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On the Study of Bi-Univalent Functions Defined by the Generalized Sălăgean Differential Operator

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Abstract. In this paper, we make use of the generalized Sălăgean differential operator to define a novel class of bi-univalent functions that is associated with the generalized hyperbolic sine function in the open unit disk $\mathbb{D}$. The prime goal of this paper to derive sharp coefficient bounds in open unit disk $\mathbb{D}$, especially the first two coefficient bounds for the functions belong to this class . The investigation also focuses on studying the classical Fekete-Szegö functional problem for functions belong to this class. Furthermore, some known corollaries are highlighted based on the unique choices of the parameters involved in this class.

2020 Mathematics Subject Classifications: 30C45, 30C50, 33C45, 33C05, 11B39

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1. Introduction

The research conducted in geometric function theory sheds a light on the intricate relationships between coefficients and the geometric properties of functions. By examining the bounds placed on the modulus of a function's coefficients, researchers can gain a deeper understanding of how these functions behave and interact within the mathematical framework. This analytical approach not only enhances our comprehension of the underlying principles governing geometric function theory but also paves the way for further exploration and discovery in this dynamic field of study.

Many operators have been used ever since the beginning of the study of analytic functions. The differential and integral operators are the most fascinating of them, using these operators has made it simpler to add new kinds of univalent and bi-univalent functions. Sălăgean introduced the differential and integral operators, that bear his name, in his 1983 publication. These operators were immensely motivating, and many mathematicians

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have used them to get novel and intriguing results in the field of geometric function theory.

In this paper, as an application of the Sălăgean operator, we introduce a new class of bi-univalent functions and discuss certain characteristic properties of this generalized function class. Consider the set $\mathcal{H}$, which consists of all functions $f(\zeta)$ that are analytic within the open unit disk denoted as $\mathbb{D} = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ and normalized by the conditions $f(0) = 0 = 1 - f'(0)$. Moreover, any function f belongs to the set $\mathcal{H}$ can be written as

$$f(\zeta) = \zeta + \sum_{n=2}^{\infty} a_n \zeta^n, \quad \text{where} \ \ \zeta \in \mathbb{D}. \tag{1}$$

Let the functions f and g be analytic in the open unit disk $\mathbb{D}$. We say that f is subordinated by g in $\mathbb{D}$, denoted as $f(z) \prec g(\zeta)$ for all $\zeta \in \mathbb{D}$, if there exists a Schwarz function w satisfying $w(0) = 0$ and $|w(\zeta)| < 1$ for all $\zeta \in \mathbb{D}$, such that $f(\zeta) = g(w(\zeta))$ for all $\zeta \in \mathbb{D}$. This relationship between f and g is a fundamental concept in complex analysis, providing a way to compare the behavior of two analytic functions within the unit disk. Notably, when the function g is univalent over $\mathbb{D}$, the condition $f(\zeta) \prec g(\zeta)$ is equivalent to $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$. This equivalence highlights the significance of the subordination principle in understanding the relationship between analytic functions. For further insights and detailed discussions on the Subordination Principle, interested readers are encouraged to explore the monographs [10], [11], [23], and [25]. These sources provide comprehensive explanations and applications of this principle in the context of complex analysis and geometric function theory.

In this paper, $\mathcal{S}$ represents the set of functions that are univalent in the open unit disk $\mathbb{D}$ and belong to the set $\mathcal{H}$. As known univalent functions are injective functions. Hence, they are invertible and the inverse functions may not be defined on the entire unit disk $\mathbb{D}$. In fact, according to Koebe one-quarter Theorem, the image of $\mathbb{D}$ under any function $f \in \mathcal{S}$ contains the disk $D(0, 1/4)$ of center 0 and radius 1/4. Accordingly, every function $f \in \mathcal{S}$ has an inverse $f^{-1} = g$ which is defined as

$$g(f(z)) = z, \quad z \in \mathbb{D}$$

$$f(g(w)) = w, \quad |w| < r(f); \ \ r(f) \geq 1/4.$$

Moreover, the inverse function is given by

$$g(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots \tag{2}$$

For this reason, we define the class Σ as follows. A function $f \in \mathcal{H}$ is said to be bi-univalent if both f and f^{-1} are univalent in $\mathbb{D}$. Therefore, let Σ denote the class of all bi-univalent functions in $\mathcal{H}$ which are given by Equation (1). For example, the following functions belong to the class Σ:

