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Exploring $\boldsymbol{\beta}$-Basic Rough Sets and Their Applications in Medicine

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Abstract. Advancements in the rough set theory of Pawlak have opened new avenues for enhancing decision-making processes, particularly in identifying disease risk factors in medical diagnoses. While traditional rough set methodologies have provided a solid foundation, improvements are continuously needed for improvements to increase accuracy and reliability. This study introduces mathematical techniques grounded in basic rough sets, incorporating $\boldsymbol{\beta}$-open concepts to enhance precision. We present nearly basic rough sets and $\boldsymbol{\beta}$-basic approximations ($\boldsymbol{\beta}_{\mathfrak{b}}$-approximations), examining their core properties and interrelationships. Our findings reveal that these novel constructs offer superior accuracy compared to traditional methods. Both theoretical analysis and practical examples support this, with our approach achieving a 100% accuracy rate in the medical diagnosis of COVID-19. This significant improvement highlights the potential of our methods to outperform existing ones in terms of precision and reliability. The introduction of $\boldsymbol{\beta}_{\mathfrak{b}}$-approximations represents a significant advancement in rough set theory, offering enhanced accuracy in decision-making applications. Our results indicate that these methods can substantially outperform traditional techniques, especially in critical areas such as medical diagnosis. Additionally, we provide a mathematical algorithm suitable for implementation in programming languages, facilitating future research and applications across various theoretical and applied fields. This work lays the groundwork for further exploring and utilizing advanced rough set methodologies in diverse domains.

2020 Mathematics Subject Classifications: 54A05, 54A10, 54H30, 68U01, 68U05, 68U35, 92C50

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1. Introduction

The rough set theory [53, 54] has recently garnered considerable attention in the fields of artificial intelligence and computer science. Originally introduced by Pawlak in 1982, this theory effectively addresses the challenges posed by incomplete knowledge, classifying objects using equivalence relations and assessing the completeness of information within a specified set. The core concepts of rough set theory are based on approximation operators and measures of accuracy, providing valuable insights into constructing boundary regions and their significance for decision-making. However, traditional applications of rough set theory are limited by its strict reliance on equivalence relations. To overcome these constraints, several extensions of the theory have been developed, utilizing binary or other types of relations.

In 1998, Yao broke free from the constraints of equivalence relations and was the first to propose a method for generalizing Pawlak's theorem using a general binary relation, without imposing conditions on the relation. Yao introduced the concept of neighborhoods derived from binary relations into the methodology of inductive set theory, pioneering this new direction. This work paved the way for many researchers to develop neighborhood systems that infer different neighborhoods from binary relations. In [65], Yao pioneered a technique based on the idea of right and left neighborhoods, corresponding to the concepts of after-sets and fore-sets [26], respectively. However, Yao demonstrated that the approximations generated through these neighborhoods did not satisfy all of Pawlak's axioms. Therefore, as Yao indicated, additional conditions must be imposed on relations to satisfy Pawlak's properties.

Since then, many researchers (for instance, [1, 3, 4, 50] have used this technique to propose extensions based on different types of binary relations, such as tolerance relations [58], similarity relations [1, 2, 27], quasi-order relations [55], and general binary relations [7, 18, 19]. Allam et al. were the first to propose the concepts of minimal-right and minimal-left neighborhoods based on right and left neighborhoods, respectively, in [19] and [18]. They also introduced innovative methods for rough sets.

Abd El-Monsef et al. [40] introduced the concept of a $\jmath$-neighborhood space (abbreviated as $\jmath$-$\boldsymbol{NS}$) in 2014, which extends the existing notion of neighborhood spaces by utilizing binary relations. They defined two new types of neighborhoods—intersections and unions of neighborhoods—using right and left neighborhoods, as defined by Yao, along with minimal right and left neighborhoods as defined by Allam et al. This approach resulted in the creation of eight distinct types of component neighborhoods based on binary relations. Their methodology offers an intriguing technique for generalizing Pawlak's method using binary relations without imposing restrictions on the relations. The approach is based on topological structures, which has paved the way for many authors to propose various methods for generalizing Pawlak rough sets using $\jmath$-$\boldsymbol{NS}$, applying several topological structures in the rough set context.

It is worth noting that the concept of intersection and union neighborhoods, introduced by Abd El-Monsef et al. [40], has been utilized in various studies. Subsequently, several new types of neighborhoods have been proposed, building upon the eight types introduced by Abd El-Monsef et al. [40], such as E-neighborhoods [13] and maximal neighborhoods [11, 14], further expanding Pawlak's rough sets through different topologies derived from $\jmath$-$\boldsymbol{NS}$. Further studies have uncovered additional operators connecting $\jmath$-$\boldsymbol{NS}$ to novel types of neighborhood systems. For example, in 2020, Atef et al. [21] developed an adhesion $\jmath$-neighborhood space, highlighting the applicability of $\jmath$-$\boldsymbol{NS}$ in the evolving concepts of adhesion sets [50]. Errors in Atef et al.'s results were later corrected by El-Bably et al. [31], providing new insights. Additionally, in 2020, Nawar et al. [51] presented adhesion $\jmath$-neighborhoods in covering-based rough sets, using the generalized covering-approximation space defined by Abd El-Monsef et al. [41]. Recently, in the research by Al-shami et al. [15], eight neighborhoods were applied based on a new type of neighborhood called 'cardinality neighborhoods,' which proposed novel rough sets based on these neighborhoods. Additionally, Al-shami and Mhemdi introduced an interesting notion called 'overlapping containment rough neighborhoods' in [16], using the concepts of $\jmath$-$\boldsymbol{NS}$ to provide new rough sets and applications in real-life problems. Consequently, all the results and medical applications explored were based on the original findings of [40]. Moreover, many researchers, inspired by Abd El-Monsef et al.'s approaches [40], have laid the groundwork for further topological applications of rough sets across diverse fields, including medicine [28, 37, 45] and economics [34, 36].

On the other hand, Abd El-Monsef et al. [40] developed a new interesting method to generate different topologies from neighborhoods—regardless of their form—directly, without relying on a basis or sub-basis. This method extended the approach of Abo Khadra et al. (2007) [47] and Abo Khadra and El-Bably (2008) [46]. The technique proposed by Abd El-Monsef et al. states that the class

$$\mathcal{T} = \{\mathcal{I} \subseteq \mathbb{S} : \mho(s) \subseteq \mathcal{I},\ \forall s \in \mathbb{S}\}$$

forms a topology on $\mathbb{S}$, where $\mathbb{S}$ represents the universe and $\mho(s)$ is the neighborhood of s. Thus, by identifying the neighborhood, we can generate different topologies on $\mathbb{S}$. This theory of generating topologies from neighborhoods has been widely adopted by researchers, such as [5, 15, 23–25], to create various topologies using different types of neighborhoods based on the eight introduced by Abd El-Monsef et al. [40]. Accordingly, this technique has opened the door to further topological applications in the rough set context.

Topological structures have significantly extended the scope of rough set theories in many research endeavors, leading to developments such as rough sets with soft sets [17, 29, 32, 33] and rough fuzzy sets [38, 39]. Additional studies [35, 42, 48, 66] have used topological features to define rough sets across various domains. Numerous research avenues have incorporated topological structures into their investigations [22, 24, 43, 44].

In 2021, El-Sayed et al. proposed for the first time the concept of "initial-neighborhoods" in the paper [34], based on the concept of right neighborhoods, defined as follows:

$$\mho_{\mathfrak{i}}(s) = \{y \in \mathbb{S} : \mho_{\mathfrak{r}}(s) \subseteq \mho_{\mathfrak{r}}(y)\},$$

where $\mho_{\mathfrak{r}}(s)$ represents the right neighborhood of s. They explored the application of initial-neighborhoods in rough set theory and their use in addressing the COVID-19 problem through generalized nano-topology. Later in 2022, Al-shami and Ciucci in [12] introduced the concept of initial-neighborhoods under the name "subset neighborhoods." They applied the concept of $\jmath$-$\boldsymbol{NS}$ to generate eight different neighborhoods of initial-neighborhoods and proposed different rough sets without exploring their topological structures.

At the same time, as a dual of "initial-neighborhoods," Abu-Gdairi et al. [8], in 2021, introduced the novel notion of "basic-neighborhoods" as follows:

$$\mho_{\mathfrak{b}}(s) = \{y \in \mathbb{S} : \mho_{\mathfrak{r}}(y) \subseteq \mho_{\mathfrak{r}}(s)\}.$$

The relationship between basic-neighborhoods and initial-neighborhoods was explained in [8] as follows:

$$\mho_{\mathfrak{b}}(s) \cap \mho_{\mathfrak{i}}(s) = \mho_{\mathbf{c}}(s),$$

where $\mho_{\mathbf{c}}(s)$ is the core-neighborhood of s, as proposed in [5, 30]. It is worth noting that the concept of "basic-neighborhoods" was later referred to as "containment neighborhoods" [10] in the same year (2021). Al-shami [10] applied the concept of $\jmath$-$\boldsymbol{NS}$ to generate various types of approximations without a detailed exploration of the associated topological properties or methods for generating corresponding topologies. In contrast, the studies by El-Gayar et al. [36] and Taher et al. [63] provide a more comprehensive analysis, investigating the relationships between the generated topologies and approximations in detail. These studies present new findings, comparing their methods with earlier techniques and demonstrating notable applications of these approximations in various decision-making contexts, particularly in the medical domain.

Therefore, the basic goal of the present manuscript is to present novel approaches based on topological constructions, specifically $\boldsymbol{\beta}$-open notions, which enhance the granulation of rough sets to support accurate medicinal analysis of COVID-19 variants. Based on prior research [8, 36, 63], the technique of basic rough sets is extended to $\boldsymbol{\beta}$-basic rough sets (briefly, $\boldsymbol{\beta}_{\mathfrak{b}}$-rough sets) to develop more precise practices. The proposed contributions introduce a novel approach for constructing topological rough sets ($\boldsymbol{\beta}_{\mathfrak{b}}$-rough sets) that do not rely on the induced topologies. This method allows the direct creation of diverse extended approximations derived from a general relation. Comparative analyses demonstrate that $\boldsymbol{\beta}_{\mathfrak{b}}$-approximations achieve higher accuracy compared to previous methods.

