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L-Convex Sublattice of an L-Lattice and Complement of an L-set

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Abstract. In this paper, the authors define and explore the notion of an L-convex sublattice in an L-lattice. The investigations in this paper lead to 'The Unique Representation Theorem' for L-convex sublattices. Also, the authors effectively use the concept of order reversing involution on the lattice L of truth values to define complement of an L-set. Further, they employ this notion in the studies of L- prime ideals and L-maximal ideals.

2020 Mathematics Subject Classifications: 06B10, 06D72, 06D75, 08A72

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1. Introduction

The literature on fuzzy algebraic structures has been growing ever since the introduction of the concept of a fuzzy subgroup by A. Rosenfeld [14] in the year 1971. Ajmal and Thomas [4–6] systematically developed the theory of fuzzy sublattices in a lattice. They introduced the notions of a fuzzy sublattice, fuzzy ideal (dual ideal), fuzzy prime ideal (dual ideal), fuzzy ideal (dual ideal) generated by a fuzzy set and studied their properties. The concept of a fuzzy convex sublattice was also introduced by Ajmal and Thomas in [4, 5], wherein the Unique Representation Theorem for convex sublattices was extended to fuzzy setting.

The concept of an L-fuzzy set was pioneered by Goguen [7] in the year 1967 .In [10], the authors studied the concept of an L-lattice. This shifts their studies from the evaluation lattice $[0, 1]$ to a more general lattice L. Moreover in [10], authors made one more transition by studying the notions of L-substructures in an L-lattice instead of fuzzy substructures of an ordinary lattice. Thus, the parent structure also shifts from a lattice to an L-lattice. It is worthwhile to mention here that under this arrangement, some notions

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such as L-maximal ideal can be defined more meaningfully in the L-setting.

In past few years, Ajmal and Jahan have successfully developed the theory of L-subgroups in [1–3, 8, 9, 12]]. They have taken the theory of L-subgroups towards completion by studying the concepts of characteristic subgroups, normalizer of a subgroup, nilpotent subgroups, solvable subgroups, normal closure of a subgroup etc., within the framework of L-groups. In [9], Jahan and Manas studied maximal and Frattini L-subgroups of an L-group.

In [10], the notions of an L-maximal ideal and L-prime ideal in an L-lattice are defined and various related results are studied. In order to take such studies further, in the present paper, we introduce the notion of an L-convex sublattice in an L-lattice.Then this notion of convex L-sublattice is used to demonstrate that the Unique Representation Theorem of classical lattice theory for convex sublattices also holds under the L-setting wherein the parent structure is an L-lattice.

In the last section of this paper, we use the notion of an order reversing involution on a lattice to define the concept of complement of an L-set. The notion of order reversing involution occurs frequently in fuzzy topological spaces and fuzzy implication algebras [11, 13, 15–17]. Thereafter, we establish some significant analogues of results of classical lattice theory to L-setting using complement of an L-set, thereby taking the theory of L-lattices to a more developed stage.

2. Preliminaries

In this work, $(M, \leq, \wedge, \vee)$ denotes a bounded lattice and $(L, \leq, \wedge, \vee)$ a complete lattice. The maximal and minimal elements of both the lattices L and M are denoted by 1 and 0 respectively. The notations '$\leq$', '$\wedge$' and '$\vee$' denote the partial order, meet and join operations respectively of both the lattices L and M. An L-subset of M is defined as a mapping $\mu : M \to L$. The collection of all L-subsets of M is denoted by L^M and is called the L-power set of M. If $\mu, \eta \in L^M$, η is said to be contained in μ(denoted by $\eta \subseteq \mu$), if $\eta(x) \leq \mu(x)$, $\forall\ x \in M$. Moreover, η is said to be properly contained in μ, if $\eta \subseteq \mu$ and there exists $x \in M$ such that $\eta(x) < \mu(x)$. If $\eta \subseteq \mu$, then η is said to be an L-subset of an L-set μ. The set of all L-subsets of μ is called the L-power set of μ and is denoted by L^μ.

If $\mu \in L^M$ and $\alpha \in L$, the level subset μ_α and the strong level subset $\mu_\alpha^>$ are defined as follows:

$$\mu_\alpha = \{x \in M / \mu(x) \geq \alpha\} \quad \text{and} \quad \mu_\alpha^> = \{x \in M / \mu(x) > \alpha\}.$$

Clearly, $\mu_\alpha^> \subseteq \mu_\alpha$, $\forall\ \alpha \in L$ and if $\alpha \leq \beta$ in L, then $\mu_\beta \subseteq \mu_\alpha$ and $\mu_\beta^> \subseteq \mu_\alpha^>$.

If $\mu \in L^M$, then $\vee_{x\in M}\mu(x)$ and $\wedge_{x\in M}\mu(x)$ are called the tip and tail of μ, respectively. The arbitrary union $\cup_{i\in I}(\mu_i)$ and intersection $\cap_{i\in I}(\mu_i)$ of a family $\{\mu_i\}_{i\in I}$ of L-subsets of M are given by:

$$(\cup_{i\in I}\mu_i)(x) = \vee_{i\in I}\mu_i(x) \text{ and } (\cap_{i\in I}\mu_i)(x) = \wedge_{i\in I}\mu_i(x).$$

Definition 1 ([4]). *Let $\mu \in L^M$. Then, μ is said to be an L-sublattice of M if $\forall\, x, y \in M$*

$$\mu(x \vee y) \geq \mu(x) \wedge \mu(y) \quad \textit{and} \quad \mu(x \wedge y) \geq \mu(x) \wedge \mu(y).$$

Let $L(M)$ denote the set of all L-sublattices of M. If $\mu \in L(M)$, μ is called an L-Lattice and is denoted by $L(\mu, M)$. If $\mu, \eta \in L(M)$ and $\eta \subseteq \mu$, then η is called an L-sublattice of the L-lattice μ. The collection of all L-sublattices of μ is denoted by $L(\mu)$. In this paper, we shall study the L-convex sublattices of an L-lattice μ rather convex sublattices of an ordinary lattice.
The following theorems provide the level subset characterizations and strong level subset characterizations of an L-sublattice of μ. For similar characterizations of L-sublattices of M, we refer to [10].

Theorem 1 ([10]). *Let $\mu, \eta \in L^M$ be such that $\eta \subseteq \mu$. Also, let $L(\mu, M)$ be an L-lattice and $a_o = tip\{\eta\}$. Then, η is an L-sublattice of μ if and only if each level subset η_α is a sublattice of μ_α, $\forall \alpha \leq a_o$. Equivalently, η is an L-sublattice of μ if and only if each nonempty level subset η_α is a sublattice of μ_α.*

Theorem 2 ([10]). *Let L be a chain. Let $\mu, \eta \in L^M$ be such that $\eta \subseteq \mu$. Also, let $L(\mu, M)$ be an L-lattice and $a_o = tip\{\eta\}$. Then, η is an L-sublattice of μ if and only if each strong level subset $\eta_\alpha^>$ is a sublattice of $\mu_\alpha^>$, $\forall\ \alpha < a_o$. Equivalently, η is an L-sublattice of μ if and only if each nonempty strong level subset $\eta_\alpha^>$ is a sublattice of $\mu_\alpha^>$.*

The notions of L-ideal, L-dual ideal in lattice M and L-ideal, L-dual ideal in an L-lattice μ are defined as follows:

Definition 2 ([10]). *Let $\mu \in L^M$. Then,*

[(i)]μ is called an L-ideal of M if $\mu \in L(M)$ and $x \leq y$ in M implies $\mu(x) \geq \mu(y)$ in L;
μ is called an L-dual ideal of M if $\mu \in L(M)$ and $x \leq y$ *in* M implies $\mu(x) \leq \mu(y)$ in L.

