EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 1, Article Number 5555
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Edge Geodetic Dominating Sets of Some Graphs

Clint Joy M. Quije[1,2], Rochelleo E. Mariano[3], Eman C. Ahmad[3,*]

[1] *Mathematics Department, College of Science and Information Technology, Ateneo De Zamboanga University, Zamboanga City, Philippines*
[2] *Institute of Arts and Sciences, Tangub City Global College, Tangub City, Philippines*
[3] *Department of Mathematics and Statistics, College of Science and Mathematics, Western Mindanao State University, Zamboanga City, Philippines*

Abstract. Let G be a simple graph. A subset D of vertices in G is a dominating set of G if every vertex not in D has at least one neighbor in D. The domination number $\gamma(G)$ of G is the minimum cardinality of a dominating set of G. An edge geodetic set of G is a set $S \subseteq V(G)$ such that every edge of G is contained in a geodetic joining some pair of vertices in S. The edge geodetic number $g_e(G)$ of G is the minimum cardinality of edge geodetic set. A set of vertices S in G is an edge geodetic dominating set of G if S is both an edge geodetic set and a dominating set. The minimum cardinality of an edge geodetic dominating set of G is its edge geodetic domination number and is denoted by $\gamma_{g_e}(G)$. In this study, we determined the edge geodetic domination number of graphs obtained through the deletion of independent edges of complete graphs and graphs resulting from the K_r-gluing of complete graphs. It is also shown that for any positive integers $2 \leq a \leq b$, there exists a connected graph G such that $g_e(G) = a$ and $\gamma_{g_e}(G) = b$.

2020 Mathematics Subject Classifications: 05C69, 05C76

Key Words and Phrases: Dominating Set, Domination Number, Edge geodetic, Complete Graphs, Deletion of Independent Edges, K_r-gluing, Realization Result or Result of Comprehension

1. Introduction

Several studies have been conducted regarding geodetic sets, geodetic bounds and edge geodetic sets in graphs. Asdain et al. [1], Chartrand et al. [3], Mariano and Canoy [8], and Santhakumaran and John [12], are some researchers who have done many results in this area. The results include determining the geodetic and edge geodetic number of graphs employing the unary and binary operations in graphs such as, the deletion of independent edges of complete graphs, K_r-gluing, join, corona, composition, and cartesian products of graphs, among others.

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v18i1.5555

Email addresses: `cjquije@gadtc.edu.ph` (C.J. Quije), `mariano.rochelleo@wmsu.edu.ph` (R. Mariano), `ahmad.eman@wmsu.edu.ph` (E. Ahmad)

Besides geodetic set and edge geodetic sets in graphs, there are also a lot of studies done due to dominating sets in graphs. In [13], it was defined that a subset D of vertices in G is called *dominating set* if every vertex not in D has at least one neighbor in D. The domination number $\gamma(G)$ of G is the minimum cardinality of a dominating set of G. The notion of domination for some classes of graphs were studied by Castaneda et. al in [2]. They considered variations of the concept of domination in a graph.

With the extensive results on the above-mentioned graph invariants, a number of researchers have ventured into making an offshoot of the study. As a consequence, the concept of geodetic domination in graphs has been coined. Apparently, this has been extended to the thought of the edge geodetic domination in graphs. Due to the former, there have been quite a number of results that were generated from the geodetic domination in graphs. Escuadra et al. [4], Vijayan et al. [15] are among those researchers who have contributed significant results in this endeavor. Only very few, however, have undergone studies on the edge geodetic domination in graphs. Stalin et al. [13] and Samodivkin [9], made some results on the edge geodetic domination in graphs. There were results with some realizations done for the edge geodetic dominating number of graphs. A number of these results were preliminaries that include edge geodetic dominating number of (special) graph in unary operation. Note, however, that the edge geodetic dominating number in the binary operation of graphs is not that extensive yet. Though there were attempts to explore on this area but the amount of results done is still meager. To cite a few, Stalin et al. [13] in their article, "Edge Geodetic Dominations in Graphs", determined the Edge geodetic number of certain classes of graphs. Necessary conditions for connected graphs of order p with edge geodetic domination number p or p-1 are given. They also had shown that for every two integers $a, b \geq 2$ with $2 \leq a < b$ and $b - a - 1 > 0$, there is a connected graph G such that $\gamma_g(G) = a$ and $\gamma_{g1}(G) = b$, where $\gamma_g(G)$ is the geodetic domination number of a graph.