The global COVID-19 pandemic has severely disrupted daily life [6, 56, 57, 64], with increasing numbers of severe cases and deaths affecting individuals' mental health. The rapid transmission, severe infection rates, and mortalities associated with numerous variants of concern, including the Alpha, Beta, Gamma, Delta, and Omicron variants, present substantial challenges to healthcare systems. The likelihood and potential mutations of these variants create significant hurdles for healthcare management. As the virus is expected to become endemic, there is a continuous need to improve methods for anticipating and diagnosing COVID-19 variants. Consequently, accurately diagnosing COVID-19 variants is vital for determining appropriate actions and preventative measures.

In conclusion, we employ these methods to develop a robust knowledge base for categorizing and forecasting COVID-19 variants, offering effective tools for decision-making in healthcare while saving valuable resources and time. By applying the proposed techniques to real-world data, we assess their efficiency in accurately identifying COVID-19 variants and determining clinically significant outcomes. Our approach represents a mathematical advancement in improving decision accuracy and revealing hidden patterns in the data, thereby contributing to ongoing efforts to tackle the challenges posed by emerging COVID-19 variants.

The central contributions and objectives of the paper are summarized below:

1. **Methodology Proposal:** We introduce an essential methodology for creating topological basic rough sets (termed $\boldsymbol{\beta}_{\mathfrak{b}}$-rough sets) that eliminates the need for induced topology. This approach allows for the direct creation of various rough sets through binary relations. The innovative idea of $\boldsymbol{\beta}_{\mathfrak{b}}$-rough sets demonstrates higher accuracy compared to previous methods, as evidenced by extensive comparisons and counterexamples.

2. **A Sorting System Based on Rules for COVID-19 Variants:** Using this approach, we establish a crucial rule-based classification method for accurately identifying and predicting COVID-19 variants. Applying this method to real-world data demonstrates the effectiveness of $\boldsymbol{\beta}_{\mathfrak{b}}$-approximations in facilitating accurate decision-making. Integrating mathematical outputs with medical diagnostics reveals hidden patterns within the dataset.

3. **Interpretation:** The proposed article represents a significant mathematical discovery, enhancing the accuracy of decision-making while providing a robust mechanism for analyzing COVID-19 variants. This strategy has the potential to save valuable time and resources for both healthcare professionals and patients.

4. **Comparative Examination with Previous Approaches:** We conduct a thorough examination of the newly proposed $\boldsymbol{\beta}_{\mathfrak{b}}$-approximations, comparing them with other established techniques in the references. These comparisons underscore the advantages and distinct features of our proposed methods.

5. **An Algorithm Suggestion:** The paper proposes a framework (in the form of a simple algorithm) tailored for applying the proposed methods to decision-making

problems, with a focus on implementation in MATLAB. It also suggests future research in financial, medical, and other theoretical contexts to enhance the applicability of these approaches. This extension aims to address future large-data challenges and broaden the scope of applied problems.

Outline of the paper: The remainder of this paper is organized as follows. In Section 2, we recall some definitions and basic results concerning $\mathfrak{b}$-neighborhoods and present properties of neighborhood systems generated by binary relations, providing a brief overview of different types of neighborhoods based on general binary relations. In Section 3, we introduce the concept of $\boldsymbol{\beta}_{\mathfrak{b}}$-rough sets as a generalized extension of rough set theory. This approach leverages topological structures rooted in related concepts while avoiding the complexities of formal topology. As a result, these methods are more accessible to those unfamiliar with advanced topological theory and broaden the application of topological ideas across various scientific fields. The characteristics and relationships of the proposed methods are explored through established results and illustrative examples. Section 4 is devoted to conducting comprehensive comparisons between the methods proposed in this manuscript and previous works, particularly those by Yao [65], Dai et al. [27], Abd El-Monsef et al. [40], and Abu-Gadairi [8]. We critically evaluate their effectiveness and applicability. In Section 5, we highlight the applications of these topological structures in the medical field, focusing on decision-making processes and prospects in rough set theory. Specifically, we apply $\boldsymbol{\beta}_{\mathfrak{b}}$-rough sets for accurate decision-making in identifying COVID-19 variants. Finally, we present our conclusions in Section 6.

2. Fundamental Ideas

The current part summarizes the key ideas from several fundamental publications on rough sets, including contributions from ([8], [27], [40], [65]). A binary relation θ on a non-empty set $\mathbb{S}$ is a subset of the Cartesian product $\mathbb{S} \times \mathbb{S}$. Any $\mathrm{s}, \mathrm{t} \in \mathbb{S}$ is said to be related to θ, which can be expressed as $\mathrm{s}\theta\mathrm{t}$.

Throughout the paper, we consider $\mathbb{S}$ to represent a finite set. The following definition determines some types of binary relations:

Definition 1. *[65] A relation θ on $\mathbb{S}$ is categorized as follows:*

1. *Reflexive: If for all $\mathrm{s} \in \mathbb{S}$, $\mathrm{s}\theta\mathrm{s}$.*
2. *Symmetric: If whenever $\mathrm{s}\theta\mathrm{t}$, it implies that $\mathrm{t}\theta\mathrm{s}$, for all $\mathrm{s}, \mathrm{t} \in \mathbb{S}$.*
3. *Transitive: If whenever $\mathrm{s}\theta\mathrm{q}$ and $\mathrm{q}\theta\mathrm{t}$ hold true for all $\mathrm{s}, \mathrm{q}, \mathrm{t} \in \mathbb{S}$, then $\mathrm{s}\theta\mathrm{t}$.*
4. *Pre-order (or quasi-order): If it is reflexive and transitive.*
5. *Similarity: If it is reflexive, and symmetric.*
6. *Equivalence: If it is reflexive, symmetric, and transitive.*

Definition 2. *By using a binary relation, different types of neighborhoods ($\jmath$-neighborhood of* $\mathrm{s} \in \mathbb{S}$*) can be produced as shown:*

1. *The right neighborhood of* s *[65], shown by* $\mho_{\mathfrak{r}}(\mathrm{s})$*, consists of all elements* $\mathrm{t} \in \mathbb{S}$ *such that* $\mathrm{s}\theta\mathrm{t}$.
2. *The minimal neighborhood of* s *[18, 19], denoted by* $\mho_{\mathfrak{m}}(\mathrm{s})$*, is the intersection of all right neighborhoods of elements* $\mathrm{t} \in \mathbb{S}$ *such that* s *is an element of each right neighborhood of* t.
3. *The basic neighborhood of* s *[8], denoted by* $\mho_{\mathfrak{b}}(\mathrm{s})$*, comprises all elements* $\mathrm{t} \in \mathbb{S}$ *such that the right neighborhood of* t *is a subset of the right neighborhood of* s.

By utilizing the above neighborhoods, different rough approximations have been constructed as follow:

Definition 3. *For any subset* $\mathcal{I} \subseteq \mathbb{S}$ *and* $\forall \jmath \in \{\mathfrak{r}, \mathfrak{m}, \mathfrak{b}\}$*, the approximation operators (lower and upper), boundary region, and measures of accuracy of* $\mathcal{I} \subseteq \mathbb{S}$ *are respectively specified by:* $\mathcal{L}_\jmath(\mathcal{I}) = \{\mathrm{s} \in \mathbb{S} : \mho_\jmath(\mathrm{s}) \subseteq \mathcal{I}\}$, $\mathcal{U}_\jmath(\mathcal{I}) = \{\mathrm{s} \in \mathbb{S} : \mho_\jmath(\mathrm{s}) \cup \mathcal{I} \neq \emptyset\}$, $\mathfrak{B}_\jmath(\mathcal{I}) = \mathcal{U}_\jmath(\mathcal{I}) - \mathcal{L}_\jmath(\mathcal{I})$, *and* $\mathcal{A}_\jmath(\mathcal{I}) = \frac{|\mathcal{L}_\jmath(\mathcal{I})|}{|\mathcal{U}_\jmath(\mathcal{I})|}$ *, where* $|\mathcal{U}_\jmath(\mathcal{I})| \neq 0$.

It should be noted that:

1. *for* $\jmath = \mathfrak{r}$*, Definition 3 represents Yao approach [65].*
2. *for* $\jmath = \mathfrak{m}$*, Definition 3 represents Dai et al. approach [27].*
3. *for* $\jmath = \mathfrak{b}$*, Definition 3 represents Abu-Gdairi et al. approach [8].*

Abd El-Monsef et al. [40] presented an interesting method to generate a general topology using eight neighborhoods extracted from a binary relation. However, we provide this method specifically for the case of right neighborhoods, as demonstrated in the following result.

Theorem 1. *[40] A relation* θ *forms the following topology on* $\mathbb{S}$*:*

$$\mathcal{T}_{\mathfrak{r}} = \{\mathcal{I} \subseteq \mathbb{S} : \mho_{\mathfrak{r}}(\mathrm{s}) \subseteq \mathcal{I},\ \forall \mathrm{s} \in \mathbb{S}\}.$$

Definition 4. *[40] Suppose that* $\mathcal{T}_{\mathfrak{r}}$ *is the topology created by* θ *on* $\mathbb{S}$*, the members of* $\mathcal{T}_{\mathfrak{r}}$ *are termed* $\mathfrak{r}$*-open sets and the complement of* $\mathfrak{r}$*-open set is* $\mathfrak{r}$*-closed. A collection* $\mathcal{C}_{\mathfrak{r}}$ *of all* $\mathfrak{r}$*-closed sets is given as* $\mathcal{C}_{\mathfrak{r}} = \{\mathcal{J} \subseteq \mathbb{S} : \mathcal{J}^c \in \mathcal{T}_{\mathfrak{r}}\}$.

By using the topological structures in Definition 4, Abd El-Monsef et al. provided interesting rough approximations given by the following definition.

