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Translation of a Bipolar Fuzzy Set in Hilbert Algebras

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Abstract. This study explores the application of bipolar fuzzy set theory within Hilbert algebras, introducing and examining the concept of bipolar fuzzy (β, α)-translations of a bipolar fuzzy set $\varphi = (\varphi^+, \varphi^-)$ in two distinct forms: Type I and Type II. Fundamental properties of these bipolar fuzzy translations are investigated in depth, alongside the introduction of bipolar fuzzy extensions and intensities, broadening the utility and flexibility of bipolar fuzzy sets in capturing nuanced bipolar information. Moreover, this work addresses the intricate relationships between the complement of a bipolar fuzzy subalgebra, bipolar fuzzy ideal, and bipolar fuzzy deductive system with respect to their level cuts. The findings significantly contribute to the broader theoretical foundation and potential applications of bipolar fuzzy logic in Hilbert algebras, offering valuable insights for managing complex bipolar information in uncertain environments.

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Key Words and Phrases: Hilbert algebra, bipolar fuzzy subalgebra, bipolar fuzzy ideal, bipolar fuzzy deductive system.

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1. Introduction

The concept of fuzzy sets was proposed by Zadeh [19]. The theory of fuzzy sets has several applications in real-life situations, and many scholars have researched fuzzy set theory. After introducing the concept of fuzzy sets, several research studies were conducted on the generalizations of fuzzy sets. In 1994, Zhang [21] initiated the concept of bipolar fuzzy sets (BFSs) as a generalization of fuzzy sets. BFSs are an extension of fuzzy sets whose membership degree range is $[-1, 1]$. In a BFS, the membership degree 0 of an element means that the element is irrelevant to the corresponding property, the membership degree $(0, 1]$ of an element indicates that the element somewhat satisfies the

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property, and the membership degree $[-1, 0)$ of an element indicates that the element somewhat satisfies the implicit counter-property. Although BFSs and intuitionistic fuzzy sets [2] look similar to each other, they are essentially different sets. In many domains, it is important to be able to deal with bipolar information. It is noted that positive information represents what is granted to be possible, while negative information represents what is considered to be impossible. This domain has recently motivated new research in several directions. In particular, fuzzy and probabilistic formalisms for bipolar information have been proposed because when we deal with spatial information in image processing or in spatial reasoning applications, this bipolarity also occurs. For instance, when we assess the position of an object in a space, we may have positive information expressed as a set of possible places and negative information expressed as a set of impossible places. As another example, let us consider the spatial relations. Human beings consider "left" and "right" in opposite directions. But this does not mean that one of them is the negation of the other. The semantics of "opposite" capture a notion of symmetry rather than a strict complementation. In particular, there may be positions considered neither to the right nor to the left of some reference object, thus leaving some room for indetermination. This corresponds to the idea that the union of positive and negative information does not cover the whole space.

The development of BFS theory has seen continuous advancements, particularly in its expansion to complex algebraic structures and applications in managing nuanced uncertainties. Recent works, such as those in [1, 14–16, 18], have played a significant role in pushing the boundaries of this field. For instance, Yang et al. [18] introduced the concept of bipolar complex fuzzy subgroups, which integrates the BFS framework within subgroup structures, facilitating refined control over positive and negative membership degrees across group operations. Following this, Rehman et al. [16] extended this approach to bipolar complex fuzzy semigroups, where bipolar fuzzy logic enables a robust representation of dual-valued uncertainties within semigroup operations, broadening the algebraic contexts where bipolar fuzzy information can be systematically applied. Further, Mahmood et al. [15] explored Γ-semigroups under the lens of bipolar complex fuzzy sets, advancing the formalism needed for multi-dimensional decision-making scenarios, where bipolar attributes play a crucial role. Similarly, research in [14] delves into bipolar soft groups, an extension that allows for flexible representation and manipulation of bipolar attributes, thus supporting fundamental operations in fuzzy group theory. Lastly, Alsuraiheed et al. [1] investigated bipolar complex fuzzy submodules, highlighting their relevance in module theory and reinforcing the versatility of BFSs in higher algebraic structures. These studies collectively contribute to a more robust framework for applying BFSs, enabling their use in increasingly complex and multi-valued systems across various mathematical domains.

The concept of Hilbert algebras was introduced in the early 50-ties by Henkin [10] for some investigations of implication in intuitionistic and other non-classical logics. In the 60-ties, these algebras were studied especially by Diego [6] from an algebraic point of view. Diego [6] proved that Hilbert algebras form a variety which is locally finite. Hilbert algebras were studied by Busneag [3, 4] and Jun [12], who recognized some of their filters as forming deductive systems. Dudek [7–9] explored the concept of fuzzification in subalgebras, ideals, and deductive systems within the framework of Hilbert algebras.