$$\frac{z}{1-z}, \quad -\log(1-z), \quad \log\sqrt{\frac{1+z}{1-z}}.$$

However, Koebe function, $\frac{2z-z^2}{2}$ and $\frac{z}{1-z^2}$ do not belong to the class Σ. For more information about univalent and bi-univalent functions we refer the readers to the articles [18], [22], [26] the monograph [11], [13], [34] and the references provided therein.

For example, within the class $\mathcal{S}$, it is established that the modulus of the coefficient a_n is bounded by the value of n. These bounds on the modulus of coefficients provide valuable insights into the geometric characteristics of these functions. Specifically, the restriction on the second coefficients of functions belonging to the class $\mathcal{S}$ offers crucial details regarding the growth and distortion bounds within this class.

The exploration of coefficient-related properties of functions within the bi-univalent class Σ commenced in the 1970s. Notably, Lewin's work, in 1967 [18], marked a significant milestone as he examined the bi-univalent function class and established a bound for the coefficient $|a_2|$. Following this, Netanyahu's research, in 1969 [26], determined that the maximum value of $|a_2|$ is $\frac{4}{3}$ for functions categorized under Σ. Furthermore, Brannan and Clunie, in 1979 [5], demonstrated that for functions in this class, the inequality $|a_2| \leq \sqrt{2}$ holds true. This foundational work has spurred numerous investigations into the coefficient bounds for various subclasses of bi-univalent functions. Despite the extensive research conducted on the coefficient bounds for bi-univalent functions, there remains a significant gap in knowledge regarding the general coefficients $|a_2|$ for cases where $n \geq 4$. The challenge of estimating the coefficients, particularly the general coefficient $|a_n|$, continues to be an unresolved issue in the field. This ongoing inquiry highlights the complexity and richness of the bi-univalent function class, suggesting that further exploration is necessary to fully understand the behavior of these coefficients in higher dimensions.

Fekete and Szegö, in 1933 [12], determined the maximum value of $|a_3 - \lambda a_2^2|$ for a univalent function f, with the real parameter $0 \leq \lambda \leq 1$. This result led to the establishment of the Fekete-Szegö problem, which involves maximizing the modulus of the functional $\Psi_\lambda(f) = a_3 - \lambda a_2^2$ for $f \in \mathcal{H}$ with any complex number λ. Numerous researchers have delved into the Fekete-Szegö functional and other coefficient estimates problems. For instance, relevant articles include [2], [3], [4], [6], [8], [12], [16], [17], [21], [22], [33], and the references provided therein. These studies have contributed to a deeper understanding of the Fekete-Szegö problem and its implications in the field of geometric function theory. Furthermore, the results presented in this paper are anticipated to yield a diverse array of results for subclasses associated with orthogonal polynomials, including Legendre, Lagrange, Laguerre, Gegenbauer, and Horadam polynomials. For more information about orthogonal polynomials, we encourage the interested readers to consult the papers [7], [9] and the related references included therein.

2. Preliminaries, Examples and Lemmas

The information presented in this section are essential for understanding the principal outcomes of this paper. Let f be an analytic function represented in the form (1). The Sălăgean differential operator, which was introduced in [35], is defined as

$$\mathcal{D}^m f(z) = z + \sum_{n=2}^{\infty} n^m a_n z^n.$$

This operator has proven to be a significant source of inspiration, leading numerous mathematicians to achieve novel and intriguing results through their applications. Many researchers, using Sălăgean operators, have developed a variety of new operators. Then, they investigated their characteristics and subsequently employed these newly defined operators to establish classes of univalent functions that exhibit exceptional properties. For examples, see the articles [15], [30], [31], [36] and the references provided therein.

