho}_1 \between (\breve{\varrho}_1 \between \breve{\varrho}_0))) \\
&\geq 2(\bar{\zeta}_+((\breve{\varrho}_0 \between \breve{\varrho}_1) \between \breve{\varrho}_2) \wedge \bar{\zeta}_+(\breve{\varrho}_2) \wedge -\frac{\varphi}{2} + \frac{\varphi^*}{2}) \\
&= -\varphi + \varphi^*.
\end{aligned}$$

Hence, $(\breve{\varrho}_0 \between (\breve{\varrho}_1 \between (\breve{\varrho}_1 \between \breve{\varrho}_0)), \breve{s} \vee \breve{t}) \in \vee(k^*, \breve{q_\varphi})\bar{\zeta}_-$ and $(\breve{\varrho}_0 \between (\breve{\varrho}_1 \between (\breve{\varrho}_1 \between \breve{\varrho}_0)), \breve{u} \wedge \breve{v}) \in \vee(k^*, \breve{q_\varphi})\bar{\zeta}_+$. So, $\bar{\zeta}$ is an $(\in, \in \vee(\varphi^*, \breve{q_\varphi}))$-$\mathcal{BFFI}$ of $\breve{\aleph}$.

Definition 9. *Let $\bar{\zeta}$ be a $\mathcal{BFS}$ of $\breve{\aleph}$ and $(\breve{s}, \breve{u}) \in [-1, 0] \times [0, 1]$, we define $U(\bar{\zeta}; \breve{s}, \breve{u}) = \{\breve{\varrho}_0 \in \breve{\aleph} \mid \bar{\zeta}_-(\breve{\varrho}_0) \leq \breve{s}$ and $\bar{\zeta}_+(\breve{\varrho}_0) \geq \breve{u}\}$ is called a $\breve{s}$-level cut of $\bar{\zeta}_-$ and $\breve{u}$-level cut of $\bar{\zeta}_+$ of the $\mathcal{BFS}$ $\bar{\zeta}$.*

Theorem 4. *A $\mathcal{BFS}$ $\bar{\zeta}$ is an $(\in, \in \vee(\varphi^*, \breve{q_\varphi}))$-$\mathcal{BFI}$ of $\breve{\aleph}$ $\Leftrightarrow$ the level subset*

$$U(\bar{\zeta}; \breve{s}, \breve{u}) = \{\breve{\varrho}_0, \breve{\varrho}_2 \in \breve{\aleph} \mid \bar{\zeta}_-(\breve{\varrho}_0 \between (\breve{\varrho}_1 \between (\breve{\varrho}_1 \between \breve{\varrho}_0))) \leq \breve{s} \text{ and } \bar{\zeta}_+(\breve{\varrho}_0 \between (\breve{\varrho}_1 \between (\breve{\varrho}_1 \between \breve{\varrho}_0))) \geq \breve{u}\}$$

is a $\mathcal{BFFI}$ of $\breve{\aleph}$ for all $\breve{s} \in [\frac{\varphi}{2} - \frac{\varphi^*}{2}, 0)$ and for all $\breve{u} \in (0, -\frac{\varphi}{2} + \frac{\varphi^*}{2}]$.

Proof. Assume that $\bar{\zeta}$ is an $(\in, \in \vee(\varphi^{*}, \breve{q_{\varphi}}))$-$\mathcal{BFI}$ of $\breve{\aleph}$. Let $(\breve{\varrho}_0 \between \breve{\varrho}_1) \between \breve{\varrho}_2, \breve{\varrho}_2 \in U(\bar{\zeta}; \breve{s}, \breve{u})$ with $\breve{s} \in [\frac{\varphi}{2} - \frac{\varphi^{*}}{2}, 0)$ and $\breve{u} \in (0, -\frac{\varphi}{2} + \frac{\varphi^{*}}{2})$. Then $\bar{\zeta}_{-}((\breve{\varrho}_0 \between \breve{\varrho}_1) \between \breve{\varrho}_2) \leq \breve{s}, \bar{\zeta}_{-}(\breve{\varrho}_2) \leq \breve{s}$ and $\bar{\zeta}_{+}((\breve{\varrho}_0 \between \breve{\varrho}_1) \between \breve{\varrho}_2) \geq \breve{u}, \bar{\zeta}_{+}(\breve{\varrho}_2) \geq \breve{u}$. Therefore from Theorem 1 that

$$\begin{aligned}\bar{\zeta}_{-}(\breve{\varrho}_0 \between (\breve{\varrho}_1 \between (\breve{\varrho}_1 \between \breve{\varrho}_0))) &\leq \bar{\zeta}_{-}((\breve{\varrho}_0 \between \breve{\varrho}_1) \between \breve{\varrho}_2) \vee \bar{\zeta}_{-}(\breve{\varrho}_2) \vee (\frac{\varphi}{2} - \frac{\varphi^{*}}{2})\\ &\leq \breve{s} \vee (\frac{\varphi}{2} - \frac{\varphi^{*}}{2})\\ &= \breve{s}\end{aligned}$$

and

$$\begin{aligned}\bar{\zeta}_{+}(\breve{\varrho}_0 \between (\breve{\varrho}_1 \between (\breve{\varrho}_1 \between \breve{\varrho}_0))) &\geq \bar{\zeta}_{+}((\breve{\varrho}_0 \between \breve{\varrho}_1) \between \breve{\varrho}_2) \wedge \bar{\zeta}_{+}(\breve{\varrho}_2) \wedge (-\frac{\varphi}{2} + \frac{\varphi^{*}}{2})\\ &\geq \breve{u} \wedge (-\frac{\varphi}{2} + \frac{\varphi^{*}}{2})\\ &= \breve{u},\end{aligned}$$

so that $\breve{\varrho}_0 \between (\breve{\varrho}_1 \between (\breve{\varrho}_1 \between \breve{\varrho}_0)) \in U(\bar{\zeta}; \breve{s}, \breve{u})$. Therefore, $U(\bar{\zeta}; \breve{s}, \breve{u})$ is a fantastic ideal of $\breve{\aleph}$.

Conversely, let $\bar{\zeta}$ be a $\mathcal{BFS}$ of $\breve{\aleph}$ be such that $U(\bar{\zeta}; \breve{s}, \breve{u}) = \{\breve{\varrho}_0 \in \breve{\aleph} \mid \bar{\zeta}_{-} \leq \breve{s} \textit{ and } \bar{\zeta}_{+} \geq \breve{u}\}$ is a fantastic ideal of $\breve{\aleph}$ for all $\breve{s} \in [\frac{\varphi}{2} - \frac{\varphi^{*}}{2}, 0)$ and $\breve{u} \in (0, -\frac{\varphi}{2} + \frac{\varphi^{*}}{2}]$. If there exist $(\breve{\varrho}_0 \between \breve{\varrho}_1) \between \breve{\varrho}_2, \breve{\varrho}_2 \in \breve{\aleph}$ such that $\bar{\zeta}_{-}(\breve{\varrho}_0 \between (\breve{\varrho}_1 \between (\breve{\varrho}_1 \between \breve{\varrho}_0))) > \bar{\zeta}_{-}((\breve{\varrho}_0 \between \breve{\varrho}_1) \between \breve{\varrho}_2) \vee \bar{\zeta}_{-}(\breve{\varrho}_2) \vee (\frac{\varphi}{2} - \frac{\varphi^{*}}{2})$ and $\bar{\zeta}_{+}(\breve{\varrho}_0 \between (\breve{\varrho}_1 \between (\breve{\varrho}_1 \between \breve{\varrho}_0))) < \bar{\zeta}_{+}((\breve{\varrho}_0 \between \breve{\varrho}_1) \between \breve{\varrho}_2) \wedge \bar{\zeta}_{+}(\breve{\varrho}_2) \wedge (-\frac{\varphi}{2} + \frac{\varphi^{*}}{2})$, then we take $\breve{s} \in (-1, 0)$ and $\breve{u} \in (0, 1)$ such that $\bar{\zeta}_{-}(\breve{\varrho}_0 \between (\breve{\varrho}_1 \between (\breve{\varrho}_1 \between \breve{\varrho}_0))) > \breve{s} > \bar{\zeta}_{-}((\breve{\varrho}_0 \between \breve{\varrho}_1) \between \breve{\varrho}_2) \vee \bar{\zeta}_{-}(\breve{\varrho}_2) \vee (\frac{\varphi}{2} - \frac{\varphi^{*}}{2})$ and $\bar{\zeta}_{+}(\breve{\varrho}_0 \between (\breve{\varrho}_1 \between (\breve{\varrho}_1 \between \breve{\varrho}_0))) < \breve{u} < \bar{\zeta}_{+}((\breve{\varrho}_0 \between \breve{\varrho}_1) \between \breve{\varrho}_2) \wedge \bar{\zeta}_{+}(\breve{\varrho}_1) \wedge (-\frac{\varphi}{2} + \frac{\varphi^{*}}{2})$.
Thus, $(\breve{\varrho}_0 \between \breve{\varrho}_1) \between \breve{\varrho}_2), \breve{\varrho}_1 \in U(\bar{\zeta}; \breve{s}, \breve{u})$ with $\breve{u} < -\frac{\varphi}{2} + \frac{\varphi^{*}}{2}$ and $\breve{s} > -\frac{\varphi}{2} + \frac{\varphi^{*}}{2}$, and so $\breve{\varrho}_0 \between (\breve{\varrho}_1 \between (\breve{\varrho}_1 \between \breve{\varrho}_0)) \in U(\bar{\zeta}; \breve{s}, \breve{u})$, i.e., $\bar{\zeta}_{-}(\breve{\varrho}_0 \between (\breve{\varrho}_1 \between (\breve{\varrho}_1 \between \breve{\varrho}_0))) \leq \breve{s} \textit{ and } \bar{\zeta}_{+}(\breve{\varrho}_0 \between (\breve{\varrho}_1 \between (\breve{\varrho}_1 \between \breve{\varrho}_0))) \geq \breve{u}$ which is a contradiction. Therefore,

