EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 17, No. 4, 2024, 4003-4013
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Solution of Well-Known Physical Problems with Fractional Conformable Derivative

Eman Abuteen[1,*], Abdessamad Ait Brahim[2], Abdelmajid El Hajaji[3], Khalid Hilal[2], Ayoub Charhabil[4]

[1] *Department of Basic Scientific Sciences, Faculty of Engineering Technology, Al-Balqa Applied University, Jordan*
[2] *AMSC Laboratory, University of Sciences and Technology , Beni Mellal, Morocco*
[3] *OEE Departement, ENCGJ, University of Chouaib Doukali, El Jadida, Morocco*
[4] *LAGA Laboratory, University Sorbonne, Paris Nord, France*

Abstract. This paper presents a new definition of fractional derivatives and integrals through the conformable derivative approach. This innovative framework offers a closer alignment with classical derivative concepts while providing a more practical and intuitive basis for fractional calculus. The new definition is applicable in two primary ranges: $0 \leq \alpha < 1$ and $n-1 \leq \alpha < n$, where n is a positive integer. It is shown that when $\alpha = 1$, this definition corresponds precisely to the classical first-order derivative. Key benefits of this approach include its improved consistency with traditional calculus and greater computational ease, making it a useful tool for both theoretical research and practical applications. By integrating fractional calculus with conventional derivative ideas, this definition simplifies the analysis and interpretation of fractional differential equations and their solutions. Additionally, we examine the definition's effects on stability and convergence in numerical methods and provide examples demonstrating its effectiveness and applicability.

2020 Mathematics Subject Classifications: 26A33, 34A08, 65L05, 45D05

Key Words and Phrases: Fractional differential equations, Conformable derivative, Falling body, Atomic Solution

---

1. Introduction

Fractional calculus generalizes the ideas of differentiation and integration to non-integer (fractional) orders. This mathematical field has its roots in the late 17th century, with an early reference found in a letter dated September 30, 1695, between Leibniz and L'Hôpital, where L'Hôpital questioned the meaning of $\frac{d^n}{dz^n} f(x)$ when $n = \frac{1}{2}$. Over time, extensive research has been devoted to fractional integrals and derivatives. Although fractional

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v17i4.5537

Email addresses: dr.eman.abuteen@bau.edu.jo (E. Abuteen), abdessamad191212@gmail.com (A. A. Brahim), a_elhajaji@yahoo.fr (A. El Hajaji), hilalkhalid2005@yahoo.fr (K. Hilal), ayoub.charhabil@gmail.com (A. Charhabil)

calculus extends classical calculus principles, its use in physics has been relatively limited historically [2–6, 8–11]. This limitation may be partly due to the difficulty in accessing foundational concepts in earlier mathematical literature. However, the applications of fractional differential equations has gained importance for accurately modeling a variety of systems in science and engineering, such as , control theory, viscoelasticity , diffusion processes, heat conduction, electrochemistry , mechanics, electricity, fractals, and chaos theory.

Lyapunov's Second or Direct Method is renowned for assessing the stability of differential equations without requiring explicit solutions. This method utilizes a Lyapunov function to examine the asymptotic behavior of solutions, which is especially useful for nonlinear systems. Its applications to non-integer order systems is significant, as it extends the use of the Lyapunov function to systems involving fractional derivatives. In this article, we explore the application of fractional-like derivatives of the Lyapunov function for analyzing stability in perturbed motion equations, presenting several theorems that parallel those of the direct Lyapunov method for specific motion equations.

Several operators of fractional derivatives are defined as follows:

1. Riemann-Liouville Fractional Derivative of order $\alpha \in [n-1, n)$:

$$^{RL}D_a^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(x)}{(t-x)^{\alpha-n+1}}\, dx,$$

2. Caputo Fractional Derivative of order $\alpha \in [n-1, n)$:

$$^{C}D_a^\alpha (f)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(x)}{(t-x)^{\alpha-n+1}}\, dx.$$

3. Caputo-Fabrizio Fractional Derivative of order $\alpha \in (0, 1)$:

$$^{CF}D_a^\alpha (f)(t) = \frac{M(\alpha)}{(1-\alpha)} \int_a^t f'(x) \exp\left[-\frac{(t-x)^\alpha}{1-\alpha}\right] dx,$$

where $M(\alpha)$ is a normalized function such that $M(0) = M(1) = 1$.

