EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 17, No. 4, 2024, 4211-4224
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Efficient Numerical Technique for Reaction-Diffusion Singularly Perturbed Boundary-Value Problems

Nawal Alzaid[1,*], Kholoud Alzahrani[1], Huda Bakodah[1]

[1] *Department of Mathematics and Statistics, Faculty of Science, University of Jeddah, Jeddah, Saudi Arabia*

Abstract. The current manuscript aims to present an efficient numerical technique for solving third-order reaction-diffusion singularly perturbed boundary-value problems. The method is based on coupling the restarted Adomian decomposition method and the shooting method. The study further provided a complete outline of the coupled numerical method and used it in tackling the governing class of third-order differential equations. The efficacy of the proposed method is demonstrated on test problems. Lastly, a high level of exactitude between the obtained approximate solution and the exact solution is achieved through comparison tables and figures.

2020 Mathematics Subject Classifications: 34A30, 34A34, 34B15

Key Words and Phrases: ODEs, BVPs, reaction-diffusion singularly perturbed problem, RADM, shooting method

1. Introduction

Singularly perturbed problems are interesting models that feature the blend of singularity and perturbation term(s) and are characterized by some concealed features of the describing physical phenomena. Certainly, "a singular perturbation problem is a problem that depends on a parameter (or parameters) in such a way that solutions behave non-uniformly as the parameter tends towards some limiting value of interest" [22]. Furthermore, "in the early twentieth century, Prandtl described singular perturbations in a seven-page report presented at the Third International Congress of Mathematics in Heidelberg in 1904. However, the term singular perturbation was first used by Friedrichs and Wasow in a paper represented at a seminar on non-linear vibrations at New York University"[4]. In light of the aforesaid, the current study makes consideration of the class of third-order reaction-diffusion singularly perturbed two-point boundary-value problems that arise in modeling various sciences and engineering applications, including reaction–diffusion processes, fluid dynamics, thermodynamics, aerodynamics, and quantum mechanics to state

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v17i4.5547

Email addresses: naalzaid@uj.edu.sa (N. Alzaid),
kalzahrani0246.stu@uj.edu.sa (K. Alzahrani), hobakodah@uj.edu.sa (H. Bakodah)

but just a few [11, 20]. In particular, reaction–diffusion processes are widely encountered in a variety of areas such as in water contamination in aquifers, spread of diseases and pollutions, ecology, chimerical processes, and transfers in dissimilar media among others [16, 17] and the references therein for more on the application of reaction–diffusion processes in real-life scenarios. Mathematically, several scientists have proposed quite a number of reliable computational techniques for the solution of such types of equations exhibiting both singularity and perturbation term(s). Thus, we begin by mentioning these methods to include the classical approach for recasting the class of second-order singularly perturbed boundary-value problem (BVP) to a system of coupled first-order differential equations [9, 14, 19].

In fact, this corresponding system of equations can easily be solved using any of the known reliable methods for first-order ordinary differential equations [12, 24], where certain members of higher-order singularly perturbed BVPs were computationally tackled using diverse approaches. Further, Howes [13] and Yao and Feng [28] made use of the asymptotic and the lower-upper solution methods, respectively, to solve some interesting singularly perturbed BVPs of third-orders. In addition, the authors in [25, 26] deployed the boundary-value technique to tackle the governing third-order problem by transforming it into a system of equations of lesser orders; read also the works of Babu and Ramanujam [7] and Cui and Geng [12], who coupled the finite element method and an analytical method, respectively, with the asymptotic approximation method to effectively tackle the class of third-order singular perturbed BVPs. Furthermore, among the notable numerical methods for the governing model include the fitted Numerov approach that recasts the third-order model to second-order by Phaneendra et al. [18], the mixture of approximations based on polynomials [21], and the DQ method by Bert and Malik [10] among others. However, the present study analyzes the class of perturbed third-order problems with the help of the restarted shooting method (RSM). Certainly, this class of perturbed equations has the perturbation parameter ϵ attached to the higher-order differential term. Besides, both the convection–diffusion and reaction–diffusion typed models can be obtained respectively from the governing problem upon reducing the order of the higher-order by one or two, correspondingly [15]. Moreover, the RSM is the mixture of the restarted Adomian decomposition method (RADM) that successfully solved various functional equations, with the elegant shooting method. In addition, the governing third-order perturbed BVP is presided over by the following ordinary differential equation [15, 29]

$$\epsilon y'''(x) + q(x)y'(x) + r(x)y(x) = s(x), \quad a \leq x \leq b, \tag{1}$$

coupled with the following boundary data

$$y(a) = \alpha, \quad y'(a) = \lambda, \quad y(b) = \beta. \tag{2}$$

where $\epsilon > 0$ is a perturbation parameter that is assumed to be very small, $q(x)$, $r(x)$, and $s(x)$ are given nice functions; with α, β, and λ as real constants. However, the main objective of this paper is to employ the restarted Adomian decomposition method and the shooting method for solving third-order reaction-diffusion singularly perturbed boundary-value problems. Certainly, via the application of RADM, which the current study among others intend to improve its convergence rate through RSM, the given problem in (1) - (2) are transformed to a system of coupled equations that are subsequently solved recurrently. In addition, the current study will seek the assistance of several computational methods to assess the computational efficiency of the RSM in solving the above model, apart from the known fourth-order Runge–Kutta method. Various tables and comparison figures will equally be used to portray the effectiveness of RSM. In addition, the exact solution, where present will be used for comparison with the corresponding RSM solution; the impact of the perturbation parameter ϵ on the solution will equally be assessed. Lastly, we arrange the present paper as follows: Section 2 gives the sketch of RSM; Section 3 presents several numerical examples for the assessment of RSM, while Section 4 gives certain concluding points.

