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Achieving Edge Consensus in Hybrid Multi-Agent Systems: Scaled Dynamics and Protocol Design

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Abstract. In contrast to existing research on consensus problems that primarily addresses the node dynamics of multi-agent systems, this study focuses on achieving consensus among the states of edges within a network. Initially, the scaled consensus of edge dynamics in hybrid multi-agent systems is investigated for both undirected and directed topologies. Subsequently, necessary and sufficient conditions are derived for both directed and undirected topologies to facilitate the design of consensus protocols aimed at resolving edge consensus challenges. Finally, numerical examples are provided to substantiate the effectiveness of the theoretical results.

2020 Mathematics Subject Classifications: 93A16, 93B70, 93C27, 93D50

Key Words and Phrases: Hybrid multi-agent systems, scaled consensus, edge consensus problem, spanning tree, directed and undirected topology

1. Introduction

In the past several decades, the study of consensus problems within multi-agent systems has garnered significant attention, owing to their extensive applicability across a diverse array of disciplines, including swarm robotics, distributed sensor networks, traffic management systems, online marketplaces, and social networks[10, 15].

Consensus is characterized as the process by which a collective of agents or entities within a system endeavors to attain an agreement or a unified decision regarding a particular quantity of interest. The achievement of consensus is critical for promoting coordinated behavior, collaboration, and decision-making among the agents. As a result, a plethora of researchers have developed consensus algorithms in recent years, which are essential for ensuring that all agents within the system converge to a common understanding or state, thereby enhancing the overall efficiency and effectiveness of the system's operation (see examples in [1, 4, 9, 11, 13, 16, 21, 22]).

The aforementioned findings predominantly addressed consensus issues pertaining to

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node dynamics. Nevertheless, in a variety of real-world scenarios, the interactions among agents are influenced not only by their individual states (nodes) but also by the relationships that exist between them (edges). The edge consensus problem is defined as the endeavor to achieve consensus or agreement among the states of edges within a network, rather than concentrating solely on the states of individual nodes or agents. In the framework of edge consensus, the objective is for the states of all edges in the network to asymptotically converge to a common value or consensus state through interactions and information exchange among neighboring edges. Given the significance of understanding and achieving consensus at the edge level for applications such as network communication, social networks, and distributed control systems, edge consensus problems have garnered considerable attention in scholarly research in recent years (see example in [6, 8, 12, 17, 18, 20]).

According to the above discussion, this paper aims to investigate scaled consensus problems of edge dynamics in hybrid multi-agent systems (HMASs) under both directed and undirected topologies. The main contributions of the paper include:

(1) Exploration of edge dynamics in HMASs: The study extends the work of Zheng et al[23], by examining consensus problems specifically related to edge dynamics in HMASs, which consist of both continuous-time and discrete-time agents.

(2) First investigation of scaled consensus problems: This research is the first to address the scaled consensus problem of edge dynamics in HMASs, providing new insights into how edge interactions can be managed to achieve consensus.

(3) Theoretical framework for edge consensus: The paper develops a theoretical framework that outlines the conditions necessary for achieving edge consensus in both directed and undirected topologies, contributing to the understanding of how edge dynamics operate within multi-agent systems.

(4) Applications to real-world scenarios: By focusing on edge dynamics, the research highlights the relevance of consensus in practical applications such as network communication and distributed control systems, emphasizing the need for algorithms that consider both node and edge interactions.

(5) Numerical simulations: The paper includes numerical simulations to validate the proposed edge consensus protocols, demonstrating their effectiveness in achieving consensus in various scenarios and reinforcing the theoretical findings.

These contributions collectively advance the understanding of consensus in multi-agent systems by integrating edge dynamics into the analysis, thereby enhancing the potential for practical applications in complex networks.

The rest of this paper is organized as follows. Some preliminaries and the problem formulation are provided in Section 2. In Section 3, the scaled consensus problems of edge dynamics in HMASs under directed and undirected topology are solved under some necessary and sufficient conditions. In Section 4, numerical examples are provided to demonstrate the effectiveness of our main results. Finally, some conclusions are drawn in Section 5.

2. Preliminaries and Problem formulations

2.1. Preliminaries

This section articulates fundamental definitions, lemmas, and notations derived from graph theory and matrix theory that are pivotal for the formulation of our principal findings (for comprehensive details, refer to[2, 7]).

In the context of this paper, the set of real numbers is denoted by $\mathbb{R}$, the set of positive integers by $\mathbb{N}$, and the n-dimensional real vector space by $\mathbb{R}^n$. Furthermore, $\mathbb{R}^{n\times n}$ signifies the collection of $n \times n$ matrices. For a matrix $A = [a_{ij}]_{n\times n} \in \mathbb{R}^{n\times n}$, the notation A^T represents its transpose. A matrix $A = [a_{ij}]_{n\times n}$ is classified as nonnegative if all its elements are nonnegative, which is denoted as $A \geq 0$. If A and B are nonnegative matrices, the relation $A \geq B$ implies that the matrix $A - B$ is also nonnegative. A matrix P is characterized as a stochastic matrix if it is nonnegative and the sum of each of its rows equals 1. Moreover, a stochastic matrix P is deemed indecomposable and aperiodic (SIA) if there exists a column vector y such that $\lim_{k\to\infty} P^k = \mathbf{1}_n y^T$, where $\mathbf{1}_n = (1, 1, \ldots, 1)^T$ is an $n \times 1$ vector.