4. Atangana-Baleanu Fractional Derivative** of order $\alpha \in (0, 1)$ in the Caputo sense:

$$^{ABC}D_a^\alpha (f)(t) = \frac{M(\alpha)}{(1-\alpha)} \int_a^t f'(x) E_\alpha \left[-\frac{(t-x)^\alpha}{1-\alpha}\right] dx,$$

where $M(\alpha)$ retains properties similar to those in the Caputo-Fabrizio derivative.

Many of these definitions deviate from the fundamental characteristics of the ordinary derivative, except for linearity. A recent study [13] introduced a novel fractional derivative that aligns with the basic principles of the ordinary derivative. This paper applies this new definition, along with the Mittag-Leffler function, to propose a new fractional derivative, examine its properties, and demonstrate its effectiveness through various examples.

2. Definition of new fractional derivative

In this section, we begin by introducing the Mittag-Leffler function, which plays a significant role in various physical processes and often emerges in the solutions to fractional differential equations.

Definition 1. *The Mittag-Leffler function E_α is defined as:*

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \tag{1}$$

where α is a positive parameter and $\Gamma(.)$ is the gamma function. The following new fractional derivative is formulated using the first two terms of the series expansion of the Mittag-Leffler function:

$$E_\alpha\left(\varepsilon t^{-\alpha}\right) = 1 + \frac{\varepsilon t^{-\alpha}}{\Gamma(\alpha+1)} + O\left(t^{-2\alpha}\right). \tag{2}$$

This approximation allows us to define a new fractional derivative that incorporates the Mittag-Leffler function's properties, providing a refined approach to modeling fractional dynamics. The term $O\left(t^{-2\alpha}\right)$ represents higher-order terms that are generally small compared to the first two terms, thus simplifying the expression while capturing the essential behavior of the Mittag-Leffler function in this context.

Definition 2. *Consider a function $f : [0,\infty) \to \mathbb{R}$. The new fractional derivative of order α, defined in the sense of the conformable fractional derivative, is given by:*

$$D^\alpha(f)(t) = \lim_{\varepsilon\to 0} \frac{f\left(tE_\alpha\left(\varepsilon t^{-\alpha}\right)\right) - f(t)}{\varepsilon}, \tag{3}$$

where $t > 0$ and $\alpha \in (0,1)$. In this context, we say that the function f is α-differentiable.

If f is α-differentiable on the interval $(0,t)$, and if the limit

$$\lim_{t\to 0^+} f^{(\alpha)}(t) \tag{4}$$

exists, then we define the fractional derivative at zero as:

$$f^{(\alpha)}(0) = \lim_{t\to 0^+} f^{(\alpha)}(t). \tag{5}$$

Theorem 1. *[6] Let f and g be α-differentiable functions at a point $t > 0$, with $0 < \alpha \leq 1$. Then the following properties hold:*

1. *Linearity: $D^\alpha(af + bg) = aD^\alpha(f) + bD^\alpha(g)$ for all $a, b \in \mathbb{R}$.*
2. *Power Function: $D^\alpha\left(t^p\right) = \frac{p^{p-\alpha}}{\Gamma(\alpha+1)}$ for all $p \in \mathbb{R}$.*
3. *Constant Function: $D^\alpha(c) = 0$ for any constant function $f(t) = c$.*
4. *Product Rule: $D^\alpha(fg) = fD^\alpha(g) + gD^\alpha(f)$.*
5. *Quotient Rule: $D^\alpha\left(\frac{f}{g}\right) = \frac{gD^\alpha(f) - fD^\alpha(g)}{g^2}$.*

Lemma 1. *[6]*

Let f be both α-differentiable and differentiable at a point $t > 0$, with $0 < \alpha \leq 1$. Then the α-fractional derivative of f is given by:

$$D^{\alpha}(f)(t) = \frac{t^{1-\alpha}}{\Gamma(\alpha+1)} f'(t), \tag{6}$$

where $f'(t)$ denotes the standard derivative of f with respect to t, and $\Gamma(\alpha+1)$ is the Gamma function evaluated at $\alpha+1$.