Definition 5. *[40] Suppose that* $\mathcal{T}_{\mathfrak{r}}$ *is the topology created by* θ *on* $\mathbb{S}$*. The* $\mathcal{T}_{\mathfrak{r}}$*-lower (resp.* $\mathcal{T}_{\mathfrak{r}}$*-upper) approximation, the* $\mathcal{T}_{\mathfrak{r}}$*-boundary region, and the* $\mathcal{T}_{\mathfrak{r}}$*-accuracy of the* $\mathcal{T}_{\mathfrak{r}}$*-approximations for* $\mathcal{I} \subseteq \mathbb{S}$ *are defined as follows:*
$\underline{\mathcal{T}}_{\mathfrak{r}}(\mathcal{I}) = \cup\{\mathcal{O} \in \mathcal{T}_{\mathfrak{r}} : \mathcal{O} \subseteq \mathcal{I}\}$, $\overline{\mathcal{T}}_{\mathfrak{r}}(\mathcal{I}) = \cap\{\mathcal{D} \in \mathcal{C}_{\mathfrak{r}} : \mathcal{I} \subseteq \mathcal{D}\}$, $Bnd_{\mathcal{T}_{\mathfrak{r}}}(\mathcal{I}) = \overline{\mathcal{T}}_{\mathfrak{r}}(\mathcal{I}) - \underline{\mathcal{T}}_{\mathfrak{r}}(\mathcal{I})$, *and* $\mathcal{A}_{\mathcal{T}_{\mathfrak{r}}}(\mathcal{I}) = \frac{|\underline{\mathcal{T}}_{\mathfrak{r}}(\mathcal{I})|}{|\overline{\mathcal{T}}_{\mathfrak{r}}(\mathcal{I})|}$ *, where* $|\overline{\mathcal{T}}_{\mathfrak{r}}(\mathcal{I})| \neq 0$.

3. $\boldsymbol{\beta}_{\mathfrak{b}}$-Rough Set Approximations

This section introduces novel extensions of fundamental rough sets, presenting alternative approaches to approximation, specifically $\boldsymbol{\beta}_{\mathfrak{b}}$-rough sets. These constructions involve topological structures rooted in related concepts, avoiding the need for formal topology. As a result, these methodologies potentially enhance accessibility for individuals not well-versed in specialized topological theory while broadening the utility of topological principles across diverse scientific fields. The characteristics and interconnections of the proposed methodologies are examined through established results and illustrative examples.

Definition 6. *Let θ be a binary relation on $\mathbb{S}$. Then, for each $\mathcal{I} \subseteq \mathbb{S}$, the $\boldsymbol{\beta}_{\mathfrak{b}}$- lower and $\boldsymbol{\beta}_{\mathfrak{b}}$- upper approximations of $\mathcal{I}$ are defined as follows:*

$$\underline{\boldsymbol{\beta}}_{\mathfrak{b}}(\mathcal{I}) = \mathcal{I} \cap \mathcal{U}_{\mathfrak{b}}[\mathcal{L}_{\mathfrak{b}}(\mathcal{U}_{\mathfrak{b}}(\mathcal{I}))] \text{ and } \overline{\boldsymbol{\beta}}_{\mathfrak{b}}(\mathcal{I}) = \mathcal{I} \cup \mathcal{L}_{\mathfrak{b}}[\mathcal{U}_{\mathfrak{b}}(\mathcal{L}_{\mathfrak{b}}(\mathcal{I}))].$$

The $\boldsymbol{\beta}_{\mathfrak{b}}$-boundary region and $\boldsymbol{\beta}_{\mathfrak{b}}$-accuracy of approximations are given respectively as:

$$\mathfrak{B}^{\boldsymbol{\beta}}_{\mathfrak{b}}(\mathcal{I}) = \overline{\boldsymbol{\beta}}_{\mathfrak{b}}(\mathcal{I}) - \underline{\boldsymbol{\beta}}_{\mathfrak{b}}(\mathcal{I}) \text{ and } \mathcal{A}_{\boldsymbol{\beta}_{\mathfrak{b}}}(\mathcal{I}) = \frac{|\underline{\boldsymbol{\beta}}_{\mathfrak{b}}(\mathcal{I})|}{|\overline{\boldsymbol{\beta}}_{\mathfrak{b}}(\mathcal{I})|}, \text{ such that } \overline{\boldsymbol{\beta}}_{\mathfrak{b}}(\mathcal{I}) \neq \emptyset.$$

The above operators are called $\boldsymbol{\beta}$-basic approximations (in briefly, $\boldsymbol{\beta}_{\mathfrak{b}}$- approximations) and it is clear that $0 \leq \mathcal{A}_{\boldsymbol{\beta}_{\mathfrak{b}}}(\mathcal{I}) \leq 1$. If $\mathcal{A}_{\boldsymbol{\beta}_{\mathfrak{b}}}(\mathcal{I}) = 1$, then $\mathcal{I}$ is named a $\boldsymbol{\beta}_{\mathfrak{b}}$-definable (or $\boldsymbol{\beta}_{\mathfrak{b}}$-exact) set. Alternatively, it is known as $\boldsymbol{\beta}_{\mathfrak{b}}$-rough.

The proposition below summarizes some aspects of the $\boldsymbol{\beta}_{\mathfrak{b}}$-approximations.

Proposition 1. *Suppose θ be a binary relation on $\mathbb{S}$. Then, for any $\mathcal{I}, \mathcal{K} \subseteq \mathbb{S}$:*

(1) $\underline{\boldsymbol{\beta}}_{\mathfrak{b}}(\mathcal{I}) \subseteq \mathcal{I} \subseteq \overline{\boldsymbol{\beta}}_{\mathfrak{b}}(\mathcal{I})$.

(2) $\underline{\boldsymbol{\beta}}_{\mathfrak{b}}(\mathbb{S}) = \overline{\boldsymbol{\beta}}_{\mathfrak{b}}(\mathbb{S}) = \mathbb{S}$, $\underline{\boldsymbol{\beta}}_{\mathfrak{b}}(\emptyset) = \overline{\boldsymbol{\beta}}_{\mathfrak{b}}(\emptyset) = \emptyset$.

(3) If $\mathcal{I} \subseteq \mathcal{K}$ then $\underline{\boldsymbol{\beta}}_{\mathfrak{b}}(\mathcal{I}) \subseteq \underline{\boldsymbol{\beta}}_{\mathfrak{b}}(\mathcal{K})$.

(4) If $\mathcal{I} \subseteq \mathcal{K}$, then $\overline{\boldsymbol{\beta}}_{\mathfrak{b}}(\mathcal{I}) \subseteq \overline{\boldsymbol{\beta}}_{\mathfrak{b}}(\mathcal{K})$.

(5) $\underline{\boldsymbol{\beta}}_{\mathfrak{b}}(\mathcal{I} \cap \mathcal{K}) = \underline{\boldsymbol{\beta}}_{\mathfrak{b}}(\mathcal{I}) \cap \underline{\boldsymbol{\beta}}_{\mathfrak{b}}(\mathcal{K})$.

(6) $\overline{\boldsymbol{\beta}}_{\mathfrak{b}}(\mathcal{I} \cup \mathcal{K}) = \overline{\boldsymbol{\beta}}_{\mathfrak{b}}(\mathcal{I}) \cup \overline{\boldsymbol{\beta}}_{\mathfrak{b}}(\mathcal{K})$.

(7) $\underline{\boldsymbol{\beta}}_{\mathfrak{b}}(\mathcal{I} \cup \mathcal{K}) \supseteq \underline{\boldsymbol{\beta}}_{\mathfrak{b}}(\mathcal{I}) \cup \underline{\boldsymbol{\beta}}_{\mathfrak{b}}(\mathcal{K})$.

(8) $\overline{\boldsymbol{\beta}}_{\mathfrak{b}}(\mathcal{I} \cap \mathcal{K}) \subseteq \overline{\boldsymbol{\beta}}_{\mathfrak{b}}(\mathcal{I}) \cap \overline{\boldsymbol{\beta}}_{\mathfrak{b}}(\mathcal{K})$.

(9) $\underline{\boldsymbol{\beta}}_{\mathfrak{b}}(\mathcal{I}) = \left[\overline{\boldsymbol{\beta}}_{\mathfrak{b}}(\mathcal{I}^c)\right]^c$, $\mathcal{I}^c$ is the complement of $\mathcal{I}$.

(10) $\overline{\boldsymbol{\beta}}_{\mathfrak{b}}(\mathcal{I}) = \left[\underline{\boldsymbol{\beta}}_{\mathfrak{b}}(\mathcal{I}^c)\right]^c$.

(11) $\underline{\beta}_{\mathfrak{b}}\left(\underline{\beta}_{\mathfrak{b}}(\mathcal{I})\right)=\underline{\beta}_{\mathfrak{b}}(\mathcal{I})$.

(12) $\overline{\beta}_{\mathfrak{b}}(\overline{\beta}_{\mathfrak{b}}(\mathcal{I}))=\overline{\beta}_{\mathfrak{b}}(\mathcal{I})$.

Proof.

(1) and (2): These are clear using Definition 6.

(3) From [8], if $\mathcal{I}\subseteq\mathcal{K}$ then $\mathcal{L}_{\mathfrak{b}}(\mathcal{I})\subseteq\mathcal{L}_{\mathfrak{b}}(\mathcal{K})$ and $\mathcal{U}_{\mathfrak{b}}(\mathcal{I})\subseteq\mathcal{U}_{\mathfrak{b}}(\mathcal{K})$.
Consequently, $\underline{\beta}_{\mathfrak{b}}(\mathcal{I})=\mathcal{I}\cap\mathcal{U}_{\mathfrak{b}}[\mathcal{L}_{\mathfrak{b}}(\mathcal{U}_{\mathfrak{b}}(\mathcal{I}))]\subseteq\mathcal{K}\cap\mathcal{U}_{\mathfrak{b}}[\mathcal{L}_{\mathfrak{b}}(\mathcal{U}_{\mathfrak{b}}(\mathcal{K}))]=\underline{\beta}_{\mathfrak{b}}(\mathcal{K})$.

(4) By using the same method of (3).

