EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 1, Article Number 5535
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

A Certain Class of Filters in Generalized Complemented Distributive Lattices

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Abstract. In this paper, we introduce K^g-filters jointly derived from the class of ideals and the class of generalized complementations in a generalized complemented distributive lattice. We obtain some algebraic properties on the obtained class, and we provide some counter-examples. Mainly, we derive some Boolean algebras (distributive lattices) through the class of K^g-filters in a generalized complemented distributive lattice. Finally, we introduce normal K^g-filters in a generalized complemented distributive lattice and then prove that the class of normal K^g-filters is a Boolean algebra.

2020 Mathematics Subject Classifications: 06D05, 06D15

Key Words and Phrases: Ideals, Filters, Distributive Lattices, Generalized Complemented Distributive Lattices, Boolean Algebras

1. Introduction

The concept of distributive lattices [2, 3] has been extensively studied by several authors by taking a unary operation, like complementation [7, 12], pseudo-complementation [14, 15], quasi-complementation [5, 13] etc., and also by considering the class of ideals (filters). Some of authors recursively studied the class of distributive lattices by taking a binary operations like generalized implementation [11], by taking median graphs [10], by considering canonical extensions [9], by using subordinations [1, 6]. The class of ideals (filters) has a major role in the theory of lattices. It is known that there is a one-to-one

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DOI: https://doi.org/10.29020/nybg.ejpam.v18i1.5535

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correspondence between the class of prime ideals and the class of prime filters in a distributive lattice. It is easy to observe that we can obtain a class of ideals (filter) from a complementation on a distributive lattice. The authors [8] introduced and studied a generalized complementation (g-complementation) on a distributive lattice with dense elements; it is a generalization of complementation and dual of pseudo-complementation in distributive lattices. In this regard, we derive filters from a generalized complementation on a distributive lattice using dense elements.

In this paper, we introduce a class of filters (K^g-filters), which are abstract combined from the class of ideals and the class of generalized complementations on a distributive lattice A through dense elements and prove several algebraic properties on them. Also, we obtain some necessary and sufficient conditions for a filter to become K^g-filter in A. We obtain some algebraic results on the class of K^g-filters in a generalized complemented distributive lattice. Mainly, we abstract some structures which are distributive lattices (Boolean algebras) [4] from the class of K^g-filters. Finally we discuss normal K^g-filters in generalized complemented distributive lattices and prove that the class of normal K^g-filters forms a Boolean algebra.

2. Some results on generalized complemented distributive lattices

In this section, we introduce a filter corresponding to an ideal in a generalized distributive lattice and obtain some properties. Also, we discuss prime filters corresponding to an ideal in a generalized complemented distributive lattice.

Definition 2.1. [8] A unary operation g on a distributive lattice A is said to be a g-complementation (generalized complementation) if for any $v, w \in A$, $v \vee v^g \in D$, and $v \vee w \in D$ if and only if $v^g \leq w$. In this case, v^g is said to be a g-complement of v, and A is called a g-complemented distributive lattice.

Let us consider A means that it is a distributive lattice with dense elements and g is a generalized complementation on A.

Proposition 2.1. *[8] For any g-complementation g on A,*

(i) $0^g \in D$

(ii) $d \in D \Rightarrow d^g = 0$

(iii) $v \leq w \Rightarrow w^g \leq v^g$

(iv) $v^{gg} \leq v$

(v) $v^{ggg} = v^g$

(vi) $0^{gg} = 0$

(vii) $v \in D \iff v^g = 0 \iff v^{gg} \in D$

(viii) $v \leq 0^g$,
for all $v, w \in A$.

Proposition 2.2. *[8] For any g-complementation g on A and for any* $v,\ w \in A$, *the following are equivalent;*

(i) $v \vee w \in D$

(ii) $v^{gg} \vee w \in D$

(iii) $v^{gg} \vee w^{gg} \in D$

(iv) $v \vee w^{gg} \in D$.

Proposition 2.3. *[8] For any* $v, w \in A$,

(i) $(v \wedge w)^g = v^g \vee w^g$

(ii) $(v \vee w)^g \leq v^g \wedge w^g$

(iii) $(v \vee w)^{gg} = v^{gg} \vee w^{gg} = (v^g \wedge w^g)^g$

(iv) $(v \wedge w)^{gg} = (v^g \vee w^g)^g = (v^{gg} \wedge w^{gg})^{gg}$.