In the year 2004, Al-Oboudi [1] has developed the generalized Sălăgean differential operator which we defined as follows. Let f be an analytic function, $m \in \mathbb{N}_0 = \{0, 1, 2, \cdots\}$, and $q \geq 0$, then we have

$$\mathcal{D}_q^0 f(z) = f(z),$$

$$\mathcal{D}_q^1 f(z) = (1-q)f(z) + qzf'(z), \quad \text{and}$$

$$\mathcal{D}_q^{m+1} f(z) = (1-q)\mathcal{D}_q^m f(z) + qz\left(\mathcal{D}_q^m f(z)\right)' = \mathcal{D}_q\left(\mathcal{D}_q^m f(z)\right).$$

Moreover, if f is in the form (1), then this operator can be written as

$$\mathcal{D}_q^m f(z) = z + \sum_{n=2}^{\infty} [1 + (n-1)q]^m a_n z^n.$$

It is clear that when $q = 1$; we have the Sălăgean differential operator [35] that is mentioned above. For more information about the generalized Sălăgean differential operator, we encourage the interested readers to consult the articles [14], [19], [20], [24], [28], [29] and the references provided therein.

Now, we aim to establish a new class which consists of bi-univalent functions that are defined using the generalized Sălăgean differential operator and that associated to the generalized hyperbolic sine function, which we denote as $\mathcal{S}^q(\lambda, m, \beta, sinh)$, which we define as follows.

Definition 1. *A function $f(z)$ belongs to the family Σ is considered to be part of the class $\mathcal{S}^q(\lambda, m, \beta, sinh)$ if it obeys the following subordination conditions:*

$$(1-\lambda)\left(\frac{\mathcal{D}_q^m f(z)}{z}\right) + \lambda\left(\mathcal{D}_q^m f(z)\right)' \prec 1 + \sinh(\beta z)$$

and

$$(1-\lambda)\left(\frac{\mathcal{D}_q^m g(w)}{w}\right)+\lambda\left(\mathcal{D}_q^m g(w)\right)' \prec 1+\sinh(\beta w),$$

where the function $g(w) = f^{-1}(w)$ *is given by the Equation (2), the parameters* $q \geq 0$, $\lambda \geq 0$, $\beta \geq 0$ *and* $m \in \mathbb{N}_0 = \{0, 1, 2, \cdots\}$.

Choosing $\lambda = 0$ and $\lambda = 1$, we get the following two subclasses of our presenting class, respectively.

Example 1. *A bi-univalent function f that is represented by Equation (1) belongs to the subclass* $\mathcal{S}_0^q(m, \beta, sinh)$ *if the following subordinations hold:*

$$\left(\frac{\mathcal{D}_q^m f(z)}{z}\right) \prec 1+\sinh(\beta z) \tag{3}$$

and

$$\left(\frac{\mathcal{D}_q^m g(w)}{w}\right) \prec 1+\sinh(\beta w), \tag{4}$$

where the function $g(w) = f^{-1}(w)$ *is given by the Equation (2), the parameters* $q \geq 0$, $\beta \geq 0$ *and* $m \in \mathbb{N}_0 = \{0, 1, 2, \cdots\}$.

Example 2. *A bi-univalent function f that is represented by Equation (1) belongs to the subclass* $\mathcal{S}_1^q(m, \beta, sinh)$ *if the following subordinations hold:*

$$\left(\mathcal{D}_q^m f(z)\right)' \prec 1+\sinh(\beta z) \tag{5}$$

and

$$\left(\mathcal{D}_q^m g(w)\right)' \prec 1+\sinh(\beta w), \tag{6}$$

where the function $g(w) = f^{-1}(w)$ *is given by the Equation (2), the parameters* $q \geq 0$, $\beta \geq 0$ *and* $m \in \mathbb{N}_0 = \{0, 1, 2, \cdots\}$.

If we take $q = 0$, $m = 0$, or $m = 1$ and $q = 0$, then $\mathcal{D}_q^m f(z) = f(z)$ for any $f \in \mathcal{H}$ that is given by Equation (1). Hence we obtain the following subclass.