Before proceeding, we present several essential definitions, lemmas, and properties as follows:

Lemma 1. *[3] If G is a connected graph, then the line graph of G, denoted $L(G)$, is also connected.*

Lemma 2. *[2] Let L represent the Laplacian matrix of an undirected graph G. The Laplacian matrix L is irreducible if and only if G is connected.*

Lemma 3. *[5] If a directed graph G contains more than one vertex and is strongly connected, then its line graph $L(G)$ is strongly connected.*

Lemma 4. *[14] A stochastic matrix possesses an algebraic multiplicity of one for the eigenvalue $\lambda = 1$ if and only if the graph associated with the matrix contains a spanning tree. Furthermore, a stochastic matrix with positive diagonal elements guarantees that $|\lambda| < 1$ for every eigenvalue other than one.*

Lemma 5. *[14] Let $A = [a_{ij}]_{n\times n}$ be a stochastic matrix. If A has an eigenvalue $\lambda = 1$ with algebraic multiplicity equal to one, and all other eigenvalues satisfy $|\lambda| < 1$, then A is classified as SIA, which implies that $\lim_{k\to\infty} A^k = \mathbf{1}_n y^T$, where y is a nonnegative vector that satisfies $A^T y = y$ and $\mathbf{1}_n^T y = 1$.*

2.2. Problem formulations

Consider an undirected network $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, with a set of N nodes and M edges, where $\mathcal{E} = \{(i, j) : \text{if there is an edge between node } i \text{ and node } j\}$ and $\mathcal{V} = \{1, 2, ..., N\}$. The topology of the network is described by the adjacency matrix $\mathcal{A} = [a_{ij}]_{N\times N}$, where

$$a_{ji} = a_{ij} = \begin{cases} 1, & \text{if } (i,j) \in \mathcal{E}; \\ 0, & \text{otherwise.} \end{cases}$$

Different from the undirected network, the directed network means that the edge $(i, j) \in \mathcal{E}$ if node i can receive information from node j. Hence, the topology of a network is described by the adjacency matrix $\mathcal{A} = [a_{ij}]_{N\times N}$, where

$$a_{ij} = \begin{cases} 1, & \text{if } (i, j) \in \mathcal{E}; \\ 0, & \text{otherwise.} \end{cases}$$

For any node i, its inbound edge (i, l) is adjacent to its outbound edge (j, i) while its outbound edge (j, i) is adjacent from its inbound edge (i, l). For better description and understanding, we say that the inbound edge (i, k) is the valid neighbor of the outbound edge (j, i).

Let $x_{ij}(t)$ and $\beta_{ij} \neq 0$ be the state and scalar scale of edge (i, j) at time t, respectively. Thus, the edge dynamics of each edge can be designed as follows:

$$\begin{cases} \beta_{ij}\dot{x}_{ij}(t) = u_{ij}(t), & \text{for } (i, j) \in \mathcal{E}_c, \\ \beta_{ij}x_{ij}(t_{k+1}) = \beta_{ij}x_{ij}(t_k) + u_{ij}(t_k), \quad t_k = kh, \quad k \in \mathbb{N} & \text{for } (i, j) \in \mathcal{E}_d, \end{cases} \tag{2.1}$$

where $u_{ij} \in \mathbb{R}$ is a consensus protocol. $\mathcal{E}_c, \mathcal{E}_d \subseteq \mathcal{E}$ are the continuous-time and discrete-time edge dynamics, respectively. In general, one says that the protocol u_{ij} in (2.1) solves the edge consensus problems if the following definition is satisfied:

Definition 1. *The hybrid system (2.1) is said to reach edge consensus if for any initial conditions,*

$$\lim_{t_k\to\infty} \|\beta_{ij}x_{ij}(t_k) - \beta_{ks}x_{ks}(t_k)\| = 0, \quad \textit{for all } (i, j), (k, s) \in \mathcal{E} \tag{2.2}$$

and

$$\lim_{t\to\infty} \|\beta_{ij}x_{ij}(t) - \beta_{ks}x_{ks}(t)\| = 0, \quad \textit{for all } (i, j), (k, s) \in \mathcal{E}_c. \tag{2.3}$$

3. scaled consensus results

3.1. Undirected communication networks

In this work, we solve scaled consensus problem of edge dynamics in HMASs by designing the consensus protocol, u_{ij}, which depends on the states of the edge (i, j) and its neighboring edges at time t_k. In addition, two edges are neighboring edges if they share exactly a common ending vertex.