Given any ideal K of A, consider a set $G(K) = \{u \in A \mid u^g \in K\}$ containing D (because $d^g = 0 \in K$, for all $d \in D$).

Lemma 2.2. *For any ideal* K *of* A, $G(K)$ *is a filter.*

Proof. Let $v, w \in G(K)$. Then $v^g, w^g \in K$. By Proposition 2.3 (i), $(v \wedge w)^g = v^g \vee w^g \in K$ (since K is ideal). Therefore $v \wedge w \in G(K)$. Given $u \in A$, we have $v,\ u \leq v \vee u$, which implies $(v \vee u)^g \leq v^g,\ u^g$ (By Proposition 2.1(iii)) and then $(v \vee u)^g \leq v^g \wedge u^g \in K$ (since $v^g \in K,\ u^g \in A$, and K is an ideal). Therefore $v \vee u \in G(K)$. Thus $G(K)$ is filter.

Remark 2.3. $D = G(0) = G((0])$ and $G(A) = A$.

Lemma 2.4. *For any ideals* K, H *of* A, *we have;*

(i) $K \subseteq H$ *implies* $G(K) \subseteq G(H)$.

(ii) $G(K) \cap G(H) = G(K \cap H)$.

(iii) $G(K) \vee G(H) \subseteq G(K \vee H)$.

Proof. (i) Suppose that $K \subseteq H$ and $u \in G(K)$. Then $u^g \in K \subseteq H$. Therefore $u \in G(H)$ and hence $G(K) \subseteq G(H)$. (ii) and (iii) follows from (i).

Lemma 2.5. *For any ideal* K *of* A, *either* $G(K) \cap K = \emptyset$ *or* $G(K) = A$.

Proof. Suppose that $G(K) \cap K \neq \phi$. Let $x \in G(K) \cap K$. Then $x^g \in K$, and $x \in K$. Therefore $x \vee x^g \in D \cap K$ (since K is an ideal). Since $0^g \leq d$ for all $d \in D$, $0^g \leq x \vee x^g$. Therefore 0 in $G(K)$ and hence $G(K) = A$ (since $G(K)$ is filter).

Lemma 2.6. *For any ideal K of A and $u \in A$,*

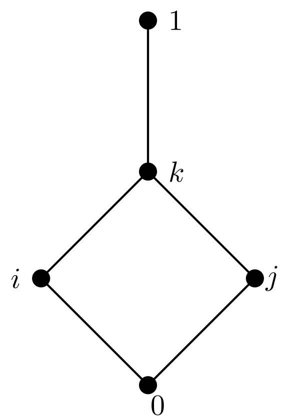
(i) $u \in G(K)$ if and only if $u^{gg} \in G(K)$

(ii) $u \in K$ implies $u^g \in G(K)$.

Proof. (i) follows from Proposition 2.1(v). (ii) follows from Proposition 2.1(iv).

The converse of Lemma 2.6(ii) need not be hold. Check out the following example.

Example 2.7. Consider a distributive lattice $A = \{0, i, j, k, 1\}$, whose Hasse diagram is showing in below;



For the ideal $K = \{0, i\}$ of A and $0^g = k, i^g = j, j^g = i, k^g = 1^g = 0$, we have $G(K) = \{j, k, 1\}$, and $k^g = 0 = 1^g \in G(K)$, but $k, 1 \notin K$.

Lemma 2.8. *For any ideal K of A, $K \cap D \neq \phi$ if and only if $A = G(K)$.*

Proof. Suppose that $K \cap D \neq \phi$. Let $v \in K \cap D$. Then $v \in K$ and $v^g = 0 \in K$. Therefore $v \in K \cap G(K)$. So that $K \cap G(K)$ is non-empty. By Lemma 2.5., $G(K) = A$. Conversely suppose that $G(K) = A$. For $0 \in A = G(K)$, we have $0^g \in K \cap D$.