(5) First, since $\mathcal{I}\cap\mathcal{K}\subseteq\mathcal{I}$ and $\mathcal{I}\cap\mathcal{K}\subseteq\mathcal{K}$. Then, by (3), $\underline{\beta}_{\mathfrak{b}}(\mathcal{I}\cap\mathcal{K})\subseteq\underline{\beta}_{\mathfrak{b}}(\mathcal{I})\cap\underline{\beta}_{\mathfrak{b}}(\mathcal{K})$.
Now, let $\mathrm{s}\in[\underline{\beta}_{\mathfrak{b}}(\mathcal{I})\cap\underline{\beta}_{\mathfrak{b}}(\mathcal{K})]$. Then $\mathrm{s}\in\underline{\beta}_{\mathfrak{b}}(\mathcal{I})$ and $\mathrm{s}\in\underline{\beta}_{\mathfrak{b}}(\mathcal{K})$ which shows
$\mathrm{s}\in[\mathcal{I}\cap\mathcal{U}_{\mathfrak{b}}[\mathcal{L}_{\mathfrak{b}}(\mathcal{U}_{\mathfrak{b}}(\mathcal{I}))]]$ and $\mathrm{s}\in[\mathcal{K}\cap\mathcal{U}_{\mathfrak{b}}[\mathcal{L}_{\mathfrak{b}}(\mathcal{U}_{\mathfrak{b}}(\mathcal{K}))]]$.
Accordingly, by [8], $\mathrm{s}\in[(\mathcal{I}\cap\mathcal{K})\cap\mathcal{U}_{\mathfrak{b}}[\mathcal{L}_{\mathfrak{b}}(\mathcal{U}_{\mathfrak{b}}(\mathcal{I}\cap\mathcal{K}))]]=\underline{\beta}_{\mathfrak{b}}(\mathcal{I}\cap\mathcal{K})$.
Therefore, $\underline{\beta}_{\mathfrak{b}}(\mathcal{I})\cap\underline{\beta}_{\mathfrak{b}}(\mathcal{K})\subseteq\underline{\beta}_{\mathfrak{b}}(\mathcal{I}\cap\mathcal{K})$.

(6) By using the same method of (5).

(7) and (8) By using the same method of (4).

(9) $[\overline{\beta}_{\mathfrak{b}}(\mathcal{I}^c)]^c=[\mathcal{I}^c\cup[\mathcal{L}_{\mathfrak{b}}(\mathcal{U}_{\mathfrak{b}}(\mathcal{L}_{\mathfrak{b}}(\mathcal{I}^c)))]]^c=\mathcal{I}\cap[\mathcal{L}_{\mathfrak{b}}(\mathcal{U}_{\mathfrak{b}}(\mathcal{L}_{\mathfrak{b}}(\mathcal{I}^c)))]^c$.
Since, from [8], $(\mathcal{L}_{\mathfrak{b}}(\mathcal{I}^c))^c=\mathcal{U}_{\mathfrak{b}}(\mathcal{I})$ and $(\mathcal{U}_{\mathfrak{b}}(\mathcal{I}^c))^c=\mathcal{L}_{\mathfrak{b}}(\mathcal{I})$.
Thus, we get $[\mathcal{L}_{\mathfrak{b}}(\mathcal{U}_{\mathfrak{b}}(\mathcal{L}_{\mathfrak{b}}(\mathcal{I}^c)))]^c=\mathcal{U}_{\mathfrak{b}}(\mathcal{L}_{\mathfrak{b}}(\mathcal{U}_{\mathfrak{b}}(\mathcal{I})))$ which implies
$[\overline{\beta}_{\mathfrak{b}}(\mathcal{I}^c)]^c=\underline{\beta}_{\mathfrak{b}}(\mathcal{I})$.

(10) By using the same method of (9).

(11) From (1), we get $\underline{\beta}_{\mathfrak{b}}\left(\underline{\beta}_{\mathfrak{b}}(\mathcal{I})\right)\subseteq\underline{\beta}_{\mathfrak{b}}(\mathcal{I})$.
Now, let $\mathrm{s}\notin\underline{\beta}_{\mathfrak{b}}(\underline{\beta}_{\mathfrak{b}}(\mathcal{I}))$ then $\mathrm{s}\notin\left[\underline{\beta}_{\mathfrak{b}}(\mathcal{I})\cap\mathcal{U}_{\mathfrak{b}}\left(\mathcal{L}_{\mathfrak{b}}\left(\mathcal{U}_{\mathfrak{b}}\left(\underline{\beta}_{\mathfrak{b}}(\mathcal{I})\right)\right)\right)\right]$ which means that $\mathrm{s}\notin\underline{\beta}_{\mathfrak{b}}(\mathcal{I})$ or $\mathrm{s}\notin\mathcal{U}_{\mathfrak{b}}\left(\mathcal{L}_{\mathfrak{b}}\left(\mathcal{U}_{\mathfrak{b}}\left(\underline{\beta}_{\mathfrak{b}}(\mathcal{I})\right)\right)\right)$.
Consequently, $\mho_{\mathfrak{b}}(\mathrm{s})\cap\underline{\beta}_{\mathfrak{b}}(\mathcal{I})=\emptyset$ and this implies $\mho_{\mathfrak{b}}(\mathrm{s})\nsubseteq\underline{\beta}_{\mathfrak{b}}(\mathcal{I})$.
Therefore, $\mathrm{s}\notin\underline{\beta}_{\mathfrak{b}}(\mathcal{I})$ and hence $\underline{\beta}_{\mathfrak{b}}(\mathcal{I})\subseteq\underline{\beta}_{\mathfrak{b}}\left(\underline{\beta}_{\mathfrak{b}}(\mathcal{I})\right)$.

(12) By using the same method of (11).

Remark 1. *The reverse relations of items (7) and (8) in Proposition 1 are generally not true as shown in the next example.*

Example 1. *If* $\mathbb{S} = \{t, u, v, w\}$ *and* $\theta = \{(t,u), (u,t), (u,v), (v,u), (v,w), (w,v)\}$. *Thus, we get* $\mho_{\mathfrak{b}}(t) = \{t\}$, $\mho_{\mathfrak{b}}(u) = \{u, w\}$, $\mho_{\mathfrak{b}}(v) = \{t, v\}$, *and* $\mho_{\mathfrak{b}}(w) = \{w\}$.
Now, let $\mathcal{I} = \{t, u\}$ *and* $\mathcal{K} = \{w\}$. *Consequently,* $\mathcal{I} \cap \mathcal{K} = \emptyset$ *and* $\mathcal{I} \cup \mathcal{K} = \{t, u, w\}$ *which implies* $\underline{\boldsymbol{\beta}}_{\mathfrak{b}}(\mathcal{I}) = \{t\}$, $\underline{\boldsymbol{\beta}}_{\mathfrak{b}}(\mathcal{K}) = \{w\}$ *and* $\underline{\boldsymbol{\beta}}_{\mathfrak{b}}(\mathcal{I} \cup \mathcal{K}) = \{t, u, w\}$. *It is clear that* $\underline{\boldsymbol{\beta}}_{\mathfrak{b}}(\mathcal{I}) \cup \underline{\boldsymbol{\beta}}_{\mathfrak{b}}(\mathcal{K}) = \{t, w\}$, *and this means that* $\underline{\boldsymbol{\beta}}_{\mathfrak{b}}(\mathcal{I}) \cup \underline{\boldsymbol{\beta}}_{\mathfrak{b}}(\mathcal{K}) \subsetneq \underline{\boldsymbol{\beta}}_{\mathfrak{b}}(\mathcal{I} \cup \mathcal{K})$. *Similarly,* $\overline{\boldsymbol{\beta}}_{\mathfrak{b}}(\mathcal{I}) = \{t, u, v\}$, $\overline{\boldsymbol{\beta}}_{\mathfrak{b}}(\mathcal{K}) = \{u, w\}$ *and* $\overline{\boldsymbol{\beta}}_{\mathfrak{b}}(\mathcal{I} \cap \mathcal{K}) = \emptyset$. *Clearly,* $\overline{\boldsymbol{\beta}}_{\mathfrak{b}}(\mathcal{I} \cap \mathcal{K}) \subsetneq \overline{\boldsymbol{\beta}}_{\mathfrak{b}}(\mathcal{I}) \cap \overline{\boldsymbol{\beta}}_{\mathfrak{b}}(\mathcal{K})$.

4. Comparisons Between $\boldsymbol{\beta}_{\mathfrak{b}}$-Approaches and the Preceding Methods

This section is dedicated to conducting comprehensive comparisons between the methods proposed in this manuscript and previous works, particularly those of Yao [65], Dai et al. [27], Abd El-Monsef et al. [40], and Abu-Gadairi [8]. By utilizing well-known results and counterexamples, we critically assess and contrast the effectiveness and applicability of these approaches. Our objective is to demonstrate the improved accuracy of the proposed approaches compared to earlier ones.

A. Comparison between Abu-Gdairi technique and the $\boldsymbol{\beta}_{\mathfrak{b}}$*-approximations*

We begin by introducing Lemma 1, which highlights the connections between the methodology advocated by Abu-Gadairi [8] and the proposed $\boldsymbol{\beta}_{\mathfrak{b}}$-approximations.

Lemma 1. *Suppose that* θ *is a binary relation on* $\mathbb{S}$. *Then, for each* $\mathcal{I} \subseteq \mathbb{S}$, *the following holds:*

(1) $\mathcal{L}_{\mathfrak{b}}(\mathcal{I}) \subseteq \mathcal{L}_{\mathfrak{b}}(\mathcal{U}_{\mathfrak{b}}(\mathcal{I}))$.

(2) $\mathcal{U}_{\mathfrak{b}}(\mathcal{L}_{\mathfrak{b}}(\mathcal{I})) \subseteq \mathcal{U}_{\mathfrak{b}}(\mathcal{I})$.

Proof. The first statement will be proven, and the second will follow naturally.
According to [8], $\mathcal{L}_{\mathfrak{b}}(\mathcal{I}) \subseteq \mathcal{U}_{\mathfrak{b}}(\mathcal{I})$. Consequently, $\mathcal{L}_{\mathfrak{b}}(\mathcal{L}_{\mathfrak{b}}(\mathcal{I})) \subseteq \mathcal{L}_{\mathfrak{b}}(\mathcal{U}_{\mathfrak{b}}(\mathcal{I}))$. Since $\mathcal{L}_{\mathfrak{b}}(\mathcal{L}_{\mathfrak{b}}(\mathcal{I})) = \mathcal{L}_{\mathfrak{b}}(\mathcal{I})$ as shown in [21], it follows that $\mathcal{L}_{\mathfrak{b}}(\mathcal{I}) \subseteq \mathcal{L}_{\mathfrak{b}}(\mathcal{U}_{\mathfrak{b}}(\mathcal{I}))$.