Example 3. *A bi-univalent function f that is represented by Equation (1) belongs to the subclass* $\mathcal{S}^*(\lambda, \beta, sinh)$ *if the following subordinations hold:*

$$(1-\lambda)\left(\frac{f(z)}{z}\right)+\lambda\left(f(z)\right)' \prec 1+\sinh(\beta z) \tag{7}$$

and

$$(1-\lambda)\left(\frac{g(w)}{w}\right)+\lambda\left(g(w)\right)' \prec 1+\sinh(\beta w), \tag{8}$$

where the function $g(w) = f^{-1}(w)$ *is given by the Equation (2), the parameters* $\lambda \geq 0$ *and* $\beta \geq 0$.

Taking $q = 1$ and $m = 1$, we get $\mathcal{D}_q^m f(z) = zf'(z)$ for any $f \in \mathcal{H}$ that is given by Equation (1). Hence we obtain the following subclass.

Example 4. *A bi-univalent function f that is represented by Equation (1) belongs to the subclass $\mathcal{S}(\lambda, \beta, sinh)$ if the following subordinations hold:*

$$f'(z) + \lambda \left((zf''(z) \right) \prec 1 + \sinh(\beta z), \tag{9}$$

and

$$g'(w) + \lambda \left(wg''(w) \right) \prec 1 + \sinh(\beta w), \tag{10}$$

where the function $g(w) = f^{-1}(w)$ is given by the Equation (2), the parameters $\lambda \geq 0$ and $\beta \geq 0$. This class of starlike functions has been studied by many reserchers, see, for example [27], [32] and the references provided therein.

The lemma outlined below is well-documented in the literature (see, for instance, [17]), is considered a fundamental principle that plays a crucial role in the research we are undertaking.

Lemma 1. *if $p(z)$ belongs to the Caratheodory class $\mathcal{P}$, then for $z \in \mathbb{D}$ the function p can be written as $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$. Moreover, $|p_n| \leq 2$ for each natural number n. In addition for any complex number ζ, we have*

$$|p_2 - \zeta p_1^2| \leq 2 \max\{1, |2\zeta - 1|\}.$$

The following lemma, which are thoroughly detailed in literature (see, for instance, [17]), are widely recognized principles that are of considerable relevance to the research we are presenting.

Lemma 2. *Let K and L be real numbers. Let p and q be complex numbers. If $|p| < r$ and $|q| < r$,*

$$|(K+L)p + (K-L)q| \leq \begin{cases} 2r|K|, & \text{if } |K| \geq |L| \\ 2r|L|, & \text{if } |K| \leq |L|. \end{cases}$$

The purpose of this article is to explore a new class of bi-univalent functions defined using the generalized Sălăgean differential operator that is related to the generalized hyperbolic sine function. The central objective is to establish estimates for the moduli of the initial coefficients of the Taylor series representation of functions within this category. Additionally, the article delves into the Fekete-Szegö functional problem pertinent to this particular class of functions, thereby enhancing the comprehension of their inherent properties.

3. Coefficient bounds of the function class $\mathcal{S}^q(\lambda, m, \beta, sinh)$

This section of the paper is devoted to explore the bounds for the modulus of the initial coefficients of functions that are part of the class $\mathcal{S}^q(\lambda, m, \beta, sinh)$, as denoted by Equation (1).

Theorem 1. *Let a function f be in the family Σ. If the function f belongs to the class $\mathcal{S}^q(\lambda, m, \beta, sinh)$ and is represented by the equation (1), then the following inequalities hold:*

$$|a_2| \leq \frac{\beta}{\sqrt{\beta(1+2\lambda)(1+2q)^m + (1+\lambda)^2(1+q)^{2m}}}, \tag{11}$$

and

$$|a_3| \leq \frac{\beta}{(1+2\lambda)(1+2q)^m} + \frac{\beta^2}{(1+\lambda)^2(1+q)^{2m}}. \tag{12}$$

Proof. Suppose a function f belongs to the class $\mathcal{S}^q(\lambda, m, \beta, sinh)$. According to the Definition 1 and the Subordination Principle, we can find two Schwarz functions $u(z)$ and $v(w)$ defined on the open unit disk $\mathbb{D}$ such that

$$(1-\lambda)\left(\frac{\mathcal{D}_q^m f(z)}{z}\right) + \lambda\left(\mathcal{D}_q^m f(z)\right)' = 1 + \sinh(\beta u(z)), \tag{13}$$

and

$$(1-\lambda)\left(\frac{\mathcal{D}_q^m g(w)}{w}\right) + \lambda\left(\mathcal{D}_q^m g(w)\right)' = 1 + \sinh(\beta v(w)). \tag{14}$$