Theorem 2.9. *Given an ideal K of A, $G(K)$ is proper if and only if either $i \notin G(K)$ or $i^g \notin G(K)$. That is., for any $i \in A$, i, i^g in $G(K)$ if and only if $A = G(K)$.*

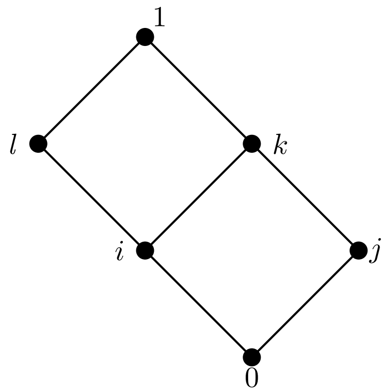
Proof. Let us consider that $G(K)$ is a proper filter of A. Let $i \in A$. If $i \in G(K)$ and $i^g \in G(K)$, then i^g in K and i^g in $G(K)$. Therefore $G(K) \cap K \neq \phi$. By Theorem 2.5., $G(K) = A$. Which is contradiction. Here $i \notin G(K)$ or $i^g \notin G(K)$. Conversely suppose that $i \notin G(K)$ or $i^g \notin G(K)$. If $G(K) = A$, then $i, i^g \in A = G(K)$. Which is contradiction. Therefore $G(K) \neq A$. Hence $G(K)$ is proper.

Theorem 2.10. *If A satisfies $(v \vee w)^g = v^g \wedge w^g$, for all $v, w \in A$ and K is a prime ideal of A, then $G(K)$ is a prime filter in A.*

Proof. Let $v, w \in A$ such that $v \vee w$ in $G(K)$. Then $(v \vee w)^g = v^g \wedge w^g$ in K. Since K is prime, $v^g \in K$ or $w^g \in K$. Therefore $v \in G(K)$ or $w \in G(K)$ and hence $G(K)$ is prime.

The converse of Theorem 2.10 need not be true.

Example 2.11. Consider the distributive lattice $A = \{0, i, j, k, l, 1\}$, whose Hasse diagram is in below;



For the ideal $K = \{0, i\}$ and $0^g = k, i^g = j, j^g = i, k^g = 1^g = 0$, $G(K) = \{j, k, 1\}$ is a prime filter, but K is not prime. Because $j \wedge l = 0 \in K$, $j \notin K$ and $l \notin K$.

Remark 2.12. The converse of Theorem 2.10. is true, provided $v^{gg} = v$, for all $v \in A$.

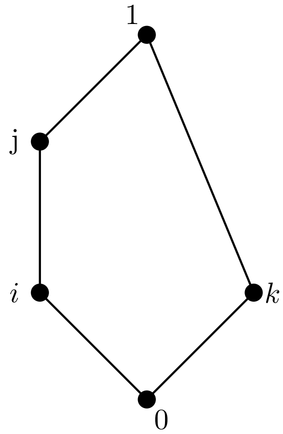
For, Suppose that A does not have a prime ideal K. In such case, $v \wedge w \in K$ such that $v \notin K$ and $w \notin K$ exist for $v, w \in A$. $(v \wedge w)^{gg} \leq v \wedge w \in K$, by Proposition 2.1 (iv). Consequently, $(v \wedge w)^{gg} \in K$. Therefore, $(v \wedge w)^g \in G(K)$ holds. $(v \wedge w)^g = v^g \vee w^g \in G(K)$, at this point. $v^g \in G(K)$ or $w^g \in G(K)$ if $G(K)$ is prime. It follows that either $w = w^{gg} \in K$ or $v = v^{gg} \in K$. It is contradictory. Therefore It is not prime, $G(K)$.

Theorem 2.13. *If $v^{gg} = v$, for all $v \in A$, then A has a unique dense element.*

Proof. Suppose that d_1 and d_2 are two dense elements in A. Now, $d_1 = d_1^{gg} = (d_1^g)^g = 0^g$ and $d_2 = d_2^{gg} = (d_2^g)^g = 0^g$ (by Proposition 2.1.(ii)). Therefore $d_1 = d_2 = 0^g$. Hence A has a unique dense element.

In a general lattice, the converse of Theorem 2.13. need not be true.

Example 2.14. Consider a lattice $A = \{0, i, j, k, 1\}$, whose Hasse diagram is in below;



and $0^g = 1, 1^g = 0, i^g = k, j^g = k, k^g = i$. Then A has a unique dense element, but $j^{gg} = i \neq j$.

3. K^g-filters in g-complemented distributive lattices

In this section, we introduce K^g-filters derived from ideals using a generalized complementation on a distributive lattice with dense elements and study some algebraic properties. We obtain some necessary conditions for a filter to become a K^g-filter. Mainly, we prove that the set of K^g-filters is a distributive lattice. Also, we obtain a Boolean algebraic structure through K^g-filters in a g-complemented distributive lattice. Finally we introduce normal K^g-filters in a generalized complemented distributive lattice and prove that the set of normal K^g-filters is a Boolean algebra.

