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Upper and Lower Rarely m-I-Continuous Multifunctions

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Abstract. In 1979, Popa [24] first introduced rarely continuous functions. In this paper, we introduce upper and lower rarely m-continuous multifunctions. Moreover, we extend this concept to a multifunction $F : (X, \tau, I) \to (Y, \sigma)$, where (X, τ, I) is an ideal topologiccal space.

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Key Words and Phrases: Rare set, m-open, m-structure, $mIO(X)$, ideal topological space, rarely continuous function, upper/lower rarely m-I-continuous, multifunction

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1. Introduction

Semi-open sets, preopen sets, α-open sets, β-open set and b-open sets play an important part in the research of generalizations of continuous functions. By using these notions, various types of continuous multifunctions are introduced and studied. As an unified form of the above generalizations of open sets, in [27] and [28] the present authors introduced minimal structures and m-spaces. We recall the notions in the section 2.

In 1979, Popa [24] first introduced the concept of rare continuity which was further studied by Long and Herrington [19] and Jafari [10], [11]. Several weak forms of rarely continuous functions, for example, rare quasi-continuity [26], rare α-continuity [13], rare pre-continuity [12] etc have been introduced and studied. Moreover these concepts are extended to multifunctions: rare continuity [25], rare quasi-continuity [15], rare α-continuity [4], rare β-continuity [14].

The purpose of this paper is to introduce the concept of upper and lower rarely m-continuous multifunctions which unifies the above stated multifunctions, that is, rare quasi-continuity, rare pre-continuity, rare α-continuity, and rare β-continuity. As generalizations of open sets, the notion of I-open sets, semi-I-open sets, pre-I-open sets, α-I-open sets, β-I-open sets and b-I-open sets are introduced and studied in an ideal topological space $(X.\tau, I)$. In the last section, we extend the results of an upper and lower rarely

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m-continuous multifunction $F : (X, m) \to (Y, \sigma)$ to a multifunction $F : (X, \tau, I) \to (Y, \sigma)$.

Throughout the present paper, (X, τ) and (Y, σ) (briefly X and Y) always denote topological spaces and $F : X \to Y$ presents a multivalued function. For a multifunction $F : X \to Y$, we shall denote the upper and lower inverse of a subset B of a space Y by $F^+(B)$ and $F^-(B)$, respectively, that is

$$F^+(B) = \{x \in X : F(x) \subset B\} \text{ and } F^-(B) = \{x \in X : F(x) \cap B \neq \emptyset\}.$$

2. Preliminaries

Let (X, τ) be a topological space and A a subset of X. The closure of A and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A subset A is said to be *regular open* (resp. *regular closed*) if $\mathrm{Int}(\mathrm{Cl}(A)) = A$ (resp. $\mathrm{Cl}(\mathrm{Int}(A))$ = A).

Definition 1. Let (X, τ) be a topological space. A subset A of X is said to be
(1) *α-open* [23] if $A \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$,
(2) *semi-open* [18] if $A \subset \mathrm{Cl}(\mathrm{Int}(A))$,
(3) *preopen* [21] if $A \subset \mathrm{Int}(\mathrm{Cl}(A))$,
(4) *β-open* [3] if $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$,
(5) *b-open* [1] if $A \subset \mathrm{Int}(\mathrm{Cl}(A)) \cup \mathrm{Cl}(\mathrm{Int}(A))$.

The family of all semi-open (resp. preopen, α-open, β-open, b-open) sets in X is denoted by $\mathrm{SO}(X)$ (resp. $\mathrm{PO}(X)$, $\alpha(X)$, $\beta(X)$, $\mathrm{BO}(X)$).

Definition 2. A subfamily m of the power set $\mathcal{P}(X)$ of a nonempty set X is called a *minimal structure* (briefly *m-structure*) [27], [28] on X if $\emptyset \in m$ and $X \in m$.

By (X, m), we denote a nonempty set X with a minimal structure m on X and call it an *m-space*. Each member of m is said to be *m-open* and the complement of an m-open set is said to be *m-closed*. By $m(x)$, we denote the family $\{U : x \in U \in m\}$.