Since limited results are done on the edge geodetic dominating number of some graphs, it is in this regard that a revisit to the foregoing topic was undertaken and some results were generated.

In doing this study, the following terminologies from [1–15] are necessary. A *simple graph* $G(V, E)$ is an undirected graph without loops or multiple edges. $V(G)$ and $E(G)$ denote the vertex and the edge sets of a graph G, respectively. Let $S \subseteq V(G)$. The *induced subgraph* $\langle S \rangle$ of G is the graph $\langle S \rangle$ with vertex set S and edge set $\{xy \in E(G) | x, y \in S\}$. The *maximum degree* of G, denoted by $\Delta(G)$ is given by $\Delta(G) = max\{deg_G(v) : v \in V(G)\}$. The *open neighborhood* of the vertex v in a graph G is the set $N(v) = \{u \in V(G) : uv \in E(G)\}$. The *closed neighborhood* of v is $N[v] = N(v) \cup \{v\}$. A subset D of vertices in G is called *dominating set* if every vertex not in D has at least one neighbor in D. The *domination number* $\gamma(G)$ of G is the minimum cardinality of a dominating set of G. A *geodetic set* of G is a set $S \subseteq V(G)$ such that every vertex of G is contained in a geodesic joining some pair of vertices in S. The geodetic number $g(G)$ of G is the *minimum order* of its geodetic sets and any geodetic set of order $g(G)$ is a *geodetic basis.* An *edge geodetic set* of G is a set $S \subseteq V(G)$ such that every edge of G is contained in a geodesic joining some pair of vertices in S. The *edge geodetic number* $g_e(G)$ of G is the minimum order of its edge geodetic sets and any edge geodetic set of order $g_e(G)$ is an edge geodetic basis of G or a g_e-set of G. A set of vertices S in G is called a *geodetic dominating set* of G if S is both geodetic set and a dominating set. The minimum cardinality of a geodetic dominating set of G is its *geodetic domination number* and is denoted by $\gamma_g(G)$. An geodetic dominating set of size $\gamma_g(G)$ is said to be a γ_g-set. Let G be a connected graph. A set of vertices S in G is called an *edge geodetic dominating set* of G if S is both edge geodetic set and a dominating set. The minimum cardinality of an edge geodetic dominating set of G is its edge *geodetic domination number* and is denoted by $\gamma_{g_e}(G)$. An edge geodetic dominating set of size $\gamma_{g_e}(G)$ is said to be a γ_{g_e}-set. A vertex v is an extreme vertex of a graph G if the subgraph induced by its neighbors is complete. The set of all extreme vertices of G is denoted by $Ext(G)$. A vertex v in a connected graph G is said to be a *semi-extreme vertex* if it has a neighbor, say u, with $N[v] \subseteq N[u]$. The set of all semi-extreme vertices of G is denoted by $Se(G)$. A *cutvertex* of a connected graph G is the vertex whose deletion increases the number of components of the subgraph of G, that is, u is a *cutvertex* if and only if $\langle G - u\rangle$ is disconnected. The *deletion* or *removal* of a proper subset S of vertices of G results in that subgraph $G \setminus S$ of G consisting of all vertices of G not in S and all edges not incident with a vertex in S. The *deletion* of a vertex u of G results in that subgraph $G \setminus u$ of G consisting of all vertices of G except u and all edges not incident with a vertex u. On the other hand, the *deletion* of a subset X of edges yields the spanning subgraph $G \setminus X$ containing all edges of G not in X. Let $G_1, G_2, \dots, G_t$ be disjoint graphs each containing a complete subgraph K_r ($r \geq 1$). Let G be the graph obtained from the union of t graphs G_i by identifying the K_r's (one from each G_i) in an arbitrary way. We call G a K_r-gluing of $G_1, G_2, \dots, G_t$. In particular, when $r = 1$ (respectively $r = 2$) we say G is a *vertex-gluing* (respectively an *edge-gluing*) of $G_1, G_2, \dots, G_t$.

2. Preliminaries

The following are the known results related to this study.