Definition 3.1. A filter N of A is called a K^g-filter if there exists an ideal K in A such that $G(K) = N$. It is easy to observe that D is a K^g-filter, because $G((0]) = D$.

Lemma 3.2. *For any $v \in A$, the principal filter generated by v^g is a K^g-filter of A. Moreover $[v^g) = G((v])$.*

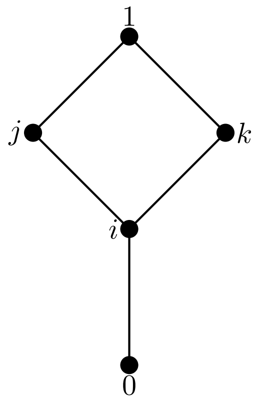
Proof. Let $u \in G((v])$. Then $u^g \in (v]$ and $u^g \leq v$. Since $u \vee u^g \in D$, $u \vee v \in D$. By Definition 2.1., $v^g \leq u$. So that $u \in [v^g)$. Therefore $G((v]) \subseteq [v^g)$. For any $u \in A$,

$$\begin{aligned} u \in [v^g) &\Rightarrow v^g \leq u \\ &\Rightarrow u^g \leq v^{gg} && \text{(Proposition 2.1(iii))} \\ &\Rightarrow u^g \leq v && \text{(since } v^{gg} \leq v) \\ &\Rightarrow u^g \in (v] \\ &\Rightarrow u \in G((v]). \end{aligned}$$

Therefore $[v^g) \subseteq G((v])$ and hence $[v^g) = G((v])$. Thus $[v^g)$ is a K^g-filter.

Now, we provide an example of a distributive lattice in which there is a K^g-filter and there is a non-K^g-filter.

Example 3.3. Consider a distributive lattice $A = \{0, i, j, k, 1\}$ whose Hasse diagram is given below;



Then $K_a = \{0\}, K_b = \{0, i\}, K_c = \{0, i, k\}, K_d = \{0, i, j\}, K_e = A$ are ideals of A and $N_a = \{1\}, N_b = \{j, 1\}, N_c = \{k, 1\}, N_d = \{i, j, k, 1\}, N_e = A$ are filters of A. Moreover N_d is a K^g-filter but others are not.

Theorem 3.4. *Let A be a distributive lattice with $v^{gg} = v$, for all $v \in A$. Then for any prime filter containing D in A is a K^g-filter.*

Proof. Let N be a prime filter containing D in A. For any $v \in N$,

$$\begin{aligned}
v^g \wedge v^{gg} &= 0 \in A - N \\
&\Rightarrow v^g \in A - N \text{ or } v^{gg} \in A - N && \text{(since } A - N \text{ is prime ideal)} \\
&\Rightarrow v^g \in A - N \text{ or } v \in A - N && \text{(since } v^{gg} = v) \\
&\Rightarrow v^g \in A - N && \text{(since } v \in N) \\
&\Rightarrow v \in G(A - N).
\end{aligned}$$

Therefore $N \subseteq G(A - N)$. Let $v \in G(A - N)$. Then $v^g \in A - N$. Since $D \subseteq N$, $v \vee v^g$ in N. Then v in N or v^g in N. Therefore v in N(since $v^g \notin N$). So that $G(A - N) \subseteq N$. Hence $G(A - N) = N$. Thus N is K^g-filter.

Corollary 3.5. *Let A be a distributive lattice with $v^{gg} = v$, for all $v \in A$. Then every maximal filter is a K^g-filter.*

Let us denote the set of K^g-filters in a distributive lattice A with dense elements by $\mathcal{GF}(A)$. Now we have the following theorem.