Definition 3. Let (X, m) be an m-space. For a subset A of X, the *m-closure* of A and the *m-interior* of A are defined in [20] as follows:
(1) $\mathrm{mCl}(A) = \cap\{F : A \subset F, X \setminus F \in m\}$,
(2) $\mathrm{mInt}(A) = \cup\{U : U \subset A, U \in m\}$.

Lemma 1. (Maki et al. [20]). *Let (X, m) be an m-space. For subsets A and B of X, the following properties hold:*
(1) $\mathrm{mCl}(X \setminus A) = X \setminus \mathrm{mInt}(A)$ *and* $\mathrm{mInt}(X \setminus A) = X \setminus \mathrm{mCl}(A)$,
(2) If $(X \setminus A) \in m$, *then* $\mathrm{mCl}(A) = A$ *and if* $A \in m$, *then* $\mathrm{mInt}(A) = A$,
(3) $\mathrm{mCl}(\emptyset) = \emptyset$, $\mathrm{mCl}(X) = X$, $\mathrm{mInt}(\emptyset) = \emptyset$ *and* $\mathrm{mInt}(X) = X$,
(4) If $A \subset B$, *then* $\mathrm{mCl}(A) \subset \mathrm{mCl}(B)$ *and* $\mathrm{mInt}(A) \subset \mathrm{mInt}(B)$,
(5) $\mathrm{mInt}(A) \subset A \subset \mathrm{mCl}(A)$,
(6) $\mathrm{mCl}(\mathrm{mCl}(A)) = \mathrm{mCl}(A)$ *and* $\mathrm{mInt}(\mathrm{mInt}(A)) = \mathrm{mInt}(A)$.

Lemma 2. (Popa and Noiri [28]). *Let (X, m) be an m-space and A a subset of X. Then $x \in \mathrm{mCl}(A)$ if and only if $U \cap A \neq \emptyset$ for every $U \in m$ containing x.*

Definition 4. An m-structure m on a nonempty set X is said to have *property $\mathcal{B}$* [20] if the union of any family of subsets belonging to m belongs to m.

Remark 1. Let (X, τ) be a topological space. Then the families τ, $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\alpha(X)$, $\mathrm{BO}(X)$ and $\beta(X)$ are m-structures and have property $\mathcal{B}$.

Lemma 3. (Popa and Noiri [29]). *For an m-structure m on a nonempty set X, the following properties are equivalent:*

(1) m has property $\mathcal{B}$;

(2) If $\mathrm{mInt}(A) = A$, then $A \in m$;

(3) If $\mathrm{mCl}(A) = A$, then A is m-closed.

Definition 5. A subset A of a topological space (X, τ) is called a *rare-set* if $\mathrm{Int}(A) = \emptyset$.

Lemma 4. *In a topological space (X, τ), $\mathrm{Int}(F \cup R) \subset F$ for every rare set R and every closet set F.*

Proof. It is obvious that $O \cap \mathrm{Cl}(A) \subset \mathrm{Cl}(O \cap A)$ of every subset A of X and any open set O of X. Hence $\mathrm{Int}(F \cup B) \subset (F \cup \mathrm{Int}(B))$ for every subset B and every closed set F. Therefore, $\mathrm{Int}(F \cup R) \subset F$ for every rare set R and every closed set F.

Definition 6. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be *rarely continuous* [24] at $x \in X$ if for any open set V of Y such that $f(x) \in V$, there exist a rare set R_V with $R_V \cap V = \emptyset$ and an open set U containing x such that $f(U) \subset V \cup R_V$.

3. Rarely m-continuous multifunctions

In this section, we define upper and lower rare m-continuity on a multifunction $F : (X, m) \to (Y, \sigma)$ and obtain their characterizations.