Theorem 2. *Consider that* θ *is a binary relation on* $\mathbb{S}$. *Then, for each* $\mathcal{I} \subseteq \mathbb{S}$:

$$\mathcal{L}_{\mathfrak{b}}(\mathcal{I}) \subseteq \underline{\boldsymbol{\beta}}_{\mathfrak{b}}(\mathcal{I}) \subseteq \mathcal{I} \subseteq \overline{\boldsymbol{\beta}}_{\mathfrak{b}}(\mathcal{I}) \subseteq \mathcal{U}_{\mathfrak{b}}(\mathcal{I}).$$

Proof. Let $w \in \mathcal{L}_{\mathfrak{b}}(\mathcal{I})$. Then $\mho_{\mathfrak{b}}(w) \subseteq \mathcal{I}$, implying $w \in \mho_{\mathfrak{b}}(w)$, which means $w \in \mathcal{I}$. By Lemma 1, $\mathcal{L}_{\mathfrak{b}}(\mathcal{I}) \subseteq \mathcal{L}_{\mathfrak{b}}(\mathcal{U}_{\mathfrak{b}}(\mathcal{I}))$, which means $w \in \mathcal{U}_{\mathfrak{b}}(\mathcal{L}_{\mathfrak{b}}(\mathcal{U}_{\mathfrak{b}}(\mathcal{I})))$. Therefore, $w \in \mathcal{I} \cap \mathcal{L}_{\mathfrak{b}}(\mathcal{U}_{\mathfrak{b}}(\mathcal{I})) = \underline{\boldsymbol{\beta}}_{\mathfrak{b}}(\mathcal{I})$, implying $\mathcal{L}_{\mathfrak{b}}(\mathcal{I}) \subseteq \underline{\boldsymbol{\beta}}_{\mathfrak{b}}(\mathcal{I})$. By a similar technique, it is easy to verify that $\overline{\boldsymbol{\beta}}_{\mathfrak{b}}(\mathcal{I}) \subseteq \mathcal{U}_{\mathfrak{b}}(\mathcal{I})$.

Corollary 1. *Suppose that* θ *is a binary relation on* $\mathbb{S}$. *Then, for each* $\mathcal{I} \subseteq \mathbb{S}$:

(1) $\mathfrak{B}_{\boldsymbol{\beta}_{\mathfrak{b}}}(\mathcal{I}) \subseteq \mathfrak{B}_{\mathfrak{b}}(\mathcal{I})$.

(2) $\mathcal{A}_{\mathfrak{b}}(\mathcal{I}) \leq \mathcal{A}_{\boldsymbol{\beta}_{\mathfrak{b}}}(\mathcal{I})$.

(3) The subset $\mathcal{I}$ *is a* $\boldsymbol{\beta}_{\mathfrak{b}}$*-exact set if it is* $\mathfrak{b}$*-exact.*

Remark 2. *According to Theorem 2 and Corollary 1, Definition 6 extends the techniques introduced by Abu-Gadairi [8]. These extensions, particularly the* $\boldsymbol{\beta}_{\mathfrak{b}}$*-approaches, demonstrate improved accuracy. However, it is important to note that Example 2 illustrates cases where this relationship does not hold in reverse.*

Example 2. *Drawing from Example 1, we compare the outcomes derived from the methodology proposed by Abu-Gadairi et al. [8] with those obtained from the suggested approach (the* $\boldsymbol{\beta}_{\mathfrak{b}}$*-approximations), as elucidated in Table 1.*

Table 1: A Comparative Analysis of Accuracy Metrics between the Technique by Abu-Gdairi et al. [8] and the Current Method.

$\mathcal{I}\subseteq\mathbb{S}$	**Abu-Gdairi et al.**	**Current methods**
	$\mathcal{A}_{\mathfrak{b}}(\mathcal{I})$	$\mathcal{A}_{\boldsymbol{\beta}_{\mathfrak{b}}}(\mathcal{I})$
$\{t\}$	0	0
$\{u\}$	$\frac{1}{2}$	1
$\{v\}$	0	1
$\{w\}$	0	1
$\{t,u\}$	$\frac{1}{2}$	1
$\{t,v\}$	0	$\frac{1}{2}$
$\{t,w\}$	0	1
$\{u,v\}$	$\frac{1}{4}$	1
$\{u,w\}$	$\frac{1}{4}$	1
$\{v,w\}$	$\frac{2}{3}$	1
$\{t,u,v\}$	$\frac{1}{4}$	1
$\{t,u,w\}$	$\frac{1}{4}$	1
$\{t,v,w\}$	$\frac{2}{3}$	1
$\{u,v,w\}$	$\frac{3}{4}$	$\frac{3}{4}$
$\mathbb{S}$	1	1

B. Comparison between the $\boldsymbol{\beta}_{\mathfrak{b}}$***-approximations and the method of Dai et al.***

This subsection aims to present a comparative study between the proposed approximations in this paper and the rough sets method by Dai et al. [27], highlighting two distinct cases.

Firstly, in the case of general binary relations, the two methods are independent and not comparable, as detailed in reference [8]. In this scenario, Dai et al.'s approximations fail to satisfy the main axioms and properties of Pawlak's theory. Conversely, the proposed $\boldsymbol{\beta}_{\mathfrak{b}}$-approximations successfully fulfill Pawlak's principles without requiring any additional

conditions, as demonstrated in Proposition 1. This represents a significant advantage of the proposed methods over the previous method.
In the second case, which involves reflexive relations where some of Pawlak's axioms are met by Dai et al.'s method, we demonstrate that our methods are more accurate than those of Dai et al., supported by counterexamples.

Lemma 2. *[8] Let θ be a reflexive relation on $\mathbb{S}$ and $\mathcal{I} \subseteq \mathbb{S}$. Then:*

(1) $\mathcal{L}_{\mathfrak{m}}(\mathcal{I}) \subseteq \mathcal{L}_{\mathfrak{b}}(\mathcal{I}) \subseteq \mathcal{I} \subseteq \mathcal{U}_{\mathfrak{b}}(\mathcal{I}) \subseteq \mathcal{U}_{\mathfrak{m}}(\mathcal{I})$.

(2) $\mathfrak{B}_{\mathfrak{b}}(\mathcal{I}) \subseteq \mathfrak{B}_{\mathfrak{m}}(\mathcal{I})$ *and* $\mathcal{A}_{\mathfrak{m}}(\mathcal{I}) \leq \mathcal{A}_{\mathfrak{b}}(\mathcal{I})$.

(3) If $\mathcal{I}$ is $\mathfrak{m}$-exact, then $\mathcal{I}$ is basic-exact.

Theorem 3. *Suppose that θ is a reflexive relation on $\mathbb{S}$. Then:*

(1) $\mathcal{L}_{\mathfrak{m}}(\mathcal{I}) \subseteq \underline{\boldsymbol{\beta}}_{\mathfrak{b}}(\mathcal{I}) \subseteq \mathcal{I} \subseteq \overline{\boldsymbol{\beta}}_{\mathfrak{b}}(\mathcal{I}) \subseteq \mathcal{U}_{\mathfrak{m}}(\mathcal{I})$.

(2) $\mathfrak{B}_{\boldsymbol{\beta}_{\mathfrak{b}}}(\mathcal{I}) \subseteq \mathfrak{B}_{\mathfrak{m}}(\mathcal{I})$ *and* $\mathcal{A}_{\mathfrak{m}}(\mathcal{I}) \leq \mathcal{A}_{\boldsymbol{\beta}_{\mathfrak{b}}}(\mathcal{I})$.

(3) If $\mathcal{I}$ is $\mathfrak{m}$-exact, then $\mathcal{I}$ is $\boldsymbol{\beta}_{\mathfrak{b}}$-exact.

Proof. From Theorem 2 and Lemma 2 the proof follows directly.

Remark 3. *It is important to notice that the converse of the above consequences is not usually legal, as illustrated in Example 3.*

Example 3. *Let $\mathbb{S} = \{t, u, v, w\}$ and consider the reflexive relation $\theta = \{(t,t), (t,v), (u,u), (u,v), (v,v), (w,w), (u,t), (t,w), (v,w), (w,v)\}$ on $\mathbb{S}$. Accordingly, the basic-neighborhoods given by: $\mho_{\mathfrak{b}}(t) = \{t, v, w\}$, $\mho_{\mathfrak{b}}(u) = \{u\}$, $\mho_{\mathfrak{b}}(v) = \{v, w\}$, and $\mho_{\mathfrak{b}}(w) = \{v, w\}$.*
Also, the $\mathfrak{m}$-neighborhoods are: $\mho_{\mathfrak{m}}(t) = X$, $\mho_{\mathfrak{m}}(u) = \{t, u\}$, $\mho_{\mathfrak{m}}(v) = \{t, v, w\}$, and $\mho_{\mathfrak{m}}(w) = \{t, v, w\}$. Therefore, a comparison among the earlier paper ($\mathfrak{m}$-approximations [27]), and the proposed technique ($\boldsymbol{\beta}_{\mathfrak{b}}$-accuracy) is given in Table 2.

C. A Comparative Analysis of the Yao Method and Nearly $\boldsymbol{\beta}_{\mathfrak{b}}$-approximations

This subsection provides a comparative investigation of the proposed approximations in this paper and Yao's rough sets method [27], focusing on two specific examples.

First, in the context of general binary relations, the two techniques are independent and not directly comparable, as discussed in [8] and Example 2. In this scenario, Yao's approximations fail to satisfy the key axioms and properties of Pawlak's theory. In contrast, the $\boldsymbol{\beta}_{\mathfrak{b}}$-approximations adhere to Pawlak's principles without additional restrictions, as shown in Proposition 1. This highlights a significant advantage of the proposed methods over the previous approach.

Lemma 3. *[8] For a reflexive relation θ on $\mathbb{S}$, $\forall w \in \mathbb{S}$: $\mho_{\mathfrak{b}}(w) \subseteq \mho_{\mathfrak{r}}(w)$.*

Table 2: Comparisons between the method of Dai et al. [27], and existing approaches.

$\mathcal{I} \subseteq \mathbb{S}$	**Dai et al.**	**Current methods**
	$\mathcal{A}_{m}(\mathcal{I})$	$\mathcal{A}_{\beta_{\mathfrak{b}}}(\mathcal{I})$
$\{t\}$	0	0
$\{u\}$	0	1
$\{v\}$	0	1
$\{w\}$	0	1
$\{t,u\}$	$\frac{1}{4}$	$\frac{1}{2}$
$\{t,v\}$	0	1
$\{t,w\}$	0	1
$\{u,v\}$	0	1
$\{u,w\}$	0	1
$\{v,w\}$	0	$\frac{2}{3}$
$\{t,u,v\}$	$\frac{1}{4}$	1
$\{t,u,w\}$	$\frac{1}{4}$	1
$\{t,v,w\}$	$\frac{2}{4}$	1
$\{u,v,w\}$	0	$\frac{3}{4}$
$\mathbb{S}$	1	1

Theorem 4. *Let θ be a reflexive relation on $\mathbb{S}$ and $\mathcal{I} \subseteq \mathbb{S}$. Then:*

(1) $\mathcal{L}_{\mathfrak{r}}(\mathcal{I}) \subseteq \mathcal{L}_{\mathfrak{b}}(\mathcal{I}) \subseteq \mathcal{I} \subseteq \mathcal{U}_{\mathfrak{b}}(\mathcal{I}) \subseteq \mathcal{U}_{\mathfrak{r}}(\mathcal{I})$.