Theorem 3.6. *$\mathcal{GF}(A)$ is a distributive lattice with the operations $G(K_a) \cap G(K_b) = G(K_a \wedge K_b)$ and $G(K_a) \sqcup G(K_b) = G(K_a \vee K_b)$, for all $G(K_a), G(K_b) \in \mathcal{GF}(A)$.*

Proof. Let $G(K_a), G(K_b) \in \mathcal{GF}(A)$, where K_a, K_b in $\mathcal{I}(A)$. By Lemma 2.4(i), $G(K_a), G(K_b) \subseteq G(K_a \vee K_b)$. Therefore $G(K_a \vee K_b)$ is an upper bound of $G(K_a), G(K_b)$. Let $G(K_c) \in \mathcal{GF}(A)$ is an upper bound of $G(K_a), G(K_b)$, for some $K_c \in \mathcal{I}(A)$. For $v \in G(K_a \vee K_b)$,

$$\begin{aligned}
v^g \in K_a \vee K_b \Rightarrow & v^g = i \vee j && \text{for some } i \in K_a \text{ and } j \in K_b \\
\Rightarrow & i^{gg} \in K_a \ and \ j^{gg} \in K_b && (\text{since } \ v^{gg} \leq v \ forall \ v \in A) \\
\Rightarrow & i^g \in G(K_a) \ and \ j^g \in G(K_b) \\
\Rightarrow & i^g \in G(K_c) \ and \ j^g \in G(K_c) && (\text{since } \ G(K_a), G(K_b) \subseteq G(K_c)) \\
\Rightarrow & i^{gg} \vee j^{gg} \in K_c && (\text{since } K_c \text{ is an ideal}) \\
\Rightarrow & (i \vee j)^{gg} \in K_c && (\text{since Proportion } \ 2.3(iii)) \\
\Rightarrow & (v^g)^{gg} \in K_c && (\text{since } v^g = i \vee j) \\
\Rightarrow & v^{ggg} \in K_c \\
\Rightarrow & v^g \in K_c && (\text{since } v^{ggg} = v^g) \\
\Rightarrow & v \in G(K_c).
\end{aligned}$$

Therefore $G(K_a \vee K_b) \subseteq G(K_c)$ and hence $G(K_a \vee K_b)$ is the least upper bound of $G(K_a)$, $G(K_b)$. Hence it is denoted by $G(K_a) \sqcup G(K_b)$. By Lemma 2.4(ii)., $G(K_a) \wedge G(K_b) = G(K_a \wedge K_b)$. Since $\mathcal{I}(A)$ is distributive, $\mathcal{GF}(A)$ is distributive. Hence $\mathcal{GF}(A)$ is a distributive lattice with the lease element $G(\{0\})$ and the greatest element $G(A)$.

Corollary 3.7. *If $\mathcal{I}(A)$ is a Boolean algebra, then $\mathcal{GF}(A)$ is a Boolean algebra, but not a sub-Boolean algebra of $\mathcal{I}(A)$.*

Lemma 3.8. *For any $v \in A, G((v^{gg}]) = G((v])$.*

Proof. Let $v \in A$. Then $v^{gg} \leq v$. Therefore $(v^{gg}] \subseteq (v]$. By Lemma 2.4(ii)., $G((v^{gg}]) \subseteq G((v])$. Let $l \in G((v])$. Then $l^g \in (v]$ and hence $l^g \leq v$. By Proposition 2.1., $l^{ggg} = l^g \leq v^{gg} \leq v$. So that $l^g \in (v^{gg}]$. Hence $l \in G((v^{gg}])$. Thus $G((v]) = G((v^{gg}])$.

Lemma 3.9. *For any v, w in A, the following (1.), and (2.) are equivalent;*

(1). $(v \wedge w)^{gg} = v^{gg} \wedge w^{gg}$

(2). $(v \vee w)^g = v^g \wedge w^g$.

Proof. $(1) \Rightarrow (2)$: Assume(1), let $v, w \in A$. Then $(v \vee w)^g = (v \vee w)^{ggg} = (v^{gg} \vee w^{gg})^g = (v^g \wedge w^g)^{gg} = v^{ggg} \wedge w^{ggg} = v^g \wedge w^g$.
$(2) \Rightarrow (1)$: Assume (2), let $v, w \in A$. Then $(v \wedge w)^{gg} = (v^g \vee w^g)^g = v^{gg} \wedge w^{gg}$.

Lemma 3.10. *For any $v, w \in A$, we have*

(i) $G((v]) = G((v^{gg}])$

(ii) $w \in N \Leftrightarrow w^{gg} \in N$ *if N is K^g-filter in A*

(iii) If N is a K^g-filter, then $N = \bigcup_{w \in N} G((w^g])$.