2. Restarted shooting method

Consider the generalized linear third-order BVP as follows

$$y'''(x) = p(x)y''(x) + q(x)y'(x) + r(x)y(x) + s(x), \quad a \leq x \leq b, \tag{3}$$

coupled with the two-point boundary data as follows

$$y(a) = \alpha, \quad y'(a) = \lambda, \quad y(b) = \beta, \tag{4}$$

where $p(x)$,$q(x)$, $r(x)$, and $s(x)$ are equally given functions; with α, β, and λ as real constants. First, the standard shooting method begins by transforming the governing model outlined in (3) - (4) into two different initial-value problems (IVPs) as follows

$$u'''(x) = p(x)u''(x) + q(x)u'(x) + r(x)u(x) + s(x), \quad a \leq x \leq b, \tag{5}$$

$$u(a) = \alpha, \quad u'(a) = \lambda, \quad u''(a) = 0, \tag{6}$$

and

$$v'''(x) = p(x)v''(x) + q(x)v'(x) + r(x)v(x), \quad a \leq x \leq b, \tag{7}$$

$$v(a) = 0, \quad v'(a) = 0, \quad v''(a) = 1. \tag{8}$$

Notably, the RADM will then be used directly to solve the aforementioned IVPs. Consequently, the classical Adomian decomposition method (ADM) must be used first, followed by the RADM. The classical ADM [1, 2, 23] is an energetic semi-analytical method for solving different forms of both linear and nonlinear ordinary and partial differential equations. Moreover, in [3, 5, 8], a combination of the Adomian decomposition method (ADM) and the shooting method are proposed to numerically examine second-order and third-order, linear and nonlinear BVPs. Solving equations (5) - (6) and (7) - (8) using the classical ADM, yields the following

$$u(x) = \phi_1(x) + L^{-1}(p(x)u''(x) + q(x)u'(x) + r(x)u(x) + s(x)), \tag{9}$$

and

$$v(x) = \phi_2(x) + L^{-1}(p(x)v''(x) + q(x)v'(x) + r(x)v(x)). \tag{10}$$

The solutions $u(x)$ and $v(x)$ should be divided into the following infinite series of components

$$u(x) = \sum_{n=0}^{\infty} u_n(x), \quad v(x) = \sum_{n=0}^{\infty} v_n(x).$$

So, the iterative relations are

$$u_0(x) = \phi_1(x) + L^{-1}(s(x)),$$

$$u_{m+1}(x) = L^{-1}(p(x)u''_m(x) + q(x)u'_m(x) + r(x)u_m(x)), \quad m \geq 0, \tag{11}$$

and

$$v_0(x) = \phi_2(x),$$

$$v_{m+1}(x) = L^{-1}(p(x)v''_m(x) + q(x)v'_m(x) + r(x)v_m(x)), \quad m \geq 0, \tag{12}$$

where the functions $\phi_1(x)$ and $\phi_2(x)$ in (11) and (12) reflect the terms originating from the integration of $L(u)$ and $L(v)$ in (5) and (7), respectively, and from applying the given conditions in (6) and (8). As a result, $L\phi_1(x) = 0$ and $L\phi_2(x) = 0$. In addition, the $m+1$-terms approximant is regarded for numerical purposes as

$$u(x) = \psi_{1,m+1}(x) = \sum_{k=0}^{m} u_k(x),$$

and

$$v(x) = \psi_{2,m+1}(x) = \sum_{k=0}^{m} v_k(x).$$

On the other hand, in reference to the alteration made by Babolian et al. [6], which modified u_0 and v_0 to initialize the RADM, one way to accomplish this is to increase the terms on both sides of (9) and (10). Therefore, assuming G_1 and G_2 to be the appropriate terms, which are determined subsequently, one obtains

$$u(x) + G_1 = \phi_1(x) + L^{-1}(p(x)u''(x) + q(x)u'(x) + r(x)u(x) + s(x)) + G_1, \tag{13}$$

and

$$v(x) + G_2 = \phi_2(x) + L^{-1}(p(x)v''(x) + q(x)v'(x) + r(x)v(x)) + G_2. \tag{14}$$

Next, upon utilizing the Wazwaz's modification of the classical ADM [27], one thus obtains the resulting recurrent scheme for (13)-(14) as follows

$$u_0(x) = G_1,$$

$$u_1(x) = \phi_1(x) + L^{-1}(p(x)u''_0(x) + q(x)u'_0(x) + r(x)u_0(x) + s(x)) - G_1,$$

$$u_{m+1}(x) = L^{-1}(p(x)u''_m(x) + q(x)u'_m(x) + r(x)u_m(x)), \quad m \geq 1, \tag{15}$$

and

$$\begin{aligned} v_0(x) &= G_2, \\ v_1(x) &= \phi_2(x) + L^{-1}(p(x)v_0''(x) + q(x)v_0'(x) + r(x)v_0(x)) - G_2, \\ v_{m+1}(x) &= L^{-1}(p(x)v_m''(x) + q(x)v_m'(x) + r(x)v_m(x)), \quad m \geq 1. \end{aligned} \tag{16}$$

Algorithm of RADM

Here, we write down an implementable algorithm for the computational implementation of RADM by first choosing some small natural numbers m,n.