2.0.1. New fractional integral

If a function f is α-differentiable in the interval (a, b), we define the α-fractional integral of f for $a \geq 0$ and $a < t < b$ as follows:

Definition 3. *The new α-fractional integral of a function f that is α-differentiable is given by:*

$$I_a^{\alpha}(f)(t) = \int_a^t \frac{\Gamma(\alpha+1)}{x^{1-\alpha}} f(x)\, dx, \tag{7}$$

where $\alpha \in (0,1)$.

One of the key results of this definition is stated in the following theorem:

Theorem 2. *If f is continuous on the domain of I^{α} for $t \geq a$, then:*

$$D^{\alpha}\left(I^{\alpha}(f)\right)(t) = f(t). \tag{8}$$

Proof: *See [1].*

Theorem 3. *[6] Let $\alpha \in (0,1]$, then:*

1) $\mathbf{D}^{\alpha}\left(\frac{\Gamma(1+\alpha)}{\alpha} t^{\alpha}\right) = 1$.

2) $\mathbf{D}^{\alpha}\left(\sin \frac{1}{\alpha} t^{\alpha}\right) = \cos(\frac{\Gamma(1+\alpha)}{\alpha} t^{\alpha})$.

3) $\mathbf{D}^{\alpha}\left(\cos \frac{1}{\alpha} t^{\alpha}\right) = -\sin(\frac{\Gamma(1+\alpha)}{\alpha} t^{\alpha})$.

4) $\mathbf{D}^{\alpha}\left(e^{\frac{1}{\alpha} t^{\alpha}}\right) = e^{\frac{\Gamma(1+\alpha)}{\alpha} t^{\alpha}}$.

3. Applications

3.1. Falling body problem:

The problem of free fall examines how objects move under the sole influence of gravity, excluding forces like air resistance. This scenario offers valuable insights into motion, acceleration, and gravity effects. During free fall, an object experiences constant acceleration due to gravity, which is approximately $9.8\,\mathrm{m/s^2}$, directed toward the center of the Earth. The behavior of a falling object can be studied using Newton's laws of motion and kinematic equations, which allow us to calculate various aspects such as the fall duration, velocity at specific times, and the distance traveled. Consider an object of mass m that begins falling from rest at a height A above the ground, with the fall starting at $t = 0$. We take downward motion as positive. At any point P along the path, the distance fallen z depends on time t. The instantaneous velocity $w(t)$ and acceleration a are derived from the distance function $z(t)$ using the following derivatives:

$$w(t) = \frac{dz(t)}{dt}, \quad a = \frac{dw(t)}{dt} = \frac{d^2 z(t)}{dt^2}. \tag{9}$$

According to Newton's Law, the force F acting on an object in free fall is given by $F = mg$, and the acceleration $\frac{dw(t)}{dt}$ is equal to g. This scenario is represented with the following differential equation and the initial conditions:

$$\frac{dw(t)}{dt} = g, \quad w(0) = 0, \quad z(0) = A. \tag{10}$$

Now, consider the fractional differential equation using the new conformable derivative:

$$\mathcal{D}^{\alpha} w(t) = g, \quad w(0) = 0, \quad z(0) = A. \tag{11}$$

Applying equation (6) yields:

$$\frac{t^{1-\alpha}}{\Gamma(1+\alpha)} w'(t) = g. \tag{12}$$

Integrating this equation results in:

$$w(t) = g\left(\frac{\Gamma(1+\alpha)}{\alpha} t^{\alpha}\right) + c. \tag{13}$$