$$\bar{\zeta}_{-}(\breve{\varrho}_0 \between (\breve{\varrho}_1 \between (\breve{\varrho}_1 \between \breve{\varrho}_0))) \leq \bar{\zeta}_{-}((\breve{\varrho}_0 \between \breve{\varrho}_1) \between \breve{\varrho}_2) \vee \bar{\zeta}_{-}(\breve{\varrho}_2) \vee (\frac{\varphi}{2} - \frac{\varphi^{*}}{2})$$

and

$$\bar{\zeta}_{+}(\breve{\varrho}_0 \between (\breve{\varrho}_1 \between (\breve{\varrho}_1 \between \breve{\varrho}_0))) \geq \bar{\zeta}_{+}((\breve{\varrho}_0 \between \breve{\varrho}_1) \between \breve{\varrho}_2) \wedge \bar{\zeta}_{+}(\breve{\varrho}_2) \wedge (-\frac{\varphi}{2} + \frac{\varphi^{*}}{2}),$$

for all $\breve{\varrho}_0, \breve{\varrho}_1, \breve{\varrho}_2 \in \breve{\aleph}$. Using the Theorem 1, we conclude that $\bar{\zeta}$ is an $(\in, \in \vee(\varphi^{*}, \breve{q_{\varphi}}))$-$\mathcal{BFFI}$ of $\breve{\aleph}$.

Theorem 5. *Let $\bar{\zeta}$ be an $(\in, \in \vee(\varphi^{*}, \breve{q_{\varphi}}))$-$\mathcal{BFFI}$ of $\breve{\aleph}$, where $\bar{\zeta}_{-}(\breve{\varrho}_0 \between (\breve{\varrho}_1 \between (\breve{\varrho}_1 \between \breve{\varrho}_0))) > \frac{\varphi}{2} - \frac{1}{2}$ and $\bar{\zeta}_{+}(\breve{\varrho}_0 \between (\breve{\varrho}_1 \between (\breve{\varrho}_1 \between \breve{\varrho}_0))) < -\frac{\varphi}{2} + \frac{\varphi^{*}}{2}$ for all $\breve{\varrho}_0, \breve{\varrho}_1, \breve{\varrho}_2 \in \breve{\aleph}$. Then $\bar{\zeta}$ is an $(\in, \in)$-$\mathcal{BFFI}$ of $\breve{\aleph}$.*

Proof. The proof is simple with theorem 1.

Theorem 6. *Let* $\bigwedge$ *be an index set and* $\{(\bar{\zeta}_{i_-}, \bar{\zeta}_{i_+}) \mid i \in \bigwedge\}$ *be a family of* $(\in, \in \vee(\varphi^*, \check{q_\varphi}))$*-* $\mathcal{BFFI}$ *of* $\check{\aleph}$*. Then* $\bar{\zeta} = \bigcap_{i\in\bigwedge}(\bar{\zeta}_{i_-}, \bar{\zeta}_{i_+})$ *is an* $(\in, \in \vee(\varphi^*, \check{q_\varphi}))$*-*$\mathcal{BFFI}$ *of* $\check{\aleph}$*.*

Proof. Let us take $(\check{\varrho}_0 \between \check{\varrho}_1) \between \check{\varrho}_2, \check{\varrho}_2 \in \check{\aleph}$ and $\check{s}_1, \check{s}_2 \in [-1, 0)$, and $\check{u}_1, \check{u}_2 \in (0, 1]$ be such that $\bar{\zeta}_-((\check{\varrho}_0 \between \check{\varrho}_1) \between \check{\varrho}_2) \leq \check{s}_1$ and $\bar{\zeta}_-((\check{\varrho}_0 \between \check{\varrho}_1) \between \check{\varrho}_2) \leq \check{s}_2$, $\bar{\zeta}_+((\check{\varrho}_0 \between \check{\varrho}_1) \between \check{\varrho}_2) \geq \check{u}_1$ and $\bar{\zeta}_+((\check{\varrho}_0 \between \check{\varrho}_1) \between \check{\varrho}_2) \geq \check{u}_2$.

Assume that $\check{\varrho}_0 \between (\check{\varrho}_1 \between (\check{\varrho}_1 \between \check{\varrho}_0))_{\check{s}_1 \vee \check{s}_2} \overline{\in \vee(\varphi^*, q)\bar{\zeta}_-}$ and $\check{\varrho}_0 \between (\check{\varrho}_1 \between (\check{\varrho}_1 \between \check{\varrho}_0))_{\check{u}_1 \wedge \check{u}_2} \overline{\in \vee(\varphi^*, q)\bar{\zeta}_+}$. Then $\bar{\zeta}_-(\check{\varrho}_0 \between (\check{\varrho}_1 \between (\check{\varrho}_1 \between \check{\varrho}_0))) > \check{s}_1 \vee \check{s}_2$ and $\bar{\zeta}_-(\check{\varrho}_0 \between (\check{\varrho}_1 \between (\check{\varrho}_1 \between \check{\varrho}_0))) + \check{s}_1 \vee \check{s}_2 \geq \varphi - \varphi^*$, and $\bar{\zeta}_+(\check{\varrho}_0 \between (\check{\varrho}_1 \between (\check{\varrho}_1 \between \check{\varrho}_0))) < \check{u}_1 \wedge \check{u}_2$ and $\bar{\zeta}_+(\check{\varrho}_0 \between (\check{\varrho}_1 \between (\check{\varrho}_1 \between \check{\varrho}_0))) + \check{u}_1 \wedge \check{u}_2 \leq -\varphi + \varphi^*$, which implies

$$\bar{\zeta}_-(\check{\varrho}_0) > \frac{\varphi}{2} - \frac{\varphi^*}{2} \text{ and } \bar{\zeta}_+(\check{\varrho}_0) < \frac{\varphi}{2} - \frac{\varphi^*}{2}. \tag{1}$$

Now, we define

$$\Delta_1 = \{i \in \bigwedge \mid \check{\varrho}_0 \between (\check{\varrho}_1 \between (\check{\varrho}_1 \between \check{\varrho}_0))_{\check{s}_1 \vee \check{s}_2} \in \bar{\zeta}_{i_{\check{t}}} \text{ and } \check{\varrho}_0 \between (\check{\varrho}_1 \between (\check{\varrho}_1 \between \check{\varrho}_0))_{\check{u}_1 \wedge \check{u}_2} \in \bar{\zeta}_{i_{\check{p}}}\}$$

and
$\Delta_2 = \{[\{i \in \bigwedge \mid \check{\varrho}_0 \between (\check{\varrho}_1 \between (\check{\varrho}_1 \between \check{\varrho}_0))_{\check{s}_1 \vee \check{s}_2}(\varphi^*, \check{q}_\varphi)\bar{\zeta}_{i_-}\} \cap \{j \in \bigwedge \mid \check{\varrho}_0 \between (\check{\varrho}_1 \between (\check{\varrho}_1 \between \check{\varrho}_0))_{\check{s}_1 \vee \check{s}_2} \overline{\in \bar{\zeta}_{i_-}}\}]$ and $[\{i \in \bigwedge \mid \check{\varrho}_0 \between (\check{\varrho}_1 \between (\check{\varrho}_1 \between \check{\varrho}_0)_{\check{u}_1 \wedge \check{u}_2} q\bar{\zeta}_{i_+}\} \cap \{j \in \bigwedge \mid \check{\varrho}_0 \between (\check{\varrho}_1 \between (\check{\varrho}_1 \between \check{\varrho}_0)_{\check{u}_1 \wedge \check{u}_2} \overline{\in \bar{\zeta}_{i_+}}\}]\}$.
Then $\bigwedge = \Delta_1 \cup \Delta_2$ and $\Delta_1 \cap \Delta_2 = \emptyset$.
If $\Delta_2 = \emptyset$, then $\check{\varrho}_0 \between (\check{\varrho}_1 \between (\check{\varrho}_1 \between \check{\varrho}_0))_{\check{s}_1 \vee \check{s}_2} \in \bar{\zeta}_{i_-}$ and $\check{\varrho}_0 \between (\check{\varrho}_1 \between (\check{\varrho}_1 \between \check{\varrho}_0))_{\check{u}_1 \wedge \check{u}_2} \in \bar{\zeta}_{i_+}$ for all $i \in \bigwedge$,
i.e., $\bar{\zeta}_{i_-}(\check{\varrho}_0 \between (\check{\varrho}_1 \between (\check{\varrho}_1 \between \check{\varrho}_0))) \leq \check{s}_1 \vee \check{s}_2$ and $\bar{\zeta}_{i_+}(\check{\varrho}_0 \between (\check{\varrho}_1 \between (\check{\varrho}_1 \between \check{\varrho}_0))) \geq \check{u}_1 \wedge \check{u}_2$ for all for all $i \in \bigwedge$, which indicate