Step I: Use the classical ADM on (5) and (7) to compute $u_0(x), u_1(x), \ldots, u_m(x)$ and $v_0(x), v_1(x), \ldots, v_m(x)$

$$\begin{aligned} w_1^1 &= u_0(x) + u_1(x) + \cdots + u_m(x), \\ w_2^1 &= v_0(x) + v_1(x) + \cdots + v_m(x). \end{aligned}$$

Step II: For $i = 2 : n$, do

$$\begin{aligned} G_1 &= w_1^{i-1}, \\ u_0(x) &= G_1, \\ u_1(x) &= \phi_1(x) + L^{-1}(p(x)u_0''(x) + q(x)u_0'(x) + r(x)u_0(x) + s(x)) - G_1, \\ u_{m+1}(x) &= L^{-1}(p(x)u_m''(x) + q(x)u_m'(x) + r(x)u_m(x)), \quad m \geq 1, \end{aligned}$$

and

$$\begin{aligned} G_2 &= w_2^{i-1}, \\ v_0(x) &= G_2, \\ v_1(x) &= \phi_2(x) + L^{-1}(p(x)v_0''(x) + q(x)v_0'(x) + r(x)v_0(x)) - G_2, \\ v_{m+1}(x) &= L^{-1}(p(x)v_m''(x) + q(x)v_m'(x) + r(x)v_m(x)), \quad m \geq 1, \\ w_1^i &= u_0(x) + u_1(x) + \cdots + u_m(x), \\ w_2^i &= v_0(x) + v_1(x) + \cdots + v_m(x). \end{aligned}$$

End.

Consequently, if we define a new function $z(x)$ that serves as the solution of the original BVP in (3)-(4)

$$z(x) = u(x) + \frac{\beta - u(b)}{v(b)} v(x), \quad v(b) \neq 0, \tag{17}$$

where $u(x)$ and $v(x)$ represent the solutions to corresponding IVPs in (5)-(6) and (7)-(8) , correspondingly. Markedly, the RADM algorithm will be applied in n steps; and in every step, m terms of the classical ADM with new u_0 and v_0 values are obtained. It should equally be noticed that the solutions $z_0, z_1, \ldots, z_m$ are used in each step; whereas for the classical ADM, mn terms are obtained, that is, using $z_0, \ldots, z_{mn}$.

3. Numerical examples

The current section carries out a computational assessment of the devised RSM on several homogeneous and non-homogeneous linear perturbed third-order BVPs for the reaction-diffusion problem of the first and second types. Indeed, the solution posed by the devised RSM has been compared with yet a coupled method, combining the shooting method with the Runge–Kutta method of the fourth-order (SRKM4), in addition to the deployment of several other efficient computational methods in the literature; specifically, the methods utilized in [15, 29]. Furthermore, the section further provides certain supportive tables and figures - Tables 1 –4 and Figures 1-6 – which report the comparison between the RSM and SRKM4 results, in addition to other results obtained using the methods utilized in [15, 29], through examining their respective error differences with those of the available exact analytical solutions. The exact solutions of the examples have been calculated using Maple 18 software. Similarly, this comparison is in line with the impactful influence of the variation in the perturbation parameter ϵ. Thus, in all the reported tables, the representation E_{RSM} is used to represent the solution obtained by the devised RSM, while E_{SRKM4} denotes the solution put forward by the beseeched SRKM4.
Example 1: Consider the non-homogeneous singular perturbation problem for reaction-diffusion of type 1 as follows:

$$\epsilon y'''(x) + 4y'(x) - 4y(x) = x^2, \quad y(0) = 0.5, \quad y'(0) = 0.5, \quad y(1) = 1.47. \tag{18}$$

Firstly, we transform the boundary value problem into two different initial-value problems (IVPs)

$$u'''(x) = -\frac{4}{\epsilon}u'(x) + \frac{4}{\epsilon}u(x) + \frac{1}{\epsilon}x^2, \quad u(0) = 0.5, \quad u'(0) = 0.5, \quad u''(0) = 0, \tag{19}$$

and

$$v'''(x) = -\frac{4}{\epsilon}v'(x) + \frac{4}{\epsilon}v(x), \quad v(0) = 0, \quad v'(0) = 0, \quad v''(0) = 1. \tag{20}$$