Using the initial condition $w(0) = 0$, we get c is zero. Similarly, we have:

$$z'(t) = g\frac{\Gamma(1+\alpha)}{\alpha} t^{\alpha}. \tag{14}$$

Thus, the function $z(t)$ is:

$$z(t) = \frac{g}{\alpha(\alpha+1)} \Gamma(\alpha+1) t^{(\alpha+1)t} + M. \tag{15}$$

Using the initial condition $z(0) = A$, we determine that:

$$z(t) = \frac{g}{\alpha(\alpha+1)} \Gamma(\alpha+1) t^{(\alpha+1)t} + A. \tag{16}$$

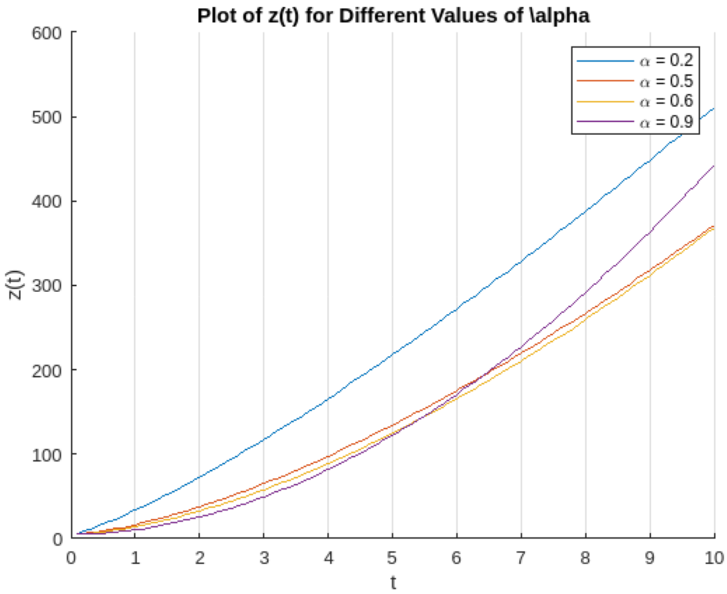

Figure 1: Comparative figures using different values of α.

This example investigates the solutions when α has the following values: $\alpha = 0.2$, $\alpha = 0.5$, $\alpha = 0.6$ and $\alpha = 0.9$, employing the recently introduced conformable fractional calculus (as detailed in [3, 4]). A comparison of our results with those obtained using other techniques illustrates how the solutions' behavior changes across these methods, shedding light on the strengths and weaknesses of the new conformable fractional calculus in comparison to more conventional approaches.

3.2. Atomic Solution

The idea of atomic solutions emerges as a solution for linear partial differential equations, both fractional and non-fractional, that cannot be resolved using the separation of variables.

Let E and F be two Banach spaces, and let E^* denote the dual space of E. For $x \in E$ and $y \in F$, consider the operator $H : E^* \to F$ defined by $H(x^*) = x^*(x)y$. This operator H is a bounded linear operator of rank one, which we denote as $x \otimes y$. Operators of this form are known as atoms.

Atoms are fundamental in the theory of tensor products and play a key role in the best approximation theory in Banach spaces [7]. An important result utilized in our paper [12] states that adding two atoms gives an atom, then either their first components or their second components are dependent. For further details on tensor products in Banach spaces, see [12].

The partial α-derivative of v with respect to x is denoted as $D_x^{\alpha} v$, and similarly, $D_x^{2\alpha} v$ represents $D_x^{\alpha} v D_x^{\alpha} v$. The same notation applies to derivatives with respect to y.

In equation (17), although the equation is linear, separating variables is not feasible. Therefore, we seek an atomic solution, which is defined as a solution of the form $v(x, t) =$

$w(x)H(t)$. We aim to solve the equation:

$$D_t^{\alpha} v + D_x^{\beta} D_x^{\beta} v = D_t^{2x} D_x^{\beta} v, \quad 0 < \alpha, \beta < 1, \tag{17}$$

with initial conditions $v(0,0) = 0$ and $D_t^{\alpha} D_x^{\beta}(0,0) = 1$.