This research delves into the application of BFS theory within Hilbert algebras, specifically introducing the bipolar fuzzy (β, α)-translations of a BFS $\varphi = (\varphi^+, \varphi^-)$ in two distinct forms: Type I and Type II. Through a detailed investigation of the fundamental properties of these translations, along with the concepts of bipolar fuzzy extensions and intensities, the study enhances the flexibility and efficacy of BFSs in representing intricate bipolar information. Additionally, it explores the complex relationships among the complements of bipolar fuzzy subalgebras, ideals, and deductive systems concerning their level cuts. Together, these findings lay a strengthened theoretical groundwork for bipolar fuzzy logic in Hilbert algebras and suggest promising applications in the management of nuanced bipolar information within uncertain systems.

2. Preliminaries

To establish a solid foundation for our discussion, we begin by formally defining the structure and properties of a Hilbert algebra.

Definition 1. *[6] A Hilbert algebra is a triplet with the formula $X = (X, \cdot, 1_X)$, where X is a nonempty set, $\cdot$ is a binary operation, and 1_X is a fixed member of X that is true according to the axioms mentioned below:*

(1) $(\forall x, y \in X)(x \cdot (y \cdot x) = 1_X)$

(2) $(\forall x, y, z \in X)((x \cdot (y \cdot z)) \cdot ((x \cdot y) \cdot (x \cdot z)) = 1_X)$

(3) $(\forall x, y \in X)(x \cdot y = 1_X, y \cdot x = 1_X \Rightarrow x = y)$

In [7], the following conclusion was established.

Lemma 1. *Let $X = (X, \cdot, 1_X)$ be a Hilbert algebra. Then*

(1) $(\forall x \in X)(x \cdot x = 1_X)$

(2) $(\forall x \in X)(1_X \cdot x = x)$

(3) $(\forall x \in X)(x \cdot 1_X = 1_X)$

(4) $(\forall x, y, z \in X)(x \cdot (y \cdot z) = y \cdot (x \cdot z))$

(5) $(\forall x, y, z \in X)((x \cdot z) \cdot ((z \cdot y) \cdot (x \cdot y)) = 1_X)$.

In a Hilbert algebra $X = (X, \cdot, 1_X)$, the binary relation $\leq$ is defined by

$$(\forall x, y \in X)(x \leq y \Leftrightarrow x \cdot y = 1_X),$$

which is a partial order on X with 1_X as the largest element.

Definition 2. *[20] A nonempty subset D of a Hilbert algebra $X = (X, \cdot, 1_X)$ is called a subalgebra of X if $x \cdot y \in D$ for all $x, y \in D$.*

Definition 3. *[5, 8] A nonempty subset D of a Hilbert algebra $X = (X, \cdot, 1_X)$ is called an ideal of X if the following conditions hold:*

(1) $1_X \in D$

(2) $(\forall x, y \in X)(y \in D \Rightarrow x \cdot y \in D)$

(3) $(\forall x, y_1, y_2 \in X)(y_1, y_2 \in D \Rightarrow (y_1 \cdot (y_2 \cdot x)) \cdot x \in D)$

Definition 4. *[8] A nonempty subset D of a Hilbert algebra $X = (X, \cdot, 1_X)$ is called a deductive system of X if*

(1) $1_X \in D$

(2) $(\forall x, y \in X)(x \cdot y \in D, x \in D \Rightarrow y \in D)$

A *fuzzy set* [19] in a nonempty set X is defined to be a function $\mu : X \to [0, 1]$, where $[0, 1]$ is the unit closed interval of real numbers.

Definition 5. *[13] A fuzzy set μ in a Hilbert algebra $X = (X, \cdot, 1_X)$ is said to be a fuzzy subalgebra of X if the following condition holds:*

$$(\forall x, y \in X)(\mu(x \cdot y) \geq \min\{\mu(x), \mu(y)\})$$

Definition 6. *[9] A fuzzy set μ in a Hilbert algebra $X = (X, \cdot, 1_X)$ is said to be a fuzzy ideal of X if the following conditions hold:*

(1) $(\forall x \in X)(\mu(1_X) \geq \mu(x))$

(2) $(\forall x, y \in X)(\mu(x \cdot y) \geq \mu(y))$

(3) $(\forall x, y_1, y_2 \in X)(\mu((y_1 \cdot (y_2 \cdot x)) \cdot x) \geq \min\{\mu(y_1), \mu(y_2)\})$