In our protocol, each edge adjusts itself based on the errors between its state and the states of its neighboring edges. The dynamical equation governing the evolution of the states of edges are designed as follows:

$$\begin{cases} u_{ij}(t) = & |\beta_{ij}|\Bigg[\sum_{s\in\mathcal{N}_i} a_{is}[\beta_{is}x_{is}(t_k) - \beta_{ij}x_{ij}(t_k)] \\ & \quad + \sum_{s\in\mathcal{N}_j} a_{js}[\beta_{js}x_{js}(t_k) - \beta_{ij}x_{ij}(t_k)]\Bigg], \quad \text{for} \quad (i,j) \in \mathcal{E}_c, \\ u_{ij}(t_{k+1}) = & h\cdot|\beta_{ij}|\Bigg[\sum_{s\in\mathcal{N}_i} a_{is}[\beta_{is}x_{is}(t_k) - \beta_{ij}x_{ij}(t_k)] \\ & \quad + \sum_{s\in\mathcal{N}_j} a_{js}[\beta_{js}x_{js}(t_k) - \beta_{ij}x_{ij}(t_k)]\Bigg], \quad \text{for} \quad (i,j) \in \mathcal{E}_d, \end{cases} \tag{3.1}$$

where $h > 0$ is the step size and $\mathcal{N}_i = \{j|(i,j) \in \mathcal{E}\}$ is the neighboring set of node i.

Remark 1. *It should be note that, in Eq.(2.1), x_{ij} and x_{ji} both denote the state of the same edge (i,j). Moreover, in protocol (3.1), we can see that an agent can interact and update information from its neighbors only at the sampling time t_k.*

Remark 2. *Obviously, if $\beta_{ij} = 1$ for all edge (i,j) and $\mathcal{E}_c = \emptyset$, the protocol (3.1) can be written as*

$$\begin{aligned} u_{ij}(t_{k+1}) =& h\Bigg[\sum_{s\in\mathcal{N}_i} a_{is}[x_{is}(t_k) - x_{ij}(t_k)] \\ & + \sum_{s\in\mathcal{N}_j} a_{js}[x_{js}(t_k) - x_{ij}(t_k)]\Bigg], \quad \text{for} \quad (i,j) \in \mathcal{E}_d, \end{aligned} \tag{3.2}$$

which was studied in [18].

Before moving on, the following assumptions are provided to obtain our main results:
(A1) The step size h is satisfied :

$$0 < h < \frac{1}{\beta_{max}\Delta},$$

where $\Delta = max\{\sum_{s\in\mathcal{N}_i, s\neq j} a_{is} + \sum_{s\in\mathcal{N}_j, s\neq i} a_{js}\}$, $\beta_{max} = \max|\beta_{ij}|$, where β_{ij} be a nonzero scalar scale of edge (i,j).

Lemma 6. *Let $\mathcal{L}$ denote the Laplacian matrix of a communication network $\mathcal{G}$ comprising N nodes and M edges. We define $\beta_{max} = \max|\beta_{ij}|$, where β_{ij} represents a nonzero scalar associated with the edge (i,j). Assuming that the step size h adheres to condition (A1) and that $|\mathcal{B}| = diag(|\beta_{ij}|)$ with $(\beta_{ij}) \in \mathbb{R}^M$, it follows that the matrix $\mathbf{I}_M + h|B|Q$ is strictly irreducible and aperiodic (SIA), where $-Q$ is the Laplacian matrix of the line graph $L(\mathcal{G})$. Specifically, the limit $\lim_{k\to\infty}[\mathbf{I}_M + h|\mathcal{B}|Q]^k = \mathbf{1}_M y^T$ holds if and only if the communication network $\mathcal{G}$ possesses a spanning tree. Moreover, it can be established that $[\mathbf{I}_M + h|\mathcal{B}|Q]^T y = y$ and $1_M^T y = 1$, where each component of the vector y is nonnegative.*