Definition 3 ([10]). *Let $\mu, \eta \in L^M$ be such that $\eta \subseteq \mu$. Also, let $L(\mu, M)$ be an L-lattice. Then,*

[(i)]η is called an L-ideal of μ if

$$\eta(x \vee y) \geq \eta(x) \wedge \eta(y) \text{ and } \eta(x \wedge y) \geq \mu(x) \wedge \eta(y); \quad \forall x, y \in M.$$

η is called an L-dual ideal of μ if

$$\eta(x \wedge y) \geq \eta(x) \wedge \eta(y) \text{ and } \eta(x \vee y) \geq \eta(x) \wedge \mu(y); \quad \forall x, y \in M.$$

It is important to note here that in a bounded lattice M, an L-ideal attains its supremum at the least element of M, whereas an L-dual ideal attains its supremum at the greatest element of M. The following theorems provide the level subset characterizations and strong level subset characterizations of an L-ideal (L-dual ideal) of μ. For similar characterizations of L-ideal (L-dual ideal) of M, refer to [10].

Theorem 3 ([10]). *Let $\mu, \eta \in L^M$ be such that $\eta \subseteq \mu$. Also, let $L(\mu, M)$ be an L-lattice and $a_o = tip\{\eta\}$. Then, η is an L-ideal(L-dual ideal) of μ if and only if each level subset η_α is an ideal(dual ideal) of μ_α, $\forall \alpha \leq a_o$. Equivalently, η is an L-ideal(L-dual ideal) of μ if and only if each nonempty level subset η_α is an ideal(dual ideal) of μ_α.*

Theorem 4 ([10]). *Let L is a chain. Let $\mu, \eta \in L^M$ be such that $\eta \subseteq \mu$. Also, let $L(\mu, M)$ be an L-lattice and $a_o = tip\{\eta\}$. Then, η is an L-ideal(L-dual ideal) of μ if and only if each strong level subset $\eta^{>}_\alpha$ $\forall \alpha < a_o$, is an ideal (dual ideal) of $\mu^{>}_\alpha$. Equivalently, η is an L-ideal(L-dual ideal) of μ if and only if each nonempty strong level subset $\eta^{>}_\alpha$ is an ideal (dual ideal) of $\mu^{>}_\alpha$.*

It can be easily verified that the intersection of an arbitrary family of L-sublattices(L-ideals, L-dual ideals) of μ is an L-sublattice(L-ideal, L-dual ideal) of μ. This leads to the definition of an L-sublattice(L-ideal, L-dual ideal) generated by an L-subset η of μ as the intersection of all L-sublattices(resp. L-ideals, L-dual ideals) of μ containing η. These are denoted by $[\eta]_\mu$, $(\eta]_\mu$ and $[\eta)_\mu$ respectively. The following result from [3] gives the structural compositions of an L-sublattice, L-ideal, L-dual ideal of an L-lattice μ generated by an L-subset η of μ in terms of level subsets.

Theorem 5 ([10]). *Let $L(\mu, M)$ be an L-lattice, $\eta \in L^M$, $\eta \subseteq \mu$ with $a_o = tip\{\eta\}$.*

[(i)]Define an L-subset η_o of M as:

$$\eta_o(x) = \bigvee_{t \leq a_o} \{t : x \in [\eta_t]\},$$

where $[\eta_t]$ is a sublattice of μ_t generated by η_t. Then, η_o is an L-sublattice of μ and $\eta_o = [\eta]_\mu$. Define an L-subset η_1 of M as:

$$\eta_1(x) = \bigvee_{t \leq a_o} \{t : x \in (\eta_t]\},$$

where $(\eta_t]$ is an ideal of μ_t generated by η_t. Then, η_1 is an L-ideal of μ and $\eta_1 = (\eta]_\mu$. Define an L-subset η_2 of M as:

$$\eta_2(x) = \bigvee_{t \leq a_o} \{t : x \in [\eta_t)\},$$

where $[\eta_t)$ is an dual ideal of μ_t generated by η_t. Then, η_2 is an L-dual ideal of μ and $\eta_2 = [\eta)_\mu$.

The concept of a maximal ideal could be meaningfully extended from classical setting to fuzzy setting by the authors in [10] by shifting the parent structure from classical lattice to an L-structure as follows:

Definition 4 ([10])**.** *Let $L(\mu, M)$ be an L-lattice.*

[(i)]A proper L-ideal η of μ is called an L-maximal ideal of μ if for any L-ideal θ of μ, whenever $\eta \subseteq \theta \subseteq \mu$, then $\theta = \eta$ or $\theta = \mu$. A proper L-dual ideal η of μ is called an L-dual maximal ideal of μ if for any L-dual ideal θ of μ, whenever $\eta \subseteq \theta \subseteq \mu$, then $\theta = \eta$ *or* $\theta = \mu$.

In [10], some characterizations of an L-maximal ideal and L-maximal dual ideal of μ were provided. Further, an L-prime ideal(L-prime dual ideal) in lattice M and an L-prime ideal(L-prime dual ideal) in an L-lattice μ are defined as follows:

Definition 5 ([4])**.** [(i)]

(i) An L-ideal μ of M is called an L-prime ideal of M if

$$\mu(x \wedge y) \leq \mu(x) \vee \mu(y); \quad \forall x, y \in M.$$

(ii) An L-dual ideal μ of M is called an L-prime dual ideal of M if

$$\mu(x \vee y) \leq \mu(x) \vee \mu(y); \quad \forall x, y \in M.$$

Definition 6 ([10])**.** *Let $L(\mu, M)$ be an L-lattice.*

[(i)]A proper L-ideal η of μ is called an L-prime ideal of μ if, $\forall\ x, y \in M$

$$\eta(x \wedge y) \wedge \mu(x) \wedge \mu(y) \leq \eta(x) \text{ or } \eta(x \wedge y) \wedge \mu(x) \wedge \mu(y) \leq \eta(y).$$

A proper L-dual ideal η of μ is called an L-dual prime ideal of μ if, $\forall\ x, y \in M$

$$\eta(x \vee y) \wedge \mu(x) \wedge \mu(y) \leq \eta(x) \quad \text{or} \quad \eta(x \vee y) \wedge \mu(x) \wedge \mu(y) \leq \eta(y).$$

In [10], the authors defined a fuzzy convex sublattice of a lattice and studied the related properties. On similar lines, an L-convex sublattice of a lattice M can be defined as follows:

Definition 7. *If $\mu \in L(M)$, then μ is called an L-convex sublattice of M if for each interval $[a, b] \subseteq M$,*

$$\mu(x) \geq \mu(a) \wedge \mu(b), \quad \forall\ x \in [a, b].$$

3. L-convex Sublattice of an L-lattice

In this section, L is taken to be a complete and completely distributive lattice in some results. The definition of a completely distributive lattice is well known in the literature and can be found in any standard text on the subject.
Let $\{J_i : i \in I\}$ be any family of subsets of a complete lattice L and F denote the set of choice functions for J_i, i.e. functions $f : I \to \prod_{i \in I} J_i$ such that $f(i) \in J_i$ for each $i \in I$. Then, we say that L is a completely distributive lattice, if

$$\bigwedge \left\{ \bigvee_{i \in I} J_i \right\} = \bigvee_{f \in F} \left\{ \bigwedge_{i \in I} f(i) \right\}.$$

The above law is known as the complete distributive law. Thus, in order theory, a complete lattice is completely distributive if arbitrary joins distribute over arbitrary meets. Note that the dual of completely distributive law is valid in a completely distributive lattice.