Definition 7. *[22] Let X be a nonempty set. A bipolar fuzzy set (BFS) φ in X is an object having the form $\varphi = \{(x, \varphi^+(x), \varphi^-(x)) \mid x \in X\}$, where $\varphi^+ : X \to [0, 1]$ and $\varphi^- : X \to [-1, 0]$ are mappings. We use the positive membership degree $\varphi^+(x)$ to denote the satisfaction degree of an element x to the property corresponding to a BFS φ, and the negative membership degree $\varphi^-(x)$ to denote the satisfaction degree of an element x to some implicit counter-property corresponding to a BFS φ. If $\varphi^+(x) \neq 0$ and $\varphi^-(x) = 0$, it is the situation that x is regarded as having only positive satisfaction for φ. If $\varphi^+(x) = 0$ and $\varphi^-(x) \neq 0$, it is the situation that x does not satisfy the property of φ but somewhat satisfies the counter property of φ. It is possible for an element x to be such that $\varphi^+(x) = 0$ and $\varphi^-(x) = 0$, when the membership function of the property overlaps that of its counter property over some portion of X. For the sake of simplicity, we shall use the symbol $\varphi = (\varphi^+, \varphi^-)$ for the bipolar fuzzy set $\varphi = \{(x, \varphi^+(x), \varphi^-(x)) \mid x \in X\}$.*

Definition 8. *[11] A BFS $\varphi = (\varphi^+, \varphi^-)$ in a Hilbert algebra $X = (X, \cdot, 1_X)$ is called a bipolar fuzzy subalgebra of X if the following condition holds:*

$$(\forall x, y \in X) \left(\begin{array}{l} \varphi^+(x \cdot y) \geq \min\{\varphi^+(x), \varphi^+(y)\} \\ \varphi^-(x \cdot y) \leq \max\{\varphi^-(x), \varphi^-(y)\} \end{array} \right) \tag{1}$$

Definition 9. *[11] A BFS $\varphi = (\varphi^+, \varphi^-)$ in a Hilbert algebra $X = (X, \cdot, 1_X)$ is called a bipolar fuzzy ideal of X if the following conditions hold:*

$$(\forall x \in X) \left(\begin{array}{l} \varphi^+(1_X) \geq \varphi^+(x) \\ \varphi^-(1_X) \leq \varphi^-(x) \end{array} \right) \tag{2}$$

$$(\forall x, y \in X) \left(\begin{array}{l} \varphi^+(x \cdot y) \geq \varphi^+(y) \\ \varphi^-(x \cdot y) \leq \varphi^-(y) \end{array} \right) \tag{3}$$

$$(\forall x, y_1, y_2 \in X) \left(\begin{array}{l} \varphi^+((y_1 \cdot (y_2 \cdot x)) \cdot x) \geq \min\{\varphi^+(y_1), \varphi^+(y_2)\} \\ \varphi^-((y_1 \cdot (y_2 \cdot x)) \cdot x) \leq \max\{\varphi^-(y_1), \varphi^-(y_2)\} \end{array} \right) \tag{4}$$

Definition 10. *[11] A BFS $\varphi = (\varphi^+, \varphi^-)$ in a Hilbert algebra $X = (X, \cdot, 1_X)$ is called a bipolar fuzzy deductive system of X if the following conditions hold:* (2) *and*

$$(\forall x, y \in X) \left(\begin{array}{l} \varphi^+(y) \geq \min\{\varphi^+(x \cdot y), \varphi^+(x)\} \\ \varphi^-(y) \leq \max\{\varphi^-(x \cdot y), \varphi^-(x)\} \end{array} \right) \tag{5}$$

3. Bipolar fuzzy (β, α)-translations in Hilbert algebras

Building upon the foundational concepts and preliminary definitions of BFSs and Hilbert algebras, we now delve into the specific framework of bipolar fuzzy (β, α)-translations. This section introduces two types of bipolar fuzzy translations, Type I and Type II, designed to extend the flexibility of BFSs in Hilbert algebras by capturing additional layers of nuanced information. Through these translations, we examine the structural adjustments within bipolar fuzzy subalgebras, ideals, and deductive systems, highlighting their adaptability in algebraic systems that require the representation of both positive and negative membership. The following analysis in Section 3 provides a detailed examination of the properties and applications of these translations, setting the stage for further exploration into their theoretical implications and practical applications.

Definition 11. *The inclusion $\subseteq$ is defined by setting, for any BFSs $\varphi = (\varphi^+, \varphi^-)$ and $\psi = (\psi^+, \psi^-)$ in a nonempty set X,*

$$\varphi \subseteq \psi \Leftrightarrow (\forall x \in X)(\varphi^+(x) \leq \psi^+(x) \text{ and } \varphi^-(x) \geq \psi^-(x)).$$

We say that $\psi = (\psi^+, \psi^-)$ is a bipolar fuzzy extension of $\varphi = (\varphi^+, \varphi^-)$, and $\varphi = (\varphi^+, \varphi^-)$ is a bipolar fuzzy intensity of $\psi = (\psi^+, \psi^-)$.