Proof. **(Sufficiency)** Since $0 < h < (\Delta\beta_{max})^{-1}$ and using the fact that $-Q = \mathcal{L}$, one obtains $\mathbf{I}_M + h|\mathcal{B}|Q = \mathbf{I}_M - h|\mathcal{B}|\mathcal{L} = (\mathbf{I}_M - h|\mathcal{B}|\mathcal{D}) + h|\mathcal{B}|\mathcal{A}$ is a stochastic matrix with positive diagonal entries, where $\mathcal{D} = diag(d_1, \ldots, d_M)$ and $\mathcal{A}$ are the degree matrix and adjacency matrix of $\mathcal{G}$, respectively. Obviously, for all $i, j \in \mathcal{I}_M$; $i \neq j$, the (i, j)th entry of $\mathbf{I}_M - h|\mathcal{B}|\mathcal{L}$ is positive if and only if $a_{ij} > 0$. Then, $\mathcal{G}$ is the graph associated with $\mathbf{I}_M - h|\mathcal{B}|\mathcal{L}$. Combining Lemma 4 and Lemma 5, gives $\lim_{k\to\infty} [\mathbf{I}_M - h|\mathcal{B}|\mathcal{L}]^k = \lim_{k\to\infty} [\mathbf{I}_M + h|\mathcal{B}|Q]^k = \mathbf{1}_M y^T$, when $\mathcal{G}$ has a spanning tree, where y is nonnegative vector. Moreover, y satisfies $[\mathbf{I}_M + h|\mathcal{B}|Q]^T y = y$, $\mathbf{1}_M^T y = 1$.
(Necessary) From Lemma 4, if $\mathcal{G}$ does not contain a spanning tree, the algebraic multiplicity of eigenvalue $\lambda = 1$ of $\mathbf{I}_M - h|\mathcal{B}|\mathcal{L}$ is $m > 1$. Then, the rank of $\lim_{k\to\infty} [\mathbf{I}_M - h|\mathcal{B}|\mathcal{L}]^k$ is not equal to 1, which is not equal to the rank of $\mathbf{1}_n y^T$. This implies that

$$\lim_{k\to\infty} [\mathbf{I}_M + h|\mathcal{B}|Q]^k = \lim_{k\to\infty} [\mathbf{I}_M - h|\mathcal{B}|\mathcal{L}]^k \neq \mathbf{1}_M y^T.$$

Therefore, $I_M + h|\mathcal{B}|Q$ is not SIA. This completes the proof.

In the following, we state the main theoretical results to solve scaled consensus of edge dynamics using the consensus protocol (3.1).

Theorem 1. *Consider an undirected network $\mathcal{G}$ with N nodes and M edges, where the edge dynamics described as (2.1). If the step size h satisfies **(A1)**, then, for any given initial states of all edges, the protocol (3.1) can asymptotically solve the edge consensus if and only if $\mathcal{G}$ is connected.*

Proof. We first show that equation (2.2) holds. From (2.1) and (3.1) we have, for $t \in (t_k, t_{k+1}]$,

$$\begin{cases} \beta_{ij}x_{ij}(t) = & \beta_{ij}x_{ij}(t_k) + (t - t_k)|\beta_{ij}| \Big[\sum_{s\in\mathcal{N}_i} a_{is}[\beta_{is}x_{is}(t_k) - \beta_{ij}x_{ij}(t_k)] \\ & \quad + \sum_{s\in\mathcal{N}_j} a_{js}[x_{js}(t_k) - x_{ij}(t_k)] \Big], \quad \text{for } (i,j) \in \mathcal{E}_c, \\ \beta_{ij}x_{ij}(t_{k+1}) = & \beta_{ij}x_{ij}(t_k) + h \cdot |\beta_{ij}| \Big[\sum_{s\in\mathcal{N}_i} a_{is}[\beta_{is}x_{is}(t_k) - \beta_{ij}x_{ij}(t_k)] \\ & \quad + \sum_{s\in\mathcal{N}_j} a_{js}[\beta_{js}x_{js}(t_k) - \beta_{ij}x_{ij}(t_k)] \Big], \quad \text{for } (i,j) \in \mathcal{E}_d. \end{cases} \tag{3.3}$$

Therefore, it follows that

$$\begin{aligned} \beta_{ij}x_{ij}(t_{k+1}) = \beta_{ij}x_{ij}(t_k) + h \cdot |\beta_{ij}| \Bigg[& \sum_{s\in\mathcal{N}_i} a_{is}[\beta_{is}x_{is}(t_k) - \beta_{ij}x_{ij}(t_k)] \\ & + \sum_{s\in\mathcal{N}_j} a_{js}[\beta_{js}x_{js}(t_k) - \beta_{ij}x_{ij}(t_k)] \Bigg], \quad \text{for } (i,j) \in \mathcal{E}_d. \end{aligned} \tag{3.4}$$

Let

$$Y = (\beta_{ij}x_{ij}) \in \mathbb{R}^M, \ i = 1, 2, \ldots, N, \quad i < j \quad \text{and} \quad \beta_{ij} \neq 0.$$