We begin this section by defining an L-convex sublattice of an L-lattice μ and study its properties.

Definition 8. *Let $L(\mu, M)$ be an L-lattice. An L-sublattice η of μ is called an L-convex sublattice of μ if*

$$\eta(x) \geq \eta(a) \wedge \eta(b) \wedge \mu(x) \text{ where } a \leq x \leq b \text{ in } M.$$

The following characterisations of an L-convex sublattice η of μ with the help of level subsets and strong level subsets of η can be verified easily.

Theorem 6. *Let $L(\mu, M)$ be an L-lattice and $\eta \in L(\mu)$ with $a_o = tip\{\eta\}$. Then, η is an L-convex sublattice of μ if and only if each level subset η_t, $\forall\ t \leq a_o$, is a convex sublattice of μ_t. Equivalently, η is an L-convex sublattice of μ if and only if each nonempty level subset η_t is a convex sublattice of μ_t.*

Theorem 7. *Let L be a chain. Let $L(\mu, M)$ be an L-lattice and $\eta \in L(\mu)$ with $a_o = tip\{\eta\}$. Then, η is an L-convex sublattice of μ if and only if each strong level subset $\eta_t^>$ is a convex sublattice of $\mu_t^>$, $\forall t < a_o$. Equivalently, η is an L-convex sublattice of μ if and only if each nonempty strong level subset $\eta_t^>$ is a convex sublattice of $\mu_t^>$.*

We now provide the following examples of L-convex sublattices in an L-lattice:

Example 1. *Let $M = \aleph$ be the chain of natural numbers and $L = P(\aleph)$, the power set of $\aleph$, be the Boolean Algebra. Define the following L-subsets of $\aleph$:*

$$\eta(n) = \begin{cases} \emptyset \text{ if } n = 1, \\ \{1, 2, ..., n-1\} \quad \forall n \geq 2; \end{cases}$$

and

$$\mu(n) = \{1, 2, ..., n\} \quad \forall n \in \aleph.$$

Then $\eta \subseteq \mu$. *Moreover, it is easy to verify that* η *and* μ *are* L*-sublattices of* $\aleph$. *Futhermore,* η *turns out to be an* L*-convex sublattice of* μ. *It is wothwhile to note here that the set of all level subsets of* η *form a chain in the lattice* $M = \aleph$. *Here, we write* $A_i = \{1, 2, \cdots, i-1\}$ $\forall i \geq 2$. *Further, note that*

$$\eta_{A_i} = A_i \text{ and } \eta_{\emptyset} = \emptyset.$$

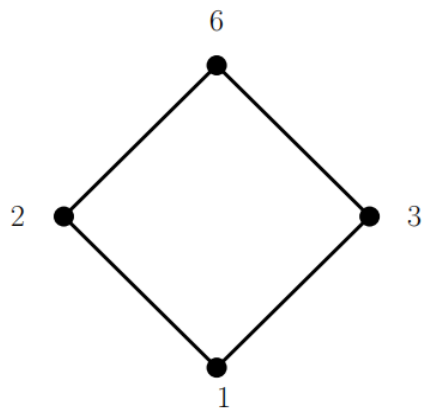
Thus, we have

$$A_1 = \emptyset \subseteq A_2 \subseteq A_3 \subseteq \cdots \subseteq A_n \subseteq \cdots \subseteq \aleph.$$

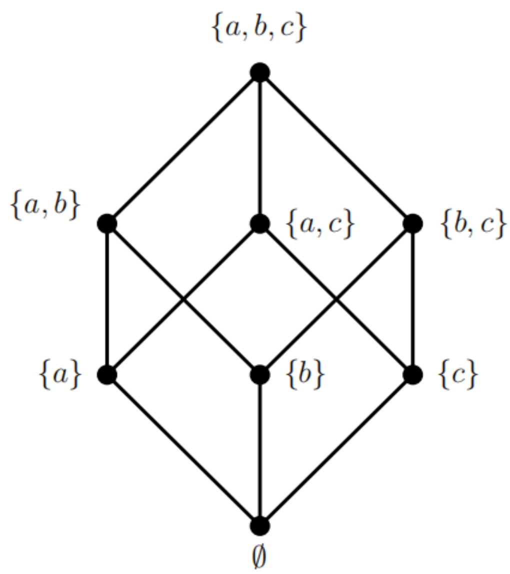
Example 2. *Let* $X = \{a, b, c\}$ *and* $L = P(X)$ *be the power set of* X. *Then* $\langle L, \cap, \cup, \prime \rangle$ *is a Boolean Algebra where* $'\cup'$, $'\cap'$ *and* $'\prime'$ *denote the ordinary intersection, union and complement of members of* L *respectively. Further, It is easy to see that* L *is Boolean Algebra with order reversing involution given by :*

$$\tau : L \longrightarrow L^*, \tau(A) = A'.$$

Let $M = \{1, 2, 3, 6\}$ *denote the set of all factors of* $'\mathbf{6}'$. *Then* $\langle M, \vee, \wedge, \prime \rangle$, *where* $a \vee b = lcm\{a, b\}$, $a \wedge b = gcd\{a, b\}$ *and* $a' = \frac{6}{a}$; $\forall\ a,\ b \in M$, *is also a Boolean Algebra. In the following diagram, (i) and (ii) represent Boolean Algebras* M *and* L *respectively.*



(i)



(ii)

Define the following L*-subsets* μ *and* η *of* M *:*

$$\mu(A) = \begin{cases} 2 & \text{if } A \in \{\emptyset, \{a\}, \{b\}, \{a, b\}\}, \\ 6 & \text{if } A = P(X) \setminus \{\emptyset, \{a\}, \{b\}, \{a, b\}\}; \end{cases}$$

and

$$\eta(A) = \begin{cases} 1 & \text{if } A \in \{\emptyset, \{b\}\}, \\ 2 & \text{if } A \in \{\{a\}, \{a,b\}\}, \\ 3 & \text{if } A \in \{c, \{\{b,c\}\}, \\ 6 & \text{if } A \in \{\{a,c\}, X\}. \end{cases}$$

Now, note that $\eta \subseteq \mu$. *The set* $\{\eta_a \; : \; a \in Im \; \eta\}$ *of all level subset of* η *is determined below :*