Proof. (i) Let $v \in A$. By Lemma 3.2., $G((v]) = [v^g) = [v^{ggg}) = G((v^{gg}])$.
(ii) Let N be a K^g filter of A. Then $N = G(K)$, for some ideal K in A. For any $w \in A$, w in $N \Leftrightarrow w \in G(K) \Leftrightarrow w^g \in K \Leftrightarrow w^{ggg} \in K \Leftrightarrow w^{gg} \in G(K) \Leftrightarrow w^{gg} \in N$. (iii) Let N be a K^g-filter of A. Then $N = G(K)$ for some ideal K in A. Let $v \in G((w^g])$ for some $w \in N$. Then $v^g \in (w^g]$. Therefore $v^g \leq w^g \Rightarrow w^{gg} \leq v^{gg}$. Now, $w \in N = G(K)$. Then $w^g \in K$. Therefore $w^{ggg} = w^g \in K$ implies $w^{gg} \in G(K) = N$. Since N is filter, $v^{gg} \in N$. So that $v \in N$. Hence $\bigcup_{w \in N} G((w^g]) \subseteq N$. Let $w \in N$. Then $w^g \in K$. Therefore $w^{ggg} = w^g \in (w^g]$ implies $w^{gg} \in G((w^g])$. So that $w \in G((w^g])$. Hence $N \subseteq \bigcup_{w \in N} G((w^g])$. Thus $N = \bigcup_{w \in N} G((w^g])$.

Let us consider a set $D(A) = \{v \in A \mid v^g \in D\}$. Then it is easy to observe that $D(A)$ is an ideal of A and $G(D(A)) = D$.

Lemma 3.11. *For any ideal K of A, and $x \in A$, we have the following equivalent conditions:*

(i) $G(K) = D$

(ii) $v \in K \Rightarrow v^g \in D$

(iii) $K \subseteq D(K) \subseteq D(A)$

Proof. (i) $\Rightarrow$ (ii); Assume (i); Let $v \in K$. Then $v^{gg} \in K$. Therefore $v^g \in G(K) = D$. Hence $v^g \in D$.
(ii) $\Rightarrow$ (iii); Assume (ii); Let $v \in K$. Then $v^g \in D$. Therefore $v \in D(A)$. Since $K \subseteq A$, $v \in D(K)$. Hence $K \subseteq D(K)$.
(iii) $\Rightarrow$ (i); Assume (iii); Now, $K \subseteq D(K) \subseteq D(A)$ implies $G(K) \subseteq G(D(K)) \subseteq G(D(A))$. Therefore $G(K) \subseteq G(D(K)) \subseteq D$. Since $D \subseteq G(K)$, $G(K) = D$.

Theorem 3.12. *The set $BG(A) = \{v \in A \mid v^{gg} = v\}$ is a Boolean algebra with the operations $\vee$ and $v_1 * v_2 = (v_1^g \vee v_2^g)^g$, for all $v_1, v_2 \in BG(A)$.*

Proof. Let $v_1, v_2 \in BG(A)$. Now, $(v_1 \vee v_2)^{gg} = v_1^{gg} \vee v_2^{gg} = v_1 \vee v_2$ (By proposition 2.3(iii)). Therefore $v_1 \vee v_2 \in BG(A)$. Now $[v_1 * v_2]^{gg} = [(v_1^g \vee v_2^g)^g]^{gg} = ((v_1^g \vee v_2^g)^g = v_1 * v_2 \in BG(A)$. Therefore $BG(A)$ is closed under $\vee$ and $*$. Now $v_1 * v_2 = (v_1^g \vee v_2^g)^g = (v_1 \wedge v_2)^{gg} \leq v_1 \wedge v_2 \leq v_1, v_2$. Therefore $v_1 * v_2$ is a lower bound of v_1, w_2. For any lower bound l of v_1, v_2 in $BG(A)$, we have $l \leq v_1, v_2 \Rightarrow v_1^g, v_2^g \leq l^g \Rightarrow v_1^g \vee v_2^g \leq l^g \Rightarrow l^{gg} \leq (v_1^g \vee v_2^g)^g \Rightarrow l \leq v_1 * v_2$ (since $l \in BG(A)$). Therefore $v_1 * v_2$ is the greatest lower bound of v_1, v_2 in $BG(A)$. Since A is distributive lattice, $BG(A)$ is form a distributive lattice. Let $b \in BG(A)$. Then there exists $b^g \in BG(A)$ such that $b \vee b^g$ is dense and $b \vee b^g = b^{gg} \vee b^g = b^{gg} \vee (b^g)^{gg} = (b \vee b^g)^{gg} = 0^g$ is the greatest element in $BG(A)$. Now, $b * b^g = (b^g \vee b^{gg})^g$. Since $b^g \vee b^{gg}$ is dense, $(b^g \vee b^{gg})^g = 0 = b * b^g$. Hence $BG(A)$ is a Boolean algebra.