Procedure: Assume $v(x,t) = w(x)H(t)$. By Substituting this into equation (17), we get:

$$w(x)H^{\alpha}(t) + w^{2\beta}(x)H(t) = w(x)H^{2\alpha x}(t). \tag{18}$$

This expression can be represented in tensor product form as:

$$w \otimes H^{\alpha} + w^{2\beta} \otimes H = w \otimes H^{2\alpha}. \tag{19}$$

Considering the initial conditions: $H(0) = 1$, $H^{\alpha}(0) = 1$, $w(0) = 0$, and $w^{\beta}(0) = 1$. In equation (17), the sum of two atomic solutions yields another atomic solution, leading to two cases:

Case (i): $H(t) = H^{\alpha}(t) = H^{2\alpha x}(t)$. Using results from [1], we find:

$$H(t) = e^{\frac{\Gamma(1+\alpha)t^{\alpha}}{\alpha}}. \tag{20}$$

Substituting this into equation (19) gives:

$$e^{\frac{\Gamma(1+\alpha)t^{\alpha}}{\alpha}} \otimes \left(w + w^{2\beta}\right) = e^{\frac{\Gamma(1+\alpha)t^{\alpha}}{\alpha}} \otimes w^{\beta}. \tag{21}$$

Thus, we obtain:

$$w^{2\beta} - w^{\beta} + w = 0. \tag{22}$$

Applying results from [6], we find:

$$w(x) = C_1 e^{\frac{1}{2}\Gamma(1+\beta)\frac{x^{\beta}}{\beta}} \frac{\sqrt{3}}{2} \frac{\Gamma(1+\beta)x^{\beta}}{\beta} + C_2 e^{\frac{1}{2}\Gamma(1+\beta)\frac{x^{\beta}}{\beta}} \sin\left(\sin\left(\frac{\sqrt{3}}{2}\frac{\Gamma(1+\beta)x^{\beta}}{\beta}\right)\right). \tag{23}$$

Using the initial conditions $w(0) = 0$ and $w^{\beta}(0) = 1$, we obtain:

$$w(x) = \frac{2}{\sqrt{3}} e^{\frac{1}{2}\Gamma(1+\beta)\frac{x^{\beta}}{\beta}} \sin\left(\sin\left(\frac{\sqrt{3}}{2}\frac{\Gamma(1+\beta)x^{\beta}}{\beta}\right)\right). \tag{24}$$

Combining this with equation (20), the atomic solution for equation (24) is:

$$v(x,t) = \left(\frac{2}{\sqrt{3}} e^{\frac{1}{2}\Gamma(1+\beta)\frac{x^{\beta}}{\beta}} \sin\left(\sin\left(\frac{\sqrt{3}}{2}\Gamma(1+\beta)\frac{x^{\beta}}{\beta}\right)\right)\right) e^{\Gamma(1+\alpha)\frac{t^{\alpha}}{\alpha}}. \tag{25}$$

Case (ii): When $w(x) = w_{\beta}(x) = w_{2\beta}(x)$, equation (19) has no solution, indicating that no atomic solution exists in this case.