Definition 12. *For any BFS $\varphi = (\varphi^+, \varphi^-)$ in a nonempty set X, we denote*

$$\top = 1 - \sup\{\varphi^+(x) \mid x \in X\},$$

$$\bot = -1 - \inf\{\varphi^-(x) \mid x \in X\}.$$

Let $\varphi = (\varphi^+, \varphi^-)$ be a BFS in a nonempty set X and $(\beta, \alpha) \in [0, \top] \times [\bot, 0]$. By a bipolar fuzzy (β, α)-translation of $\varphi = (\varphi^+, \varphi^-)$ of type I, we mean a BFS $\varphi^{T_1}_{(\beta,\alpha)} = (\varphi^+_{(\beta,T_1)}, \varphi^-_{(\alpha,T_1)})$, where

$$\varphi^+_{(\beta,T_1)} : X \to [0, 1], x \mapsto \varphi^+(x) + \beta,$$

$$\varphi^-_{(\alpha,T_1)} : X \to [-1, 0], x \mapsto \varphi^-(x) + \alpha.$$

Theorem 1. *If a BFS $\varphi = (\varphi^+, \varphi^-)$ in a Hilbert algebra $X = (X, \cdot, 1_X)$ is a bipolar fuzzy subalgebra of X, then for all $(\beta, \alpha) \in [0, \top] \times [\bot, 0]$, the bipolar fuzzy (β, α)-translation $\varphi^{T_1}_{(\beta,\alpha)} = (\varphi^+_{(\beta,T_1)}, \varphi^-_{(\alpha,T_1)})$ of $\varphi = (\varphi^+, \varphi^-)$ is a bipolar fuzzy subalgebra of X.*

Proof. Assume that $\varphi = (\varphi^+, \varphi^-)$ is a bipolar fuzzy subalgebra of X. For any $(\beta, \alpha) \in [0, \top] \times [\bot, 0]$ and for all $x, y \in X$, we have

$$\begin{aligned}
\varphi^+_{(\beta,T_1)}(x \cdot y) &= \varphi^+(x \cdot y) + \beta \\
&\geq \min\{\varphi^+(x), \varphi^+(y)\} + \beta \\
&= \min\{\varphi^+(x) + \beta, \varphi^+(y) + \beta\} \\
&= \min\{\varphi^+_{(\beta,T_1)}(x), \varphi^+_{(\beta,T_1)}(y)\},
\end{aligned}$$

$$\begin{aligned}
\varphi^-_{(\alpha,T_1)}(x \cdot y) &= \varphi^-(x \cdot y) + \alpha \\
&\leq \max\{\varphi^-(x), \varphi^-(y)\} + \alpha \\
&= \max\{\varphi^-(x) + \alpha, \varphi^-(y) + \alpha\} \\
&= \max\{\varphi^-_{(\alpha,T_1)}(x), \varphi^-_{(\alpha,T_1)}(y)\}.
\end{aligned}$$

Hence, $\varphi^{T_1}_{(\beta,\alpha)} = (\varphi^+_{(\beta,T_1)}, \varphi^-_{(\alpha,T_1)})$ is a bipolar fuzzy subalgebra of X.

Theorem 2. *If there exists $(\beta, \alpha) \in [0, \top] \times [\bot, 0]$ such that the bipolar fuzzy (β, α)-translation $\varphi^{T_1}_{(\beta,\alpha)} = (\varphi^+_{(\beta,T_1)}, \varphi^-_{(\alpha,T_1)})$ of a BFS $\varphi = (\varphi^+, \varphi^-)$ is a bipolar fuzzy subalgebra of a Hilbert algebra $X = (X, \cdot, 1_X)$, then $\varphi = (\varphi^+, \varphi^-)$ is a bipolar fuzzy subalgebra of X.*

Proof. Assume that $\varphi^{T_1}_{(\beta,\alpha)} = (\varphi^+_{(\beta,T_1)}, \varphi^-_{(\alpha,T_1)})$ is a bipolar fuzzy subalgebra of X for $(\beta, \alpha) \in [0, \top] \times [\bot, 0]$. For all $x, y \in X$, we have

$$\begin{aligned}
\varphi^+(x \cdot y) + \beta &= \varphi^+_{(\beta,T_1)}(x \cdot y) \\
&\geq \min\{\varphi^+_{(\beta,T_1)}(x), \varphi^+_{(\beta,T_1)}(y)\} \\
&= \min\{\varphi^+(x) + \beta, \varphi^+(y) + \beta\} \\
&= \min\{\varphi^+(x), \varphi^+(y)\} + \beta,
\end{aligned}$$

$$\begin{aligned}
\varphi^-(x \cdot y) + \alpha &= \varphi^-_{(\alpha,T_1)}(x \cdot y) \\
&\leq \max\{\varphi^-_{(\alpha,T_1)}(x), \varphi^-_{(\alpha,T_1)}(y)\} \\
&= \max\{\varphi^-(x) + \alpha, \varphi^-(y) + \alpha\} \\
&= \max\{\varphi^-(x), \varphi^-(y)\} + \alpha.
\end{aligned}$$

Thus, $\varphi^+(x\cdot y)\geq\min\{\varphi^+(x),\varphi^+(y)\}$ and $\varphi^-(x\cdot y)\leq\max\{\varphi^-(x),\varphi^-(y)\}$. Hence, $\varphi=(\varphi^+,\varphi^-)$ is a bipolar fuzzy subalgebra of X.