$$\bar{\zeta}_{i_-}(\check{\varrho}_0 \between (\check{\varrho}_1 \between (\check{\varrho}_1 \between \check{\varrho}_0))) \leq \check{s}_1 \vee \check{s}_2 \text{ and } \bar{\zeta}_{i_+}(\check{\varrho}_0 \between (\check{\varrho}_1 \between (\check{\varrho}_1 \between \check{\varrho}_0))) \geq \check{u}_1 \wedge \check{u}_2.$$

This is a contrary. Hence, for every $i \in \Delta_2$, and so, $\Delta_2 \neq \emptyset$, we have

$$\bar{\zeta}_{i_-}(\check{\varrho}_0 \between (\check{\varrho}_1 \between (\check{\varrho}_1 \between \check{\varrho}_0))) > \check{s}_1 \vee \check{s}_2 \text{ and } \bar{\zeta}_{i_-}(\check{\varrho}_0 \between (\check{\varrho}_1 \between (\check{\varrho}_1 \between \check{\varrho}_0))) + \check{s}_1 \vee \check{s}_2 < \varphi - \varphi^*, \text{ and}$$
$$\bar{\zeta}_{i_+}(\check{\varrho}_0 \between (\check{\varrho}_1 \between (\check{\varrho}_1 \between \check{\varrho}_0))) < \check{u}_1 \wedge \check{u}_2 \text{ and } \bar{\zeta}_{i_+}(\check{\varrho}_0 \between (\check{\varrho}_1 \between (\check{\varrho}_1 \between \check{\varrho}_0))) + \check{u}_1 \wedge \check{u}_2 > -\varphi + \varphi^*.$$

It follows that $\check{s}_1 \vee \check{s}_2 < \frac{\varphi}{2} - \frac{\varphi^*}{2}$ and $\check{u}_1 \wedge \check{u}_2 > -\frac{\varphi}{2} + \frac{\varphi^*}{2}$.
Now, $\check{\varrho}_0 \between (\check{\varrho}_1 \between (\check{\varrho}_1 \between \check{\varrho}_0))_{\check{s}_1} \in \bar{\zeta}_-$ and $\check{\varrho}_0 \between (\check{\varrho}_1 \between (\check{\varrho}_1 \between \check{\varrho}_0))_{\check{u}_1} \in \bar{\zeta}_+ \Rightarrow \bar{\zeta}_-(\check{\varrho}_0 \between (\check{\varrho}_1 \between (\check{\varrho}_1 \between \check{\varrho}_0))) \leq \check{s}_1$ and $\bar{\zeta}_+(\check{\varrho}_0 \between (\check{\varrho}_1 \between (\check{\varrho}_1 \between \check{\varrho}_0))) \geq \check{u}_1$, and thus,

$$\begin{aligned}\bar{\zeta}_{i_-}((\check{\varrho}_0 \between \check{\varrho}_1) \between \check{\varrho}_2) &\leq \bar{\zeta}_-((\check{\varrho}_0 \between \check{\varrho}_1) \between \check{\varrho}_2)\\ &\leq \check{s}_1\\ &\leq \check{s}_1 \vee \check{s}_2\\ &< \frac{\varphi}{2} - \frac{\varphi^*}{2}\end{aligned}$$

and

$$\begin{aligned}\bar{\zeta}_{i_{\check{p}}}((\check{\varrho}_0 \between \check{\varrho}_1) \between \check{\varrho}_2) &\geq \bar{\zeta}_+((\check{\varrho}_0 \between \check{\varrho}_1) \between \check{\varrho}_2)\\ &\geq \check{u}_1\\ &\geq \check{u}_1 \wedge \check{u}_2\\ &> -\frac{\varphi}{2} + \frac{\varphi^*}{2}\end{aligned}$$

for all $i \in \bigwedge$.

Similarly, we get $\bar{\zeta}_{i_-}((\check{\varrho}_0 \between \check{\varrho}_1) \between \check{\varrho}_2) < \frac{\varphi}{2} - \frac{\varphi^*}{2}$ and $\bar{\zeta}_{i_+}(\check{\varrho}_2) > -\frac{\varphi}{2} + \frac{\varphi^*}{2}$ for all $i \in \bigwedge$.

We suppose that $\check{s} = \bar{\zeta}_{i_-}((\check{\varrho}_0 \between \check{\varrho}_1) \between \check{\varrho}_2) > \frac{\varphi}{2} - \frac{\varphi^*}{2}$ and $\check{u} = \bar{\zeta}_{i_+}(\check{\varrho}_2) < -\frac{\varphi}{2} + \frac{\varphi^*}{2}$. Taking that $\check{s} > \check{t} > \frac{\varphi}{2} - \frac{\varphi^*}{2}$ and $\check{u} < \check{v} < -\frac{\varphi}{2} + \frac{\varphi^*}{2}$, we get

$$((\check{\varrho}_0 \between \check{\varrho}_1) \between \check{\varrho}_2)_{\check{t}} \in \bar{\zeta}_{i_-}(\check{\varrho}_0 \between (\check{\varrho}_1 \between (\check{\varrho}_1 \between \check{\varrho}_0)) \text{ and } \check{\varrho}_{2\check{t}} \in \bar{\zeta}_{i_-}(\check{\varrho}_0 \between \check{\varrho}_2), \text{ but}$$
$$\check{\varrho}_0 \between (\check{\varrho}_1 \between (\check{\varrho}_1 \between \check{\varrho}_0))_{\check{t}\vee\check{t}} = (\check{\varrho}_0 \between (\check{\varrho}_1 \between (\check{\varrho}_1 \between \check{\varrho}_0)))_{\check{t}} \overline{\in \vee(\varphi^*, q)}\bar{\zeta}_{i_-}$$

and

$$((\check{\varrho}_0 \between \check{\varrho}_1) \between \check{\varrho}_2)_{\check{v}} \in \bar{\zeta}_{i_{\check{p}}}(\check{\varrho}_0 \between (\check{\varrho}_1 \between (\check{\varrho}_1 \between \check{\varrho}_0))) \text{ and } \check{\varrho}_{1\check{v}} \in \bar{\zeta}_{i_{\check{p}}}(\check{\varrho}_0 \between (\check{\varrho}_1 \between (\check{\varrho}_1 \between \check{\varrho}_0))), \text{ but}$$
$$\check{\varrho}_0 \between (\check{\varrho}_1 \between (\check{\varrho}_1 \between \check{\varrho}_0))_{\check{v}\wedge\check{v}} = \check{\varrho}_{0\check{v}} \overline{\in \vee(\varphi^*, q)}\bar{\zeta}_{i_{\check{p}}}.$$

This contradicts that $\bar{\zeta}$ is an $(\in, \in \vee(\varphi^*, \check{q_\varphi})) - \mathcal{BFFI}$ of $\check{\aleph}$. Hence, $\bar{\zeta}_{i_+}(\check{\varrho}_0 \between (\check{\varrho}_1 \between (\check{\varrho}_1 \between \check{\varrho}_0))) \leq \frac{\varphi}{2} - \frac{\varphi^*}{2}$ and $\bar{\zeta}_{i_+}(\check{\varrho}_2) \geq -\frac{\varphi}{2} + \frac{\varphi^*}{2}$ for all $i \in \bigwedge$, so $\bar{\zeta}_-(\check{\varrho}_2) \leq \frac{\varphi}{2} - \frac{\varphi^*}{2}$ and $\bar{\zeta}_+(\check{g}) \geq (-\frac{\varphi}{2} + \frac{\varphi^*}{2})$ which contradicts 1. Therefore, $\check{\varrho}_0 \between (\check{\varrho}_1 \between (\check{\varrho}_1 \between \check{\varrho}_0))_{\check{s}_1\vee\check{s}_2} \in \vee(\varphi^*, q)\bar{\zeta}_-$ and $\check{\varrho}_0 \between (\check{\varrho}_1 \between (\check{\varrho}_1 \between \check{\varrho}_0))_{\check{u}_1\wedge\check{u}_2} \in \vee(\varphi^*, q)\bar{\zeta}_+$ and consequently, $\bar{\zeta}$ is an $(\in, \in \vee(\varphi^*, \check{q_\varphi}))$-$\mathcal{BFFI}$ of $\check{\aleph}$.