Definition 7. Let (X, m) be an m-space and (Y, σ) a topological space. A multifunction $F : (X, m) \to (Y, \sigma)$ is said to be

(1) *upper rarely m-continuous* at a point $x \in X$ if for each open set V containing $F(x)$, there exist a rare set R_V with $R_V \cap V = \emptyset$ and an m-open set $U \in m(x)$ such that $F(U) \subset V \cup R_V$,

(2) *lower rarely m-continuous* at a point $x \in X$ if for each open set V meeting $F(x)$, there exist a rare set R_V with $R_V \cap V = \emptyset$ and an m-open set $U \in m(x)$ such that $F(u) \cap (V \cup R_V) \neq \emptyset$ for each $u \in U$,

Theorem 1. *For a multifunction $F : (X, m) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is upper rarely m-continuous at $x \in X$;

(2) for each open set V of Y containing $F(x)$, there exists a rare set R_V with $V \cap R_V = \emptyset$ such that $x \in \mathrm{mInt}(F^+(V \cup R_V))$;

(3) for each open set V of Y containing $F(x)$, there exists a rare set R_V with $\mathrm{Cl}(V) \cap R_V = \emptyset$ such that $x \in \mathrm{mInt}(F^+(\mathrm{Cl}(V) \cup R_V))$;

(4) for each regular open set V of Y containing $F(x)$, there exists a rare set R_V with $V \cap R_V = \emptyset$ such that $x \in \mathrm{mInt}(F^+(V \cup R_V))$;

(5) for each open set V of Y containing $F(x)$, there exists $U \in m(x)$ such that $\mathrm{Int}[F(U) \cap (Y \setminus V)] = \emptyset$,

(6) for each open set V of Y containing $F(x)$, there exists $U \in m(x)$ such that $\mathrm{Int}(F(U)) \subset \mathrm{Cl}(V)$.

Proof. (1) $\Rightarrow$ (2): Let V be any open set of Y containing $F(x)$. By (1), there exist a rare set R_V with $R_V \cap V = \emptyset$ and an m-open set $U \in m(x)$ such that $F(U) \subset V \cup R_V$. Hence $x \in U \subset F^+(V \cup R_V)$. Since U is m-open, $x \in \mathrm{mInt}(F^+(V \cup R_V))$.

(2) $\Rightarrow$ (3): Let V be any open set of Y such that $F(x) \subset V$. Then by (2), there exists a rare set R_V with $V \cap R_V = \emptyset$ such that $x \in \mathrm{mInt}(F^+(V \cup R_V))$. Let $S_V = R_V \cap (Y \setminus \mathrm{Cl}(V))$, then $S_V \cap \mathrm{Cl}(V) = \emptyset$ and S_V is a rare set. Since $\mathrm{Cl}(V) \cup S_V = \mathrm{Cl}(V) \cup [R_V \cap (Y \setminus \mathrm{Cl}(V))] = \mathrm{Cl}(V) \cup R_V \supset V \cup R_V$. Therefore, $x \in \mathrm{mInt}(F^+(V \cup R_V)) \subset \mathrm{mInt}(F^+(\mathrm{Cl}(V) \cup S_V))$.

(3) $\Rightarrow$ (4): Let V be any regular open set of Y containing $F(x)$. By (3), there exists a rare set R_V with $\mathrm{Cl}(V) \cap R_V = \emptyset$ such that $x \in \mathrm{mInt}(F^+(\mathrm{Cl}(V) \cup R_V))$. Let $S_V = R_V \cup (\mathrm{Cl}(V) \setminus V)$. Then by Lemma 4, S_V is a rare set and $S_V \cap V = \emptyset$. Therefore, $x \in \mathrm{mInt}(F^+(V \cup S_V))$.

(4) $\Rightarrow$ (5): V be any open set of Y containing $F(x)$. Then $F(x) \subset V \subset \mathrm{Int}(\mathrm{Cl}(V))$ and $\mathrm{Int}(\mathrm{Cl}(V))$ is regular open. By (4), there exists a rare set R_V with $R_V \cap \mathrm{Int}(\mathrm{Cl}(V)) = \emptyset$ and $x \in \mathrm{mInt}(F^+(\mathrm{Int}(\mathrm{Cl}(V)) \cup R_V))$. Hence there exists $U \in m(x)$ such that $x \in U \subset F^+(\mathrm{Int}(\mathrm{Cl}(V)) \cup R_V)$. Thus, $F(U) \subset \mathrm{Int}(\mathrm{Cl}(V)) \cup R_V$. Therefore, by using Lemma 4, we have

$$\begin{aligned}\mathrm{Int}[F(U) \cap (Y \setminus V)] &= \mathrm{Int}(F(U)) \cap \mathrm{Int}(Y \setminus V) \subset \mathrm{Int}(\mathrm{Cl}(V) \cup R_V) \cap (Y \setminus \mathrm{Cl}(V)) \subset \\ &(\mathrm{Cl}(V) \cup \mathrm{Int}(R_V)) \cap (Y \setminus \mathrm{Cl}(V)) = \mathrm{Cl}(V) \cap (Y \setminus \mathrm{Cl}(V)) = \emptyset.\end{aligned}$$