Theorem 3. *If a BFS $\varphi=(\varphi^+,\varphi^-)$ in a Hilbert algebra $X=(X,\cdot,1_X)$ is a bipolar fuzzy ideal (resp., deductive system) of X, then for all $(\beta,\alpha)\in[0,\top]\times[\bot,0]$, the bipolar fuzzy (β,α)-translation $\varphi^{T_1}_{(\beta,\alpha)}=(\varphi^+_{(\beta,T_1)},\varphi^-_{(\alpha,T_1)})$ of $\varphi=(\varphi^+,\varphi^-)$ is a bipolar fuzzy ideal (resp., deductive system) of X.*

Proof. The proof is similar to Theorem 1.

Theorem 4. *If there exists $(\beta,\alpha)\in[0,\top]\times[\bot,0]$ such that the bipolar fuzzy (β,α)-translation $\varphi^{T_1}_{(\beta,\alpha)}=(\varphi^+_{(\beta,T_1)},\varphi^-_{(\alpha,T_1)})$ of a BFS $\varphi=(\varphi^+,\varphi^-)$ is a bipolar fuzzy ideal (resp., deductive system) of a Hilbert algebra $X=(X,\cdot,1_X)$, then $\varphi=(\varphi^+,\varphi^-)$ is a bipolar fuzzy ideal (resp., deductive system) of X.*

Proof. The proof is similar to Theorem 2.

Remark 1. *If $\varphi=(\varphi^+,\varphi^-)$ is a BFS in a nonempty set X, then for all $(\beta,\alpha)\in[0,\top]\times[\bot,0]$, $\varphi^+_{(\beta,T_1)}(x)=\varphi^+(x)+\beta\geq\varphi^+(x)$ and $\varphi^-_{(\alpha,T_1)}(x)=\varphi^-(x)+\alpha\leq\varphi^-(x)$ for all $x\in X$. Hence, the bipolar fuzzy (β,α)-translation $\varphi^{T_1}_{(\beta,\alpha)}=(\varphi^+_{(\beta,T_1)},\varphi^-_{(\alpha,T_1)})$ of $\varphi=(\varphi^+,\varphi^-)$ is a bipolar fuzzy extension of $\varphi=(\varphi^+,\varphi^-)$ for all $(\beta,\alpha)\in[0,\top]\times[\bot,0]$.*

Definition 13. *For any BFS $\varphi=(\varphi^+,\varphi^-)$ in a nonempty set X, we denote*

$$\mp=\inf\{\varphi^+(x)\mid x\in X\},$$

$$\pm=\sup\{\varphi^-(x)\mid x\in X\}.$$

Let $\varphi=(\varphi^+,\varphi^-)$ be a BFS in a nonempty set X and $(\beta,\alpha)\in[0,\mp]\times[\pm,0]$. By a bipolar fuzzy (β,α)-translation of $\varphi=(\varphi^+,\varphi^-)$ of Type II, we mean a BFS $\varphi^{T_2}_{(\beta,\alpha)}=(\varphi^+_{(\beta,T_2)},\varphi^-_{(\alpha,T_2)})$, where

$$\varphi^+_{(\beta,T_2)}:X\to[0,1],x\mapsto\varphi^+(x)-\beta,$$

$$\varphi^-_{(\alpha,T_2)}:X\to[-1,0],x\mapsto\varphi^-(x)-\alpha.$$

Theorem 5. *If a BFS $\varphi=(\varphi^+,\varphi^-)$ in a Hilbert algebra $X=(X,\cdot,1_X)$ is a bipolar fuzzy subalgebra of X, then for all $(\beta,\alpha)\in[0,\mp]\times[\pm,0]$, the bipolar fuzzy (β,α)-translation $\varphi^{T_2}_{(\beta,\alpha)}=(\varphi^+_{(\beta,T_2)},\varphi^-_{(\alpha,T_2)})$ of $\varphi=(\varphi^+,\varphi^-)$ is a bipolar fuzzy subalgebra of X.*