Now, using those Schwarz functions, we define two new analytic functions $h(z)$ and $k(w)$ as follow:

$$h(z) = \frac{1+u(z)}{1-u(z)} \quad \text{and} \quad k(w) = \frac{1+v(w)}{1-v(w)}.$$

It is clear that, these functions $h(z)$ and $k(w)$ are analytic in the open unit disk $\mathbb{D}$ and belong to the Caratheodory class. Thus, we can write them as follows

$$h(z) = \frac{1+u(z)}{1-u(z)} = 1 + h_1 z + h_2 z^2 + \cdots$$

and

$$k(w) = \frac{1+v(w)}{1-v(w)} = 1 + k_1 w + k_2 w^2 + \cdots$$

Moreover, $h(0) = 1 = k(0)$, they have positive real parts, $|h_j| \leq 2$ and $|k_j| \leq 2$ for all $j \in \mathbb{N}$.

Equivalently, we get the following representations of $u(z)$ and $v(w)$

$$u(z) = \frac{h(z)-1}{h(z)+1} = \frac{1}{2}\left[h_1 z + \left(h_2 - \frac{h_1^2}{2}\right) z^2 + \cdots\right], \tag{15}$$

and

$$v(w) = \frac{k(w) - 1}{k(w) + 1} = \frac{1}{2}\left[k_1 w + \left(k_2 - \frac{k_1^2}{2}\right) w^2 + \cdots\right]. \tag{16}$$

On one hand, by consulting Equation (15), the right-hand sides of Equations (13) can be written as:

$$\begin{aligned} 1 + \sinh(\beta u(z)) = 1 &+ \frac{\beta h_1}{2} z + \beta\left(\frac{h_2}{2} - \frac{h_1^2}{4}\right) z^2 \\ &+ \left(\frac{\beta h_1^3}{8} - \frac{\beta h_1 h_2}{2} + \frac{\beta h_3}{2} + \frac{\beta^3 h_1^3}{48}\right) z^3 \\ &+ \left(\frac{3\beta h_1^2 h_2}{8} - \frac{\beta h_1 h_3}{2} - \frac{\beta h_1^4}{16} - \frac{\beta h_2^2}{4} - \frac{\beta^3 h_1^4}{32} + \frac{\beta h_4}{2} + \frac{\beta^3 h_1^2 h_2}{16}\right) z^4 + \cdots \end{aligned} \tag{17}$$

Thus, considering Equation (17) then comparing coefficients on both sides of Equation (13), we get the following two equations

$$(1 + \lambda)(1 + q)^m a_2 = \frac{\beta}{2} h_1, \tag{18}$$

and

$$(1 + 2\lambda)(1 + 2q)^m a_3 = \beta\left(\frac{h_2}{2} - \frac{h_1^2}{4}\right). \tag{19}$$

On the other hand, by consulting Equation (16), the right-hand side of Equation (14) can be written as:

$$\begin{aligned} 1 + \sinh(\beta v(w)) = 1 &+ \frac{\beta k_1}{2} w + \beta\left(\frac{k_2}{2} - \frac{k_1^2}{4}\right) w^2 \\ &+ \left(\frac{\beta k_1^3}{8} - \frac{\beta k_1 k_2}{2} + \frac{\beta k_3}{2} + \frac{\beta^3 k_1^3}{48}\right) w^3 \\ &+ \left(\frac{3\beta k_1^2 k_2}{8} - \frac{\beta k_1 k_3}{2} - \frac{\beta k_1^4}{16} - \frac{\beta k_2^2}{4} - \frac{\beta^3 k_1^4}{32} + \frac{\beta k_4}{2} + \frac{\beta^3 k_1^2 k_2}{16}\right) w^4 + \cdots \end{aligned} \tag{20}$$