(2) $\mathfrak{B}_{\mathfrak{b}}(\mathcal{I}) \subseteq \mathfrak{B}_{\mathfrak{r}}(\mathcal{I})$ *and* $\mathcal{A}_{\mathfrak{r}}(\mathcal{I}) \leq \mathcal{A}_{\mathfrak{b}}(\mathcal{I})$.

(3) If $\mathcal{I}$ is $\mathfrak{r}$-exact, then $\mathcal{I}$ is $\mathfrak{b}$-exact.

Proof. We will start by proving the first statement; following proofs will use similar approaches. Let $w \in \mathcal{L}_{\mathfrak{r}}(\mathcal{I})$, implying $\mho_{\mathfrak{r}}(w) \subseteq \mathcal{I}$. Thus, by Lemma 3, $\mho_{\mathfrak{b}}(w) \subseteq \mathcal{I}$, implying $w \in \mathcal{L}_{\mathfrak{b}}(\mathcal{I})$. Therefore, $\mathcal{L}_{\mathfrak{r}}(\mathcal{I}) \subseteq \mathcal{L}_{\mathfrak{b}}(\mathcal{I})$. Similarly, applying the same reasoning, we deduce $\mathcal{U}_{\mathfrak{b}}(\mathcal{I}) \subseteq \mathcal{U}_{\mathfrak{r}}(\mathcal{I})$.

Note: Theorem 4 illustrates the relationships between Yao's approach ($\mathfrak{r}$-approximations) and Abu-Gdairi approach ($\mathfrak{b}$-approximations) in the case of reflexivity of the relation.

Theorem 5. *Suppose that θ is a reflexive relation on $\mathbb{S}$. Then:*

(1) $\mathcal{L}_{\mathfrak{r}}(\mathcal{I}) \subseteq \underline{\boldsymbol{\beta}}_{\mathfrak{b}}(\mathcal{I}) \subseteq \mathcal{I} \subseteq \overline{\boldsymbol{\beta}}_{\mathfrak{b}}(\mathcal{I}) \subseteq \mathcal{U}_{\mathfrak{r}}(\mathcal{I})$.

(2) $\mathfrak{B}_{\beta_{\mathfrak{b}}}(\mathcal{I}) \subseteq \mathfrak{B}_{\mathfrak{r}}(\mathcal{I})$ *and* $\mathcal{A}_{\mathfrak{r}}(\mathcal{I}) \leq \mathcal{A}_{\beta_{\mathfrak{b}}}(\mathcal{I})$.

(3) If $\mathcal{I}$ is $\mathfrak{r}$-exact, then $\mathcal{I}$ is $\boldsymbol{\beta}_{\mathfrak{b}}$-exact.

Proof. By using Theorem 2 and Theorem 4, the proof is straightforward.

Remark 4. *The opposite of the previous consequences is not necessarily correct, as confirmed in Example 4.*

Example 4. *Referring to Example 3, where the relation is reflexive on $\mathbb{S}$, we obtain the $\mathfrak{r}$-neighborhoods as follows: $\mho_{\mathfrak{r}}(t)=\{t,v,w\}$, $\mho_{\mathfrak{r}}(u)=\{t,u\}$, $\mho_{\mathfrak{r}}(v)=\{v,w\}$, and $\mho_{\mathfrak{r}}(w)=\{v,w\}$. Therefore, a comparison among the preceding Yao's approach ($\mathfrak{r}$-approximations [65]) and the proposed method ($\boldsymbol{\beta}_{\mathfrak{b}}$-accuracy) is explained in Table 3.*

Table 3: Comparisons between the technique of Yao [65] and the current technique.

$\mathcal{I}\subseteq\mathbb{S}$	**Yao method**	**Existing method**
	$\mathcal{A}_{\mathfrak{r}}(\mathcal{I})$	$\mathcal{A}_{\boldsymbol{\beta}_{\mathfrak{b}}}(\mathcal{I})$
$\{t\}$	0	0
$\{u\}$	0	1
$\{v\}$	0	1
$\{w\}$	0	1
$\{t,u\}$	$\frac{1}{2}$	$\frac{1}{2}$
$\{t,v\}$	0	1
$\{t,w\}$	0	1
$\{u,v\}$	0	1
$\{u,w\}$	0	1
$\{v,w\}$	$\frac{2}{3}$	$\frac{2}{3}$
$\{t,u,v\}$	$\frac{1}{4}$	1
$\{t,u,w\}$	$\frac{1}{4}$	1
$\{t,v,w\}$	$\frac{3}{4}$	1
$\{u,v,w\}$	$\frac{1}{2}$	$\frac{3}{4}$
$\mathbb{S}$	1	1

D. Comparing the method of Abd El-Monsef et al. with the provided approach.

This subsection conducts a comparative analysis between the technique of Abd El-Monsef et al. [40] and the proposed technique. Both the established theorem and the illustrative example demonstrate that the proposed techniques exhibit greater accuracy than those of Abd El-Monsef et al.

Theorem 6. *Let θ be a reflexive relation on $\mathbb{S}$ and $\mathcal{T}_{\mathfrak{r}}$ is the topology created by θ. Then, $\forall\mathcal{I}\subseteq\mathbb{S}$:*

(1) $\underline{\mathcal{T}}_{\mathfrak{r}}(\mathcal{I})\subseteq\underline{\boldsymbol{\beta}}_{\mathfrak{b}}(\mathcal{I})\subseteq\mathcal{I}\subseteq\overline{\boldsymbol{\beta}}_{\mathfrak{b}}(\mathcal{I})\subseteq\overline{\mathcal{T}}_{\mathfrak{r}}(\mathcal{I})$.

(2) $\mathfrak{B}_{\boldsymbol{\beta}_{\mathfrak{b}}}(\mathcal{I})\subseteq\mathfrak{B}_{\mathcal{T}_{\mathfrak{r}}}(\mathcal{I})$ *and* $\mathcal{A}_{\mathcal{T}_{\mathfrak{r}}}(\mathcal{I})\leq\mathcal{A}_{\boldsymbol{\beta}_{\mathfrak{b}}}(\mathcal{I})$.

(3) If $\mathcal{I}$ is $\mathfrak{r}$-exact, then $\mathcal{I}$ is $\boldsymbol{\beta}_{\mathfrak{b}}$-exact.

Proof. The proof follows straightforwardly from Theorem 4.

Remark 5. *The converse of the previous consequences does not generally apply, as demonstrated in Example 5.*

Example 5. *Continuing with Example 4, we obtain the topology $\mathcal{T}_{\mathfrak{r}}$ created by θ and the family $\mathcal{C}_{\mathfrak{r}}$ of all closed sets as follows: $\mathcal{T}_{\mathfrak{r}} = \{\mathbb{S}, \emptyset, \{v,w\}, \{t,v,w\}\}$ and $\mathcal{C}_{\mathfrak{r}} = \{\mathbb{S}, \emptyset, \{t,u\}, \{u\}\}$.*

Accordingly, we compare the preceding approaches (Abd El-Monsef et al. [40]) ($\mathcal{T}_{\mathfrak{r}}$-approximations [6]) and the provided technique ($\boldsymbol{\beta}_{\mathfrak{b}}$-accuracy) as clarified in Table 4.

Table 4: Comparing Abd El-Monsef et al.'s technique to the planned method.

$\mathcal{I} \subseteq \mathbb{S}$	**Abd El-Monsef approaches**	**Existing methods**
	$\mathcal{A}_{\mathcal{T}_{\mathfrak{r}}}(\mathcal{I})$	$\mathcal{A}_{\boldsymbol{\beta}_{\mathfrak{b}}}(\mathcal{I})$
$\{t\}$	0	0
$\{u\}$	0	1
$\{v\}$	0	1
$\{w\}$	0	1
$\{t,u\}$	0	1
$\{t,v\}$	0	$\frac{1}{2}$
$\{t,w\}$	0	1
$\{u,v\}$	0	1
$\{u,w\}$	0	1
$\{v,w\}$	$\frac{1}{2}$	1
$\{t,u,v\}$	0	1
$\{t,u,w\}$	0	1
$\{t,v,w\}$	$\frac{3}{4}$	1
$\{u,v,w\}$	0	$\frac{3}{4}$
$\mathbb{S}$	1	1

Concluding Remarks: From the previous comparisons, several key observations can be made:

1. **High Accuracy:** The proposed approach ($\boldsymbol{\beta}_{\mathfrak{b}}$-rough sets) surpasses preceding approaches (Yao [65], Dai et al. [27], and Abu-Gadairi [8]) in terms of accuracy and approximation operators. It extends Pawlak's properties to applied problems without requiring extra conditions.

2. **Easy Topological Techniques:** These methods leverage induced topological structures, making them valuable for applying topological ideas in rough set approaches. Their accessibility extends beyond specialists in topology, as they simplify the presentation of topological approximations individually, enabling the use of rough approximations ($\boldsymbol{\beta}_{\mathfrak{b}}$-rough sets) without requiring advanced concepts. This simplification democratizes the application of rough-set philosophies, making them available to researchers without extensive studies in topology. While topology-based methods

are effective for developing specialized algorithms, integrating topological concepts into specific algorithms enhances their usability.

3. **Easy Techniques for Big Datasets:** The present paradigm reduces significant modeling constraints, thereby enabling a wider representation of issues, especially when handling large datasets. This adaptability makes it useful for evaluating and defining a wide range of real-world scenarios, such as infectious diseases like coronavirus, where sample size has a direct impact on the accuracy of decision-making, as shown in the next part.

5. Using $\beta_\flat$-Rough Sets for Accurate Decision Making in Identifying COVID-19 Variants

The COVID-19 pandemic, which began in January 2020, has worsened with new, more deadly strains, affecting global mental well-being. The countries of the Middle East have identified numerous variants, including "Alpha," "Delta," and "Omicron." We offer a rule-based framework to categorize and predict these variants.