Proof. Assume that $\varphi=(\varphi^+,\varphi^-)$ is a bipolar fuzzy subalgebra of X. For any $(\beta,\alpha)\in[0,\mp]\times[\pm,0]$ and for all $x,y\in X$, we have

$$\begin{aligned}\varphi^+_{(\beta,T_2)}(x\cdot y)&=\varphi^+(x\cdot y)-\beta\\&\geq\min\{\varphi^+(x),\varphi^+(y)\}-\beta\\&=\min\{\varphi^+(x)-\beta,\varphi^+(y)-\beta\}\\&=\min\{\varphi^+_{(\beta,T_2)}(x),\varphi^+_{(\beta,T_2)}(y)\},\end{aligned}$$

$$\begin{aligned}\varphi^-_{(\alpha,T_2)}(x\cdot y) &= \varphi^-(x\cdot y)-\alpha\\ &\le \max\{\varphi^-(x),\varphi^-(y)\}-\alpha\\ &= \max\{\varphi^-(x)-\alpha,\varphi^-(y)-\alpha\}\\ &= \max\{\varphi^-_{(\alpha,T_2)}(x),\varphi^-_{(\alpha,T_2)}(y)\}.\end{aligned}$$

Hence, $\varphi^{T_2}_{(\beta,\alpha)} = (\varphi^+_{(\beta,T_2)}, \varphi^-_{(\alpha,T_2)})$ is a bipolar fuzzy subalgebra of X.

Theorem 6. *If there exists $(\beta,\alpha) \in [0,\mp] \times [\pm,0]$ such that the bipolar fuzzy (β,α)-translation $\varphi^{T_2}_{(\beta,\alpha)} = (\varphi^+_{(\beta,T_2)}, \varphi^-_{(\alpha,T_2)})$ of $\varphi = (\varphi^+,\varphi^-)$ is a bipolar fuzzy subalgebra of a Hilbert algebra $X = (X,\cdot,1_X)$, then $\varphi = (\varphi^+,\varphi^-)$ is a bipolar fuzzy subalgebra of X.*

Proof. Assume that $\varphi^{T_2}_{(\beta,\alpha)} = (\varphi^+_{(\beta,T_2)}, \varphi^-_{(\alpha,T_2)})$ is a bipolar fuzzy subalgebra of X for $(\beta,\alpha) \in [0,\mp] \times [\pm,0]$. For all $x,y \in X$, we have

$$\begin{aligned}\varphi^+(x\cdot y)-\beta &= \varphi^+_{(\beta,T_2)}(x\cdot y)\\ &\ge \min\{\varphi^+_{(\beta,T_2)}(x),\varphi^+_{(\beta,T_2)}(y)\}\\ &= \min\{\varphi^+(x)-\beta,\varphi^+(y)-\beta\}\\ &= \min\{\varphi^+(x),\varphi^+(y)\}-\beta,\end{aligned}$$

$$\begin{aligned}\varphi^-(x\cdot y)-\alpha &= \varphi^-_{(\alpha,T_2)}(x\cdot y)\\ &\le \max\{\varphi^-_{(\alpha,T_2)}(x),\varphi^-_{(\alpha,T_2)}(y)\}\\ &= \max\{\varphi^-(x)-\alpha,\varphi^-(y)-\alpha\}\\ &= \max\{\varphi^-(x),\varphi^-(y)\}-\alpha.\end{aligned}$$

Thus, $\varphi^+(x\cdot y) \ge \min\{\varphi^+(x),\varphi^+(y)\}$ and $\varphi^-(x\cdot y) \le \max\{\varphi^-(x),\varphi^-(y)\}$. Hence, $\varphi = (\varphi^+,\varphi^-)$ is a bipolar fuzzy subalgebra of X.

Theorem 7. *If a BFS $\varphi = (\varphi^+,\varphi^-)$ in a Hilbert algebra $X = (X,\cdot,1_X)$ is a bipolar fuzzy ideal (resp., deductive system) of X, then for all $(\beta,\alpha) \in [0,\mp] \times [\pm,0]$, the bipolar fuzzy (β,α)-translation $\varphi^{T_2}_{(\beta,\alpha)} = (\varphi^+_{(\beta,T_2)}, \varphi^-_{(\alpha,T_2)})$ of $\varphi = (\varphi^+,\varphi^-)$ is a bipolar fuzzy ideal (resp., deductive system) of X.*

Proof. The proof is similar to Theorem 5.

Theorem 8. *If there exists $(\beta,\alpha) \in [0,\mp] \times [\pm,0]$ such that the bipolar fuzzy (β,α)-translation $\varphi^{T_2}_{(\beta,\alpha)} = (\varphi^+_{(\beta,T_2)}, \varphi^-_{(\alpha,T_2)})$ of $\varphi = (\varphi^+,\varphi^-)$ is a bipolar fuzzy ideal (resp., deductive system) of a Hilbert algebra $X = (X,\cdot,1_X)$, then $\varphi = (\varphi^+,\varphi^-)$ is a bipolar fuzzy ideal (resp., deductive system) of X.*

Proof. The proof is similar to Theorem 6.