More over, considering Equation (20), then comparing coefficients on both sides of Equation (14) we get the following two equations

$$-(1 + \lambda)(1 + q)^m a_2 = \frac{\beta}{2} k_1, \tag{21}$$

and

$$(1 + 2\lambda)(1 + 2q)^m (2a_2^2 - a_3) = \beta\left(\frac{k_2}{2} - \frac{k_1^2}{4}\right). \tag{22}$$

Now, using Equation (18) and Equation (21), we get the following equation

$$a_2 = \frac{\beta h_1}{2(1+\lambda)(1+q)^m} = \frac{-\beta k_1}{2(1+\lambda)(1+q)^m}. \tag{23}$$

Hence, the last equation gives the following equation

$$\beta^2(h_1^2 + k_1^2) = 8(1+\lambda)^2(1+q)^{2m} a_2^2. \tag{24}$$

Moreover, adding Equation (19) to Equation (22), we obtain the following equation

$$\beta(h_1^2 + k_1^2) + 8(1+2\lambda)(1+2q)^m a_2^2 = 2\beta(h_2 + k_2).$$

Therefore, consulting Equation (24), the last equation can be written as

$$a_2^2 = \frac{\beta^2(h_2 + k_2)}{4\beta(1+2\lambda)(1+2q)^m + 4(1+\lambda)^2(1+2q)^{2m}}. \tag{25}$$

Therefore, considering Equation (25), then using constraints $|h_2| \leq 2$ and $|k_2| \leq 2$, we get

$$|a_2|^2 \leq \frac{\beta^2}{\beta(1+2\lambda)(1+2q)^m + (1+\lambda)^2(1+2q)^{2m}}, \tag{26}$$

which gives the desired inequality (11) that represents the coefficient estimate of $|a_2|$.

In the next step, we are looking to determine the coefficient estimate for $|a_3|$. Subtracting Equation (22) from Equation (19), we easily get the following equation

$$4(1+2\lambda)(1+2q)^m(a_3 - a_2^2) = \beta(h_2 - k_2) - \frac{\beta(h_1^2 - k_1^2)}{2}.$$

Hence, consulting Equation (24), we get $\beta(h_1^2 - k_1^2) = 0$. Therefore, the last equation can be written as

$$a_3 = \frac{\beta(h_2 - k_2)}{4(1+2\lambda)(1+2q)^m} + a_2^2. \tag{27}$$

Moreover, consulting Equation (24), Equation (27) can be written as

$$a_3 = \frac{\beta(h_2 - k_2)}{4(1+2\lambda)(1+2q)^m} + \frac{\beta^2(h_1^2 + k_1^2)}{8(1+\lambda)^2(1+q)^{2m}}. \tag{28}$$

Thus, using the constraints $|h_j| \leq 2$ and $|k_j| \leq 2$ for all $j \in \mathbb{N}$, simple calculations of Equation (28) gives the required estimation of $|a_3|$. Consequently, the proof of Theorem 1 is now concluded.

The following corollaries come out directly from Theorem 1, they are corresponding to the examples presented in the previous section, respectively. The methods used in establishing these corollaries bear a strong resemblance to those used in the proof of the previous Theorem 1, which is why we have opted to omit the comprehensive proofs' details.

Corollary 1. *Let f be a bi-univalent function of the form (1). If the function f satisfies the subordinations (3) and (4), then the following hold*

$$|a_2| \leq \frac{\beta}{\sqrt{\beta(1+2q)^m + (1+q)^{2m}}},$$

and

$$|a_3| \leq \frac{\beta}{(1+2q)^m} + \frac{\beta^2}{(1+q)^{2m}}.$$

Corollary 2. *Let f be a bi-univalent function of the form (1). If the function f satisfies the subordinations (5) and (6), then it can be concluded that*

$$|a_2| \leq \frac{\beta}{\sqrt{3\beta(1+2q)^m + 4(1+q)^{2m}}},$$

and

$$|a_3| \leq \frac{\beta}{3(1+2q)^m} + \frac{\beta^2}{4(1+q)^{2m}}.$$

Corollary 3. *Let f be a bi-univalent function of the form (1). If the function f satisfies the subordinations (7) and (8), then the following hold*