Therefore, we have $\mathrm{Int}[F(U) \cap (Y \setminus V)] = \emptyset$.

(5) $\Rightarrow$ (6): For each open set V of Y containing $F(x)$, there exists $U \in m(x)$ such that $\mathrm{Int}[F(U) \cap (Y \setminus V)] = \emptyset$. Hence $\mathrm{Int}[F(U)) \cap (Y \setminus \mathrm{Cl}(V)] = \emptyset$ and $\mathrm{Int}(F(U)) \subset \mathrm{Cl}(V)$.

(6) $\Rightarrow$ (1): Let V be any open set of Y containing $F(x)$. By (6), there exists $U \in m(x)$ such that $\mathrm{Int}(F(U)) \subset \mathrm{Cl}(V)$. Let $M = F(U) \cap (Y \setminus V)$. Then $\mathrm{Int}(M) \subset \mathrm{Int}(F(U)) \cap \mathrm{Int}(Y \setminus V) = \mathrm{Int}(F(U)) \cap (Y \setminus \mathrm{Cl}(V)) = \emptyset$. Hence M is a rare set and $M \cap V = \emptyset$. Let $N = \mathrm{Cl}(V) \setminus V$. Then N is a closed rare set such that $N \cap V = \emptyset$. Therefore, $R_V = M \cup N$ is a rare set and $R_V \cap V = \emptyset$ by using Lemma 4: $\mathrm{Int}(R_V) = \mathrm{Int}(\mathrm{Int}(R_V)) = \mathrm{Int}(\mathrm{Int}(M \cup N)) \subset \mathrm{Int}(N) = \emptyset$. Now, we have the following:

$$\begin{aligned}F(U) &= [F(U) \setminus \mathrm{Int}(F(U)] \cup \mathrm{Int}(F(U))] \\ &\subset [F(U) \setminus \mathrm{Int}(F(U)] \cup \mathrm{Cl}(V) \\ &= [(F(U)) \cap (V \cup (Y \setminus V)) \setminus \mathrm{Int}(F(U)] \cup [(\mathrm{Cl}(V) \setminus V) \cup V] \\ &= (F(U) \cap V) \cup (F(U) \cap (Y \setminus V)) \setminus \mathrm{Int}(F(U)] \cup [N \cup V] \\ &\subset [V \cup (F(U) \cap (Y \setminus V))] \cup [N \cup V] \\ &= V \cup (M \cup N)\end{aligned}$$

$= V \cup R_V$.
Consequently, there exists a rare set $R_V = M \cup N$ such that $R_V \cap V = \emptyset$ and $F(U) \subset V \cup R_V$. Thus, F is upper rarely m-continuous at $x \in X$.

Theorem 2. *For a multifunction $F : (X, m) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is lower rarely m-continuous at $x \in X$;

(2) for each open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists a rare set R_V with $V \cap R_V = \emptyset$ such that $x \in \mathrm{mInt}(F^-(V \cup R_V))$;

(3) for each open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists a rare set R_V with $\mathrm{Cl}(V) \cap R_V = \emptyset$ such that $x \in \mathrm{mInt}(F^-(\mathrm{Cl}(V) \cup R_V))$;

(4) for each regular open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists a rare set R_V with $V \cap R_V = \emptyset$ such that $x \in \mathrm{mInt}(F^-(V \cup R_V))$.

Proof. The proofs of (1) $\Rightarrow$ (2), (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4) are similar with Theorem 1.