Remark 2. *If $\varphi = (\varphi^+,\varphi^-)$ is a BFS in a nonempty set X, then for all $(\beta,\alpha) \in [0,\mp] \times [\pm,0]$, $\varphi^+_{(\beta,T_2)}(x) = \varphi^+(x)-\beta \le \varphi^+(x)$ and $\varphi^-_{(\alpha,T_2)}(x) = \varphi^-(x)-\alpha \ge \varphi^-(x)$ for all $x \in X$. Hence, the bipolar fuzzy (β,α)-translation $\varphi^{T_2}_{(\beta,\alpha)} = (\varphi^+_{(\beta,T_2)}, \varphi^-_{(\alpha,T_2)})$ of $\varphi = (\varphi^+,\varphi^-)$ is a bipolar fuzzy intensity of $\varphi = (\varphi^+,\varphi^-)$ for all $(\beta,\alpha) \in [0,\mp] \times [\pm,0]$.*

Definition 14. *Let $\varphi = (\varphi^+, \varphi^-)$ be a BFS in a nonempty set X. The BFS $\overline{\varphi} = (\overline{\varphi^+}, \overline{\varphi^-})$ in X defined by: for all $x \in X$,*

$$\overline{\varphi^+}(x) = 1 - \varphi^+(x),$$

$$\overline{\varphi^-}(x) = -1 - \varphi^-(x),$$

is called the complement of $\varphi = (\varphi^+, \varphi^-)$ in X.

Definition 15. *[17] Let $\varphi = (\varphi^+, \varphi^-)$ be a BFS in a nonempty set X. For $(t^+, t^-) \in [0, 1] \times [-1, 0]$, the sets*

$$P_L(\varphi, t^+) = \{x \in X \mid \varphi^+(x) \leq t^+\},$$

$$P_U(\varphi, t^+) = \{x \in X \mid \varphi^+(x) \geq t^+\}$$

are called the positive lower t^--cut and the positive upper t^+-cut of $\varphi = (\varphi^+, \varphi^-)$, respectively. The sets

$$N_L(\varphi, t^-) = \{x \in X \mid \varphi^-(x) \leq t^-\},$$

$$N_U(\varphi, t^-) = \{x \in X \mid \varphi^-(x) \geq t^-\}$$

are called the negative lower t^--cut and the negative upper t^+-cut of $\varphi = (\varphi^+, \varphi^-)$, respectively.

Theorem 9. *Let $\varphi = (\varphi^+, \varphi^-)$ be a BFS in a Hilbert algebra $X = (X, \cdot, 1_X)$. Then $\overline{\varphi} = (\overline{\varphi^+}, \overline{\varphi^-})$ is a bipolar fuzzy subalgebra of X if and only if for all $(t^+, t^-) \in [0, 1] \times [-1, 0]$, $P_L(\varphi, t^+)$ and $N_U(\varphi, t^-)$ are subalgebras of X if $P_L(\varphi, t^+)$ and $N_U(\varphi, t^-)$ are nonempty.*

Proof. Assume that $\overline{\varphi} = (\overline{\varphi^+}, \overline{\varphi^-})$ is a bipolar fuzzy subalgebra of X. Let $(t^+, t^-) \in [0, 1] \times [-1, 0]$ be such that $P_L(\varphi, t^+)$ and $N_U(\varphi, t^-)$ are nonempty.

Let $x, y \in P_L(\varphi, t^+)$. Then $\varphi^+(x) \leq t^+$ and $\varphi^+(y) \leq t^+$, so t^+ is an upper bound of $\{\varphi^+(x), \varphi^+(y)\}$. By (1), we have $\overline{\varphi^+}(x \cdot y) \geq \min\{\overline{\varphi^+}(x), \overline{\varphi^+}(y)\}$. So,

$$1 - \varphi^+(x \cdot y) \geq \min\{1 - \varphi^+(x), 1 - \varphi^+(y)\} = 1 - \max\{\varphi^+(x), \varphi^+(y)\}.$$

Thus,

$$\varphi^+(x \cdot y) \leq \max\{\varphi^+(x), \varphi^+(y)\} \leq t^+$$

and so $x \cdot y \in P_L(\varphi, t^+)$. Therefore, $P_L(\varphi, t^+)$ is a subalgebra of X.

Let $x, y \in N_U(\varphi, t^-)$. Then $\varphi^-(x) \geq t^-$ and $\varphi^-(y) \geq t^-$, so t^- is a lower bound of $\{\varphi^-(x), \varphi^-(y)\}$. By (1), we have $\overline{\varphi^-}(x \cdot y) \leq \max\{\overline{\varphi^-}(x), \overline{\varphi^-}(y)\}$. So,

$$-1 - \varphi^-(x \cdot y) \leq \max\{-1 - \varphi^-(x), -1 - \varphi^-(y)\} = -1 - \min\{\varphi^-(x), \varphi^-(y)\}.$$

Thus,

$$\varphi^-(x \cdot y) \geq \min\{\varphi^-(x), \varphi^-(y)\} \geq t^-$$

and so $x \cdot y \in N_U(\varphi, t^-)$. Therefore, $N_U(\varphi, t^-)$ is a subalgebra of X.