Next, guesses are chosen for $n = 2$ and $m = 30$ when using the RADM for (19)-(20) . What is more, the recursive relations are then constructed using the ADM in the first step as follows

$$\begin{cases} u_0(x) = 0.5 + 0.5x + \frac{x^5}{60\epsilon}, \\ u_{m+1}(x) = -\frac{4}{\epsilon}L^{-1}(u'_m(x)) + \frac{4}{\epsilon}L^{-1}(u_m(x)), \quad m \geq 0. \end{cases}$$

and

$$\begin{cases} v_0(x) = \frac{x^2}{2}, \\ v_{m+1}(x) = -\frac{4}{\epsilon}L^{-1}(v'_m(x)) + \frac{4}{\epsilon}L^{-1}(v_m(x)), \quad m \geq 0. \end{cases}$$

where the approximate solutions of (19)-(20) in first step are acquired in a series expression as follows

$$u(x) = G_1 = \sum_{m=0}^{30} u_m(x),$$

and

$$v(x) = G_2 = \sum_{m=0}^{30} v_m(x).$$

In the second step, the RADM is applied, and thus gives the resulting recursive relations as follows

$$\begin{cases} u_0(x) = G_1, \\ u_1(x) = 0.5 + 0.5x + \frac{x^5}{60\epsilon} - \frac{4}{\epsilon}L^{-1}(u_0'(x)) + \frac{4}{\epsilon}L^{-1}(u_0(x)) - G_1, \\ u_{m+1}(x) = -\frac{4}{\epsilon}L^{-1}(u_m'(x)) + \frac{4}{\epsilon}L^{-1}(u_m(x)), \quad m \geq 1. \end{cases}$$

and

$$\begin{cases} v_0(x) = G_2, \\ v_1(x) = \frac{x^2}{2} - \frac{4}{\epsilon}L^{-1}(v_0'(x)) + \frac{4}{\epsilon}L^{-1}(v_0(x)) - G_2, \\ v_{m+1}(x) = -\frac{4}{\epsilon}L^{-1}(v_m'(x)) + \frac{4}{\epsilon}L^{-1}(v_m(x)), \quad m \geq 1. \end{cases}$$

Then, the solutions of the IVPs in (19) and (20) are thus obtained from the above schemes upon taking the respective series summations. Finally, the approximate solution $z(x)$ with $m = 60$, when $h = \frac{1}{60}$, is numerically computed in Tables 1 and 2 by using

$$z(x_k) = u(x_k) + \frac{1.47 - u(1)}{v(1)} v(x_k).$$

where $x_k = kh$ for $k = 0, 1, \dots, m$.

Table 1: The absolute errors for RSM and SRKM4 with different values of ϵ.

x	$\epsilon = 2^{-3}$		$\epsilon = 2^{-6}$		$\epsilon = 2^{-9}$	
	E_{SRKM4}	E_{RSM}	E_{SRKM4}	E_{RSM}	E_{SRKM4}	E_{RSM}
0	1.0×10^{-39}	1.0×10^{-39}	6.0×10^{-40}	6.0×10^{-40}	1.2×10^{-39}	1.2×10^{-39}
$\frac{5}{60}$	8.2×10^{-9}	1.1×10^{-39}	2.6×10^{-7}	8.1×10^{-38}	3.9×10^{-5}	6.4×10^{-6}
$\frac{10}{60}$	2.5×10^{-8}	4.0×10^{-40}	4.0×10^{-7}	2.0×10^{-37}	4.0×10^{-6}	3.0×10^{-6}
$\frac{15}{60}$	4.2×10^{-8}	1.3×10^{-39}	1.9×10^{-7}	2.0×10^{-37}	5.3×10^{-5}	3.5×10^{-6}
$\frac{20}{60}$	5.2×10^{-8}	3.5×10^{-39}	5.2×10^{-7}	9.9×10^{-38}	3.1×10^{-6}	7.1×10^{-6}
$\frac{25}{60}$	4.9×10^{-8}	4.6×10^{-39}	3.2×10^{-7}	7.7×10^{-38}	4.3×10^{-5}	2.4×10^{-6}
$\frac{30}{60}$	3.2×10^{-8}	4.7×10^{-39}	1.2×10^{-6}	1.7×10^{-37}	4.8×10^{-5}	7.8×10^{-6}
$\frac{35}{60}$	6.1×10^{-9}	4.6×10^{-39}	4.5×10^{-7}	2.6×10^{-37}	6.6×10^{-6}	5.3×10^{-6}
$\frac{40}{60}$	2.1×10^{-8}	4.0×10^{-39}	8.8×10^{-7}	2.3×10^{-37}	1.0×10^{-4}	6.0×10^{-6}
$\frac{45}{60}$	4.2×10^{-8}	3.0×10^{-39}	6.3×10^{-7}	1.2×10^{-37}	1.6×10^{-5}	9.1×10^{-6}
$\frac{50}{60}$	4.6×10^{-8}	2.0×10^{-39}	1.1×10^{-6}	1.1×10^{-37}	8.0×10^{-5}	5.6×10^{-6}
$\frac{55}{60}$	3.2×10^{-8}	2.0×10^{-39}	1.6×10^{-6}	6.6×10^{-38}	6.3×10^{-5}	1.0×10^{-5}
1	3.0×10^{-39}	3.0×10^{-39}	3.0×10^{-39}	3.0×10^{-39}	3.0×10^{-39}	3.0×10^{-39}

In this regard, Table 1 reports the resulting absolute error differences incurred by the devised RSM method and the beseeched SRKM4 when compared with the available exact solution for different values of the perturbation parameter ϵ. From Table 1, it is observed

Table 2: Comparison between different methods with different values of ϵ.