Then, equation (3.4) can be written as

$$Y(t_{k+1}) = [\mathbf{I}_M + h|\mathcal{B}|Q]Y(t_k), \tag{3.5}$$

where h is a step size, I_M is an identity matrix, $|\mathcal{B}| = diag(|\beta_{ij}|)$ and Q is a zero-row-sum symmetric matrix with nonnegative off-diagonal elements and the diagonal elements are $-\{\sum_{s\in\mathcal{N}_i, s\neq j} a_{is} + \sum_{s\in\mathcal{N}_j, s\neq i} a_{js}\}$. Since the step size h satisfies **(A1)** once obtains that $[\mathbf{I}_M + h\mathcal{B}]$ is a doubly stochastic matrix. Moreover, since $\mathcal{G}$ is an undirected and connected, which implies that $\mathcal{G}$ contains a spanning tree. Hence, by lemma 6, there exists a column vector v such that $\lim_{k\to\infty}[\mathbf{I}_M + h|\mathcal{B}|Q]^k = \mathbf{1}_M v^T$. Thus,

$$\lim_{k\to\infty} Y(t_k) = \lim_{k\to\infty}[\mathbf{I}_M + h|\mathcal{B}|Q]^k Y(t_0) = \mathbf{1}_M v^T Y(t_0), \tag{3.6}$$

which implies that the equation (2.2) holds. Now, we will show that

$$\lim_{t\to\infty} \|\beta_{ij}x_{ij}(t) - \beta_{ks}x_{ks}(t)\| = 0, \quad \text{for all} \ \ (i,j),(k,s) \in \mathcal{E}_c.$$

Consider, for $(i,j),(k,s) \in \mathcal{E}_c$,

$$\begin{aligned}\|\beta_{ij}x_{ij}(t) - \beta_{ks}x_{ks}(t)\| \leq \|\beta_{ij}x_{ij}(t) - \beta_{ij}x_{ij}(t_k)\| + \|\beta_{ij}x_{ij}(t_k) - \beta_{ks}x_{ks}(t_k)\| \\ + \|\beta_{ks}x_{ks}(t_k) - \beta_{ks}x_{ks}(t)\|.\end{aligned} \tag{3.7}$$

From equation (3.3), one obtains, for $t \in (t_k, t_{k+1}]$,

$$\begin{aligned}\|\beta_{ij}x_{ij}(t) - \beta_{ij}x_{ij}(t_k)\| \leq h \cdot |\beta_{ij}| \Bigg[\sum_{s\in\mathcal{N}_i} a_{is}\|\beta_{is}x_{is}(t_k) - \beta_{ij}x_{ij}(t_k)\| \\ + \sum_{s\in\mathcal{N}_j} a_{js}\|\beta_{js}x_{js}(t_k) - \beta_{ij}x_{ij}(t_k)\| \Bigg].\end{aligned}$$

As $t \to \infty$, we have $t_k \to \infty$. Thus,

$$\lim_{t\to\infty} \|\beta_{ij}x_{ij}(t) - \beta_{ij}x_{ij}(t_k)\| = 0 \qquad \forall \ \ (i,j) \in \mathcal{E}_c.$$

Taking the limit as $t \to \infty$ on both sides of equation (3.7), one obtains

$$\lim_{t\to\infty} \|\beta_{ij}x_{ij}(t) - \beta_{ks}x_{ks}(t)\| = 0, \qquad \forall (i,j),(k,s) \in \mathcal{E}_c.$$

This implies that equation (2.3) holds. Therefore, the system (2.1) under protocol (3.1) reaches edge consensus.

3.2. Directed communication networks

To address the edge consensus problems, we employ a transformation of the original node graph into its corresponding line graph, utilizing the established concept of line graphs [5]. According to Reference [5], the line graph $L(G)$ of a directed graph G is itself a directed graph characterized by $\sum_{i=1}^{N} d_{in}^{i}$ vertices and $\sum_{i=1}^{N} d_{in}^{i} d_{out}^{i}$ edges, where d_{in}^{i} and d_{out}^{i} denote the in-degree and out-degree of node i, respectively. The transformation process from a directed graph to its line graph is delineated in three distinct steps [19]:

Step 1: An edge (i, j) in the directed graph is interpreted as originating from vertex j to vertex i.

Step 2: Each directed edge (i, j) is converted into a vertex referred to as (i, j), termed a generated node.

Step 3: If the directed edge (i_1, j_i) is a valid neighbor of the directed edge (i_2, j_2) in the original node graph, a new directed edge is established between the generated nodes (i_1, j_i) and (i_2, j_2), with (i_1, j_i) serving as the initial node and (i_2, j_2) as the terminal node (refer to examples in Figures 1-3).