Remark 1. *[8] Let G be a nontrivial connected graph. Then $V(G)$ is an edge geodetic dominating set of G.*

Theorem 1. *[13] If G has at least two vertices of degree $n-1$, then $\gamma_{g_e}(G) = n$*

Theorem 2. *[13] For the complete graph K_n with $n \geq 2$, $\gamma_{g_e}(K_n) = n$.*

Theorem 3. *[12] Each extreme vertex of G belongs to every edge geodetic cover of G. In particular, each end vertex of G belongs to every edge geodetic cover of G*

Theorem 4. *[13] If G has exactly one vertex of degree $n-1$, then $\gamma_{g_e}(G) = n-1$.*

3. Main Results

3.1. Deletion of Independent Edges of Complete Graphs

Theorem 5. *Let G be a complete graph of order $n \geq 4$. Let $S \subseteq V(H)$, where H is a connected subgraph of G obtained by deleting $m \leq \lfloor \frac{n}{2} \rfloor$ independent edges in G.*

i. If $m < \lfloor \frac{n}{2} \rfloor$, then S is an edge geodetic dominating set of H if and only if $S = V(H)$.

ii. If $m = \lfloor \frac{n}{2} \rfloor$ and n is odd, then S is an edge geodetic dominating set of H if and only if $S = V(H) \setminus \{v\}$ or $S = V(H)$, v is the vertex of degree $n-1$ in H.

iii. If $m = \lfloor \frac{n}{2} \rfloor$ and n is even, then S is an edge geodetic dominating set of H if and only if $S = V(H) \setminus \{u, v\}$ or $S = V(H) \setminus \{v\}$ or $S = V(H) \setminus \{u\}$ or $S = V(H)$, where vertices u and v are not adjacent in H.

Proof. Let G be a complete graph of order $n \geq 4$. Suppose H is a connected subgraph of G obtained by deleting m independent edges in G. If $1 \leq m < \lfloor \frac{n}{2} \rfloor$, then the subgraph H contains two or more vertices v with $\Delta(\langle N(v) \rangle) = |N(v)| - 1$ for all $v \in V(H)$. By Theorem 1 and Remark 1, S is an edge geodetic dominating set of H if and only if $S = V(H)$. This proves (i).

If $m = \lfloor \frac{n}{2} \rfloor$ and n is odd, then $m = \lfloor \frac{n}{2} \rfloor = \frac{n-1}{2}$. Hence, the subgraph H contains a unique vertex $v \in V(H)$ such that $deg_G(v) = n - 1$. Thus, by Remark 1, Theorem 1 and Theorem 3, S is an edge geodetic set of H and consequently, an edge geodetic dominating set of H if and only if $S = V(H) \setminus \{v\}$ or $S = V(H)$, where v is the vertex of degree $n-1$ in H. This proves (ii).

Suppose $m = \lfloor \frac{n}{2} \rfloor$ and n is even. Consider a pair (u, v) of vertices in H such that u is not adjacent to v in H. Then we claim that $S = V(H) \setminus \{u, v\}$ is a minimum edge geodetic dominating set. To prove this claim, let $x, y \in V(H)$ such that $xy \in E(H)$ and consider the following cases:

Case 1. Suppose $x, y \in S$. Then xy is contained in the $x - y$ geodesic.

Case 2. Suppose $x = u$ or $x = v$ and $y \in S$. Without loss of generality, assume $x = u$. Pick $z \in V(H) \setminus \{y\}$ such that $zy \notin E(H)$. Then $uz \in E(H)$. Therefore, $[z, u, y]$ is a $z - y$ geodesic containing $uy \in E(H)$. Hence, $S = V(H) \setminus \{u, v\}$, where $uv \notin E(H)$, is an edge geodetic set. In addition, $y \in S$ is adjacent to u and v, hence, S is an edge geodetic dominating set.