Numerical Methods	Maximum Error $\epsilon = 2^{-3}$	$\epsilon = 2^{-6}$	$\epsilon = 2^{-9}$
RSM	6.0×10^{-39}	1.9×10^{-36}	1.1×10^{-5}
SRKM4	5.3×10^{-8}	1.7×10^{-6}	1.3×10^{-4}
IVT [15]	6.2×10^{-2}	3.7×10^{-3}	2.3×10^{-3}
DQ [29]	3.2×10^{-20}	9.0×10^{-11}	3.9×10^{-4}

that RSM greatly outperformed the comparing SRKM4 everywhere except on the end-point boundaries, which both pose the same error difference. In the same way, Table 2 further shops for more computational methods to equally measure the efficacy of the devised approach for solving the governing reaction-diffusion problem of type 1. Notably, the proposed RSM equally happens to be the best method among the four competing methods, and then followed by the DQ method [29] across all the variational effects of the perturbation parameter ϵ. Moreover, Figures 1-3 depict the solution curves, making a comparison between the exact solution, the acquired RSM solution, and the comparing SRKM4 solution for different values of the perturbation constant. However, all the curves are noted to be in good agreement with the exact solution; besides, the obtained RSM solution aligns more with the exact solution on a bigger scale.

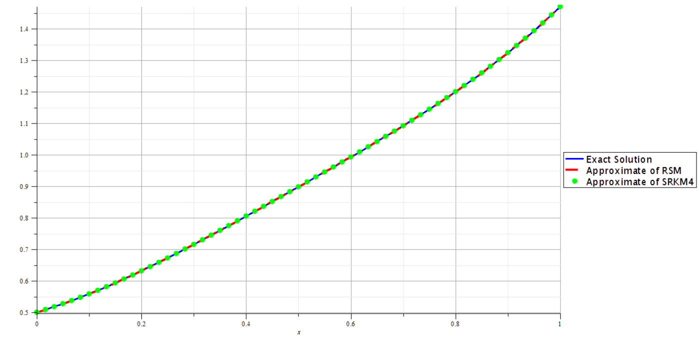

Figure 1: Graphical comparison, depicting the exact and contending approximate solutions with $\epsilon = 2^{-3}$.

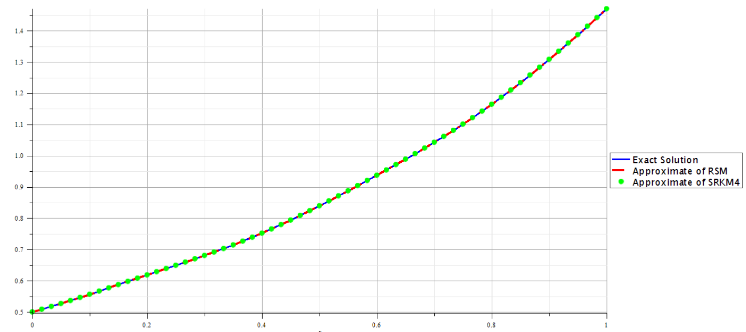

Figure 2: Graphical comparison, depicting the exact and contending approximate solutions with $\epsilon = 2^{-6}$.

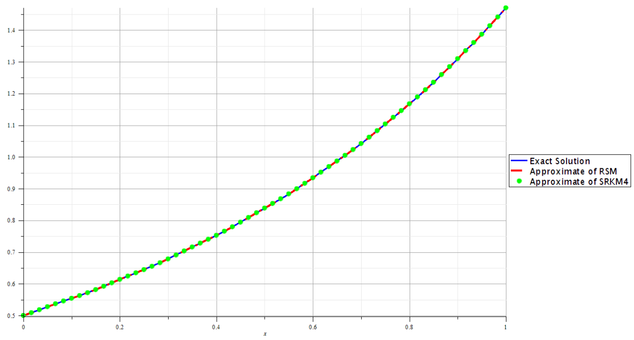

Figure 3: Graphical comparison, depicting the exact and contending approximate solutions with $\epsilon = 2^{-9}$.

Example 2: Consider the homogeneous singular perturbation problem for reaction-diffusion of type 2 as follows:

$$\epsilon y'''(x) + \left(1 + \frac{x}{2}\right) y'(x) - \frac{1}{2} y(x) = 0, \quad y(0) = 0.6, \quad y'(0) = 0.23, \quad y(1) = 0.9. \tag{21}$$

Accordingly, the following two IVPs are considered:

$$u'''(x) = -\frac{1}{\epsilon}\left(1 + \frac{x}{2}\right) u'(x) + \frac{1}{2}\epsilon u(x), \quad u(0) = 0.6, \quad u'(0) = 0.23, \quad u''(0) = 0, \tag{22}$$

and

$$v'''(x) = -\frac{1}{\epsilon}\left(1 + \frac{x}{2}\right) v'(x) + \frac{1}{2}\epsilon v(x), \quad v(0) = 0, \quad v'(0) = 0, \quad v''(0) = 1. \tag{23}$$