(4) $\Rightarrow$ (1): Let V be an open set in Y such that $F(x) \cap V \neq \emptyset$. Then $F(x) \cap \mathrm{Int}(\mathrm{Cl}(V)) \neq \emptyset$. By (4), there exists a rare set R_V with $\mathrm{Int}(\mathrm{Cl}(V)) \cap R_V = \emptyset$ such that $x \in \mathrm{mInt}(F^-(\mathrm{Int}(\mathrm{Cl}(V)) \cup R_V))$. Since $\mathrm{Cl}(V) \setminus V$ is a closed rare set, by Lemma 4, $(\mathrm{Cl}(V) \setminus V) \cup R_V$ is a rare set. Therefore, $S_V = [\mathrm{Int}(\mathrm{Cl}(V)) \setminus V] \cup R_V$ is a rare set. And $\mathrm{Int}(\mathrm{Cl}(V)) \cup R_V = V \cup [\mathrm{Int}(\mathrm{Cl}(V)) \setminus V] \cup R_V = V \cup S_V$. Therefore, $x \in \mathrm{mInt}(F^-(V \cup S_V))$. Hence there exists $U \in m(x)$ such that $U \subset F^-(V \cup S_V)$ and $F(u) \cap (V \cup S_V) \neq \emptyset$ for every $u \in U$.

Remark 2. If $m = \tau$ (resp. $\alpha(X), \beta(X), SO(X)$), then we have characterizations in [25] (resp. [4], [14], [15]).

By Theorem 1, we have the following characterizations of rare m-continuity for a function $f : (X, m) \to (Y, \sigma)$

Corollary 1. *For a function $f : (X, m) \to (Y, \sigma)$, the following properties are equivalent:*

(1) f is rarely m-continuous at $x \in X$;

(2) for each open set V of Y containing f(x), there exists a rare set R_V with $V \cap R_V = \emptyset$ such that $x \in \mathrm{mInt}(f^{-1}(V \cup R_V))$;

(3) for each open set V of Y containing f(x), there exists a rare set R_V with $\mathrm{Cl}(V) \cap R_V = \emptyset$ such that $x \in \mathrm{mInt}(f^{-1}(\mathrm{Cl}(V) \cup R_V))$;

(4) for each regular open set V of Y containing f(x), there exists a rare set R_V with $V \cap R_V = \emptyset$ such that $x \in \mathrm{mInt}(F^{-1}(V \cup R_V))$;

(5) for each open set V of Y containing f(x), there exists $U \in m(x)$ such that $\mathrm{Int}[f(U) \cap (Y \setminus V)] = \emptyset$,

(6) for each open set V of Y containing f(x), there exists $U \in m(x)$ such that $\mathrm{Int}(f(U)) \subset \mathrm{Cl}(V)$.

Remark 3. If $m = \tau$ (resp. α, PO(X), SO(X)), then by Corollary 1, we obtain the characterizations of rare continuity [19] (resp. rare α-continuity [13], rare pre-continuity [12], rare quasicontinuity [26]).

4. Ideal Topological Spaces

Definition 8. A nonempty collection I of subsets of a set X is called an *ideal* on X [17], [30] if it satisfies the following two conditions:
(1) $A \in I$ and $B \subset A$ implies $B \in I$,
(2) $A \in I$ and $B \in I$ implies $A \cup B \in I$.

A topological space (X, τ) with an ideal I on X is called an *ideal topological space* and is denoted by (X, τ, I). Let (X, τ, I) be an ideal topological space. For any subset A of X, $A^\star(I, \tau) = \{x \in X : U \cap A \notin I$ for every $U \in \tau(x)\}$, where $\tau(x) = \{U \in \tau : x \in U\}$, is called the *local function* of A with respect to τ and I [16]. Hereafter $A^\star(I, \tau)$ is simply denoted by $A^\star$. It is well known that $\mathrm{Cl}^\star(A) = A \cup A^\star$ defines a Kuratowski closure operator on X and the topology generated by $\mathrm{Cl}^\star$ is denoted by $\tau^\star$.