Next, let $D \subseteq V(H)$ and let $T = V(H) \setminus D$, where $|D| \geq 3$ or $\langle D \rangle$ contains K_2. By definition of independent edges, there exists $u, v \in D$ such that $uv \in E(H)$. Then there exist a unique a and a unique b in S such that $au, vb \notin E(H)$. Since $ux \in E(H)$ for all $x \in V(H) \setminus \{a\}$ and $vz \in E(H)$ for all $z \in V(H) \setminus \{b\}$, it follows that $[u, v], [a, v, u]$ and $[b, u, v]$ are the only geodesics containing uv. Thus, T is not an edge geodetic set of H. Therefore, S is an edge geodetic dominating set of H if and only if $S = V(H) \setminus \{u, v\}$ or $S = V(H) \setminus \{v\}$ or $S = V(H)$, where vertices u and v are not adjacent in H. This proves (iii). $\square$

Corollary 1. *Let G be a complete graph of order n. If H is a connected subgraph of G obtained by deleting m independent edges in G, then*

$$\gamma_{g_e}(H) = \begin{cases} n & \text{if} \quad m < \lfloor \frac{n}{2} \rfloor \\ n-1 & \text{if} \quad m = \lfloor \frac{n}{2} \rfloor \quad \text{and } n \text{ is odd} \\ n-2 & \text{if} \quad m = \lfloor \frac{n}{2} \rfloor \quad \text{and } n \text{ is even} \end{cases}$$

3.2. K_r-gluing of Complete Graphs

Theorem 6. *Let p, q, r be positive integers such that $1 \le r \le p \le q$. Let G be the K_r-gluing of K_p and K_q and $S \subseteq V(G)$.*

i If $1 = r < p \le q$, then $S = V(G) \setminus V(K_r)$ is the edge geodetic dominating basis of G.

ii If $1 < r < p \le q$, then $V(G)$ is the edge geodetic dominating basis of G.

iii If $1 < r = p \le q$, then $V(G)$ is the edge geodetic dominating basis of G.

Proof.

i Since $V(K_r) = \{v\}$, then all the elements of $K_p \setminus \{v\}$ and $K_q \setminus \{v\}$ are neighbors of v in G. This implies that v is the only vertex in G with $deg_G(v) = |V(G)| - 1$. Thus, by Theorem 4, $\gamma_{g_e}(G) = |V(G)| - 1$ with $V(G) \setminus \{V(K_r)\}$ is the unique edge geodetic dominating basis of G.

ii If $1 < r < p \le q$, then every vertex $v \in V(K_r)$ has degree $|V(G)| - 1$. Since $|V(K_r)| = r > 1$, then G contains more than one vertex of degree $|V(G)| - 1$. Hence, by Theorem 1, $\gamma_{g_e}(G) = |V(G)|$ and $V(G)$ is the edge geodetic dominating basis of G.

iii If $1 < r = p \le q$, then $G = K_q$. Since K_q is a complete graph, by Theorem 2, $\gamma_{g_e}(G) = |V(G)|$ with $V(G)$ as the edge geodetic dominating basis of G. □

Corollary 2. *Let p, q, r be positive integers such that $1 \le r \le p \le q$. Let G be the K_r-gluing of K_p and K_q and $S \subseteq V(G)$, then*

$$\gamma_{g_e}(G) = \begin{cases} |V(G)| - 1 & \text{if} \quad 1 = r < p \le q \\ |V(G)| & \text{if} \quad 1 < r < p \le q \\ |V(G)| & \text{if} \quad 1 < r = p \le q \end{cases}$$

3.3. Result of Comprehension

Theorem 7. *For any positive integers $2 \le a \le b$, there exists a connected graph G such that $g_e(G) = a$ and $\gamma_{g_e}(G) = b$.*

Proof. If $a = b$, consider the complete graph $G = K_a$. Then by Theorem 2, $\gamma_{g_e}(G) = a$. If $2 = a < b$ with $g_e(G) = 2$ and $\gamma_{g_e}(G) = 3$, consider the path P: $u_1, u_2, \ldots, u_7$. For $b \geq 4$, let G be the graph obtained from the path on seven vertices P: $u_1, u_2, \ldots, u_7$ by adding $(b-3)$ sets of three new vertices v_{ij}, namely, $\{v_{11}, v_{21}, v_{31}\}, \{v_{12}, v_{22}, v_{32}\}, \ldots, \{v_{1(b-3)}, v_{2(b-3)}, v_{3(b-3)}\}$ for $i = 1, 2, 3$ and $1 \leq j \leq b-3$, with the paths $P_1 : u_2, v_{11}, v_{21}, v_{31}, u_6, P_2 : u_2, v_{12}, v_{22}, v_{32}, u_6, \ldots, P_{(b-3)} : u_2, v_{1(b-3)}, v_{2(b-3)}, v_{3(b-3)}, u_6$ such that v_{ik} is not adjacent to v_{ij}, where $j \neq k$ and $j, k = 1, 2, 3, \ldots, b-3$. The graph is shown in Figure 1.