For any $\mathcal{BFS}$ $\bar{\zeta}$ in $\check{\aleph}$, where $\check{s} \in [1, 0)$ and $\check{u} \in (0, 1]$, we denote

$$\bar{\zeta}_{\check{s}_-} = \{\check{\varrho}_0, \check{\varrho}_2 \in \check{\aleph} \mid \check{\varrho}_0 \between (\check{\varrho}_1 \between (\check{\varrho}_1 \between \check{\varrho}_0))_{\check{s}} \in (\varphi^*, \check{q_\varphi})\bar{\zeta}_-\},$$
$$\bar{\zeta}_{\check{u}_+} = \{\check{\varrho}_0, \check{\varrho}_2 \in \check{\aleph} \mid \check{\varrho}_0 \between (\check{\varrho}_1 \between (\check{\varrho}_1 \between \check{\varrho}_0))_{\check{u}} \in (\varphi^*, \check{q_\varphi})\bar{\zeta}_+\},$$

and

$$[\bar{\zeta}]_{(\check{s},\check{u})} = \{\check{\varrho}_0, \check{\varrho}_2 \in \check{\aleph} \mid \check{\varrho}_0 \between (\check{\varrho}_1 \between (\check{\varrho}_1 \between \check{\varrho}_0)_{\check{s}} \in (\varphi^*, \check{q}_\varphi\bar{\zeta}_- \textit{ and } \check{\varrho}_0 \between (\check{\varrho}_1 \between (\check{\varrho}_1 \between \check{\varrho}_0)_{\check{u}} \in (\varphi^*, \check{q}_\varphi\bar{\zeta}_+\}.$$

Then it is obvious that $[\bar{\zeta}]_{(\check{s},\check{u})} = U(\bar{\zeta}; \check{s}, \check{u}) \cup \bar{\zeta}_{\check{s}_{\check{t}}} \cup \bar{\zeta}_{\check{v}_{\check{p}}}$. Here, $[\bar{\zeta}]_{(\check{s},\check{u})}$ is an $(\in, \in \vee(\varphi^*, \check{q_\varphi}))$-level fantastic ideal of $\bar{\zeta}$.

Theorem 7. *Let $\bar{\zeta}$ be a $\mathcal{BFS}$ in $\check{\aleph}$. Then $\bar{\zeta}$ is an $(\in, \in \vee(\varphi^*, \check{q_\varphi}))$-$\mathcal{BFFI}$ of $\check{\aleph}$ if and only if $[\bar{\zeta}]_{(\check{s},\check{u})}$ is a fantastic ideal of $\check{\aleph}$, for all $\check{s} \in [-1, 0)$ and $\check{u} \in (0, 1]$.*

Proof. Suppose that $\bar{\zeta}$ is an $(\in, \in \vee(\varphi^*, \check{q_\varphi}))$-$\mathcal{BFFI}$ of $\check{\aleph}$ and let $\check{\varrho}_0, \check{\varrho}_1, \check{\varrho}_2 \in [\bar{\zeta}]_{(\check{s},\check{u})}$ for $\check{s} \in [-1, 0)$ and $\check{u} \in (0, 1]$. Then $(\check{\varrho}_0 \circledast \check{\varrho}_1) \circledast \check{\varrho}_{2\check{s}} \in \check{q_\varphi}\bar{\zeta}_-, \check{\varrho}_{2\check{s}} \in (\varphi^*, \check{q}_\varphi)\bar{\zeta}_-$ and $(\check{\varrho}_0 \circledast \check{\varrho}_1) \circledast \check{\varrho}_{2\check{u}} \in (\varphi^*, \check{q}_\varphi)\bar{\zeta}_+, \check{\varrho}_{2\check{u}} \in (\varphi^*, \check{q}_\varphi)\bar{\zeta}_+$. That is $\bar{\zeta}_-((\check{\varrho}_0 \circledast \check{\varrho}_1) \circledast \check{\varrho}_2) \leq \check{s}$ or

$\bar{\zeta}_-((\breve{\varrho}_0 \divideontimes \breve{\varrho}_1) \divideontimes \breve{\varrho}_2) + \breve{s} < \varphi - \varphi^{\divideontimes}$, $\bar{\zeta}_-(\breve{\varrho}_2) \leq \breve{s}$ or $\bar{\zeta}_-(\breve{\varrho}_2) + \breve{s} < \varphi - \varphi^{\divideontimes}$ and $\bar{\zeta}_+((\breve{\varrho}_0 \divideontimes \breve{\varrho}_1) \divideontimes \breve{\varrho}_2) \geq \breve{u}$ or $\bar{\zeta}_+((\breve{\varrho}_0 \divideontimes \breve{\varrho}_1) \divideontimes \breve{\varrho}_2) + \breve{u} > -\varphi + \varphi^{\divideontimes}$, $\bar{\zeta}_+(\breve{\varrho}_2) \geq \breve{u}$ or $\bar{\zeta}_+(\breve{\varrho}_2) + \breve{u} > -\varphi + \varphi^{\divideontimes}$. Using the Theorem 1, we get,

$$\bar{\zeta}_-(\breve{\varrho}_0 \between (\breve{\varrho}_1 \between (\breve{\varrho}_1 \between \breve{\varrho}_0))) \leq \bar{\zeta}_-((\breve{\varrho}_0 \between \breve{\varrho}_1) \between \breve{\varrho}_2) \vee \bar{\zeta}_-(\breve{\varrho}_2) \vee (\frac{\varphi}{2} - \frac{\varphi^{\divideontimes}}{2})$$

and

$$\bar{\zeta}_+(\breve{\varrho}_0 \between (\breve{\varrho}_1 \between (\breve{\varrho}_1 \between \breve{\varrho}_0))) \geq \bar{\zeta}_+((\breve{\varrho}_0 \between \breve{\varrho}_1) \between \breve{\varrho}_2) \wedge \bar{\zeta}_+(\breve{\varrho}_2) \wedge (-\frac{\varphi}{2} + \frac{\varphi^{\divideontimes}}{2})$$

<u>*Case 1*</u>. $\bar{\zeta}_-((\breve{\varrho}_0 \between \breve{\varrho}_1) \between \breve{\varrho}_2) \leq \breve{s}$, $\bar{\zeta}_-(\breve{\varrho}_2) \leq \breve{s}$ and $\bar{\zeta}_+((\breve{\varrho}_0 \between \breve{\varrho}_1) \between \breve{\varrho}_2) \geq \breve{u}$, $\bar{\zeta}_+(\breve{\varrho}_2) \geq \breve{u}$. If $\breve{s} < \frac{\varphi}{2} - \frac{\varphi^{\divideontimes}}{2}$ and $\breve{u} > -\frac{\varphi}{2} + \frac{\varphi^{\divideontimes}}{2}$, then

$$\begin{aligned}\bar{\zeta}_-(\breve{\varrho}_0 \between (\breve{\varrho}_1 \between (\breve{\varrho}_1 \between \breve{\varrho}_0))) &\leq \bar{\zeta}_-((\breve{\varrho}_0 \between \breve{\varrho}_1) \between \breve{\varrho}_2) \vee \bar{\zeta}_-(\breve{\varrho}_2) \vee (\frac{\varphi}{2} - \frac{\varphi^{\divideontimes}}{2}) \\ &= \frac{\varphi}{2} - \frac{\varphi^{\divideontimes}}{2}\end{aligned}$$

and

$$\begin{aligned}\bar{\zeta}_+(\breve{\varrho}_0 \between (\breve{\varrho}_1 \between (\breve{\varrho}_1 \between \breve{\varrho}_0))) &\geq \bar{\zeta}_+((\breve{\varrho}_0 \between \breve{\varrho}_1) \between \breve{\varrho}_2) \wedge \bar{\zeta}_+(\breve{\varrho}_1) \wedge (-\frac{\varphi}{2} + \frac{\varphi^{\divideontimes}}{2}) \\ &= -\frac{\varphi}{2} + \frac{\varphi^{\divideontimes}}{2}.\end{aligned}$$