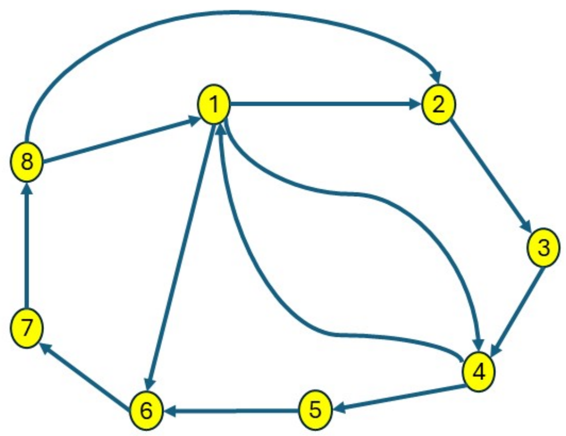

Figure 1: A communication network $\mathcal{G}$

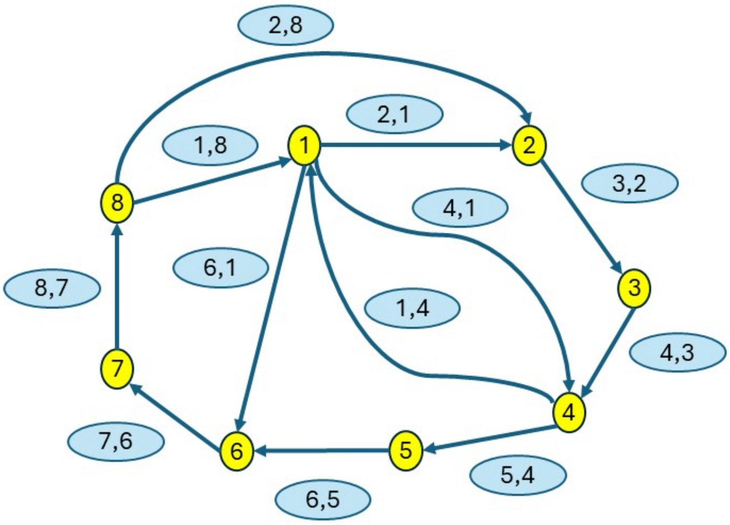

Figure 2: The evolution of the directed graph to its line graph (Step 2)

Assume that all agents can communicate and update their neighbors at the sampling time t_k, then the edge consensus protocol for directed topology can be designed as

$$\begin{cases} u_{ij}(t) &= |\beta_{ij}|\left[\sum_{s\in\mathcal{N}_j} a_{js}[\beta_{is}x_{is}(t_k) - \beta_{ij}x_{ij}(t_k)]\right], \quad \forall (i,j) \in \mathcal{E}_c, \\ u_{ij}(t_{k+1}) &= h \cdot |\beta_{ij}|\left[\sum_{s\in\mathcal{N}_j} a_{js}[\beta_{is}x_{is}(t_k) - \beta_{ij}x_{ij}(t_k)]\right], \quad \forall (i,j) \in \mathcal{E}_d, \end{cases} \tag{3.8}$$

where all variables are defined as in the previous section.

Remark 3. *If $\beta_{ij} = 1$ for all edge (i,j) and $\mathcal{E}_c = \emptyset$, the protocol (3.8) can be written as*

$$u_{ij}(t_{k+1}) = h \cdot \left[\sum_{s\in\mathcal{N}_j} a_{js}[x_{is}(t_k) - x_{ij}(t_k)]\right], \quad \forall (i,j) \in \mathcal{E}_d, \tag{3.9}$$

and if $\mathcal{E}_d = \emptyset$ together with $\beta_{ij} = 1$ for all edge (i,j), one obtains

$$u_{ij}(t) = \left[\sum_{s\in\mathcal{N}_j} a_{js}[x_{is}(t_k) - x_{ij}(t_k)]\right], \quad \forall (i,j) \in \mathcal{E}_c, \tag{3.10}$$

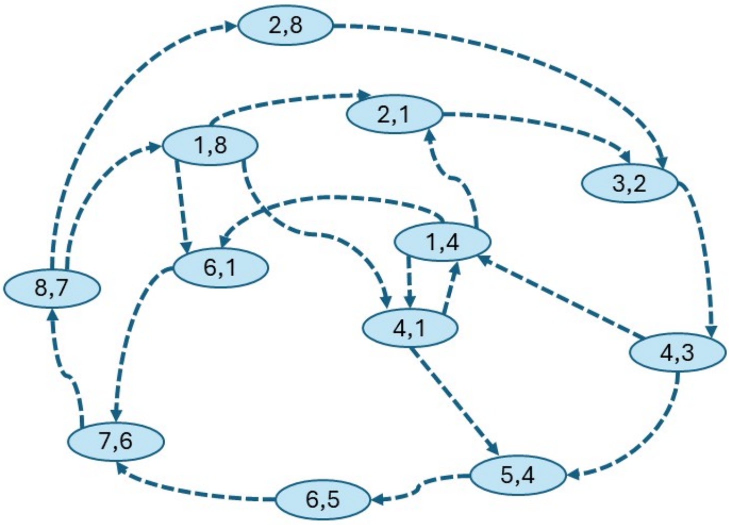

Figure 3: The evolution of the directed graph to its line graph (Step 3)

which were studied in [19]. This shows the generalization of our protocols.