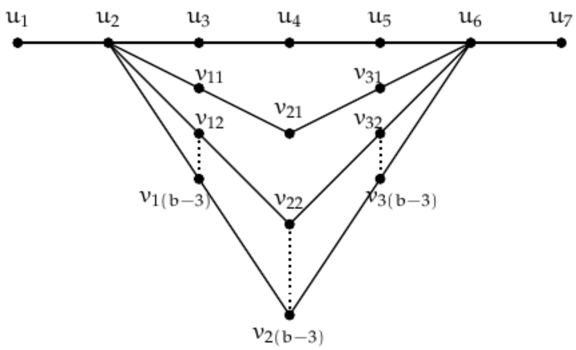

Figure 1: A graph G

From the figure, $\{u_1, u_7\}$ is the edge geodetic basis of G, so that $g_e(G) = 2$.

Moreover, since $\{u_1, u_7\}$ is an edge geodetic basis, we need to include this to any edge geodetic dominating basis of G. In addition, u_2 and u_6 are dominated by u_1 and u_7, respectively. Thus, we are left with u_3, u_4, u_5 and v_{ij} for $i = 1, 2, 3$ and $1 \leq j \leq b-3$ which are nondominated vertices. To get the minimum cardinality of an edge geodetic dominating set, we have to choose those vertices at the center, hence, we pick u_4 and $v_{2j} (1 \leq j \leq b-3)$. Therefore, $\gamma_{g_e}(G) = g_e(G) + (b-3) + 1 = 2 + b - 3 + 1 = b$.

If $2 < a < b$, for $b = a+1$, consider the graph obtained from the path on eight vertices $P : u_1, u_2, u_3, \ldots, u_8$ by adding $a-2$ vertices, $w_1, w_2, \ldots w_{a-2}$ and joining each $w_i (1 \leq i \leq a-2)$ with u_3. However, for $b > a+1$, let G be the graph obtained from the path on eight vertices $P : u_1, u_2, u_3, \ldots, u_8$ by adding $a-2$ vertices, $w_1, w_2, \ldots w_{a-2}$ and $(b-a-1)$ sets of three new vertices, namely, $\{v_{11}, v_{21}, v_{31}\}, \{v_{12}, v_{22}, v_{32}\}, \ldots, \{v_{1(b-a-1)}, v_{2(b-a-1)}, v_{3(b-a-1)}\}$ for $i = 1, 2, 3$, with the paths $P_1 : u_3, v_{11}, v_{21}, v_{31}, u_7, P_2 : u_3, v_{12}, v_{22}, v_{32}, u_7, \ldots, P_{b-a-1} : u_3, v_{1(b-a-1)}, v_{2(b-a-1)}, v_{3(b-a-1)}, u_7$ such that v_{ik} is not adjacent to $v_{ij}, j \neq k$ and $j, k = 1, 2, 3, \ldots, b-a-1$ and joining each $w_i (1 \leq i \leq a-2)$ with u_3. The graph G is shown in Figure 2.

Let $S = \{u_1, u_8, w_1, w_2, \ldots, w_{a-2}\}$. It is clear that this is an edge geodetic basis of G, so that $g_e(G) = a - 2 + 2 = a$. Moreover, u_2, u_7, and u_3 are dominated by u_1, u_8 and w_i, respectively. Hence, we are left with u_4, u_5, u_6, v_{ij} for $i = 1, 2, 3$ and $1 \leq j \leq b-a-1$ which are nondominated vertices. To get the minimum cardinality of edge geodetic dominating set, we need to choose u_5, and v_{2j} for $(1 \leq j \leq b-a-1)$. Therefore, $\gamma_{g_e}(G) = g_e(G) + (b-a-1) + 1 = a + b - a - 1 + 1 = b$.