Conversely, assume that for all $(t^+, t^-) \in [0,1] \times [-1,0]$, $P_L(\varphi, t^+)$ and $N_U(\varphi, t^-)$ are subalgebras of X if $P_L(\varphi, t^+)$ and $N_U(\varphi, t^-)$ are nonempty.

Let $x, y \in X$. Then $\varphi^+(x), \varphi^+(y) \in [0,1]$. Choose $t^+ = \max\{\varphi^+(x), \varphi^+(y)\}$. Thus, $\varphi^+(x) \leq t^+$ and $\varphi^+(y) \leq t^+$, so $x, y \in P_L(\varphi, t^+) \neq \emptyset$. By the assumption, we have $P_L(\varphi, t^+)$ is a subalgebra of X and so $x \cdot y \in P_L(\varphi, t^+)$. Thus,

$$\varphi^+(x \cdot y) \leq t^+ = \max\{\varphi^+(x), \varphi^+(y)\}.$$

So,

$$\begin{aligned}\overline{\varphi^+}(x \cdot y) &= 1 - \varphi^+(x \cdot y) \\ &\geq 1 - \max\{\varphi^+(x), \varphi^+(y)\} \\ &= \min\{1 - \varphi^+(x), 1 - \varphi^+(y)\} \\ &= \min\{\overline{\varphi^+}(x), \overline{\varphi^+}(y)\}.\end{aligned}$$

Let $x, y \in X$. Then $\varphi^-(x), \varphi^-(y) \in [-1,0]$. Choose $t^- = \min\{\varphi^-(x), \varphi^-(y)\}$. Thus, $\varphi^-(x) \geq t^-$ and $\varphi^-(y) \geq t^-$, so $x, y \in N_U(\varphi, t^-) \neq \emptyset$. By the assumption, we have $N_U(\varphi, t^-)$ is a subalgebra of X and so $x \cdot y \in N_U(\varphi, t^-)$. Thus, $\varphi^-(x \cdot y) \geq t^- = \min\{\varphi^-(x), \varphi^-(y)\}$. So,

$$\begin{aligned}\overline{\varphi^-}(x \cdot y) &= -1 - \varphi^-(x \cdot y) \\ &\leq -1 - \min\{\varphi - (x), \varphi^-(y)\} \\ &= \max\{-1 - \varphi^-(x), -1 - \varphi^-(y)\} \\ &= \max\{\overline{\varphi^-}(x), \overline{\varphi^-}(y)\}.\end{aligned}$$

Hence, $\overline{\varphi} = (\overline{\varphi^+}, \overline{\varphi^-})$ is a bipolar fuzzy subalgebra of X.

Theorem 10. *Let $\varphi = (\varphi^+, \varphi^-)$ be a BFS in a Hilbert algebra $X = (X, \cdot, 1_X)$. Then $\overline{\varphi} = (\overline{\varphi^+}, \overline{\varphi^-})$ is a bipolar fuzzy ideal (resp., deductive system) of X if and only if for all $(t^+, t^-) \in [0,1] \times [-1,0]$, $P_L(\varphi, t^+)$ and $N_U(\varphi, t^-)$ are ideals (resp., deductive systems) of X if $P_L(\varphi, t^+)$ and $N_U(\varphi, t^-)$ are nonempty.*

Proof. The proof is similar to Theorem 9.

4. Conclusion

This study has advanced the application of BFS theory within Hilbert algebras by introducing and rigorously analyzing bipolar fuzzy (β, α)-translations of a BFS $\varphi = (\varphi^+, \varphi^-)$ in two distinct forms: Type I and Type II. The in-depth investigation of the fundamental properties of these translations, along with the development of bipolar fuzzy extensions and intensities, has broadened the theoretical and practical utility of BFSs in capturing complex bipolar information. Further, the study elucidates intricate relationships among the complement of a bipolar fuzzy subalgebra, bipolar fuzzy ideals, and bipolar fuzzy deductive systems through their level cuts, offering a deeper structural understanding of Hilbert algebras under bipolar fuzzy logic. These findings contribute significantly to the theoretical framework supporting bipolar fuzzy systems, which hold promise for handling nuanced and uncertain information across diverse applications. Building on this work, future research can explore the extended use of BFSs within complex structures such as bipolar complex fuzzy subgroups and semigroups, as demonstrated in [1, 14–16, 18]. These directions will likely open new pathways in the algebraic modeling of systems characterized by multifaceted uncertainties, further enriching the versatility of bipolar fuzzy logic in advanced mathematical and applied contexts.

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