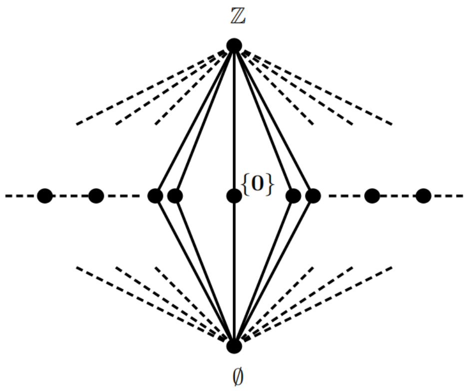
$$\eta_1 = M,\; \eta_2 = \{\{a\}, \{a,b\}, \{a,c\}, X\},\; \eta_3 = \{\{c\}, \{b,c\}, \{a,c\}, X\} \text{ and } \eta_6 = \{\{a,c\}, X\}.$$

Further, the set $\{\mu_a \; : \; a \in Im \; \mu\}$ *of all level subset of* μ *is determined below :*

$$\mu_2 = M,\; \text{and } \mu_6 = \{\{c\}, \{b,c\}\{a,c\}, X\}.$$

Now, it is easy to see that η *and* μ *are L-sublattices of* M. *Futhermore,* η *forms an L-convex sublattice of* μ. *Observe that in this example the set of all level subsets of* η *does not form a chain. Infact, the set of all level subsets* $\{\eta_a : a \in Im \; \eta\}$ *turns out to be only a poset under the ordering of usual set theoretic containment.*

Example 3. *Let* $M = \emptyset \cup \mathbb{Z} \cup \{\{n\} : n \in \mathbb{Z}\}$. *Then* M *is a Boolean Algebra with the following Hasse Diagram :*



Further, let $L = \{A \subseteq \mathbb{R} : \text{either } A \text{ or } A' \text{ is finite}\}$. *Here* A' *is complement of* A *in* $\mathbb{R}$. *It is easy to see that* L *is Boolean Algebra with order reversing involution given by :*

$$\tau : L \longrightarrow L^*, \tau(A) = A'.$$

Define the following L-subsets of M *:*

$$\eta(A) = \begin{cases} \emptyset & \text{if } A = \mathbb{Z}, \\ \mathbb{R} & \text{if } A = \emptyset, \\ \{n\} & \text{if } n \in \mathbb{Z}; \end{cases}$$

and

$$\mu(A) = \begin{cases} \emptyset & \text{if } A = \mathbb{Z}, \\ \mathbb{R} & \text{if } A = \emptyset, \\ \{n, -n\} & \text{if } n \in \mathbb{Z}. \end{cases}$$

Now, note that $\eta \subseteq \mu$. The set $\{\eta_a \ : \ a \in Im\ \eta\}$ of all level subset of η is determined below :

$$\eta_{\mathbb{R}} = \emptyset, \eta_{\{n\}} = \{\mathbb{Z}, \{n\}\} \text{ and } \eta_{\emptyset} = M.$$

Further, the set $\{\mu_a \ : \ a \in Im\ \mu\}$ of all level subset of μ is determined below :

$$\mu_{\mathbb{R}} = \emptyset, \mu_{\{\pm\ n\}} = \{\mathbb{Z}, \{n\}\}, \text{ and } \mu_{\emptyset} = M.$$

Now, it is easy to see that η and μ are L-sublattices of M. Futhermore, η forms an L-convex sublattice of μ. Observe that in this example the Hesse Diagram of set of all level subsets of both η and μ coincide with that of Hasse Diagram of the lattice M given above. Infact, the set of all level subsets $\{\eta_a : a \in Im\ \eta\}$ turns out to be lattice under the usual set theoretic containment.

The following result is also straightforward.

Theorem 8. *The intersection of an arbitrary family of L-convex sublattices of L-lattice μ is an L-convex sublattice of μ.*

The above result is instrumental in defining an L-convex sublattice of μ generated by an L-subset η of μ.

Definition 9. *An L-convex sublattice of L-lattice μ generated by an L-subset η of μ is defined as the intersection of all L-convex sublattices of μ containing η and is denoted by $[\eta]^c_\mu$. Thus,*

$$[\eta]^c_\mu = \bigcap \{\eta_i : \eta_i \text{ is an } L\text{-convex sublattice of } \mu,\ \eta \subseteq \eta_i,\ \forall\ i \in I\}.$$

The next result provides a complete structure of L-convex sublattice generated by L-subset η of μ in terms of level subsets.

Theorem 9. *Let L be a complete and completely distributive lattice and $L(\mu, M)$ be an L-lattice. Let $\eta \in L^M$ with $\eta \subseteq \mu$ and $a_0 = tip\{\eta\}$. Define an L-subset η' of M as:*

$$\eta'(x) = \bigvee_{t \leq a_o} \{t : x \in [\eta_t]_c\},\ \forall\ x \in M;$$

where $[\eta_t]_c$ is the convex sublattice of lattice μ_t generated by η_t. Then, η' is an L-convex sublattice of μ and $\eta' = [\eta]^c_\mu$.

Proof. Since $\eta \subseteq \mu$, $\eta_t \subseteq \mu_t$, $\forall\ t \in L$. As μ is an L-lattice, μ_t is a sublattice of M, $\forall$ $t \leq tip\{\mu\}$. Moreover, $[\eta_t]_c \subseteq \mu_t$, as $[\eta_t]_c$ is the convex sublattice of μ_t generated by η_t. Thus,

$$\begin{aligned}\eta'(x) &= \bigvee_{t \leq a_o} \{t : x \in [\eta_t]_c\} \\ &\leq \bigvee_{t \leq tip\{\mu\}} \{t : x \in \mu_t\} \\ &= \mu(x).\end{aligned}$$

We thus have $\eta' \subseteq \mu$. Further, to prove that $\eta \subseteq \eta'$, let $x \in M$ and let $\eta(x) = \alpha \leq a_0$. Then,

$$x \in \eta_\alpha \subseteq [\eta_\alpha]_c.$$

Therefore, by definition of $\eta', \alpha \leq \eta'(x)$. That is, $\eta(x) \leq \eta'(x)$. Thus, $\eta \subseteq \eta'$. We now prove that η' is an L-sublattice of μ. For any $z \in M$, define a subset $L_\eta(z)$ of L as follows:

$$L_\eta(z) = \{t \in L/t \leq a_0, z \in [\eta_t]_c\}.$$

Clearly, $\eta'(x) = \bigvee L_\eta(x)$. Let $x, y \in M$, $a \in L_\eta(x)$ and $b \in L_\eta(y)$. We claim that $a \wedge b \in L_\eta(x \vee y)$. First note that $\eta_a \cup \eta_b \subseteq \eta_{a \wedge b}$. Since $a \in L_\eta(x)$ and $b \in L_\eta(y)$, we have $a, b \leq a_0$, $x \in [\eta_a]_c$, $y \in [\eta_b]_c$. Therefore, $x = p\{x_i\}$ (a lattice polynomial in variables x_i's. where $x_i \in \eta_a$, $\forall i$). Similarly, $y = q\{y_j\}$ (a lattice polynomial in variables y_j's. where $y_j \in \eta_b$, $\forall j$). Thus, $x \vee y$ is also a lattice polynomial in variables x_i's and y_j's, where $x_i, y_j \in \eta_a \cup \eta_b \subseteq \eta_{a \wedge b}$. That is,

$$x \vee y \in [\eta_{a \wedge b}]_c.$$

We also have $a \wedge b \leq a_0$. Thus, $a \wedge b \in L_\eta(x \vee y)$. This implies that

$$\eta'(x \vee y) \geq a \wedge b; \quad \forall\ a \in L_\eta(x) \text{ and } b \in L_\eta(y).$$

Consequently,

$$\begin{aligned}\eta'(x \vee y) &\geq \vee\{a \wedge b/a \in L_\eta(x), b \in L_\eta(y)\} \\ &= \{\vee\{a/a \in L_\eta(x)\}\} \wedge \{\vee\{b/b \in L_\eta(y)\}\} \\ &\quad \text{(as } L \text{ is a completely distributive lattice)} \\ &= \eta'(x) \wedge \eta'(y).\end{aligned}$$

Similarly, it can be proved that

$$\eta'(x \wedge y) \geq \eta'(x) \wedge \eta'(y); \quad \forall x, y \in M.$$

Thus, η' is an L-sublattice of μ. Further, to establish that η' is an L-convex sublattice of μ, we shall prove that,

(i) $\eta'(w) \geq \eta'(x) \wedge \eta'(y) \wedge \mu(w)$ where $x \leq w \leq y$ in M.