5.1. Experimental Outcomes and Medical Decision Table

This section provides the medical results of an early experimental examination with ten patients (see Table 5) are provided. It should be noted that the present application is based on existing data, as described in references [6, 56]. Table 5 presents the outcomes of eight variants of COVID-19 (Alpha, Delta, and Omicron) observed in these patients. The symptoms associated with each variant, categorized as "Alpha," "Delta," and "Omicron," are detailed below:

- Alpha variants includes (FE, SB, BP, DC, HE, ST, CP).
- Delta variants include (FE, SB, BP, HE, ST, CP, CO, LoT, LoS, MY, FA, and RH).
- Omicron variants include (FE, HE, CO, CL, BA, WE, FA, NS, SN, LBP, LoA).

The above symptoms are interpreted as follows:

FE = fever
SB = shortness of breath
BP = body pain
DC = dry cough
HE = headache
ST = sore throat
CP = chest pain
CO = cough
SN = sneezing
MY = myalgias
CL = cold
BA = body ache
WE = weakness
FA = fatigue
NS = night sweats
RH = rhinorrhea
LoT = loss of taste
LoA = loss of appetite
LoS = loss of smell
LBP = lower back pain

Table 5 describes the attributes (or symptoms) of individuals infected with the Alpha, Delta, and Omicron variants. Notably, the Omicron variant shares several symptoms with the Alpha and Delta variants—such as fever, cough, shortness of breath, exhaustion, loss of taste, loss of smell, sore throat, and headache—but also exhibits some unique symptoms.

Table 5: [6, 56] Patient Data System for COVID Variant Infections.

Person	symptoms								COVID-19
	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	Decision
p_1	High	No	Yes	Yes	No	Yes	No	Yes	Alpha
p_2	High	No	Yes	Yes	Yes	Yes	No	Yes	Omicron
p_3	High	No	No	Yes	No	No	No	No	No
p_4	Normal	No	No	No	No	No	Yes	No	No
p_5	Normal	Yes	No	Yes	No	Yes	No	Yes	Delta
p_6	High	Yes	No	Yes	No	No	No	No	Delta
p_7	High	Yes	Yes	Yes	No	No	No	No	Delta
p_8	Normal	No	No	Yes	No	No	No	No	No

5.2. The Utilization of β_6-Rough Set Techniques in Decision-Making Concerning COVID-19 Variations

This process involves several steps. First, the attributes of symptoms (conditions), represented as $\mathfrak{C} = \{a_1, a_2, a_3, \ldots, a_8\}$ for the patient set $\mathbb{S} = \{p_1, p_2, p_3, \ldots, p_8\}$, are transformed into qualitative terms. Then, we proceed to compute the similarity of symptoms among patients, as depicted in Table 7, using the following formula:

$$\Upsilon(x, y) = \frac{\sum_{\ell=1}^{n} [a_\ell(x) = a_\ell(y)]}{n},$$

where $\Upsilon(x, y)$ is the degree of similarity between patients x and y, and "n" denotes the total number of disease attributes.

Thus, Table 6 shows the calculation of similarity across patients based on their condition attributes:

From data presented in Table 6, the patient set $\mathbb{S}$ is segmented into two distinct groups:

- The group of persons afflicted by COVID-19: $\mathcal{Q} = \{p_1,\ p_2, p_5,\ p_6, p_7\}$.
- The group of persons unaffected by COVID-19: $\mathcal{W} = \{p_3,\ p_4, p_8\}$.

Furthermore, in the subset of COVID-19-infected persons, there are additional categorizations:

- The infected persons with Omicron: $\mathcal{W}_1 = \{p_1\}$.

Table 6: Symptom Similarities Between Eight Patients.

	p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8
p_1	1	0.875	0.625	0.25	0.625	0.5	0.625	0.5
p_2	0.875	1	0.5	0.125	0.5	0.375	0.5	0.375
p_3	0.625	0.5	1	0.625	0.5	0.875	0.75	0.875
p_4	0.25	0.125	0.5	1	0.375	0.5	0.375	0.75
p_5	0.625	0.5	0.5	0.375	1	0.625	0.5	0.625
p_6	0.5	0.375	0.875	0.5	0.625	1	0.875	0.75
p_7	0.625	0.5	0.75	0.375	0.5	0.875	1	0.625
p_8	0.5	0.375	0.875	0.75	0.625	0.625	0.75	1

- The infected persons with Alpha: $\mathcal{W}_2 = \{p_2\}$.
- The infected persons with Delta: $\mathcal{W}_3 = \{p_5, p_6, p_7\}$.

The next phases include creating customized associations to suit the scheme's needs. We create r-neighborhoods and b-neighborhoods for all patients, as shown in Table 6, using a relation appropriate for the task at hand. Based on expert advice from doctors, the relation is defined as:

$$x\theta y \Leftrightarrow \Upsilon(x,y) \geq 0.75.$$

The proposed relation mentioned above, characterized by a threshold value of 0.75, indicates the degree of similarity, where larger values indicate greater similarity and increased accuracy of results. This relation can be adjusted according to the perspectives of system specialists. Moreover, this relation represents a similarity relation that is non-transitive, meaning it does not satisfy the properties of an equivalence relation. Consequently, Pawlak's technique inadequately addresses and describes this issue.

Table 7: $\mathfrak{r}$-neighborhoods, $\mathfrak{m}$-neighborhoods, and $\mathfrak{b}$-neighborhoods of each patient.

x	$\mho_{\mathfrak{r}}(x)$	$\mho_{\mathfrak{m}}(x)$	$\mho_{\mathfrak{b}}(x)$
p_1	$\{p_1, p_2\}$	$\{p_1, p_2\}$	$\{p_1, p_2\}$
p_2	$\{p_1, p_2\}$	$\{p_1, p_2\}$	$\{p_1, p_2\}$
p_3	$\{p_3, p_6, p_7, p_8\}$	$\{p_3, p_4, p_6, p_7, p_8\}$	$\{p_3, p_6, p_7\}$
p_4	$\{p_4, p_8\}$	$\{p_3, p_4, p_6, p_8\}$	$\{p_4\}$
p_5	$\{p_5\}$	$\{p_5\}$	$\{p_5\}$
p_6	$\{p_3, p_6, p_7\}$	$\{p_3, p_4, p_6, p_7, p_8\}$	$\{p_6\}$
p_7	$\{p_3, p_6, p_7, p_8\}$	$\{p_3, p_6, p_7, p_8\}$	$\{p_3, p_6, p_7\}$
p_8	$\{p_3, p_4, p_6, p_8\}$	$\{p_3, p_4, p_6, p_7, p_8\}$	$\{p_4, p_8\}$

Next, we use the proposed method ($\boldsymbol{\beta}_{\mathfrak{b}}$-rough sets) to calculate the approximations (resp. accuracies) for the subsets $\mathcal{Q}$, $\mathcal{W}$, $\mathcal{W}_1$, $\mathcal{W}_2$, and $\mathcal{W}_3$.

Furthermore, comparisons of these results to those obtained using prior methodologies are provided in publications [65], [27], and [8]. This comparative analysis is intended to demonstrate the importance of the supplied approaches in the medical identification of COVID-19 variations, as confirmed in Table 8.

Table 8: Comparisons of Yao [3], Dai et al, [27], Abu-Gadairi [8] methodologies with the current method.

$\mathcal{I}$	**Yao' technique**			**Dai et al.' technique**		
	$\mathcal{L}_{\mathfrak{r}}(\mathcal{I})$	$\mathcal{U}_{\mathfrak{r}}(\mathcal{I})$	$\mathcal{A}_{\mathfrak{r}}(\mathcal{I})$	$\mathcal{L}_{\mathfrak{m}}(\mathcal{I})$	$\mathcal{U}_{\mathfrak{m}}(\mathcal{I})$	$\mathcal{A}_{\mathfrak{m}}(\mathcal{I})$
$\mathcal{Q}$	$\{p_1, p_2, p_5\}$	$\mathbb{S}-\{p_4\}$	42%	$\{p_1, p_2, p_5\}$	$\mathbb{S}-\{p_5\}$	42%
$\mathcal{W}$	$\{p_4\}$	$\mathbb{S}-\{p_1, p_2, p_5\}$	20%	$\{p_4\}$	$\mathbb{S}-\{p_1, p_2, p_5\}$	20%
$\mathcal{W}_1$	φ	$\{p_1, p_2\}$	0	φ	$\{p_1, p_2\}$	0
$\mathcal{W}_2$	φ	$\{p_1, p_2\}$	0	φ	$\{p_1, p_2\}$	0
$\mathcal{W}_3$	$\{p_5\}$	$\{p_3, p_5, p_6, p_7, p_8\}$	20%	$\{p_7\}$	$\{p_3, p_6, p_7, p_8\}$	25%
$\mathcal{I}$	**Abu-Gdairi method**			**Current technique**		
	$\mathcal{L}_{\mathfrak{b}}(\mathcal{I})$	$\mathcal{U}_{\mathfrak{b}}(\mathcal{I})$	$\mathcal{A}_{\mathfrak{b}}(\mathcal{I})$	$\underline{\boldsymbol{\beta}}_{\mathfrak{b}}(\mathcal{I})$	$\overline{\boldsymbol{\beta}}_{\mathfrak{b}}(\mathcal{I})$	$\mathcal{A}_{\boldsymbol{\beta}_{\mathfrak{b}}}(\mathcal{I})$
$\mathcal{Q}$	$\{p_1, p_2, p_5, p_6\}$	$\mathbb{S}-\{p_4, p_8\}$	66%	$\mathcal{Q}$	$\mathbb{S}-\{p_4\}$	83%
$\mathcal{W}$	$\{p_4, p_8\}$	$\{p_3, p_4, p_7, p_8\}$	50%	$\{p_4, p_8\}$	$\mathcal{W}$	66%
$\mathcal{W}_1$	φ	$\{p_1, p_2\}$	0	$\mathcal{W}_1$	$\mathcal{W}_1$	100%
$\mathcal{W}_2$	φ	$\{p_1, p_2\}$	0	$\mathcal{W}_2$	$\mathcal{W}_2$	100%
$\mathcal{W}_3$	$\{p_5, p_6\}$	$\{p_3, p_5, p_7, p_8\}$	50%	$\mathcal{W}_3$	$\mathcal{W}_3$	100%

Discussions: Upon reviewing the data presented in Table 8, several significant observations arise:

- The results generated by the proposed method ($\boldsymbol{\beta}_{\mathfrak{b}}$-rough sets) accurately match the medical results of the medical decision table (Table 5), achieving accuracies of up to 100%.