Lemma 5. *Let (X, τ, I) be an ideal topological space and A, B be subsets of X. Then the following properties hold:*
(1) $A \subset B$ implies $\mathrm{Cl}^\star(A) \subset \mathrm{Cl}^\star(B)$,
(2) $\mathrm{Cl}^\star(X) = X$ and $\mathrm{Cl}^\star(\emptyset) = \emptyset$,
(3) $\mathrm{Cl}^\star(A) \cup \mathrm{Cl}^\star(B) \subset \mathrm{Cl}^\star(A \cup B)$.

Definition 9. Let (X, τ, I) be an ideal topological space. A subset A of X is said to be
(1) *α-I-open* [8] if $A \subset \mathrm{Int}(\mathrm{Cl}^\star(\mathrm{Int}(A)))$,
(2) *semi-I-open* [8] if $A \subset \mathrm{Cl}^\star(\mathrm{Int}(A))$,
(3) *pre-I-open* [2] if $A \subset \mathrm{Int}(\mathrm{Cl}^\star(A))$,
(4) *b-I-open* [5] if $A \subset \mathrm{Int}(\mathrm{Cl}^\star(A)) \cup \mathrm{Cl}^\star(\mathrm{Int}(A))$,
(5) *β-I-open* [9] if $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}^\star(A)))$,
(6) *weakly semi-I-open* [6] if $A \subset \mathrm{Cl}^\star(\mathrm{Int}(\mathrm{Cl}(A)))$,
(7) *weakly b-I-open* [22] if $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}^\star(A))) \cup \mathrm{Cl}^\star(\mathrm{Int}(\mathrm{Cl}(A)))$,
(8) *strongly β-I-open* [7] if $A \subset \mathrm{Cl}^\star(\mathrm{Int}(\mathrm{Cl}^\star(A)))$.

Among the sets in Definition 9, we have the following relations:

DIAGRAM

$$\begin{array}{ccccccc}
\text{open} \Rightarrow & \alpha\text{-}I\text{-open} & \Rightarrow & \text{semi-}I\text{-open} & \Rightarrow & \text{weakly semi-}I\text{-open} \\
 & \Downarrow & & \Downarrow & & \Downarrow \\
 & \text{pre-}I\text{-open} & \Rightarrow & b\text{-}I\text{-open} & \Rightarrow & \text{weakly } b\text{-}I\text{-open} \\
 & & & \Downarrow & & \Uparrow \\
 & & & \text{strongly } \beta\text{-}I\text{-open} & \Rightarrow & \beta\text{-}I\text{-open}
\end{array}$$

The family of all α-I-open (resp. semi-I-open, pre-I-open, b-I-open, β-I-open, weakly semi-I-open, weakly b-I-open, strongly β-I-open) sets in an ideal topological space (X, τ, I) is denoted by $\alpha\mathrm{IO}(X)$ (resp. $\mathrm{SIO}(X)$, $\mathrm{PIO}(X)$, $\mathrm{BIO}(X)$, $\beta\mathrm{IO}(X)$, $\mathrm{WSIO}(X)$, $\mathrm{WBIO}(X)$, $\mathrm{S}\beta\mathrm{IO}(X)$).

Remark 4. If $I = \{\emptyset\}$, then $A^\star = \mathrm{Cl}(A)$ and $\mathrm{Cl}^\star(A) = A^\star \cup A = \mathrm{Cl}(A)$. Therefore,

(1) $\tau^\star = \tau$, $\alpha\mathrm{IO}(X) = \alpha(X)$, $\mathrm{SIO}(X) = \mathrm{SO}(X)$, $\mathrm{PIO}(X) = \mathrm{PO}(X)$, $\mathrm{BIO}(X) = \mathrm{BO}(X)$ and $\beta\mathrm{IO}(X) = \beta(X)$.

(2) $\mathrm{WSIO}(X)$, $\mathrm{WBIO}(X)$, $\mathrm{S}\beta\mathrm{IO}(X)$ and $\beta\mathrm{IO}(X)$ are coincide with $\beta(X)$.

Definition 10. By $\mathrm{mIO}(X)$, we denote each one of the families $\tau^\star$, $\alpha\mathrm{IO}(X)$, $\mathrm{SIO}(X)$, $\mathrm{PIO}(X)$, $\mathrm{BIO}(X)$, $\beta\mathrm{IO}(X)$, $\mathrm{WSIO}(X)$, $\mathrm{WBIO}(X)$, $\mathrm{S}\beta\mathrm{IO}(X)$.