Hence,

$$\bar{\zeta}_-(\breve{\varrho}_0 \between (\breve{\varrho}_1 \between (\breve{\varrho}_1 \between \breve{\varrho}_0))) + \breve{s} < \frac{\varphi}{2} - \frac{\varphi^{\divideontimes}}{2} + \frac{\varphi}{2} - \frac{\varphi^{\divideontimes}}{2} = \varphi - \varphi^{\divideontimes},$$

and

$$\bar{\zeta}_+(\breve{\varrho}_0 \between (\breve{\varrho}_1 \between (\breve{\varrho}_1 \between \breve{\varrho}_0))) + \breve{u} > -\frac{\varphi}{2} + \frac{\varphi^{\divideontimes}}{2} - \frac{\varphi}{2} + \frac{\varphi^{\divideontimes}}{2} = -\varphi + \varphi^{\divideontimes},$$

and so, $\breve{\varrho}_0 \between (\breve{\varrho}_1 \between (\breve{\varrho}_1 \between \breve{\varrho}_0)) \in (\varphi^{\divideontimes}, \breve{q_{\varphi}})\bar{\zeta}_-$ and $\breve{\varrho}_0 \between (\breve{\varrho}_1 \between (\breve{\varrho}_1 \between \breve{\varrho}_0)) \in (\varphi^{\divideontimes}, \breve{q_{\varphi}})\bar{\zeta}_+$. If $\breve{s} \geq \frac{\varphi}{2} - \frac{\varphi^{\divideontimes}}{2}$ and $\breve{u} \leq -\frac{\varphi}{2} + \frac{\varphi^{\divideontimes}}{2}$, then

$$\bar{\zeta}_-(\breve{\varrho}_0 \between (\breve{\varrho}_1 \between (\breve{\varrho}_1 \between \breve{\varrho}_0))) \leq \bar{\zeta}_-((\breve{\varrho}_0 \between \breve{\varrho}_1) \between \breve{\varrho}_2) \vee \bar{\zeta}_-(\breve{\varrho}_2) \vee (\frac{\varphi}{2} - \frac{\varphi^{\divideontimes}}{2}) \leq \breve{s}$$

and

$$\bar{\zeta}_+(\breve{\varrho}_0 \between (\breve{\varrho}_1 \between (\breve{\varrho}_1 \between \breve{\varrho}_0))) \geq \bar{\zeta}_+((\breve{\varrho}_0 \between \breve{\varrho}_1) \between \breve{\varrho}_2) \wedge \bar{\zeta}_+(\breve{\varrho}_2) \wedge (-\frac{\varphi}{2} + \frac{\varphi^{\divideontimes}}{2}) \geq \breve{u}.$$

Thus, $\breve{\varrho}_0 \between (\breve{\varrho}_1 \between (\breve{\varrho}_1 \between \breve{\varrho}_0))_{\breve{s}} \in \vee(\varphi^{\divideontimes}, \breve{q_{\varphi}})\bar{\zeta}_-$ and $\breve{\varrho}_0 \between (\breve{\varrho}_1 \between (\breve{\varrho}_1 \between \breve{\varrho}_0))_{\breve{u}} \in \vee(\varphi^{\divideontimes}, \breve{q_{\varphi}})\bar{\zeta}_+$. Therefore, $[\bar{\zeta}]_{(\breve{s}, \breve{u})}$.

<u>*Case 2*</u>. $\bar{\zeta}_-((\check{\varrho}_0 \between \check{\varrho}_1) \between \check{\varrho}_2) \leq \check{s}$, $\bar{\zeta}_-(\check{\varrho}_2) + \check{s} < \varphi - \varphi^*$ and $\bar{\zeta}_+((\check{\varrho}_0 \between \check{\varrho}_1) \between \check{\varrho}_2) \geq \check{u}$, $\bar{\zeta}_+(\check{\varrho}_2) + \check{u} > -\varphi + \varphi^*$. If $\check{s} < \frac{\varphi}{2} - \frac{\varphi^*}{2}$ and $\check{u} > -\frac{\varphi}{2} + \frac{\varphi^*}{2}$, then

$$\begin{aligned}
\bar{\zeta}_-(\check{\varrho}_0 \between (\check{\varrho}_1 \between (\check{\varrho}_1 \between \check{\varrho}_0))) &\leq \bar{\zeta}_-((\check{\varrho}_0 \between \check{\varrho}_1) \between \check{\varrho}_2) \vee \bar{\zeta}_-(\check{\varrho}_2) \vee (\frac{\varphi}{2} - \frac{\varphi^*}{2}) \\
&= \bar{\zeta}_-(\check{\varrho}_2) \vee (\frac{\varphi}{2} - \frac{\varphi^*}{2}) \\
&= (\varphi - \varphi^* - \check{s}) \vee (\frac{\varphi}{2} - \frac{\varphi^*}{2}) \\
&= \varphi - \varphi^* - \check{s}
\end{aligned}$$

and

$$\begin{aligned}
\bar{\zeta}_+(\check{\varrho}_0 \between (\check{\varrho}_1 \between (\check{\varrho}_1 \between \check{\varrho}_0))) &\geq \bar{\zeta}_+((\check{\varrho}_0 \between \check{\varrho}_1) \between \check{\varrho}_2) \wedge \bar{\zeta}_+(\check{\varrho}_2) \wedge (-\frac{\varphi}{2} + \frac{\varphi^*}{2}) \\
&= \bar{\zeta}_+(\check{\varrho}_1) \wedge (-\frac{\varphi}{2} + \frac{\varphi^*}{2}) \\
&= (-\varphi + \varphi^* - \check{u}) \wedge (-\frac{\varphi}{2} + \frac{\varphi^*}{2}) \\
&= -\varphi + \varphi^* - \check{u}.
\end{aligned}$$

Hence,

$$\bar{\zeta}_-(\check{\varrho}_0 \between (\check{\varrho}_1 \between (\check{\varrho}_1 \between \check{\varrho}_0))) + \check{s} < -\frac{\varphi}{2} + \frac{\varphi^*}{2} + -\frac{\varphi}{2} + \frac{\varphi^*}{2} = \varphi - \varphi^*$$

and

$$\bar{\zeta}_+(\check{\varrho}_0 \between (\check{\varrho}_1 \between (\check{\varrho}_1 \between \check{\varrho}_0))) + \check{u} > -\frac{\varphi}{2} + \frac{\varphi^*}{2} - \frac{\varphi}{2} + \frac{\varphi^*}{2} = -\varphi + \varphi^*,$$

and so, $\check{\varrho}_0 \between (\check{\varrho}_1 \between (\check{\varrho}_1 \between \check{\varrho}_0)) \in (\varphi^*, \check{q}_\varphi)\bar{\zeta}_-$ and $\check{\varrho}_0 \between (\check{\varrho}_1 \between (\check{\varrho}_1 \between \check{\varrho}_0)) \in (\varphi^*, \check{q}_\varphi)\bar{\zeta}_+$. If $\check{s} \geq \frac{\varphi}{2} - \frac{\varphi^*}{2}$ and $\check{u} \leq -\frac{\varphi}{2} + \frac{\varphi^*}{2}$, then

$$\begin{aligned}
\bar{\zeta}_-(\check{\varrho}_0 \between (\check{\varrho}_1 \between (\check{\varrho}_1 \between \check{\varrho}_0))) &\leq \bar{\zeta}_-((\check{\varrho}_0 \between \check{\varrho}_1) \between \check{\varrho}_2) \vee \bar{\zeta}_-(\check{\varrho}_2) \vee (\frac{\varphi}{2} - \frac{\varphi^*}{2}) \\
&\leq \check{s} \vee (\varphi - \varphi^* - \check{s}) \vee (\frac{\varphi}{2} - \frac{\varphi^*}{2}) = \check{s}.
\end{aligned}$$

and

$$\begin{aligned}
\bar{\zeta}_+(\check{\varrho}_0 \between (\check{\varrho}_1 \between (\check{\varrho}_1 \between \check{\varrho}_0))) &\geq \bar{\zeta}_+((\check{\varrho}_0 \between \check{\varrho}_1) \between \check{\varrho}_2) \wedge \bar{\zeta}_+(\check{\varrho}_2) \wedge (-\frac{\varphi}{2} + \frac{\varphi^*}{2}) \\
&\geq \check{u} \wedge (-\varphi + \varphi^* - \check{u}) \wedge (-\frac{\varphi}{2} + \frac{\varphi^*}{2}) = \check{u}.
\end{aligned}$$

Thus, $\check{\varrho}_0 \between (\check{\varrho}_1 \between (\check{\varrho}_1 \between \check{\varrho}_0))_{\check{s}} \in \vee(\varphi^*, \check{q}_\varphi)\bar{\zeta}_-$ and $\check{\varrho}_0 \between (\check{\varrho}_1 \between (\check{\varrho}_1 \between \check{\varrho}_0))_{\check{u}} \in \vee(\varphi^*, \check{q}_\varphi)\bar{\zeta}_+$. Therefore, $[\bar{\zeta}]_{(\check{s},\check{u})}$.