For any $z \in M$, let
$$L_\eta(z) = \{t \in L/t \leq a_0, z \in [\eta_t]_c\}.$$
Then, $\eta'(x) = \bigvee L_\eta(x)$. Also let

$$L_\mu(z) = \{t \in L/\ t \leq tip\ \{\mu\},\ z \in \mu_t = [\mu_t]\}.$$

Let $r \in L_\mu(w)$, $s \in L_\eta(x)$ and $t \in L_\eta(y)$. Then,

$$r \leq tip\ \{\mu\};\ s, t \leq a_0,\ w \in [\mu_r] = \mu_r,\ x \in [\eta_s]_c \text{ and } y \in [\eta_t]_c.$$

Hence, $s \wedge t \wedge r \leq a_0$ and we have

$$\begin{aligned} &x \in [\eta_s]_c \subseteq \mu_s \subseteq \mu_{s\wedge t\wedge r};\\ &y \in [\eta_t]_c \subseteq \mu_t \subseteq \mu_{s\wedge t\wedge r};\ \text{and}\\ &w \in [\mu_r] = \mu_r \subseteq \mu_{s\wedge t\wedge r}.\end{aligned}$$

Thus, $x, y, w \in \mu_{s\wedge t\wedge r}$. Moreover,

$$x \in [\eta_s]_c \subseteq [\eta_{s\wedge t\wedge r}]_c \text{ and } y \in [\eta_t]_c \subseteq [\eta_{s\wedge t\wedge r}]_c;$$

and $[\eta_{s\wedge t\wedge r}]_c$ is a convex sublattice of $\mu_{s\wedge t\wedge r}$ generated by the L-set $\eta_{s\wedge t\wedge r}$. Therefore, we have $w \in [\eta_{s\wedge t\wedge r}]_c$ (as $x \leq w \leq y$ in M). This implies

$$\eta'(w) \geq s \wedge t \wedge r;\ \forall\ r \in L_\mu(w),\ s \in L_\eta(x)\ \text{ and } t \in L_\eta(y).$$

That is,

$$\begin{aligned}\eta'(w) &\geq \vee\{s \wedge t \wedge r/r \in L_\mu(w), s \in L_\eta(x) \text{ and } t \in L_\eta(y)\}\\ &= \{\vee\{s/s \in L_\eta(x)\}\} \wedge \{\vee\{t/t \in L_\eta(y)\}\} \wedge \{\vee\{r/r \in L_\mu(w)\}\}\\ &\qquad \text{(as } L \text{ is a completely distributive lattice)}\\ &= \eta'(x) \wedge \eta'(y) \wedge \mu(w).\end{aligned}$$

Hence, η' is an L-convex sublattice of μ. Now it is left to prove that η' is the smallest L-convex sublattice of μ containing η. For this, suppose θ is an L-convex sublattice of μ such that $\eta \subseteq \theta$. Then, $\eta_t \subseteq \theta_t$, $\forall\ t \in L$. Since θ is an L-convex sublattice of μ, therefore by Theorem 6, each nonempty θ_t is a convex sublattice of μ_t. Therefore, $[\theta_t]_c = \theta_t$. This implies that
$$[\eta_t]_c \subseteq \theta_t,\ \ \forall\ t \leq a_0.$$
Also, $a_0 = tip\{\eta\} \leq tip\{\theta\}$. Thus, $\forall\ x \in M$, we have

$$\begin{aligned}\eta'(x) &= \bigvee_{t\leq a_o} \{t : x \in [\eta_t]_c\}\\ &\leq \bigvee_{t\leq tip\{\theta\}} \{t : x \in \theta_t\}\end{aligned}$$

$$= \theta(x).$$

That is, $\eta' \subseteq \theta$. Hence, $\eta' = [\eta]^c_\mu$.

The next result is significant for establishing the Unique Representation Theorem for L-convex sublattice of an L-lattice μ. The proof being trivial is omitted.

Theorem 10. *Let $L(\mu, M)$ be an L-lattice and $\eta \subseteq \mu$ be an L-ideal (L-dual ideal) of μ. Then, η is an L-convex sublattice of μ.*

Before discussing the Unique Representation Theorem for L-convex sublattices in an L-lattice μ, we provide the structural composition of an L-ideal of μ generated by an L-subset η of μ in terms of strong level subsets of η. The following result is proved by taking L to be a dense chain. Note that a dense chain is a completely distributive lattice.

Theorem 11. *Let L be a dense chain and $L(\mu, M)$ be an L-lattice. Let $\eta \in L^M$, $\eta \subseteq \mu$ and $a_0 = tip\{\eta\}$. Define an L-set $\hat{\eta}$ of M as follows:*

$$\widehat{\eta}(x) = \vee_{t<a_0}\{t : x \in (\eta_t^>]\},$$

where $(\eta_t^>]$ is an ideal of $\mu_t^>$ generated by $\eta_t^>$. Then, $\hat{\eta} = (\eta]_\mu$.

Proof. Since $\eta \subseteq \mu$, $\eta_t^> \subseteq \mu_t^>$, $\forall\ t \in L$. As μ is an L-lattice, $\mu_t^>$ is a sublattice of M, $\forall$ $t < tip\{\mu\}$. Also, as $(\eta_t^>]$ is an ideal of $\mu_t^>$ generated by $\eta_t^>$, we have

$$(\eta_t^>] \subseteq \mu_t^>, \quad \forall\ t < a_0.$$

Thus,

$$\begin{aligned}\widehat{\eta}(x) &= \vee_{t<a_0}\{t : x \in (\eta_t^>]\}\\ &\leq \vee_{t<tip\{\mu\}}\{t : x \in \mu_t^>\}\\ &\leq \mu(x).\end{aligned}$$

We thus have $\hat{\eta} \subseteq \mu$. To establish that $\eta \subseteq \hat{\eta}$, we prove that $\eta_\alpha^> \subseteq (\hat{\eta})_\alpha^>$, $\forall\ \alpha \in L$. Let $\alpha \in L$ and $x \in \eta_\alpha^>$. Then, $\eta(x) > \alpha$. Since L is a dense chain, $\exists\ \beta \in L$ such that $\eta(x) > \beta > \alpha$. This implies $x \in \eta_\beta^>$ and hence $x \in (\eta_\beta^>]$. Consequently,

$$\begin{aligned}\widehat{\eta}(x) &= \vee_{t<a_0}\{t : x \in (\eta_t^>]\}\\ &\geq \beta > \alpha.\end{aligned}$$