Theorem 3.13. *There is an onto homomorphism between $\mathcal{I}(A)$ to $\mathcal{GF}(A)$.*

Proof. Define a map Φ from $\mathcal{I}(A)$ to $\mathcal{GF}(A)$ by $\Phi(K) = G(K)$ for all $K \in \mathcal{I}(A)$. Let $K_a, K_b \in \mathcal{GF}(A)$. Now, $K_a = K_b$. Then $\Phi(K_a) = \Phi(K_b)$. Therefore Φ is well defined. Now, $\Phi(K_a \cap K_b) = G(K_a \cap K_b) = G(K_a) \wedge G(K_b) = \Phi(K_a) \wedge \Phi(K_b)$, $\Phi(K_a \vee K_b) = G(K_a \vee K_b) = G(K_a) \sqcup G(K_b) = \Phi(K_a) \sqcup \Phi(K_b)$ and $\Phi(\{0\}) = G(\{0\}) = D$. Therefore Φ is an onto homomorphism.

Remark 3.14. The above homomorphism need not be one-one.
In Example 3.3., let $K_b = \{0, i\}, K_c = \{0, i, k\}$, then $G(K_b) = A = G(K_c)$. That is $\Phi(K_b) = \Phi(K_c)$. But $K_b \neq K_c$. Hence Φ is not one-one.

Lemma 3.15. *Let Φ be a homomorphism between $\mathcal{I}(A)$ and $\mathcal{GF}(A)$. Then kernel of Φ is an ideal of $\mathcal{I}(A)$.*

Proof. Let $K_a, K_b \in ker(\Phi)$. Then $\Phi(K_a) = D = \Phi(K_b)$. Therefore $G(K_a) = D = G(K_b)$. Now, $\Phi(K_a \vee K_b) = G(K_a \vee K_b) = G(K_a) \sqcup G(K_b) = D \sqcup D = D$. Therefore $K_a \vee K_b \in ker(\Phi)$. For any $K \in \mathcal{I}(A), \Phi(K \wedge K_a) = G(K \wedge K_a) = G(K) \wedge G(K_a) = G(K) \wedge D = D$. Therefore $K \wedge K_a \in ker(\Phi)$. Hence $ker(\Phi)$ is an ideal of $\mathcal{I}(A)$.

Theorem 3.16. *If Φ is a homomorphism from $\mathcal{I}(A)$ to $\mathcal{GF}(A)$ defined by $\Phi(K) = G(K)$, for all $K \in \mathcal{I}(A)$, then the following are equivalent;*

(i) $ker(\Phi) = \{0\}$

(ii) $v^{gg} = v$*, for all* $v \in A$.

(iii) Φ *is one-one.*

Proof. (ii) $\Rightarrow$ (iii); Suppose that $v^{gg} = v$ for all $v \in A$. Let $K_a, K_b \in \mathcal{I}(A)$. Now, $\Phi(K_a) = \Phi(K_b)$. Then $G(K_a) = G(K_b)$. For any $k \in A$,

$$\begin{aligned} k \in K_a &\Leftrightarrow k^{gg} \in K_a && \text{(since } k^{gg} = k) \\ &\Leftrightarrow k^{g} \in G(K_a) = G(K_b) \\ &\Leftrightarrow k^{g} \in G(K_b) \\ &\Leftrightarrow k^{gg} \in K_b \\ &\Leftrightarrow k \in K_b. && \text{(since } k^{gg} = k) \end{aligned}$$