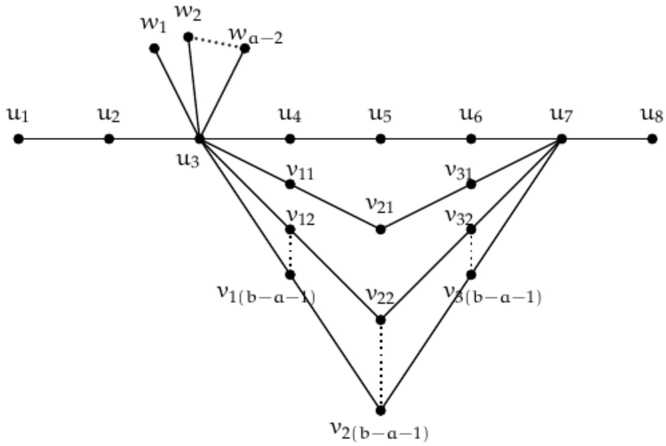

Figure 2: A graph G

4. Conclusion and Recommendation

This study introduced and investigated the concept of edge geodetic dominating set S of the graph G. The authors primary focus has been on the following areas: deletion of independent edges of complete graph, the K_r-gluing of complete graphs, and result of comprehension.

Researchers who are interested in this concept can further generate results in the various graph operations that include cartesian product and composition of graphs, among many others. Furthermore, they may explore and analyze the bounds in relation to other well-established parameters in graph theory.

Acknowledgements

The authors extend their sincere gratitude to the panel of reviewers for their insightful comments and recommendations, which significantly enhanced the quality of this paper. Additionally, the authors express their appreciation to Ateneo de Zamboanga University, Tangub City Global College, and Western Mindanao State University for supporting this research.

References

[1] A. Asdain, J. I. Salim, and R. G. Artes Jr. Geodetic bounds in graphs. *International Journal of Mathematics and Computer Science*, 18(4):767–771, 2023.

[2] A. T. Castaneda, M. A. Medina, and L. Ruivivar. Notions of domination for some classes of graphs. *DLSU Research Congress*, 2016.

[3] G. Chartrand, F. Harary, and P. Zhang. Geodetic sets in graphs. *Discussioness Mathematicae Graph Theory*, 20(1):129–138, 2000.

[4] H. Escuadro, R. Gera, A. Hansberg, N. Jafari Rad, and L. Volkmann. Geodetic domination in graphs. *Journal of Combinatorial Mathematics and Combinatorial Computing*, 77:89–101, 2011.

[5] A. Gamorez and S. Canoy Jr. Monophonic eccentric domination numbers of graphs. *European Journal of Pure and Applied Mathematics*, 15(2):635–645, 2022.

[6] A. Hansberg and L. Volkmann. On the geodetic and geodetic domination number of a graph. *Discrete Mathematics*, 310(15–16):2140–2146, 2010.

[7] J. Harris, J. Hirst, and M. Mossinghoff. *Combinatorics and Graph Theory.* Springer Science + Business Media, Lawrence University, 2008.

[8] R. Mariano and S. Canoy Jr. Edge geodetic cover in graphs. *International Mathematical Forum*, 4:2301–2310, 2009.

[9] V. Samodivkin. On the edge geodetic and edge geodetic domination number of a graph. *Communications in Combinatorics and Optimization*, 5:41–54, 2020.

[10] E. Sandueta and S. Canoy Jr. Weakly connected domination in graphs resulting from some graph operations. *International Mathematical Forum*, 6:1031–1035, 2011.

[11] A. P. Santhakumaran. Comment on edge geodetic cover in graphs. *Proyecciones Journal of Mathematics*, 34:343–350, 2015.

[12] A. P. Santhakumaran and J. John. Edge geodetic number of a graph. *Journal of Discrete Mathematical Sciences and Cryptography*, 10:415–432, 2007.

[13] D. Stalin and J. John. Edge geodetic dominations in graphs. *International Journal of Pure and Applied Mathematics*, 116:31–40, 2017.

[14] P. A. P. Sudhahar, A. Ajitha, and A. Subramanian. Edge geodetic domination number of a graph. *International Journal of Mathematics and its Applications*, 4:45–50, 2016.

[15] A. Vijayan and N. Jaspin Beaula. Geodetic dominating sets and geodetic dominating polynomials of cycles. *International Journal of Engineering Science and Computing*, 6:3774–3779, 2016.