That is, $x \in (\hat{\eta})_\alpha^>$. This proves that $\eta \subseteq \hat{\eta}$. We now prove that $\hat{\eta}$ is an L-ideal of μ. For any $z \in M$, define a set

$$L_\eta(z) = \{t \in L / t < a_0, z \in (\eta_t^>]\}.$$

Then, $\widehat{\eta}(x) = \bigvee L_\eta(x)$. Let $x, y \in M$. We claim that for any $a \in L_\eta(x)$ and $b \in L_\eta(y)$, $a \wedge b \in L_\eta(x \vee y)$. Suppose, $a \in L_\eta(x)$ and $b \in L_\eta(y)$. Then,

$$a < a_0, \quad b < b_0, \quad x \in (\eta_a^>], \quad y \in (\eta_b^>].$$

Since L is a chain, $a \wedge b < a_0$. Now, $x \in (\eta_a^>]$, $y \in (\eta_b^>]$ implies that

$$x \leq x_1 \vee \ldots \vee x_n, \quad x_i \in \eta_a^>, \quad \forall\ i; \text{ and}$$
$$y \leq y_1 \vee \ldots \vee y_m, \quad y_j \in \eta_b^>, \quad \forall\ j.$$

Then, we have $x \vee y \leq (\vee x_i) \vee (\vee y_j)$, which is a finite join of elements of $\eta_a^> \cup \eta_b^>$ and $\eta_a^> \cup \eta_b^> \subseteq \eta_{a\wedge b}^>$. Therefore,

$$x \vee y \in (\eta_{a\wedge b}^>] \text{ and } a \wedge b < a_0.$$

Thus, $a \wedge b \in L_\eta(x \vee y)$. Consequently, $\widehat{\eta}(x \vee y) \geq a \wedge b;\ \forall\ a \in L_\eta(x)$ and $b \in L_\eta(y)$. Hence,

$$\begin{aligned}\widehat{\eta}(x \vee y) &\geq \vee\{a \wedge b / a \in L_\eta(x), b \in L_\eta(y)\}\\ &= \{\vee\{a/a \in L_\eta(x)\}\} \wedge \{\vee\{b/b \in L_\eta(y)\}\}\\ &\quad \text{(as } L \text{ is a completely distributive lattice)}\\ &= \eta'(x) \wedge \eta'(y).\end{aligned}$$

Now, to verify that $\widehat{\eta}(x \wedge y) \geq \mu(x) \wedge \widehat{\eta}(y)$, we again define the following subsets of L for $z \in M$:

$$L_\eta(z) = \{t \in L/t < a_0, z \in (\eta_t^>]\} \text{ and } L_\mu(z) = \{t \in L/t \leq tip\{\mu\}, z \in \mu_t = [\mu_t]\}.$$

Thus, $\widehat{\eta}(x) = \bigvee L_\eta(x)$ and $\mu(x) = \bigvee L_\mu(x)$. If $a \in L_\mu(x)$ and $b \in L_\eta(y)$, then $a \leq tip\ \{\mu\}$, $b < a_0$, $x \in \mu_a = [\mu_a]$ and $y \in (\eta_b^>]$. Therefore,

$$a \wedge b < a_0 \text{ and } y \leq y_1 \vee \ldots \vee y_m,\ y_j \in \eta_b^>, \quad \forall\ j.$$

This implies

$$x \wedge y \leq y_1 \vee \ldots \vee y_m,\ y_j \in \eta_b^> \subseteq \eta_{a\wedge b}^>, \quad \forall\ j.$$

We thus have $x \wedge y \in (\eta_b^>] \subseteq (\eta_{a\wedge b}^>]$ and therefore,

$$a \wedge b \in L_\eta(x \wedge y); \quad \forall\ a \in L_\mu(x) \text{ and } b \in L_\eta(y).$$

That is,

$$\widehat{\eta}(x \wedge y) \geq a \wedge b; \quad \forall\ a \in L_\mu(x) \text{ and } b \in L_\eta(y).$$

Consequently,

$$\begin{aligned}\widehat{\eta}(x \wedge y) &\geq \vee\{a \wedge b / a \in L_\mu(x) \text{ and } b \in L_\eta(y)\}\\ &= \{\vee\{a/a \in L_\mu(x)\}\} \wedge \{\vee\{b/b \in L_\eta(y)\}\}\\ &\quad \text{(as } L \text{ is a completely distributive lattice)}\\ &= \mu(x) \wedge \widehat{\eta}(y).\end{aligned}$$

We thus get that $\hat{\eta}$ is an L-ideal of μ. Finally, to prove that $\hat{\eta}$ is the smallest L-ideal of μ containing η, suppose θ is an L-ideal of μ such that $\eta \subseteq \theta$. Then, $\eta_t^{>} \subseteq \theta_t^{>}$ and $\theta_t^{>}$ is an ideal of $\mu_t^{>}$. Therefore, $(\theta_t^{>}] = \theta_t^{>}$. We thus have:

$$\begin{aligned}\widehat{\eta}(x) &= \vee_{t<a_0}\{t : x \ \in (\eta_t^{>}]\} \\ &\leq \bigvee_{t \leq tip\{\theta\}} \{t : x \in \theta_t^{>}\} \\ &\leq \theta(x).\end{aligned}$$

That is, $\hat{\eta} \subseteq \theta$. Hence, $\hat{\eta} = (\eta]_\mu$.

A similar result holds for L-dual ideal generated by η.

Theorem 12. *Let L be a dense chain and $L(\mu, M)$ be an L-lattice. Let $\eta \in L^M$, $\eta \subseteq \mu$ and $a_0 = tip\{\eta\}$. Define an L-subset $\check{\eta}$ of M as follows:*

$$\check{\eta}(x) = \vee_{t<a_0}\{t : x \in [\eta_t^{>})\},$$

where $[\eta_t^{>})$ is a dual ideal of $\mu_t^{>}$ generated by $\eta_t^{>}$. Then, $\check{\eta} = [\eta)_\mu$.

We now establish the Unique Representation Theorem for L-convex sublattices in an L-lattice. In the following theorem, O represents the constant L-subset with all truth values equal to 0 of lattice L.