<u>*Case 3*</u>. $\bar{\zeta}_-((\check{\varrho}_0 \between \check{\varrho}_1) \between \check{\varrho}_2) + \check{s} < \varphi - \varphi^*$, $\bar{\zeta}_-(\check{\varrho}_2) \leq \check{s}$ and $\bar{\zeta}_+((\check{\varrho}_0 \between \check{\varrho}_1) \between \check{\varrho}_2) + \check{u} > -\varphi + \varphi^*$, $\bar{\zeta}_+(\check{\varrho}_2) \geq \check{u}$. If $\check{s} < \frac{\varphi}{2} - \frac{\varphi^*}{2}$ and $\check{u} > -\frac{\varphi}{2} + \frac{\varphi^*}{2}$, then

$$\begin{aligned}\bar{\zeta}_-(\check{\varrho}_0 \between (\check{\varrho}_1 \between (\check{\varrho}_1 \between \check{\varrho}_0))) &\leq \bar{\zeta}_-((\check{\varrho}_0 \between \check{\varrho}_1) \between \check{\varrho}_2) \vee \bar{\zeta}_-(\check{\varrho}_2) \vee (\frac{\varphi}{2} - \frac{\varphi^*}{2}) \\ &= (\varphi - \varphi^* - \check{s}) \vee (\frac{\varphi}{2} - \frac{\varphi^*}{2}) \\ &= \varphi - \varphi^* - \check{s}\end{aligned}$$

and

$$\begin{aligned}\bar{\zeta}_+(\check{\varrho}_0 \between (\check{\varrho}_1 \between (\check{\varrho}_1 \between \check{\varrho}_0))) &\geq \bar{\zeta}_+((\check{\varrho}_0 \between \check{\varrho}_1) \between \check{\varrho}_2) \wedge \bar{\zeta}_+(\check{\varrho}_2) \wedge (-\frac{\varphi}{2} + \frac{\varphi^*}{2}) \\ &= (-\varphi + 1 - \check{u}) \wedge (-\frac{\varphi}{2} + \frac{\varphi^*}{2}) \\ &= -\varphi + \varphi^* - \check{u}.\end{aligned}$$

Hence,

$$\begin{aligned}\bar{\zeta}_-(\check{\varrho}_0 \between (\check{\varrho}_1 \between (\check{\varrho}_1 \between \check{\varrho}_0))) + \check{s} &< \frac{\varphi}{2} - \frac{\varphi^*}{2} + \frac{\varphi}{2} - \frac{\varphi^*}{2} \\ &= \varphi - \varphi^*\end{aligned}$$

and

$$\begin{aligned}\bar{\zeta}_+(\check{\varrho}_0 \between (\check{\varrho}_1 \between (\check{\varrho}_1 \between \check{\varrho}_0))) + \check{u} &> -\frac{\varphi}{2} + \frac{\varphi^*}{2} - \frac{\varphi}{2} + \frac{\varphi^*}{2} \\ &= -\varphi + \varphi^*,\end{aligned}$$

and so, $\check{\varrho}_0 \between (\check{\varrho}_1 \between (\check{\varrho}_1 \between \check{\varrho}_0)) \in (\varphi^*, \check{q}_\varphi)\bar{\zeta}_-$ and $\check{\varrho}_0 \between (\check{\varrho}_1 \between (\check{\varrho}_1 \between \check{\varrho}_0)) \in (\varphi^*, \check{q}_\varphi)\bar{\zeta}_+$. If $\check{s} \geq \frac{\varphi}{2} - \frac{\varphi^*}{2}$ and $\check{u} \leq -\frac{\varphi}{2} + \frac{\varphi^*}{2}$, then

$$\begin{aligned}\bar{\zeta}_-(\check{\varrho}_0 \between (\check{\varrho}_1 \between (\check{\varrho}_1 \between \check{\varrho}_0))) &\leq \bar{\zeta}_-((\check{\varrho}_0 \between \check{\varrho}_1) \between \check{\varrho}_2) \vee \bar{\zeta}_-(\check{\varrho}_2) \vee (\frac{\varphi}{2} - \frac{\varphi^*}{2}) \\ &\leq (\varphi - \varphi^* - \check{s}) \vee \check{s} \vee (\frac{\varphi}{2} - \frac{\varphi^*}{2}) = \check{s}\end{aligned}$$

and

$$\begin{aligned}\bar{\zeta}_+(\check{\varrho}_0 \between (\check{\varrho}_1 \between (\check{\varrho}_1 \between \check{\varrho}_0))) &\geq \bar{\zeta}_+((\check{\varrho}_0 \between \check{\varrho}_1) \between \check{\varrho}_2) \wedge \bar{\zeta}_+(\check{\varrho}_2) \wedge (-\frac{\varphi}{2} + \frac{\varphi^*}{2}) \\ &\geq (-\varphi + \varphi^* - \check{u}) \wedge \check{u} \wedge (-\frac{\varphi}{2} + \frac{\varphi^*}{2}) = \check{u}.\end{aligned}$$

Thus, $\check{\varrho}_0 \between (\check{\varrho}_1 \between (\check{\varrho}_1 \between \check{\varrho}_0))_{\check{s}} \in \vee(\varphi^*, \check{q}_\varphi)\bar{\zeta}_-$ and $\check{\varrho}_0 \between (\check{\varrho}_1 \between (\check{\varrho}_1 \between \check{\varrho}_0))_{\check{u}} \in \vee(\varphi^*, \check{q}_\varphi)\bar{\zeta}_+$. Therefore, $[\bar{\zeta}]_{(\check{s},\check{u})}$.

Case 4. $\bar{\zeta}_-((\breve{\varrho}_0 \between \breve{\varrho}_1) \between \breve{\varrho}_2) + \check{s} < \varphi - \varphi^*$, $\bar{\zeta}_-(\breve{\varrho}_2) + \check{s} < \varphi - \varphi^*$ and $\bar{\zeta}_+((\breve{\varrho}_0 \between \breve{\varrho}_1) \between \breve{\varrho}_2) + \check{u} > -\varphi + \varphi^*$, $\bar{\zeta}_+(\breve{\varrho}_2) + \check{u} > -\varphi + \varphi^*$. If $\check{s} < \frac{\varphi}{2} - \frac{\varphi^*}{2}$ and $\check{u} > -\frac{\varphi}{2} + \frac{\varphi^*}{2}$, then

$$\begin{aligned}\bar{\zeta}_-(\breve{\varrho}_0 \between (\breve{\varrho}_1 \between (\breve{\varrho}_1 \between \breve{\varrho}_0))) &\leq \bar{\zeta}_-((\breve{\varrho}_0 \between \breve{\varrho}_1) \between \breve{\varrho}_2) \vee \bar{\zeta}_-(\breve{\varrho}_2) \vee (\frac{\varphi}{2} - \frac{\varphi^*}{2})\\ &= (\varphi - \varphi^* - \check{s}) \vee (\frac{\varphi}{2} - \frac{\varphi^*}{2})\\ &= \varphi - \varphi^* - \check{s}\end{aligned}$$

and

$$\begin{aligned}\bar{\zeta}_+(\breve{\varrho}_0 \between (\breve{\varrho}_1 \between (\breve{\varrho}_1 \between \breve{\varrho}_0))) &\geq \bar{\zeta}_+((\breve{\varrho}_0 \between \breve{\varrho}_1) \between \breve{\varrho}_2) \wedge \bar{\zeta}_+(\breve{\varrho}_1) \wedge (-\frac{\varphi}{2} + \frac{\varphi^*}{2})\\ &= (-\varphi + \varphi^* \check{u}) - \wedge(-\frac{\varphi}{2} + \frac{\varphi^*}{2})\\ &= -\varphi + \varphi^* - \check{u}.\end{aligned}$$