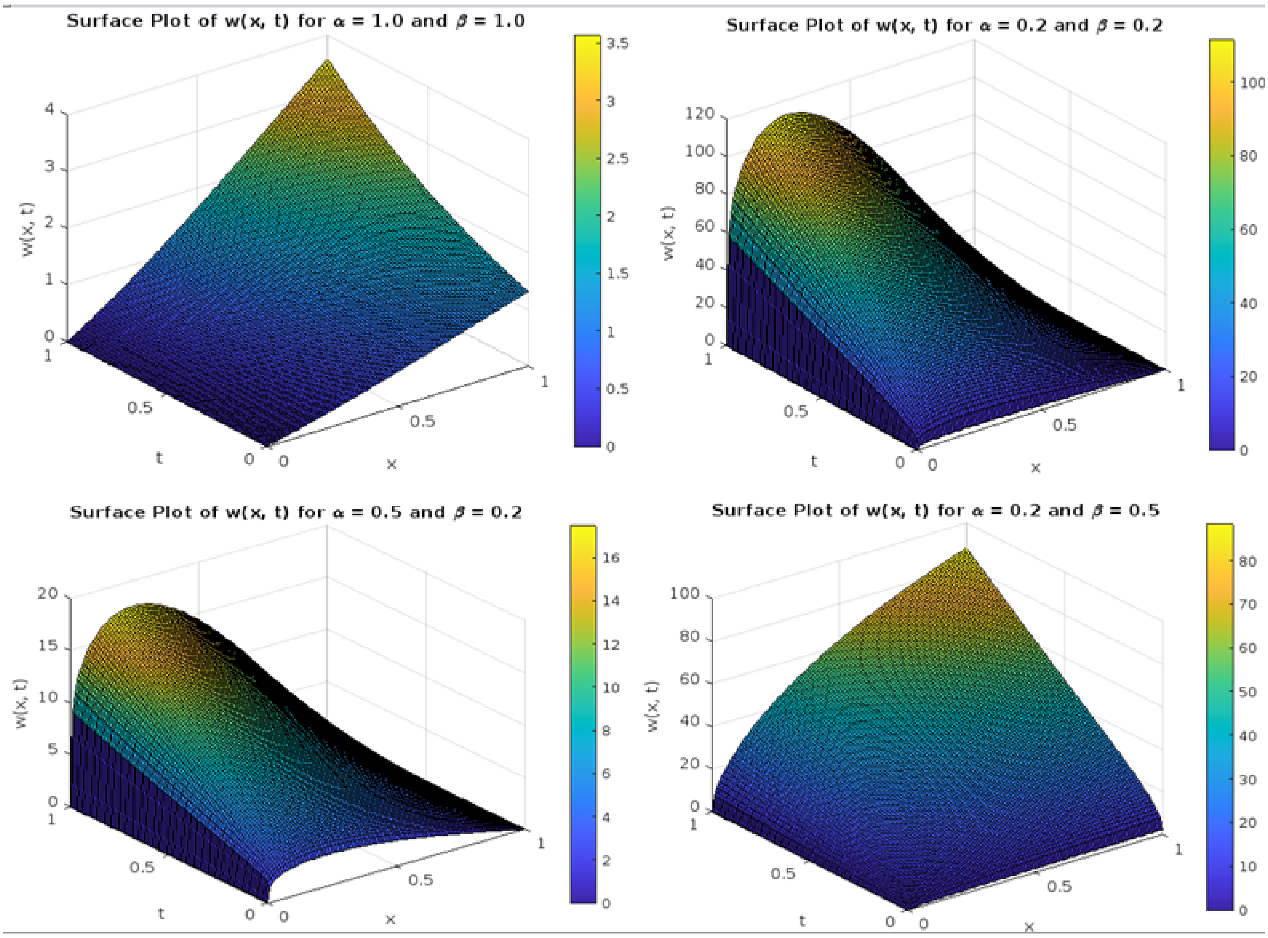

Figure 2: The exact and approximate solution of v(x,t) for equation (25), at varying values of α and β.