Therefore $K_a = K_b$. Hence Φ is one-one.
(iii) $\Rightarrow$ (i); Suppose that Φ is one-one. Let $K \in ker(\Phi)$. Then $\Phi(K) = D$. Therefore $\Phi(K) = G(K) = D = G((0]) = \Phi((0])$. Since Φ is one-one, $K = (0]$. Hence $ker(\Phi) = \{0\}$.
(ii) $\Rightarrow$ (i); Suppose that $v^{gg} = v$, for all $v \in A$. Let $K \in ker(\Phi)$. Then $\Phi(K) = D$. Therefore $G(K) = D$. Let $k \in K$. Then $k^{gg} \in K$. Therefore $k^{g} \in G(K) = D$. For this $k^{g} \in D, k^{gg} = 0$. Since $k^{gg} = k = 0$. Hence $K = \{0\}$. Therefore $ker(\Phi) = \{0\}$.
(iii) $\Rightarrow$ (ii); Suppose that Φ is one-one. Let $v \in A$. By Lemma 3.3., $G((v^{gg}]) = G((v])$. Then $\Phi((v^{gg}]) = \Phi((v])$. Therefore $(v^{gg}] = (v]$. Since $v^{gg} \in (v^{gg}] = (v]$, $(v^{gg}] \leq v$. Similarly $v \leq v^{gg}$. Hence $v^{gg} = v$, for all $v \in A$.

For any $i \in A$, $G((i])$ is a K^g-filter of A and it is called a normal K^g-filter. The set of all normal K^g-filters of A denoted by $\mathcal{NG}(A) = \{G((i]) = [i^g) | i \in A\}$ and it forms a sublattice of $\mathcal{GF}(A)$. Moreover $\mathcal{NG}(A)$ is a Boolean algebra.

Theorem 3.17. *The set $\mathcal{NG}(A)$ forms a sublattice of $\mathcal{GF}(A)$ with the operations $G((i]) \wedge G((j]) = G((i \wedge j])$ and $G((i]) \sqcup G((j]) = G((i \vee j])$, for all $G((i]), G(j]) \in \mathcal{NG}(A)$, for some $i, j \in A$. Moreover $\mathcal{NG}(A)$ is a Boolean algebra.*

Proof. Let $G((i]), G((j]) \in \mathcal{NG}(A)$. By Lemma 2.4., $G((i]) \wedge G((j]) = G((i] \wedge (j]) = G((i \wedge j])$ and by Theorem 3.9., $G((i]) \sqcup G((j]) = G((i] \vee (j]) = G((i \vee j])$. Therefore $G((i]) \wedge G((j]) \in \mathcal{NG}(A)$ and $G((i]) \sqcup G((j]) \in \mathcal{NG}(A)$. Hence $\mathcal{NG}(A)$ is a sublattice of distributive lattice $\mathcal{GF}(A)$ with the least element $G((0])$ and the greatest element $G((d])$ where d is dense in A. For any $G((i])$ in $\mathcal{NG}(A)$, there exists $G((i^g])$ in $\mathcal{NG}(A)$ such that $G((i]) \wedge G((i^g]) = G((i \wedge i^g])$. Let $x \in G((i \wedge i^g]) = G((i]) \wedge G((i^g])$. Then we have $x^g \in (i \wedge i^g]$. So, $x^g \leq i \wedge i^g$. Hence $(i \wedge i^g)^g = i^g \vee i^{gg} \leq x^{gg} \leq x$. Since $i^g \vee i^{gg}$ is dense, x is dense. Then $G((i] \wedge G(i^g]) \subseteq D$. Hence $G((i] \wedge G(i^g]) = D$. Now, $G((i]) \sqcup G((i^g]) = G((i \vee i^g]) = [(i \vee i^g)^g)$. Since $(i \vee i^g)$ is dense, $(i \vee i^g)^g = 0$. Therefore $G((i \vee i^g]) = A$. Hence $G((i]) \sqcup G((i^g]) = A$. Thus $\mathcal{NG}(A)$ is a Boolean algebra.

Theorem 3.18. *For any normal K^g-filter N of A, there exists a prime filter Q of A such that $N \subseteq Q$.*

Proof. Let N be a K^g-filter of A. Then there exists an ideal K of A such that $G(K) = N$. Now, $N \cap K = \Phi$. Then there exists a prime filter Q of A such that $Q \cap K = \Phi$ and $N \subseteq Q$.

4. Conclusions

This paper exponentially enrich algebraic properties of a certain class of filters (K^g-filters) generated by a generalized complementation on a distributive lattice with dense elements. Also, we prove the class of K^g-filters forms a distributive lattice which is not induced. Further, we derive normal K^g-filters in generalized complemented distributive lattices and proven that the class of normal K^g-filters is a Boolean algebra which is not induced. We can classify K^g-filters and normal K^g-filters in different class of lattice.

Conflict of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper

Acknowledgements

The authors wish to thank the anonymous reviewers for their valuable suggestions. This work was supported by Directorate of Research and Innovation, Walter Sisulu University, South Africa.

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