$$|a_2| \leq \frac{\beta}{\sqrt{\beta(1+2\lambda) + (1+\lambda)^2}},$$

and

$$|a_3| \leq \frac{\beta}{(1+2\lambda)} + \frac{\beta^2}{(1+\lambda)^2}.$$

Corollary 4. *Let f be a bi-univalent function of the form (1). If the function f satisfies the subordinations (9) and (10), then it can be concluded that*

$$|a_2| \leq \frac{\beta}{\sqrt{3\beta(1+2\lambda) + 4(1+\lambda)^2}},$$

and

$$|a_3| \leq \frac{\beta}{3(1+2\lambda)} + \frac{\beta^2}{4(1+\lambda)^2}.$$

4. Fekete-Szegö problem of the function class $\mathcal{S}^q(\lambda, m, \beta, sinh)$

In this section, we will establish the Fekete-Szegö inequalities for functions that are members of our class $\mathcal{S}^q(\lambda, m, \beta, sinh)$ and some of its subclasses.

Theorem 2. *If a function f is a member of the class $\mathcal{S}^q(\lambda, m, \beta, sinh)$ and is represented by equation (1), then for $\beta \neq 0$ and for a real number ζ the following inequality holds*

$$|a_3-\zeta a_2^2| \leq \begin{cases} \frac{\beta}{(1+2\lambda)(1+2q)^m}, & if \ \ \zeta \in [\zeta_1,\zeta_2] \\ \frac{\beta^2|1-\zeta|}{\beta A+B^2}, & if \ \ \zeta \notin [\zeta_1,\zeta_2], \end{cases} \tag{29}$$

where

$$A=(1+2\lambda)(1+2q)^m, \quad B=(1+\lambda)(1+q)^m,$$

$$\zeta_1=\frac{-B^2}{\beta A}, \quad and \ \ \zeta_2 = 2-\zeta_1.$$

Proof. For any real number ζ, using Equation (27), we get the following equation

$$a_3-\zeta a_2^2=\frac{\beta(h_2-k_2)}{4(1+2\lambda)(1+2q)^m}+(1-\zeta)a_2^2. \tag{30}$$

Therefore, by consulting Equation (25), the last equation can be written as follows

$$\begin{aligned} a_3-\zeta a_2^2 =& \frac{\beta(h_2-k_2)}{4(1+2\lambda)(1+2q)^m} \\ &+ \frac{\beta^2(1-\zeta)(h_2+k_2)}{4\beta(1+2\lambda)(1+2q)^m+4(1+\lambda)^2(1+2q)^{2m}}. \end{aligned} \tag{31}$$

Moreover, the last equation can be written as follows

$$a_3-\zeta a_2^2=\left(\Delta+\frac{\beta}{4A}\right)h_2+\left(\Delta-\frac{\beta}{4A}\right)k_2, \tag{32}$$

where

$$\Delta=\frac{\beta^2(1-\zeta)}{4\beta(1+2\lambda)(1+2q)^m+4(1+\lambda)^2(1+2q)^{2m}}.$$

Now, by applying Lemma 2 on Equation (32), we easily arrive the following inequality

$$|a_3-\zeta a_2^2| \leq \begin{cases} \frac{\beta}{(1+2\lambda)(1+2q)^m}, & \text{if} \ \ |\Delta| \leq \frac{\beta}{4A} \\ \frac{\beta^2|1-\zeta|}{\beta A+B^2}, & \text{if} \ \ |\Delta| \geq \frac{\beta}{4A}. \end{cases} \tag{33}$$

Now, considering the following inequality

$$\left|\frac{\beta^2(1-\zeta)}{4\beta(1+2\lambda)(1+2q)^m+4(1+\lambda)^2(1+2q)^{2m}}\right| \leq \frac{\beta}{4A}.$$

Then, for $\beta \neq 0$, simple calculations gives us the following inequality

$$\frac{-B^2}{\beta A} \leq \zeta \leq \frac{2\beta A+B^2}{\beta A}.$$

Therefore, taking $\zeta_1=\dfrac{-B^2}{\beta A}$ and $\zeta_2=\dfrac{2\beta A+B^2}{\beta A}$, we easily get the desired inequality that represented by (29). This completes the proof.