As proceed, guesses are chosen for $n = 2$ and $m = 30$ when using the RADM on (22)–(23), leading to the recursive relations via the classical ADM follows

$$u_0(x) = 0.6 + 0.23x,$$

$$u_{m+1}(x) = -\frac{1}{\epsilon}\left(1 + \frac{x}{2}\right) L^{-1}\left(u'_m(x)\right) + \frac{1}{2}\epsilon L^{-1}\left(u_m(x)\right), \quad m \geq 0$$

and

$$v_0(x) = \frac{x^2}{2},$$

$$v_{m+1}(x) = -\frac{1}{\epsilon}\left(1 + \frac{x}{2}\right) L^{-1}\left(v'_m(x)\right) + \frac{1}{2}\epsilon L^{-1}\left(v_m(x)\right), \quad m \geq 0,$$

which approximate solutions of (22)–(23) in first step. Further, the attained approximate solutions in a series form are as follows

$$u(x) = G_1 = \sum_{m=0}^{30} u_m(x),$$

and

$$v(x) = G_2 = \sum_{m=0}^{30} v_m(x).$$

In the second step, the RADM is applied, and thus gives the respective recursive relations as follows

$$\begin{cases} u_0(x) = G_1, \\ u_1(x) = 0.6 + 0.23x - \frac{1}{\epsilon}\left(1+\frac{x}{2}\right) L^{-1}\left(u_0'(x)\right) + \frac{1}{2}\epsilon L^{-1}(u_0(x)) - G_1, \\ u_{m+1}(x) = -\frac{1}{\epsilon}\left(1+\frac{x}{2}\right) L^{-1}(u_m'(x)) + \frac{1}{2}\epsilon L^{-1}(u_m(x)), \quad m \geq 1 \end{cases}$$

$$\begin{cases} v_0(x) = G_2, \\ v_1(x) = \frac{x^2}{2} - \frac{1}{\epsilon}\left(1+\frac{x}{2}\right) L^{-1}(v_0'(x)) + \frac{1}{2}\epsilon L^{-1}(v_0(x)) - G_2, \\ v_{m+1}(x) = -\frac{1}{\epsilon}\left(1+\frac{x}{2}\right) L^{-1}(v_m'(x)) + \frac{1}{2}\epsilon L^{-1}(v_m(x)), \quad m \geq 1 \end{cases}$$

Moreover, the solutions of the IVPs in (22) and (23) are thus obtained from the above schemes upon taking the respective series summations. Finally, the approximate solution $z(x)$ with $m = 60$, when $h = \frac{1}{60}$, is computed in Table 3 and 4 by using the relation

$$z(x_k) = u(x_k) + \frac{0.9 - u(1)}{v(1)} v(x_k)$$

In the same vein, Table 3 outlines the absolute error differences existing between the exact

Table 3: The absolute errors for RSM and SRKM4 with different values of ϵ.

x	$\epsilon = 2^{-3}$		$\epsilon = 2^{-6}$		$\epsilon = 2^{-9}$	
	E_{SRKM4}	E_{RSM}	E_{SRKM4}	E_{RSM}	E_{SRKM4}	E_{RSM}
0	4.3×10^{-39}	9.3×10^{-38}	2.0×10^{-36}	2.5×10^{-36}	4.1×10^{-35}	1.8×10^{-34}
$\frac{5}{60}$	2.5×10^{-10}	9.7×10^{-38}	3.2×10^{-8}	2.6×10^{-36}	1.9×10^{-5}	3.0×10^{-33}
$\frac{10}{60}$	5.1×10^{-10}	1.0×10^{-37}	7.7×10^{-8}	2.7×10^{-36}	2.7×10^{-5}	4.1×10^{-33}
$\frac{15}{60}$	7.5×10^{-10}	1.0×10^{-37}	1.1×10^{-7}	2.8×10^{-36}	2.8×10^{-6}	6.8×10^{-34}
$\frac{20}{60}$	9.2×10^{-10}	1.1×10^{-37}	1.2×10^{-7}	2.9×10^{-36}	1.5×10^{-5}	2.5×10^{-33}
$\frac{25}{60}$	1.0×10^{-9}	1.1×10^{-37}	1.2×10^{-7}	3.0×10^{-36}	3.2×10^{-5}	4.4×10^{-33}
$\frac{30}{60}$	9.7×10^{-10}	1.1×10^{-37}	1.3×10^{-7}	3.1×10^{-36}	4.5×10^{-6}	1.3×10^{-33}
$\frac{35}{60}$	8.4×10^{-10}	1.2×10^{-37}	1.6×10^{-7}	3.2×10^{-36}	1.6×10^{-5}	2.8×10^{-33}
$\frac{40}{60}$	6.2×10^{-10}	1.2×10^{-37}	2.0×10^{-7}	3.3×10^{-36}	3.5×10^{-5}	4.4×10^{-33}
$\frac{45}{60}$	3.7×10^{-10}	1.3×10^{-37}	2.0×10^{-7}	3.4×10^{-36}	2.9×10^{-6}	1.6×10^{-33}
$\frac{50}{60}$	1.4×10^{-10}	1.3×10^{-37}	1.6×10^{-7}	3.5×10^{-36}	2.3×10^{-5}	3.5×10^{-33}
$\frac{55}{60}$	1.2×10^{-11}	1.4×10^{-37}	7.4×10^{-8}	3.6×10^{-36}	3.4×10^{-5}	4.3×10^{-33}
1	6.6×10^{-39}	1.4×10^{-37}	3.0×10^{-36}	3.7×10^{-36}	6.2×10^{-35}	2.7×10^{-34}

solution and the devised RSM method, and on the other hand, between the exact solution and the beseeched SRKM4 for different values of ϵ. Certainly, Table 3 revealed that the proposed RSM is very efficient across the whole solution domain with the exception of the two boundary end points, which SRKM4 is noted to reveal the least error difference. In addition, while deploying more efficient computational methods, Table 4 further compares the devised RSM with SRKM4 and the initial value problem [15] and DQ [29] methods, the proposed RSM for the solution of the governing reaction-diffusion problem of type 2

Table 4: Comparison between different methods with different values of ϵ.