- The proposed methodology exceeded earlier approaches (Yao [65], Dai et al. [27], and Abu-Gadairi [8]) in performance.

- Previous techniques faced difficulties in consistently classifying COVID-19 infections (resp. variants), despite access to pertinent information from the decision table.

- Our technique enhances approximation operators (resp. accuracy measures) by expanding Pawlak's concepts to applied situations without introducing new restrictions.

- These approaches influence induced topological structures, offering advantages in topological applications involving rough set approaches and their extensions. Designed for ease of use, these methods are accessible to non-topology experts. By presenting topological approximations independently, we establish rough approximations ($\boldsymbol{\beta}_{\mathfrak{b}}$-rough sets) without the need for complex concepts. This simplification aids rough set philosophies, making them more accessible to scholars without extensive backgrounds in topology and enhancing their practical utility. While topology-based techniques excel in specific algorithmic expansions, integrating topological concepts into specific algorithms enhances their real-world performance.

Lastly, we introduce Algorithm 1 and a flowchart (**Figure 1**) outlining the use of our proposed techniques (specifically, $\boldsymbol{\beta}_{\mathfrak{b}}$-approximations) to assist in decision-making tasks. This algorithm is easily implementable using simple programming languages like MATLAB, facilitating high-precision medical diagnosis of COVID-19 and its variants. Moreover, it has versatile applications beyond medical analysis, extending to domains such as economics and machine learning.

Algorithm 1. *Framework for Using $\boldsymbol{\beta}_{\mathfrak{b}}$-Approximations in Decision-Making Problems*

1. Input Data:

- *Obtain the dataset relevant to the decision-making problem.*
- *Identify the binary relation θ on the set $\mathbb{S}$.*

2. Compute Similarities:

- *Calculate the degrees of similarity $\Upsilon\ (x,\ y)$ between attributes for each object via the formula:* $\Upsilon\ (x,\ y) = \dfrac{\sum_{\ell=1}^{n}[a_\ell(x) = a_\ell(y)]}{n}$, *where n is the number of condition attributes. Generate a table to display the similarity measures among attributes for all objects.*

3. Establish Binary Relation:

- *Define the binary relation as $(x,\ y) \in \theta \Leftrightarrow \Upsilon\ (x,\ y) \geq E$, where E is the threshold degree of similarity specified by expert requirements.*

4. Calculate $\boldsymbol{\beta}_{\mathfrak{b}}$-Approximations:

- *For each subset $\mathcal{I} \subseteq \mathbb{S}$, compute:*
  - $\underline{\boldsymbol{\beta}}_{\mathfrak{b}}(\mathcal{I}) = \mathcal{I} \cap \mathcal{U}_{\mathfrak{b}}[\mathcal{L}_{\mathfrak{b}}(\mathcal{U}_{\mathfrak{b}}(\mathcal{I}))]$; *and*
  - $\overline{\boldsymbol{\beta}}_{\mathfrak{b}}(\mathcal{I}) = \mathcal{I} \cup \mathcal{L}_{\mathfrak{b}}[\mathcal{U}_{\mathfrak{b}}(\mathcal{L}_{\mathfrak{b}}(\mathcal{I}))]$.

5. Apply Decision-Making Criteria:

- *Utilize $\boldsymbol{\beta}_{\mathfrak{b}}$-approximations to classify or make decisions based on the problem requirements using Definition 6 and Flowchart (**Figure 1**).*

6. Decision Outcome:

- *Generate and analyze the decision outcomes.*
- *Compare with known benchmarks or criteria to validate accuracy.*

7. Adjust Parameters (if necessary):

- *Refine the binary relation θ on the set $\mathbb{S}$ based on decision outcomes and repeat the process.*

8. Output Results:

- *Present the results and insights derived from the $\boldsymbol{\beta}_{\mathfrak{b}}$-approximations in the context of the decision-making problem.*

The subsequent figure (**Figure 1**) characterizes a simple flowchart of the accuracy measures induced from the above algorithm.

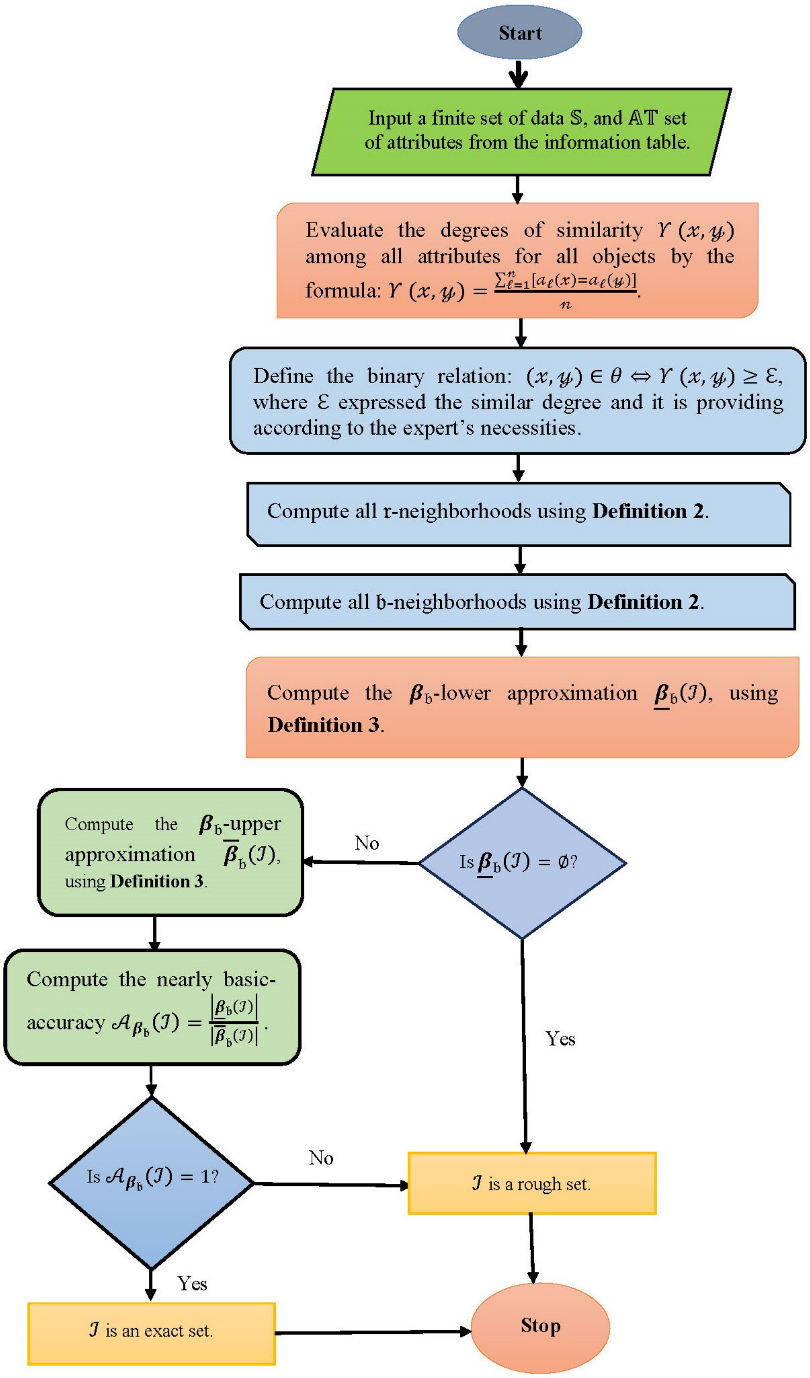

Figure 1: Flowchart to use β_b-approximations in decision-making problems.

6. Conclusions and Future Works

This manuscript introduces and explores different generalizations of rough sets known as β-basic approximations (or β_b-approximations), utilizing topological ideas. These methods are derived directly from neighbors and do not require advanced topology, making them accessible to non-specialists. We present five distinct initial approximations that extend Pawlak's theory and its extensions. These methods significantly improve approximation operators (representing accuracy measures) for any binary relation, surpassing previous approaches. Furthermore, they preserve Pawlak's concepts without additional requirements (see Proposition 1), thereby enhancing their applicability to various real-world problems.

Comparative analyses with previous methods (Yao [65], Dai et al. [27], Abd El-Monsef et al. [40], and Abu-Gadairi [8]) show that our proposed methods are more accurate, as demonstrated by established results (Theorems 2, 3, 4, 5, 6, and their corollaries) and explanatory examples. As a result, these approaches are particularly successful in demonstrating roughness and exactness.

Advantages of the Proposed Approaches:

- **Supple Modeling:** Our methods provide greater flexibility in modeling several challenges by reducing preliminary restrictions, thereby enhancing our ability to represent real-world complexities, especially with large datasets.
- **Easy Application:** Using basic neighborhoods to define $\boldsymbol{\beta}_{\mathfrak{b}}$-approximations instead of topologies simplifies Abu-Gadairi's technique for large datasets. This version, demonstrated in medical applications concerning eight patients, facilitates usage for non-topology professionals.
- **Enhanced Accuracy:** Our methodology improves accuracy, enabling more precise decisions in practical settings such as medical diagnosis (e.g., COVID-19 variants, machine learning applications, decision-making challenges). Furthermore, it achieved up to 100% accuracy in diagnosing COVID-19 variants related to clinical data in a medical decision table (Table 5), surpassing previous research.
- **Algorithm Execution:** Using a flowchart and simulated data, our algorithm offers a user-friendly approach in MATLAB that outperforms existing methodologies.

Future Works:

Future studies involve extending the presented approaches ($\boldsymbol{\beta}_{\mathfrak{b}}$-rough sets) to other sectors, such as the medical fields [6, 23, 28, 63] and economic fields [34, 36], in order to assess real-world applicability. We aim to use $\boldsymbol{\beta}_{\mathfrak{b}}$-approximations in several contexts, including approaches in soft rough sets [29, 33], rough fuzzy sets [35, 38, 39], fuzzy soft sets [9, 20, 59, 61], fuzzy topological spaces [49, 60, 62], and graph theory and ideal applications [5, 25, 45, 52].

Conflict of interest: The authors declare that there is no conflict of interest.

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