Lemma 6. *Let (X, τ, I) be an ideal topological space. Then* $\mathrm{mIO}(X)$ *is an m-structure and has property $\mathcal{B}$.*

Proof. The proof follows from Lemma 5(1)(2). As an example, we shall show that $\alpha\mathrm{IO}(X)$ has property $\mathcal{B}$. Let A_α be an α-I-open set for each $\alpha \in \Lambda$. Then $A_\alpha \subset \mathrm{Int}(\mathrm{Cl}^\star(\mathrm{Int}(A_\alpha))) \subset \mathrm{Int}(\mathrm{Cl}^\star(\mathrm{Int}(\cup_{\alpha\in\Lambda} A_\alpha)))$ for each $\alpha \in \Lambda$ and hence $\cup_{\alpha\in\Lambda} A_\alpha \subset \mathrm{Int}(\mathrm{Cl}^\star(\mathrm{Int}(\cup_{\alpha\in\Lambda} A_\alpha)))$. Therefore, $\cup_{\alpha\in\Lambda} A_\alpha$ is α-I-open.

Remark 5. It is shown in Theorem 3.4 of [8] (resp. Theorem 2.10 of [2], Theorem 2.1 of [6], Theorem 2.7 of [22], Proposition 3 of [7]) that $\mathrm{SIO}(X)$ (resp. $\mathrm{PIO}(X)$, $\mathrm{WSIO}(X)$, $\mathrm{WBIO}(X)$, $\mathrm{S}\beta\mathrm{IO}(X)$) has property $\mathcal{B}$.

Definition 11. Let (X, τ, I) be an ideal topological space. For a subset A of X, the *$mIO(X)$-closure* $\mathrm{mCl_I}(A)$ and the *$mIO(X)$-interior* $\mathrm{mInt_I}(A)$ are defined as follows:

(1) $\mathrm{mCl_I}(A) = \cap\{F : A \subset F, X \setminus F \in \mathrm{mIO}(X)\}$,

(2) $\mathrm{mInt_I}(A) = \cup\{U : U \subset A, U \in \mathrm{mIO}(X)\}$.

Let (X, τ, I) be an ideal topological space and $\mathrm{mIO}(X)$ the m-structure on X. If $\mathrm{mIO}(X) = \alpha\mathrm{IO}(X)$ (resp. $\mathrm{SIO}(X)$, $\mathrm{PIO}(X)$, $\mathrm{BIO}(X)$, $\beta\mathrm{IO}(X)$, $\mathrm{WSIO}(X)$, $\mathrm{WBIO}(X)$, $\mathrm{S}\beta\mathrm{IO}(X)$), then we have

(1) $\mathrm{mCl_I}(A) = \alpha\mathrm{Cl_I}(A)$ (resp. $\mathrm{sCl_I}(A)$, $\mathrm{pCl_I}(A)$, $\mathrm{bCl_I}(A)$, $\beta\mathrm{Cl_I}(A)$, $\mathrm{wsCl_I}(A)$, $\mathrm{wbCl_I}(A)$, $\mathrm{s}\beta\mathrm{Cl_I}(A)$),

(2) $\mathrm{mInt_I}(A) = \alpha\mathrm{Int_I}(A)$ (resp. $\mathrm{sInt_I}(A)$, $\mathrm{pInt_I}(A)$, $\mathrm{bInt_I}(A)$, $\beta\mathrm{Int_I}(A)$, $\mathrm{wsInt_I}(A)$, $\mathrm{wbInt_I}(A)$, $\mathrm{s}\beta\mathrm{Int_I}(A)$).

5. Rarely m-I-continuous multifunctions

In this section, by using the results in Section 3, we obtain properties of upper/lower rarely m-I-continuous multifunction $F : (X, \tau, I) \to (Y, \sigma)$.