The following corollaries are natural generated from the previously Theorem 2 under the conditions presented in the Examples that are given in the second section, respectively. The approach used to establish this corollary is quite similar to that of the earlier theorem; hence, we have chosen to omit the comprehensive proof for this corollary.

Corollary 5. *Let f be a bi-univalent function of the form (1). If the function f satisfies the subordinations (3) and (4), then for a real number ζ the following holds*

$$|a_3 - \zeta a_2^2| \leq \begin{cases} \frac{\beta}{(1+2q)^m}, & if \ \ \zeta \in [\zeta_1, \zeta_2] \\ \frac{\beta^2|1-\zeta|}{\beta(1+2q)^m+(1+q)^{2m}}, & if \ \ \zeta \notin [\zeta_1, \zeta_2], \end{cases}$$

where

$$\zeta_1 = \frac{-(1+q)^{2m}}{\beta(1+2q)^m}, \quad and \ \ \zeta_2 = 2 - \zeta_1.$$

Corollary 6. *Let f be a bi-univalent function of the form (1). If the function f satisfies the subordinations (5) and (6), then for a real number ζ the following holds*

$$|a_3 - \zeta a_2^2| \leq \begin{cases} \frac{\beta}{3(1+2q)^m}, & if \ \ \zeta \in [\zeta_1, \zeta_2] \\ \frac{\beta^2|1-\zeta|}{3\beta(1+2q)^m+4(1+q)^{2m}}, & if \ \ \zeta \notin [\zeta_1, \zeta_2], \end{cases}$$

where

$$\zeta_1 = \frac{-4(1+q)^{2m}}{3\beta(1+2q)^m}, \quad and \ \ \zeta_2 = 2 - \zeta_1.$$

Corollary 7. *Let f be a bi-univalent function of the form (1). If the function f satisfies the subordinations (7) and (8), then for a real number ζ the following holds*

$$|a_3 - \zeta a_2^2| \leq \begin{cases} \frac{\beta}{(1+2\lambda)}, & if \ \ \zeta \in [\zeta_1, \zeta_2] \\ \frac{\beta^2|1-\zeta|}{\beta(1+2\lambda)+(1+\lambda)^2}, & if \ \ \zeta \notin [\zeta_1, \zeta_2], \end{cases}$$

where

$$\zeta_1 = \frac{-(1+\lambda)^2}{\beta(1+2\lambda)}, \quad and \ \ \zeta_2 = 2 - \zeta_1.$$

Corollary 8. *Let f be a bi-univalent function of the form (1). If the function f satisfies the subordinations (9) and (10), then for a real number ζ the following holds*

$$|a_3 - \zeta a_2^2| \leq \begin{cases} \frac{\beta}{3(1+2\lambda)}, & if \ \ \zeta \in [\zeta_1, \zeta_2] \\ \frac{\beta^2|1-\zeta|}{3\beta(1+2\lambda)+4(1+\lambda)^2}, & if \ \ \zeta \notin [\zeta_1, \zeta_2], \end{cases}$$

where

$$\zeta_1 = \frac{-4(1+\lambda)^2}{3\beta(1+2\lambda)}, \quad and \ \ \zeta_2 = 2 - \zeta_1.$$

5. Conclusion

This research paper explored a novel class of bi-univalent functions characterized by the generalized Sălăgean differential operator, which is linked to the generalized hyperbolic sine function. The author has established estimates for the initial coefficients of the Taylor-Maclaurin series for functions within this class and has developed the Fekete-Szegö inequalities relevant to these functions and their various subclasses. The findings of this study are expected to yield numerous results for subclasses defined through orthogonal polynomials, such as Legendre, Lagrange, Laguerre, Gegenbauer, and Horadam polynomials. Furthermore, the presented work in this paper will inspire researchers to extend its concepts to harmonic functions and symmetric q-calculus.

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6. Conflicts of interest

The author confirms that there are no relevant conflicts of interest that are pertinent to the content of this article.

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