Theorem 13. *Let L be a dense chain, $L(\mu, M)$ be an L-lattice, η, $\theta \subseteq \mu$ such that η is an L-ideal of μ and θ is an L-dual ideal of μ. Then, $\eta \cap \theta$ is an L-convex sublattice of μ if $\eta \cap \theta \neq O$. Further, every L-convex sublattice of μ can be expressed in this form in one and only one way.*

Proof. Let η be an L-ideal of μ and θ be an L-dual ideal of μ. Then by Theorem 10, η and θ are L-convex sublattices of μ. Since intersection of L-convex sublattices of μ is an L-convex sublattice of μ, therefore $\eta \cap \theta$ is an L-convex sublattice of μ provided $\eta \cap \theta \neq O$ (i.e., $\exists\ x \in M$ such that $(\eta \cap \theta)(x) > 0$).
Next, let γ be an L-convex sublattice of μ. We take

$$\eta = (\gamma]_\mu \text{ and } \theta = [\gamma)_\mu.$$

We prove that $\gamma = \eta \cap \theta$. Clearly, $\gamma \subseteq \eta$ and $\gamma \subseteq \theta$. Therefore, $\gamma \subseteq \eta \cap \theta$. Suppose, $\gamma \subsetneq \eta \cap \theta$. Then, $\exists\ x \in M$ such that $\gamma(x) < \eta(x) \wedge \theta(x)$. Let $\gamma(x) = t$. Then, $\eta(x) > t$ and $\theta(x) > t$. That is, $x \in \eta_t^{>}$ and $x \in \theta_t^{>}$. By Theorem 4, $\eta_t^{>}$ is an ideal of $\mu_t^{>}$ and $\theta_t^{>}$ is a dual ideal of $\mu_t^{>}$. This implies

$$\eta_t^{>} = (\eta_t^{>}] \ \text{ and } \ \theta_t^{>} = [\theta_t^{>}).$$

Since $\eta = (\gamma]_\mu$, $\theta = [\gamma)_\mu$ and $\eta(x) > t$, $\theta(x) > t$, therefore by Theorem 11, 12, $\exists\ r, s < a_0$, $x \in (\gamma_r^{>}]$, $x \in [\gamma_s^{>})$, such that $t < r$ and $t < s$. Thus we have,

$$\begin{aligned}&x \in (\gamma_r^{>}] \subseteq (\gamma_{r\wedge s}^{>}], \quad t < r; \quad \text{and} \\ &x \in [\gamma_s^{>}) \subseteq [\gamma_{r\wedge s}^{>}), \quad t < s.\end{aligned}$$

That is, $t \leq r \wedge s$. Moreover,

$$\exists\ x_1, \ldots, x_n \in \gamma^{>}_{r\wedge s} \subseteq \mu^{>}_{r\wedge s}\ \ \exists\ y_1, \ldots, y_m \in \gamma^{>}_{r\wedge s} \subseteq \mu^{>}_{r\wedge s} \text{ such that}$$

$$a = y_1 \wedge \ldots \wedge y_m \leq x \leq x_1 \vee \ldots \vee x_n = b.$$

Since $\mu^{>}_{r\wedge s}$ is a sublattice of M and $(\gamma^{>}_{r\wedge s}]$ is an ideal of $\mu^{>}_{r\wedge s}$, we have $a, b, x \in \mu^{>}_{r\wedge s}$. We also have $a, b \in \gamma^{>}_{r\wedge s}$ as $\gamma^{>}_{r\wedge s}$ is a sublattice of $\mu^{>}_{r\wedge s}$. Now, γ being an L-convex sublattice of μ, $\gamma^{>}_{r\wedge s}$ is a convex sublattice of $\mu^{>}_{r\wedge s}$. Consequently, $x \in \gamma^{>}_{r\wedge s}$. That is, $t = \gamma(x) > r \wedge s$, which contradicts the fact that $t < r$ and $t < s$ (as L is a chain). Hence,

$$\gamma(x) = (\eta \cap \theta)(x), \quad \forall\ x \in M.$$

That is $\gamma = \eta \cap \theta$. Thus each convex sublattice γ of μ can be expressed in this form. To prove the uniqueness of this representation, suppose there exists an L-ideal η of μ and an L-dual ideal θ of μ such that $\gamma = \eta \cap \theta$. We prove that $\eta = (\gamma]_\mu$ and $\theta = [\gamma)_\mu$. Since $\gamma \subseteq \eta$, therefore $(\gamma]_\mu \subseteq \eta$.
For reverse inclusion, let $x \in M$ and $\eta(x) = t$. Then clearly, $t \leq a_0$. Since $\gamma \subseteq \eta$, therefore $\gamma_t \subseteq \eta_t$. If $y \in \gamma_t$, then $x, y \in \eta_t$. This implies $x \vee y \in \eta_t$. We also have $y \in \gamma_t \subseteq [\gamma_t)$, where $[\gamma_t)$ is a dual ideal of μ_t and $y \leq x \vee y$ in μ_t. Therefore, $x \vee y \in [\gamma_t)$ and $t \leq a_0$. This implies

$$t \leq [\gamma)(x \vee y) = \theta(x \vee y).$$

We also have $\eta(x \vee y) \geq t$. Therefore, $\theta(x \vee y) \wedge \eta(x \vee y) \geq t$. That is, $\gamma(x \vee y) \geq t$. Thus,

$$x \vee y \in \eta_t \subseteq (\gamma_t].$$

Note that $(\gamma_t]$ is an ideal of μ_t and $x \leq x \vee y$. Hence, $x \in (\gamma_t]$ and $t \leq a_0$. Thus,

$$\eta(x) = t \leq \vee_{r \leq a_0}\{r : x \in (\gamma_r]\} = (\gamma](x).$$

Hence, $\eta \subseteq (\gamma]$ and therefore, $\eta = (\gamma]$. Similarly, $\theta = [\gamma)$. Consequently, we get the uniqueness of the representation of γ.

4. Complement of an L-set and L-prime ideal and L-maximal ideal of an L-lattice

In this section, the important concept of an order reversing involution on a lattice is discussed. Based on this notion, the complement of an L-lattice is defined. These notions occur frequently in Lattice Implication Algebras and L-topological spaces [11, 13, 16, 17]. If $(L, \leq, \wedge, \vee)$ is a lattice, then $L^*(= L)$ is also a lattice with respect to reverse order "$\geq$", where $y \geq x$ in L^* if and only if $x \leq y$ in L. An order reversing involution on a lattice L is defined as a bijection $\tau : L \to L^*$ satisfying $\tau(\tau(x)) = x$, $\forall\ x \in L$ and $x \leq y$ in L if and only if $\tau(y) \leq \tau(x)$ in $L = L^*$.

It is interesting to note that in all the examples provided in Section 3, there is an order reversing involution on the lattice L of truth values.
The following result displays an inherent property of an order reversing involution.

Lemma 1. *If L and L^* are lattices and $\tau : L \to L^*$ is an order reversing bijection, then*

$$\tau(a \vee b) = \tau(a) \wedge \tau(b) \quad \text{and} \quad \tau(a \wedge b) = \tau(a) \vee \tau(b); \quad \forall\ a, b \in L.$$

An order reversing involution defined on a lattice L of truth values leads to the definition of complement of an L-set as follows:

Definition 10. *Let $\mu \in L^M$ and τ be an order reversing involution on L, i.e., $\tau : L \to L^*$ is a bijection satisfying $\tau(\tau(x)) = x$, $\forall\ x \in L$ and $x \leq y$ in L if and only if $\tau(y) \leq \tau(x)$ in $L = L^*$. Define an L-set $\mu' : M^* \to L^*$ as*

$$\mu'(x) = \tau(\mu(x)),\ \forall\ x \in M^*(= M).$$

Then, $\mu' \in L^M$ and μ' is called the complement of μ in L^M.

The following lemma establishes the De Morgan's Laws in L^M:

Lemma 2. *Let $\mu, \eta \in L^M$ and τ be an order reversing involution on L. Then,*

$$(\mu \cup \eta)' = \mu' \cap \eta' \quad \text{and} \quad (\mu \cap \eta)' = \mu' \cup \eta'.$$

Proof. Let $x \in M$. Then,

$$\begin{aligned}(\mu \cup \eta)'(x) &= \tau[(\mu \cup \eta)(x)] \\ &= \tau[\mu(x) \vee \eta(x)] \\ &= \tau(\mu(x)) \wedge \tau(\eta(x)) \\ &= \mu'(x) \wedge \eta'(x) \\ &= (\mu' \cap \eta')(x).\end{aligned}$$

Hence, $(\mu \cup \eta)' = \mu' \cap \eta'$. The proof of the other part follows similarly.