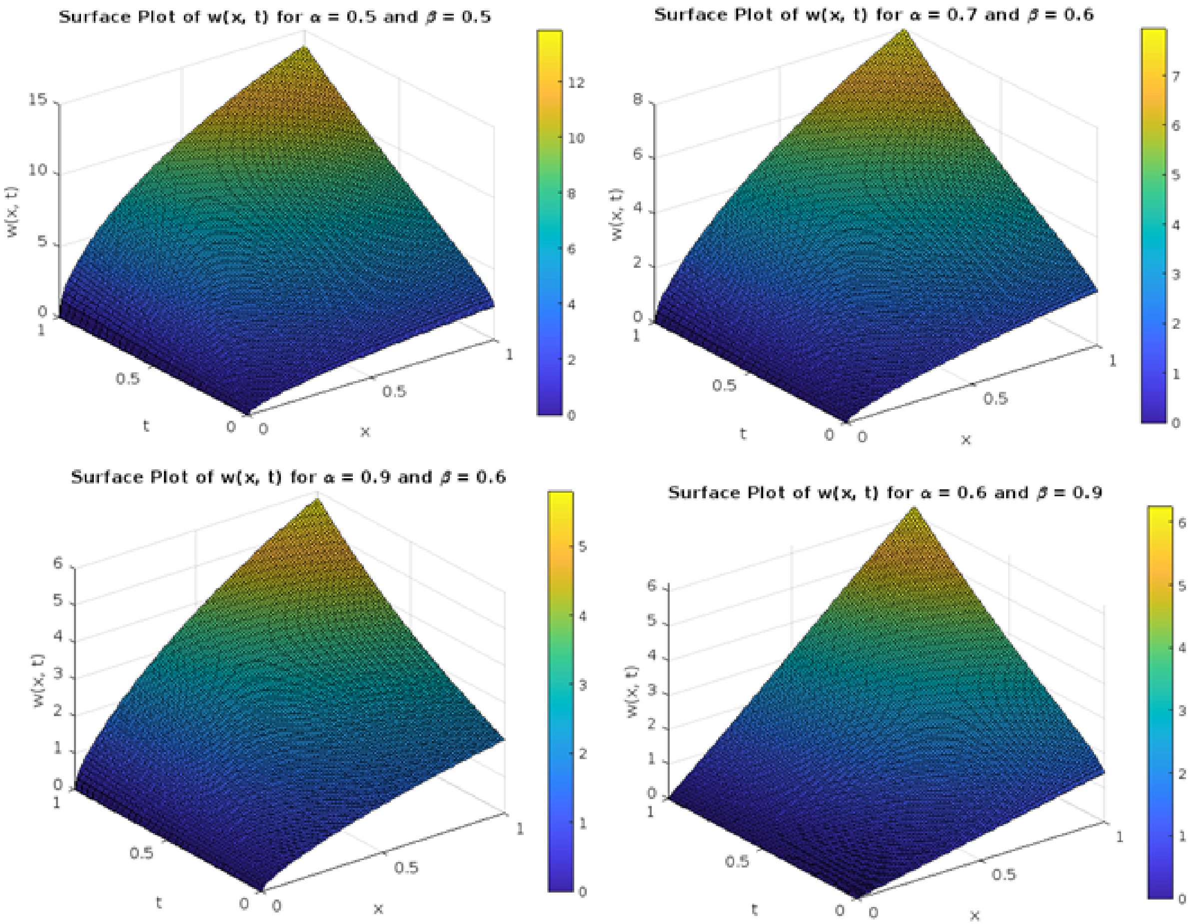

Figure 3: The exact and approximate solution of v(x,t) for equation (25), at varying values of α and β.

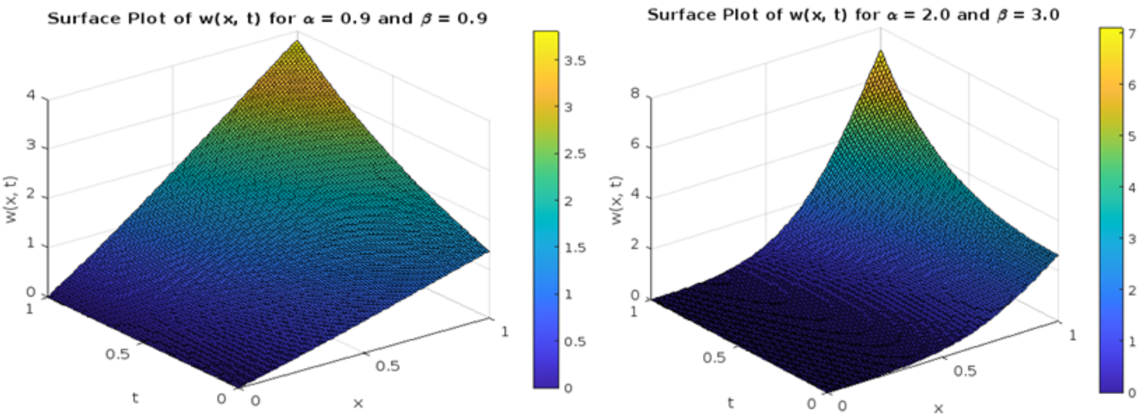

Figure 4: The exact and approximate solution of v(x,t) for equation (25), at varying values of α and β.