Theorem 2. *Consider a directed communication network $\mathcal{G}$, where the edge dynamics described as in (2.1). Then, the consensus protocol (3.8) solves scaled consensus of edge dynamics if and only if, for any initial state, the following conditions are satisfied:*

$(A1)$ *The step size h is satisfies*

$$0 < h < \frac{1}{max_{i,j}\{\sum_{j\in\mathcal{N}_j, s\neq i} a_{js}\}\beta_{max}}, \tag{3.11}$$

$(A2)$ *the communication network $\mathcal{G}$ contains a spanning tree.*

Proof. From the system (2.1) and protocol (3.8), we have, for $t \in (t_k, t_{k+1}]$,

$$\begin{cases} \beta_{ij}x_{ij}(t) = & \beta_{ij}x_{ij}(t_k) + (t-t_k)|\beta_{ij}|\Bigg[\sum_{s\in\mathcal{N}_j} a_{js}[\beta_{is}x_{is}(t_k) - \beta_{ij}x_{ij}(t_k)]\Bigg], \quad \text{for } (i,j) \in \mathcal{E}_c, \\ \beta_{ij}x_{ij}(t_{k+1}) = & \beta_{ij}x_{ij}(t_k) + h\cdot|\beta_{ij}|\Bigg[\sum_{s\in\mathcal{N}_j} a_{js}[\beta_{is}x_{is}(t_k) - \beta_{ij}x_{ij}(t_k)]\Bigg], \quad \text{for } (i,j) \in \mathcal{E}_d. \end{cases} \tag{3.12}$$

Therefore, it follows that

$$\beta_{ij}x_{ij}(t_{k+1}) = \beta_{ij}x_{ij}(t_k) + h \cdot |\beta_{ij}| \cdot \Bigg[\sum_{s\in\mathcal{N}_i} a_{is}[\beta_{is}x_{is}(t_k) - \beta_{ij}x_{ij}(t_k)] + \sum_{s\in\mathcal{N}_j} a_{js}[\beta_{js}x_{js}(t_k) - \beta_{ij}x_{ij}(t_k)] \Bigg], \quad \text{for} \quad (i,j) \in \mathcal{E}. \tag{3.13}$$

Let

$$Y = (\beta_{ij}x_{ij}) \in \mathbb{R}^{(\sum_{i=1}^{N} d(in)_i)\times 1}, \quad i = 1,2,\dots,N, j \in \mathcal{N}_i, \beta_{ij} \neq 0.$$

Then, the system (2.1) with protocol (3.8) can be written as:

$$Y(t_{k+1}) = [I + h|\mathcal{B}|Q]Y(t_k), \tag{3.14}$$

where $I \in \mathbb{R}^{(\sum_{i=1}^{N} d(in)_i)\times(\sum_{i=1}^{N} d(in)_i)}$, h is a step size, $\mathcal{B} = diag(\beta_{ij})$ and Q is a zero-row-sum symmetric matrix with nonnegative off-diagonal elements and diagonal elements $-\sum_{s\in\mathcal{N}_j, s\neq i} a_{js}\}$. It follows from $(A2)$ that $\mathcal{G}$ contains a spanning tree and the step size h is satisfied $(A1)$. Then, by Lemma 6, $[I + h|\mathcal{B}|Q]$ is SIA. Hence, there exists a column vector u such that $\lim_{k\to\infty} [\mathbf{I}_M + h|\mathcal{B}|Q]^k = \mathbf{1}_M u^T$. Thus,

$$\lim_{k\to\infty} Y(t_k) = \lim_{k\to\infty} [\mathbf{I}_M + h|\mathcal{B}|Q]^k Y(t_0) = \mathbf{1}_M u^T Y(t_0), \tag{3.15}$$

which implies that (2.2) holds. On the other hand, the proof of (2.3) can be cut since the proof is similar to the proof of Theorem 1.

4. Simulations and discussion

In order to demonstrate the effectiveness of theoretical results in this work, the following example is provided.

Example 1. *Assume that there are* 8 *agents denoted by* $1-8$*, where the directed network of communications* $\mathcal{G}$ *is given as in Figure 1. It is easy to see that the network* $\mathcal{G}$ *consists of* 8 *nodes and* 12 *edges.*

In order to solve scaled consensus, we first transform the graph $\mathcal{G}$ to its line graph, denoted by $\mathbb{L}(\mathcal{G})$, as shown in Figure 3. According to the communication network $\mathcal{G}$, the adjacency matrix of $\mathcal{G}$ is denoted by $\mathcal{A}$ and the Laplacian matrix of its line graph is $-Q$, where

$$\mathcal{A} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

and

$$Q = \begin{bmatrix} -2 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & -2 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}.$$

It can be seen that the line graph $\mathbb{L}(\mathcal{G})$ has 12 nodes and 19 edges as shown in Figure 3. Let $\mathcal{B} = (\beta_{ij}) = (0.2, -0.2, 1, -1, 1.5, -1.5, 0.5, -0.5, 1.1, -1.1, 0.7, -0.7)^T \in \mathbb{R}^{12}$ and the initial values are $(x_{ij}(t_0)) = (1, -1, 0.5, -0.5, 1.5, -1.5, -2, 2, 3, -3, 4, -4)^T \in \mathbb{R}^{12}$. Then, by choosing $h = 0.125$ such that

$$0 < h < \frac{1}{max_{i,j}\{\sum_{j\in\mathcal{N}_j, s\neq i} a_{js}\}\beta_{max}} = \frac{1}{3}.$$

Obviously, $\mathbb{L}(\mathcal{G})$ contains a spanning tree, and hence, by Theorem 2, the scaled edge consensus problems are solved (see Figure 4). However, if h does not satisfy (3.11), our protocol cannot guarantee reaching scaled edge consensus as shown in Figure 4 for $h = 0.7$.