Numerical Methods	Maximum Error $\epsilon = 2^{-3}$	$\epsilon = 2^{-6}$	$\epsilon = 2^{-9}$
RSM	1.4×10^{-37}	3.7×10^{-36}	4.9×10^{-33}
SRKM4	1.0×10^{-9}	2.1×10^{-7}	4.1×10^{-5}
IVT [15]	4.5×10^{-2}	1.6×10^{-2}	7.9×10^{-3}
DQ [29]	2.0×10^{-24}	2.2×10^{-15}	2.8×10^{-7}

is equally noted to be the best, followed by the DQ method across all the chosen values ϵ. What is more, Figures 4-6 depict the solution curves, comparing the exact solution, the acquired RSM solution, and the contending SRKM4 solution for different values of the perturbation constant. Besides, the curves are noted to be in good agreement with the exact solution, with the RSM solution aligning more on a magnified scale.

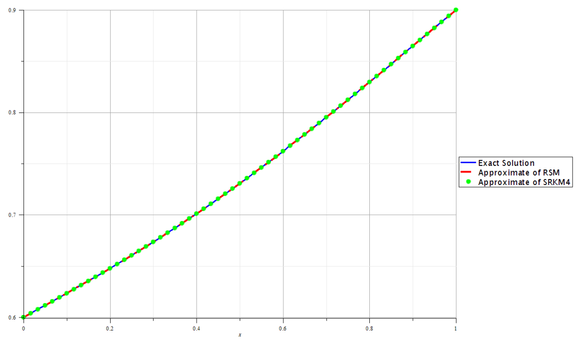

Figure 4: Graphical comparison, depicting the exact and contending approximate solutions with $\epsilon = 2^{-3}$.

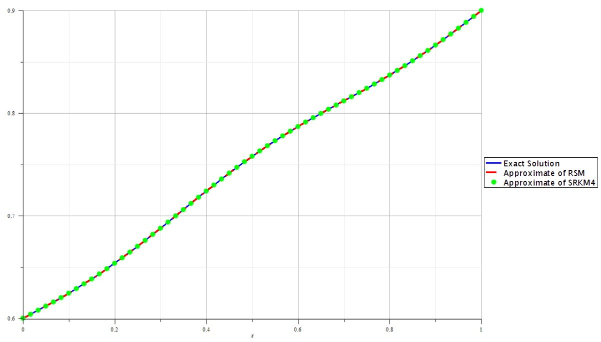

Figure 5: Graphical comparison, depicting the exact and contending approximate solutions with $\epsilon = 2^{-6}$.

4. Conclusions

In conclusion, the current manuscript presents a proficient numerical method to address a class of singularly perturbed third-order linear BVPs. Specifically; we used the modified RADM in conjunction with the iterative shooting method to devise a highly effective

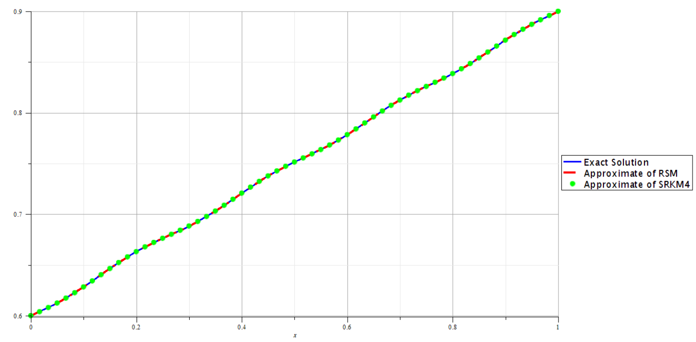

Figure 6: Graphical comparison, depicting the exact and contending approximate solutions with $\epsilon = 2^{-9}$.

strategy called the RSM. The RSM as a reliable computational method has then been demonstrated on the reaction-diffusion singularly perturbed problems of the first and second types, respectively. Following the method's implementation on the demonstrated models, it was discovered that the current approach outperformed many of the previous approaches, including the SRKM4 and the computational techniques identified in the literature [15, 29]; however, SRKM4 outperformed the devised method on the two end-points of the boundary – see the given accommodating tables. The effectiveness of the devised method was further evaluated taking into account the noted speedier convergence and the level of exactitude with the exact analytical solutions in comparison with the other references. Additionally, one can broadly say that the devised technique can be applicable in solving various BVPs of both higher orders and perturbation terms. Aside, nonlinear BVPs could equally be tackled with the proposed methodology when the related Adomian polynomials are systematically computed for the involving nonlinear terms.

References

[1] M. Adel, M. Khader, and H. Ahmad. MHD nanofluid flow and heat transfer caused by a stretching sheet that is heated convectively: an approximate solution using ADM. *Case Studies in Thermal Engineering*, 60:1–11, 2024.

[2] G. Adomian. *Solving frontier problems of physics: the decomposition method.* Kluwer Academic Publishers, Boston, 1994.

[3] N. Al-Zaid, K. Alzahrani, H. Bakodah, and M. Al-Mazmumy. Efficient decomposition shooting method for solving third-order boundary value problems. *International Journal of Modern Nonlinear Theory and Application*, 12(3):81–98, 2023.