Definition 12. Let (X, τ, I) be an ideal topological space and (Y, σ) a topological space. A multifunction $F : (X, \tau, I) \to (Y, \sigma)$ is said to be

(1) *upper rarely m-I-continuous* at a point $x \in X$ if for each open set V containing $F(x)$, there exist a rare set R_V with $R_V \cap V = \emptyset$ and an mI-open set $U \in mIO(X)$ containing x such that $F(U) \subset V \cup R_V$,

(2) *lower rarely m-I-continuous* at a point $x \in X$ if for each open set V meeting $F(x)$, there exist a rare set R_V with $R_V \cap V = \emptyset$ and an m-open set $U \in mIO(X)$ containing x

such that $F(u) \cap (V \cup R_V) \neq \emptyset$ for each $u \in U$,

Lemma 7. *A multifunction $F : (X, \tau, I) \to (Y, \sigma)$ is upper/lower rarely m-I-continuous at $x \in X$ if and only if a multifunction $F : (X, mIO(X)) \to (Y, \sigma)$ is upper/lower rarely m-continuous at $x \in X$.*

Proof. This is obvious from Definitions 7 and 12

Theorem 3. *For a multifunction $F : (X, \tau, I) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is upper rarely m-I-continuous at $x \in X$;

(2) for each open set V of Y containing F(x), there exists a rare set R_V with $V \cap R_V = \emptyset$ such that $x \in \mathrm{mInt_I}(F^+(V \cup R_V))$;

(3) for each open set V of Y containing F(x), there exists a rare set R_V with $\mathrm{Cl}(V) \cap R_V = \emptyset$ such that $x \in \mathrm{mInt_I}(F^+(\mathrm{Cl}(V) \cup R_V))$;

(4) for each regular open set V of Y containing F(x), there exists a rare set R_V with $V \cap R_V = \emptyset$ such that $x \in \mathrm{mInt_I}(F^+(V \cup R_V))$;

(5) for each open set V of Y containing F(x), there exists $U \in mIO(X)$ containing x such that $\mathrm{Int}[F(U) \cap (Y \setminus V)] = \emptyset$,

(6) for each open set V of Y containing F(x), there exists $U \in mIO(X)$ containing x such that $\mathrm{Int}(F(U)) \subset \mathrm{Cl}(V)$.

Theorem 4. *For a multifunction $F : (X, \tau, I) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is lower rarely m-I-continuous at $x \in X$;

(2) for each open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists a rare set R_V with $V \cap R_V = \emptyset$ such that $x \in \mathrm{mInt_I}(F^-(V \cup R_V))$;

(3) for each open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists a rare set R_V with $\mathrm{Cl}(V) \cap R_V = \emptyset$ such that $x \in \mathrm{mInt_I}(F^-(\mathrm{Cl}(V) \cup R_V))$;

(4) for each regular open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists a rare set R_V with $V \cap R_V = \emptyset$ such that $x \in \mathrm{mInt_I}(F^-(V \cup R_V))$.

By Theorem 3, we have the following characterizations of rare m-I-continuity of function $f : (X, \tau, I) \to (Y, \sigma)$

Corollary 2. *For a function $f : (X, \tau, I) \to (Y, \sigma)$, the following properties are equivalent:*

(1) f is rarely m-I-continuous at $x \in X$;

(2) for each open set V of Y containing f(x), there exists a rare set R_V with $V \cap R_V = \emptyset$ such that $x \in \mathrm{mInt_I}(f^{-1}(V \cup R_V))$;

(3) for each open set V of Y containing f(x), there exists a rare set R_V with $\mathrm{Cl}(V) \cap R_V = \emptyset$ such that $x \in \mathrm{mInt_I}(f^{-1}(\mathrm{Cl}(V) \cup R_V))$;

(4) for each regular open set V of Y containing f(x), there exists a rare set R_V with $V \cap R_V = \emptyset$ such that $x \in \mathrm{mInt_I}(f^{-1}(V \cup R_V))$;

(5) for each open set V of Y containing f(x), there exists $U \in mIO(X)$ containing x

such that $\mathrm{Int}[f(U) \cap (Y \setminus V)] = \emptyset$,

(6) for each open set V of Y containing f(x), there exists $U \in mIO(X)$ *containing* x *such that* $\mathrm{Int}(f(U)) \subset \mathrm{Cl}(V)$.

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