In the next result, it is proved that the complement of an L-prime ideal in M is an L-dual prime ideal in M.

Theorem 14. *Let τ be an order reversing involution on lattice L and μ be an L-prime ideal of M. Then μ', the complement of μ in L^M, is an L-dual prime ideal of M.*

Proof. Let $x, y \in M$. Since μ is an L-prime ideal of M, we have

$$\mu(x \wedge y) \leq \mu(x) \vee \mu(y).$$

This implies, $\tau(\mu(x \wedge y)) \geq \tau[\mu(x) \vee \mu(y)]$ as τ is an order reversing involution. That is,

$$\begin{aligned}\mu'(x \wedge y) &\geq \tau[\mu(x) \vee \mu(y)] \\ &= \tau(\mu(x)) \wedge \tau(\mu(y)) \\ &= \mu'(x) \wedge \mu'(y). \qquad (1)\end{aligned}$$

Further, if $x \leq y$ in M, then $\mu(x) \geq \mu(y)$ in L (as μ is an L-ideal of M). This implies

$$\tau(\mu(x)) \leq \tau(\mu(y)) \text{ in } L^*.$$

Thus,

$$\mu'(x) \leq \mu'(y)\,. \tag{2}$$

Moreover, as μ is an L-ideal of M, $x \leq x \vee y$ and $y \leq x \vee y$ in M implies

$$\mu(x) \geq \mu(x \vee y) \text{ and } \mu(y) \geq \mu(x \vee y).$$

Thus, $\mu'(x) \leq \mu'(x \vee y)$ and $\mu'(y) \leq \mu'(x \vee y)$ and hence

$$\mu'(x) \wedge \mu'(y) \leq \mu'(x) \leq \mu'(x \vee y)\,. \tag{3}$$

By (1), (2) and (3), we get that μ' is an L-dual ideal of M. To establish that μ' is an L-dual prime ideal of M, note that $\mu(x \vee y) \geq \mu(x) \wedge \mu(y)$. This implies

$$\begin{aligned} \mu'(x \vee y) &= \tau(\mu(x \vee y)) \\ &\leq \tau[\mu(x) \wedge \mu(y)] \\ &= \tau(\mu(x)) \vee \tau(\mu(y)) \\ &= \mu'(x) \vee \mu'(y). \end{aligned}$$

Consequently, μ' is an L-dual prime ideal of M.

By the above theorem, it can be concluded that if L is a lattice with an order reversing involution τ, then μ is an L-prime ideal of M if and only if μ' is an L-dual prime ideal of M. The next theorem, combined with Theorem 14, leads to the following interesting analogue of a result from classical lattice theory:

An L-ideal η of M is an L-prime ideal of M if and only if η' is an L-dual ideal of M. In fact, η' is an L-dual prime ideal of M.

Theorem 15. *Let τ be an order reversing involution on the lattice L and η be an L-ideal of M such that η' is an L-dual ideal of M. Then, η and η' are L-prime ideals of M.*

Proof. Suppose η is an L-ideal of M such that η' is an L-dual ideal of M. We have

$$\eta'(x \wedge y) \geq \eta'(x) \wedge \eta'(y); \quad \forall\, x, y \in M.$$

This implies

$$\begin{aligned} \eta(x \wedge y) &= \tau[\eta'(x \wedge y)] \\ &\leq \tau[\eta'(x) \wedge \eta'(y)] \\ &= \tau(\eta'(x)) \vee \tau(\eta'(y)) \\ &= \eta(x) \vee \eta(y). \end{aligned}$$

Thus, η is an L-prime ideal of M. Similarly, we have

$$\eta(x \wedge y) \geq \eta(x) \wedge \eta(y); \quad \forall\, x, y \in M.$$

This implies

$$\begin{aligned}\eta'(x \wedge y) &= \tau[\eta(x \wedge y)] \\ &\leq \tau[\eta(x) \wedge \eta(y)] \\ &= \tau(\eta(x)) \vee \tau(\eta(y)) \\ &= \eta'(x) \vee \eta'(y).\end{aligned}$$

Hence, η and η' are L-prime ideals of M.
In fact, in the above theorem, it can be proved that η' is an L-dual prime ideal of M.

We conclude this paper by discussing an analogue of another well known fact in classical lattice theory that,

In a distributive lattice with the maximal element 1, every proper ideal is contained in a maximal ideal.

The next theorem proves a similar result in an L-lattice μ.

Theorem 16. *Let $L(\mu, M)$ be an L-lattice, where L is a completely distributive lattice. Let $\eta \subseteq \mu$ be an L-ideal of μ. Then, there exists an L-maximal ideal θ of μ such that $\eta \subseteq \theta$.*

Proof. Let $I = \{\gamma/\gamma$ is an L-ideal of μ such that $\eta \subseteq \gamma\}$. Let $\Phi = \{\gamma_i\}_{i \in \Lambda}$ be a chain in I. Then clearly,

$$\eta \subseteq \cup\{\gamma_i\} \text{ and } \cup\{\gamma_i\} \subseteq \mu.$$

If $x, y \in M$,

$$\begin{aligned}\{\cup\gamma_i\}(x \vee y) &= \vee\gamma_i(x \vee y) \\ &\geq \vee[\gamma_i(x) \wedge \gamma_i(y)] \\ &\qquad \text{(as each } \gamma_i \text{ is an } L\text{-ideal of } \mu) \\ &= [\vee\gamma_i(x)] \wedge [\vee\gamma_i(y)] \\ &= (\cup\gamma_i)(x) \wedge (\cup\gamma_i)(y).\end{aligned}$$

Moreover,

$$\begin{aligned}\{\cup\gamma_i\}(x \wedge y) &= \vee\gamma_i(x \wedge y) \\ &\geq \vee[\mu(x) \wedge \gamma_i(y)] \\ &\qquad \text{(as each } \gamma_i \text{ is an } L\text{-ideal of } \mu) \\ &= \mu(x) \wedge [\vee\gamma_i(y)] \\ &= \mu(x) \wedge (\cup\gamma_i)(y).\end{aligned}$$

Thus, $\{\cup\gamma_i\}$ is an L-ideal of μ containing η. That is, $\{\cup\gamma_i\} \in I$. Thus, every chain in I has an upper bound in I. Therefore, by Zorn's Lemma, I has a maximal element. That is, $\exists$ a maximal L-ideal γ of μ such that $\eta \subseteq \gamma$. Hence, γ is the required L-maximal ideal of μ containing η.

Conclusion

In the present work, the concept of an L-convex sublattice in an L-lattice is studied in detail and the unique representation theorem for L-convex sublattices is established. Moreover, the concept of order reversing involution is utilized on the lattice L of truth values to define the complement of an L-set. The notion of complementation plays a significant role in the theory of Boolean Algebras, lattice implication Algebras and topological spaces. In this paper, it is established that the concept of complementation of an L-set leads to proving some significant results in L-lattice theory. This notion is further worthy of attention as it may lead to some remarkable development in the theory of L-substructures of an L-

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