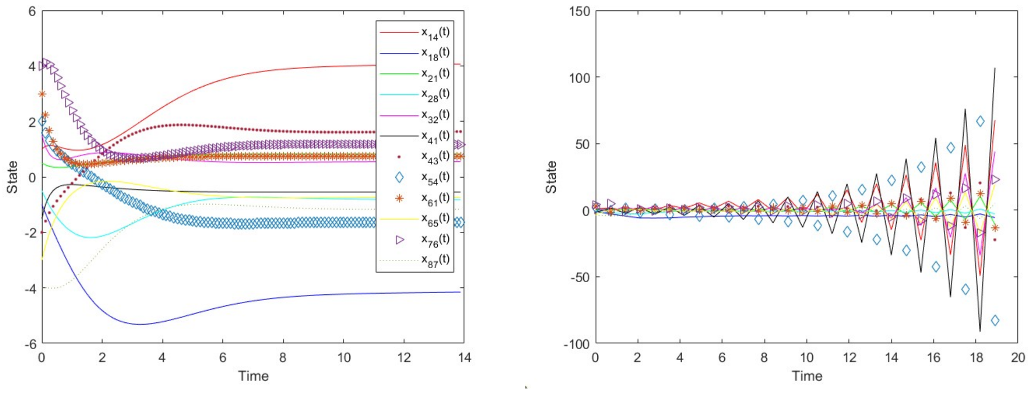

Figure 4: The state trajectory of each edge using protocol(3.8) with $h = 0.125$ and $h = 0.7$, respectively.

In addition, when $(\beta_{ij}) = (1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)^T$ the scaled consensus problems are the usual consensus problems(see Figure 5)

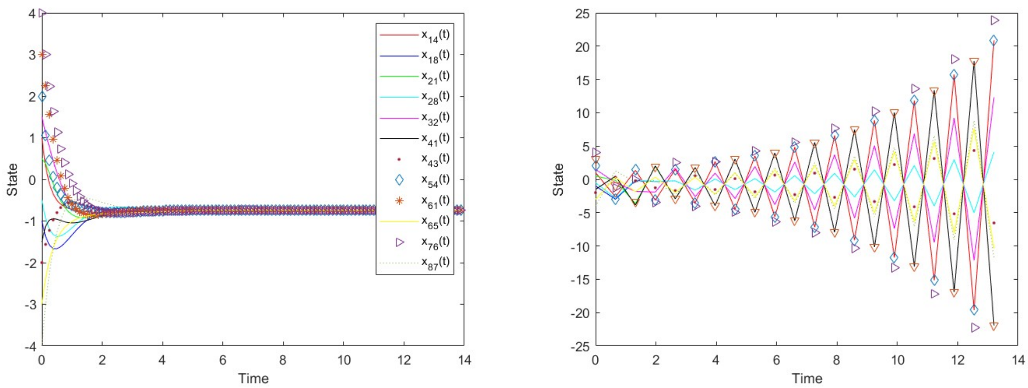

Figure 5: The state of each edge under protocol (3.8) with $(\beta_{ij}) = (1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)^T$, $h = 0.125$ and $h = 0.7$, respectively

It can be seen that for undirected network, the protocol (3.1) solves scaled consensus problems if the communication network is connected and the step size is small enough. On the other hand, for directed networks, our protocol (3.8) proves scaled consensus problems when the the communication network contains a spanning tree under the small step size see Figure 4 for the scalar scale $\beta_{ij} \neq 1$. Moreover, if the scalar scale $\beta_{ij} = 1$, the simulations results show the effectiveness and generalization of our theorems compared with the results of [18, 19] (see Figure 5).

5. Conclusion

This research investigates the scaled consensus problems of edge dynamics in hybrid multi-agent systems (HMASs), focusing on both directed and undirected topologies. By shifting the emphasis from node dynamics to edge interactions, the study derives necessary and sufficient conditions for achieving edge consensus and proposes effective protocols tailored to specific network structures. The findings demonstrate that these protocols can successfully resolve edge consensus challenges, provided that the network maintains appropriate connectivity and structure. Numerical simulations validate the robustness and effectiveness of the proposed methodologies, underscoring their relevance in practical applications such as network communication, distributed control systems, and social networks. This work contributes to a deeper understanding of cooperative behavior in complex networks, enhancing the potential for improved coordination and decision-making among agents. In future work, the proposed scheme will be taken into account for time delay with disturbances.

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