Hence,

$$\bar{\zeta}_-(\breve{\varrho}_0 \between (\breve{\varrho}_1 \between (\breve{\varrho}_1 \between \breve{\varrho}_0))) + \check{s} < \frac{\varphi}{2} - \frac{\varphi^*}{2} + \frac{\varphi}{2} - \frac{\varphi^*}{2} = \varphi - \varphi^*$$

and

$$\bar{\zeta}_+(\breve{\varrho}_0 \between (\breve{\varrho}_1 \between (\breve{\varrho}_1 \between \breve{\varrho}_0))) + \check{u} > -\frac{\varphi}{2} + \frac{\varphi^*}{2} - \frac{\varphi}{2} + \frac{\varphi^*}{2} = -\varphi + \varphi^*,$$

and so, $\breve{\varrho}_0 \between (\breve{\varrho}_1 \between (\breve{\varrho}_1 \between \breve{\varrho}_0)) \in \breve{q_\varphi}\bar{\zeta}_-$ and $\breve{\varrho}_0 \between (\breve{\varrho}_1 \between (\breve{\varrho}_1 \between \breve{\varrho}_0)) \in \breve{q_\varphi}\bar{\zeta}_+$. If $\check{s} \geq \frac{\varphi}{2} - \frac{\varphi^*}{2}$ and $\check{u} \leq -\frac{\varphi}{2} + \frac{\varphi^*}{2}$, then

$$\begin{aligned}\bar{\zeta}_-(\breve{\varrho}_0 \between (\breve{\varrho}_1 \between (\breve{\varrho}_1 \between \breve{\varrho}_0))) &\leq \bar{\zeta}_-((\breve{\varrho}_0 \between \breve{\varrho}_1) \between \breve{\varrho}_2) \vee \bar{\zeta}_-(\breve{\varrho}_2) \vee (\frac{\varphi}{2} - \frac{\varphi^*}{2})\\ &\leq (\varphi - \varphi^* - \check{s}) \vee (\frac{\varphi}{2} - \frac{\varphi^*}{2}) = (\frac{\varphi}{2} - \frac{\varphi^*}{2}) \leq \check{s}\end{aligned}$$

and

$$\begin{aligned}\bar{\zeta}_+(\breve{\varrho}_0 \between (\breve{\varrho}_1 \between (\breve{\varrho}_1 \between \breve{\varrho}_0))) &\geq \bar{\zeta}_+((\breve{\varrho}_0 \between \breve{\varrho}_1) \between \breve{\varrho}_2) \wedge \bar{\zeta}_+(\breve{\varrho}_2) \wedge (-\frac{\varphi}{2} + \frac{\varphi^*}{2})\\ &\geq (-\varphi + \varphi^* - \check{u}) \wedge (-\frac{\varphi}{2} + \frac{\varphi^*}{2}) = (-\frac{\varphi}{2} + \frac{\varphi^*}{2}) \geq \check{u}.\end{aligned}$$

Therefore, $\breve{\varrho}_0 \between (\breve{\varrho}_1 \between (\breve{\varrho}_1 \between \breve{\varrho}_0))_{\check{s}} \in \vee(\varphi^*, \breve{q_\varphi})\bar{\zeta}_-$ and $\breve{\varrho}_0 \between (\breve{\varrho}_1 \between (\breve{\varrho}_1 \between \breve{\varrho}_0))_{\check{u}} \in \vee(\varphi^*, \breve{q_\varphi})\bar{\zeta}_+$. Hence, $[\bar{\zeta}]_{(\check{s},\check{u})}$. Therefore, $[\bar{\zeta}]_{(\check{s},\check{u})}$ is a fantastic ideal of $\breve{\aleph}$.

Conversely, let $\bar{\zeta}$ be a $\mathcal{BFS}$ in $\bar{\zeta}$ and $\check{s} \in [-1, 0), \check{u} \in (0, 1]$. Then $\bar{\zeta}$ is an $(\in, \in \vee(\varphi^*, \breve{q_\varphi}))$-$\mathcal{BFFI}$ of $\breve{\aleph}$ be such that $[\bar{\zeta}]_{(\check{s},\check{u})}$ is a fantastic ideal of $\breve{\aleph}$. If possible, let

$$\bar{\zeta}_-(\breve{\varrho}_0 \between (\breve{\varrho}_1 \between (\breve{\varrho}_1 \between \breve{\varrho}_0))) > \check{s} \geq \bar{\zeta}_-((\breve{\varrho}_0 \between \breve{\varrho}_1) \between \breve{\varrho}_2) \vee \bar{\zeta}_-(\breve{\varrho}_2) \vee (\frac{\varphi}{2} - \frac{\varphi^*}{2})$$

and

$$\bar{\zeta}_{+}(\breve{\varrho}_0 \between (\breve{\varrho}_1 \between (\breve{\varrho}_1 \between \breve{\varrho}_0))) < \breve{u} \leq \bar{\zeta}_{+}((\breve{\varrho}_0 \between \breve{\varrho}_1) \between \breve{\varrho}_2) \wedge \bar{\zeta}_{+}(\breve{\varrho}_2) \wedge (-\frac{\varphi}{2} + \frac{\varphi^{*}}{2}),$$

for some $\breve{s} \in (-1,0), \breve{u} \in (0, \breve{v})$. Then $(\breve{\varrho}_0 \between \breve{\varrho}_1) \between \breve{\varrho}_2, \breve{\varrho}_2 \in U(\bar{\zeta}; \breve{s}, \breve{u}) \subseteq [\bar{\zeta}]_{(\breve{s},\breve{u})}$, which indicate $\breve{\varrho}_0 \between (\breve{\varrho}_1 \between (\breve{\varrho}_1 \between \breve{\varrho}_0)) \in [\bar{\zeta}]_{(\breve{s},\breve{u})}$. Thus, $\bar{\zeta}_{-}((\breve{\varrho}_0 \between \breve{\varrho}_1) \between \breve{\varrho}_2) \leq \breve{s}$ or $\bar{\zeta}_{-}((\breve{\varrho}_0 \between \breve{\varrho}_1) \between \breve{\varrho}_2) + \breve{s} < \varphi - \varphi^{*}$, $\bar{\zeta}_{-}(\breve{\varrho}_2) \leq \breve{s}$ or $\bar{\zeta}_{-}(\breve{\varrho}_2) + \breve{s} < \varphi - \varphi^{*}$ and $\bar{\zeta}_{+}((\breve{\varrho}_0 \between \breve{\varrho}_1) \between \breve{\varrho}_2) \geq \breve{u}$ or $\bar{\zeta}_{+}((\breve{\varrho}_0 \between \breve{\varrho}_1) \between \breve{\varrho}_2) + \breve{u} > -\varphi + \varphi^{*}$, $\bar{\zeta}_{+}(\breve{\varrho}_2) \geq \breve{u}$ or $\bar{\zeta}_{+}(\breve{\varrho}_2) + \breve{u} > -\varphi + \varphi^{*}$, and these are a contradiction. Hence,

$$\bar{\zeta}_{-}(\breve{\varrho}_0 \between (\breve{\varrho}_1 \between (\breve{\varrho}_1 \between \breve{\varrho}_0))) \leq \bar{\zeta}_{-}((\breve{\varrho}_0 \between \breve{\varrho}_1) \between \breve{\varrho}_2) \vee \bar{\zeta}_{-}(\breve{\varrho}_2) \vee (\frac{\varphi}{2} - \frac{\varphi^{*}}{2})$$

and

$$\bar{\zeta}_{+}(\breve{\varrho}_0 \between (\breve{\varrho}_1 \between (\breve{\varrho}_1 \between \breve{\varrho}_0))) \geq \bar{\zeta}_{+}((\breve{\varrho}_0 \between \breve{\varrho}_1) \between \breve{\varrho}_2) \wedge \bar{\zeta}_{+}(\breve{\varrho}_2) \wedge (-\frac{\varphi}{2} + \frac{\varphi^{*}}{2}),$$

for all $\breve{\varrho}_0, \breve{\varrho}_1, \breve{\varrho}_2 \in \breve{\aleph}$. Now, by using the Theorem 1, we conclude that $\bar{\zeta}$ is an $(\in, \in \vee(\varphi^{*}, \breve{q_{\varphi}}))$-$\mathcal{BFFI}$ of $\breve{\aleph}$.

5. Conclusion

The concept of an $(\in, \in \vee(\varphi^{*}, \breve{q_{\varphi}}))$-bipolar fuzzy set combines elements from fuzzy sets and bipolar fuzzy sets. In this paper, we investigated $(\in, \in \vee(\varphi^{*}, \breve{q_{\varphi}}))$-bipolar fuzzy ideals and $(\in, \in \vee(\varphi^{*}, \breve{q_{\varphi}}))$-bipolar fuzzy fantastic ideals and discussed their essential properties. We examined the connection between $(\in, \in \vee(\varphi^{*}, \breve{q_{\varphi}}))$-bipolar fuzzy fantastic ideals and fuzzy fantastic ideals. In our future study of the bipolar fuzzy structure, we may consider the following topics: (i) bipolar complex fuzzy q-ideals in BCK/BCI-algebra; (ii) bipolar complex intuitionistic fuzzy commutative ideals in BG-algebras, BE-algebras; and (iii) complex picture fuzzy ideals in BCC-algebras, JU-algebras etc.

Acknowledgements

The authors gratefully acknowledge the funding of the Deanship of Graduate Studies and Scientific Research, Jazan University, Saudi Arabia, through Project Number: GSSRD-24.

References

[1] E.A. Abuhijileh, M. Massa'deh, A. Sheimat, and A. Alkouri. Complex fuzzy groups based on rosenfeld's approach. *WSEAS Trans. Math.*, 20:368–377, 2021.

[2] M. Akram and A. Farooq. m-polar fuzzy lie ideals of lie algebras. *Quasigroups Related Systems*, 24(2):141–150, 2016.

[3] M. Akram, A. Farooq, and K.P. Shum. On m-polar fuzzy lie subalgebras. *Ital. J. Pure Appl. Math.*, 36:445–454, 2016.