4. Conclusion

This method presents numerous benefits, notably its alignment with classical calculus principles and its ease of computation. These features significantly improve its usefulness for both theoretical studies and practical applications. By integrating fractional calculus concepts with traditional derivatives, our definition simplifies the analysis and interpretation of fractional differential equations and their solutions. This fusion not only clarifies these complex equations but also facilitates more intuitive and effective problem-solving.

Furthermore, we explore the broader implications of this definition across various fields. This includes assessing its impact on the stability and convergence of numerical methods, which are essential for achieving accurate and dependable results in computational tasks. Through specific examples, we demonstrate how our approach can be applied effectively in different scenarios, showcasing its practical benefits and flexibility. By emphasizing these applications, we highlight the definition's potential to advance both theoretical research and practical problem-solving in fractional calculus.

References

[1] T. Abdeljawad and D. Baleanu. Integration by parts and its applications of a new nonlocal fractional derivative with Mittag-Leffler nonsingular kernel. *Journal of Nonlinear Sciences and Applications (JNSA)*, 10(3), 2017.

[2] D. Baleanu and H. K. Jassim. A modification fractional homotopy perturbation method for solving Helmholtz and coupled Helmholtz equations on Cantor sets. *Fractal and Fractional*, 3(30):1–8, 2019.

[3] A. Ait Brahim, J. El Ghordaf, A. El Hajaji, K. Hilal, and J. E. Nápoles Valdes. A comparative analysis of Conformable, non-conformable, Riemann-Liouville, and

Caputo fractional derivatives. *European Journal of Pure and Applied Mathematics*, July 2024.

[4] A. Ait Brahim, A. El Hajaji, K. Hilal, and J. El Ghordaf. On a novel fractional calculus and its applications to well-known problems. *European Journal of Pure and Applied Mathematics*, April 2024.

[5] P. Cui and H. K. Jassim. Local fractional Sumudu decomposition method to solve fractal PDEs arising in mathematical physics. *Fractals*, 32(4):1–6, 2024.

[6] Reza Danaei. New definition of fractional derivative included Mittag-Leffler function of conformable type. *Journal Mathematics and Computational Sciences*, 2024.

[7] W. Deeb and R. Khalil. Best approximation in L(X, Y). *Mathematical Proceedings of the Cambridge Philosophical Society*, 104(3):527–531, 1988.

[8] H. Jafari, H. K. Jassim, A. Ansari, and V. T. Nguyen. Laplace decomposition method for solving the two-dimensional diffusion problem in fractal heat transfer. *Fractals*, 32(4):1–6, 2024.

[9] H. Jafari, H. K. Jassim, A. Ansari, and V. T. Nguyen. Local fractional variational iteration transform method: A tool for solving local fractional partial differential equations. *Fractals*, 32(4):1–8, 2024.

[10] H. Jafari, H. K. Jassim, and D. Baleanu. On the existence and uniqueness of solutions for local differential equations. *Entropy*, 18:1–9, 2016.

[11] H. Jafari, M. Y. Zayir, and H. K. Jassim. Analysis of fractional Navier-Stokes equations. *Heat Transfer*, 52(3):2859–2877, 2023.

[12] Roshdi Khalil. Isometries of Lp $\otimes Lp. TamkangJournalofMathematics, 16 : 77 - -85, 1985.$

[13] K. M. Owolabi and A. Atangana. *Numerical methods for fractional differentiation*, volume 54. Springer Singapore, 2019.