[4] M. Alquran and B. Al-Khamaiseh. Algorithms to solve nonlinear singularly perturbed two point boundary value problems. *Applied Mathematical Sciences*, 4(57):2809–2827, 2010.

[5] K. Alzahrani, N. Alzaid, H. Bakodah, and M. Almazmumy. Computational approach

to third-order nonlinear boundary value problems via efficient decomposition shooting method. *Axioms*, 13(4):248, 2024.

[6] E. Babolian, H. Goghary, S. Javadi, and M. Ghasemi. Restarted adomian method for nonlinear differential equations. *International Journal of Computer Mathematics*, 82(1):97–102, 2005.

[7] A. Babu and N Ramanujam. An asymptotic finite element method for singularly perturbed third and fourth order ordinary differential equations with discontinuous source term. *Applied Mathematics and Computation*, 191(2):372–380, 2007.

[8] H. Bakodah, K. Alzahrani, N. Alzaid, and M. Almazmumy. Efficient decomposition shooting method for tackling two-point boundary value models. *Journal of Umm Al-Qura University for Applied Sciences*, 1–11, 2024.

[9] R. Bawa and V. Kumar. An ε-uniform initial value technique for convection-diffusion singularly perturbed problems. In *Proceedings of International MultiConference of Engineers and Computer Scientists*, 2008.

[10] C. Bert and M. Malik. Differential quadrature method in computational mechanics: A review. *Appl Mech Rev*, 49(1):1–28, 1996.

[11] J. Biazar and H. Ghazvini. He's variational iteration method for solving linear and non-linear systems of ordinary differential equations. *Applied Mathematics and Computation*, 191(1):287–297, 2007.

[12] M. Cui and F. Geng. A computational method for solving third-order singularly perturbed boundary-value problems. *Applied Mathematics and Computation*, 198(2):896–903, 2008.

[13] F. Howes. The asymptotic solution of a class of third-order boundary value problems arising in the theory of thin film flows. *SIAM Journal on Applied Mathematics*, 43(5):993–1004, 1983.

[14] M. Kadalbajoo and D. Kumar. Initial value technique for singularly perturbed two point boundary value problems using an exponentially fitted finite difference scheme. *Computers & Mathematics with Applications*, 57(7):1147–1156, 2009.

[15] M. Kumar and S. Tiwari. An initial-value technique to solve third-order reaction–diffusion singularly perturbed boundary-value problems. *International Journal of Computer Mathematics*, 89(17):2345–2352, 2012.

[16] J. Mo and W. Lin. The nonlinear singularly perturbed initial boundary value problems of nonlocal reaction diffusion systems. *Acta Mathematicae Applicatae Sinica*, 22(2):277–286, 2006.

[17] K. Mukherjee and S. Natesan. Optimal error estimate of upwind scheme on shishkin-type meshes for singularly perturbed parabolic problems with discontinuous convection coefficients. *BIT Numerical Mathematics*, 51:289–315, 2011.

[18] K. Phaneendra, Y. Reddy, and GBSL. Soujanya. Asymptotic-numerical method for third-order singular perturbation problems. *International Journal of Applied Science and Engineering*, 10(3):241–248, 2012.

[19] H. Prasad and Y. Reddy. Initial-value technique for singularly perturbed two-point boundary value problems using differential quadrature method. *International Journal of Mathematical Sciences and Engineering Applications (IJMSEA)*, 5(IV):407–431, 2011.

[20] J. Ramos. A smooth locally-analytical technique for singularly perturbed two-point boundary-value problems. *Applied mathematics and computation*, 163(3):1123–1142, 2005.

[21] C. Shu. *Differential Quadrature and Its Application in Engineering*. Springer-Verlag, 2000.

[22] D. Smith. *Singular-perturbation theory: an introduction with applications*. Cambridge University Press, 1985.

[23] N. Sweilam and M. Khader. Approximate solutions to the nonlinear vibrations of multiwalled carbon nanotubes using adomian decomposition method. *Applied Mathematics and Computation*, 217(2):495–505, 2010.

[24] T. Valanarasu and T. Ramanujam. Asymptotic numerical method for singularly perturbed third order ordinary differential equations with a discontinuous source term. *Novi Sad Journal of Mathematics*, 37(2):41–57, 2007.

[25] S. Valarmathi and N. Ramanujam. An asymptotic numerical method for singularly perturbed third-order ordinary differential equations of convection-diffusion type. *Computers & Mathematics with Applications*, 44(5-6):693–710, 2002.

[26] S. Valarmathi and N. Ramanujam. Boundary value technique for finding numerical solution to boundary value problems for third order singularly perturbed ordinary differential equations. *International journal of computer mathematics*, 79(6):747–763, 2002.

[27] A. Wazwaz. A reliable modification of adomian decomposition method. *Applied mathematics and computation*, 102(1):77–86, 1999.

[28] Q. Yao and Y. Feng. The existence of solution for a third-order two-point boundary value problem. *Applied Mathematics Letters*, 15(2):227–232, 2002.

[29] G. Yiğit and M. Bayram. Chebyshev differential quadrature for numerical solutions of third-and fourth-order singular perturbation problems. *Proceedings of the National Academy of Sciences, India Section A: Physical Sciences*, 90(17):429–436, 2020.