[4] D. Al-Kadi and G. Muhiuddin. Bipolar fuzzy bci-implicative ideals of bci-algebras. *Ann. Commun. Math*, 1(3):88–96, 2020.

[5] A. Al-Masarwah and A. G. Ahmad. Doubt bipolar fuzzy subalgebras and ideals in bck/bci-algebras. *J. Math. Anal.*, 9(3):9–27, 2018.

[6] A. Al-Masarwah and A. G. Ahmad. Novel concepts of doubt bipolar fuzzy h-ideals of bck/bci-algebras. *International Journal of Innovative Computing, Information and Control*, 14(06):2025–2041, 2018.

[7] A. Al-Masarwah and A. G. Ahmad. Subalgebras of type (α, β) based on m-polar fuzzy points in bck/bci-algebras. *AIMS Mathematics*, 5(2):1035–1050, 2020.

[8] A. Al-Masarwah and A. G. Ahmad. A new interpretation of multi-polarity fuzziness subalgebras of bck/bci-algebras. *Fuzzy Information and Engineering*, 14(3):243–254, 2022.

[9] A. Al-Masarwah, A. G. Ahmad, G. Muhiuddin, and D. Al-Kadi. Generalized m-polar fuzzy positive implicative ideals of bck-algebras. *Journal of Mathematics*, 2021(1):6610009, 2021.

[10] A. Al-Masarwah and A.G. Ahmad. On some properties of doubt biplar fuzzy h-ideals in bck/bci-algebras. *Eur. J. Pure Appl. Math.*, 11(3):652–670, 2018.

[11] A. Al-Masarwah and A.G. Ahmad. m-polar (α, β)-fuzzy ideals in bck/bci-algebras. *Symmetry*, 11(1):44, 2019.

[12] A. Al-Masarwah and A.G. Ahmad. m-polar fuzzy ideals of bck/bci-algebras. *J. King Saud Univ.-Sci.*, 31(4):1220–1226, 2019.

[13] A. Al-Masarwah and A.G. Ahmad. A new form of generalized m-pf ideals in bck/bci-algebras. *Ann. Commun. Math.*, 2(1):11–16, 2019.

[14] A. Al-Masarwah and A.G. Ahmad. On (complete) normality of m-pf subalgebras in bck/bci-algebras. *AIMS Math.*, 4(3):740–750, 2019.

[15] Moin A. Ansari. Rough set theory applied to ju-algebras. *Int. J. Math. Comput. Sc.*, 16:1371–1384, 2021.

[16] Moin A. Ansari, Ali N. A. Koam, and A. Haider. Intersection soft ideals and their quotients on ku-algebras. *AIMS Mathematics*, 6(11):12077–12084, 2021.

[17] M. Aslam, S. Abdullah, and M. Masood. Bipolar fuzzy ideals in la-semigroups. *World Appl. Sci. J.*, 17(12):1769–1782, 2012.

[18] M. Balamurugan, N. Alessa, K. Loganathan, and N. Amar Nath. $(\acute{\in}, \acute{\in} \vee \acute{q}_{\tilde{k}})$-uni-intuitionistic fuzzy soft h-ideals in subtraction bg-algebras. *Mathematics*, 11(10):2296, 2023.

[19] M. Balamurugan, T. Ramesh, A. Al-Masarwah, and K. Alsager. A new approach of complex fuzzy ideals in bck/bci-algebras. *Mathematics*, 12(10):1583, 2024.

[20] S.K. Bhakat and P. Dasi. $(\in, \in \vee q)$-fuzzy subgroup. *Fuzzy Sets and Systems*, 30(3):359–368, 1996.

[21] J. Chen, S. Li, S. Ma, and X. Wang. m-polar fuzzy sets: An extension of bipolar fuzzy sets. *Sci. World J.*, 2014.

[22] A. Farooq, G. Ali, and M.Akram. On m-polar fuzzy groups. *International Journal of Algebra and Statistics*, 5(2):115–127, 2016.

[23] A. Iamapan, M. Balamurugan, and V. Govindan. $(\in, \in \vee q_{\tilde{k}})$-anti-intuitionistic fuzzy soft b-ideals in bck/bci-algebras. *Mathematics and Statistics*, 10(3):515–522, 2022.

[24] M. Ibrar, A. Khan, and B. Davvazi. Characterizations of regular ordered semigroups in terms of (α, β)-bipolar fuzzy generalized bi-ideals. *Journal of Intelligent & Fuzzy Systems*, 33(1):365–376, 2017.

[25] Y. Imai and K. Isék. On axiom systems of propositional calculi. xiv. *Proc. Japan Acad.*, 42(1):19–22, 1966.

[26] K. Isék. An algebra related with a propositional calculus. *Linear Algebra and its Applications*, 42(1):26–29, 1996.

[27] A. Jaleel, T. Mahmood, W. Emam, and S. Yin. Interval valued bipolar complex fuzzy soft sets and their applications in decision making. *Scientific Reports*, 14:1–9, 2024.

[28] C. Jana, M. Pal, and A.B. Saiedi. $(\in, \in \vee q)$-bipolar fuzzy bck-algebras. *Missouri Journal of Mathematical Sciences*, 29(2):139–160, 2017.

[29] K. Kawila, C. Udomsetchai, and A. Iampan. Bipolar fuzzy up-algebras. *Math. Comput. Appl.*, 23(4):69, 2018.

[30] Ali N. A. Koam, Azeem Haider, and Moin A. Ansari. On an extension of ku-algebras. *AIMS Mathematics*, 6(2):1249–1257, 2021.

[31] K. J. Lee. Bipolar fuzzy subalgebras and bipolar fuzzy ideals of bck/bci-algebras. *Bull. Malays. Math. Sci. Soc.*, 32(3):361–373, 2009.

[32] K. J. Lee and Y. B. Jun. Bipolar fuzzy a-ideals of bci-algebras. *Commun. Korean Math. Soc.*, 26(4):531–542, 2011.

[33] K. M. Lee. Bipolar-valued fuzzy sets and their basic operations. *Proc. Int. Conf. on Intelligent Technologies, Bangkok, Thailand*, pages 307–312, 2000.

[34] T. Mahmood, U. Rehman, A. Jaleel, J. Ahmmad, and R. Chinram. Bipolar complex fuzzy soft sets and their applications in decision-making. *Mathematics*, 10:1048, 2022.

[35] M.T. Mohseni, S.S. Ahn, R.A. Borzooei, and Y.B. Jun. Multipolar fuzzy p-ideals of bci-algebras. *Mathematics*, 7(11):1094, 2019.

[36] G. Muhiuddin. Bipolar fuzzy ku-subalgebras/ideals of ku-algebras. *Annals of Fuzzy Mathematics and Informatics*, 8(3):409–418, 2014.

[37] G. Muhiuddin, N. Abughazalah, A. Aljuhani, and M. Balamurugan. Tripolar picture fuzzy ideals of bck-algebras. *Symmetry*, 14(8):1562, 2022.

[38] G. Muhiuddin, D. Al-Kadi, and M. Balamurugan. Anti-intuitionistic fuzzy soft a-ideals applied to bci-algebras. *Axioms*, 9(3):79, 2020.

[39] G. Muhiuddin, D. Al-Kadi, A. Mehboob, and K.P. Shum. New types of bipolar fuzzy ideals of bck-algebras. *International Journal of Analysis and Applications*, 18(5):859–875, 2020.

[40] G. Muhiuddin, H. Harizavi, and Y.B. Jun. Bipolar-valued fuzzy soft hyper bck-ideals in bck-algebras. *Discrete Mathematics Algorithms and Applications*, 12(2):295998, 2019.

[41] G. Muhiuddin, M.M. Takallo, R.A. Borzooei, and Y.B. Jun. m-polar fuzzy q-ideals in bci-algebras. *Journal of King Saud Univeristy Science*, 32(6):2803–2809, 2020.

[42] M. Mursaleen, M. Balamurugan, K. Loganathan, and K.S. Nisar. $(\in, \in \vee \check{q})$-bipolar fuzzy-ideals of bck/bci-algebras. *Journal of Function Spaces*, (10):6615288, 2021.

[43] A. Rosenfeld. Fuzzy groups. *J. Math. Anal. Appl.*, 35:512–517, 1971.

[44] A.B. Saeid. Bipolar-valued fuzzy bck/bci-algebras. *World Applied Sci. J.*, 7(11):1404–1411, 2009.

[45] L.A. Zadeh. Fuzzy sets. *Information and Control*, 8:338–353, 1965.

[46] W. R. Zhang, L. Zhang, and Y. Yang. Bipolar logic and bipolar fuzzy logic. *Inform. Sci.*, 165(3):265–287, 2004.

[47] W.R. Zhang. Bipolar fuzzy sets and relations: A computational framework for cognitive and modeling and multiagent decision analysis. *Proc. of IEEE Conf.*, 2:305–309, 1994.