EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 1, Article Number 5559
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Some Properties of the Bell-Based Apostol-Frobenius-Type Tangent Polynomials

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Abstract. In this paper, we introduce a new class of Frobenius-Tangent polynomials, derived from the Bell numbers and Apostol-type functions. We conduct a detailed investigation into the properties of these polynomials, utilizing various analytical techniques. By employing generating functions for Bell-based Apostol-Frobenius-Type Tangent polynomials of higher order, we obtain both explicit and implicit summation formulas and its relation to Appell polynomials.

2020 Mathematics Subject Classifications: 05A15, 11B68, 11B73, 26C05, 33B10
Key Words and Phrases: Tangent polynomials, Bell polynomials, Apostol-Frobenius- Type poly-Tangent polynomials, Bell-based Apostol-Frobenius-Type poly-Tangent polynomials, Stirling numbers, Appell polynomials

1. Introduction

In mathematical analysis, special polynomials play a pivotal role due to their extensive applications across various domains. Among these, the Frobenius-Tangent polynomials have garnered significant attention. These polynomials, denoted as $T_n(x)$, are defined by the generating function ([11]),[12])

$$\sum_{n=0}^{\infty} T_n(x)\frac{z^n}{n!} = \left(\frac{2}{e^{2z}+1}e^{xz}\right), \tag{1}$$

where $T_n(0) = T_n$, the Tangent numbers defined coefficient of the following series expansion of the tangent function ([13])

$$\tan z = \sum_{n=0}^{\infty}(-1)^{n+1}T_{2n+1}\frac{z^{2n+1}}{(2n+1)!}, \tag{2}$$

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DOI: https://doi.org/10.29020/nybg.ejpam.v18i1.5559

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with $T_0 = 1$ and $T_{2n} = 0$, $n \in \mathbb{N}$.

They emerge naturally in the study of differential equations and have applications in numerical analysis and approximation theory.

Parallel to this, the Apostol polynomials, particularly the Apostol-Bernoulli and Apostol-Euler polynomials, have been extensively studied. The Apostol-Bernoulli polynomials $B_n^{(a)}(x)$ are defined via the generating function:

$$\sum_{n=0}^{\infty} B_n^{(a)}(x)\frac{t^n}{n!} = \frac{te^{xt}}{ae^t - 1}, \tag{3}$$

where a is a non-zero parameter. These polynomials generalize the classical Bernoulli polynomials and have applications in number theory and combinatorics.

Bell polynomials, denoted as $B_n(x)$, are another important class, defined by the generating function:

$$\sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} = e^{x(e^t-1)}. \tag{4}$$

They are instrumental in the study of combinatorial structures and have applications in the theory of partitions and moments of probability distributions.

In the framework of orthogonal polynomials, it is noteworthy that certain classes of Apostol and Bell polynomials exhibit orthogonality properties under specific weight functions. For instance, the study by Luo and Srivastava [8] looks into some generalizations of Apostol-Bernoulli and Apostol-Euler polynomials, exploring their orthogonality and other properties. Similarly, the work by Kurt [7] introduces new families of polynomials associated with the Bell numbers and polynomials, discussing their potential orthogonality under certain conditions. Additionally, recent research by Khan and Riaz [6] investigates certain subclasses of Apostol-type polynomials, providing insights into their structural properties within the orthogonal polynomial framework. Further studies have expanded the landscape of these polynomials. Dattoli et al. [5] introduced a family of hybrid polynomials that exhibit characteristics of both Hermite and Laguerre polynomials, enriching the theory of special functions. Ramírez and Cesarano [9] explored new classes of degenerated generalized Apostol-Bernoulli, Apostol-Euler, and Apostol-Genocchi polynomials, deriving explicit expressions and recurrence relations. In a subsequent work, Ramírez et al. [10] presented new results for these degenerated polynomials, establishing algebraic relationships and recurrence formulas.

The study of special numbers such as the tangent numbers, Bernoulli numbers, Euler numbers, and Genocchi numbers has become an interesting area for many mathematicians ([2],[4],[14]). Tangent numbers and polynomials possess many significant properties that can be found in mathematics and physics. Analogues and symmetric properties for tangent polynomials are derived in [12] and [11]. Building upon these foundational studies, this paper aims to explore higher-order bivariate Bell-based Apostol-Frobenius-type poly-Tangent polynomials. We will derive explicit representations and investigate their structural properties.

The **Bell-based Apostol-Frobenius-Type Tangent Polynomials** ${}_BT_n(x, y, u, \lambda)$ is defined by the generating function

$$\sum_{n=0}^{\infty} {}_BT_n(x, y, u, \lambda)\frac{t^n}{n!} = \left(\frac{1-u}{\lambda e^{2t} - u}\right) e^{xt+y(e^t-1)}.$$

This paves way to our working definition of the **Bell-based Apostol-Frobenius-Type Tangent Polynomials of higher order** ${}_BT_n^r(x, y, u, \lambda)$ defined by the generating function

$$\sum_{n=0}^{\infty} {}_BT_n^{(r)}(x, y; u, \lambda)\frac{t^n}{n!} = \left(\frac{1-u}{\lambda e^{2t} - u}\right)^r e^{xt+y(e^t-1)}. \tag{5}$$

The next three functions are special cases of (5); when $x = 0$ and $y \neq 0$, we obtain ${}_BT_n^{(r)}(y, u, \lambda)$ defined by

$$\sum_{n=0}^{\infty} {}_BT_n^{(r)}(y, u, \lambda)\frac{t^n}{n!} = \left(\frac{1-u}{\lambda e^{2t} - u}\right)^r e^{y(e^t-1)}, \tag{6}$$

known as **Bell-based Apostol-Frobenius-Type Tangent numbers of higher order**. When $y = 0$ and $x \neq 0$ we have the polynomial $T_n^r(x, u, \lambda)$ defined by

$$\sum_{n=0}^{\infty} T_n^r(x, u, \lambda)\frac{t^n}{n!} = \left(\frac{1-u}{\lambda e^{2t} - u}\right)^r e^{xt}, \tag{7}$$

known as **Apostol-Frobenius-Type Tangent polynomials of higher order**. Lastly, when both $x = y = 0$ we have the polynomial $T_n^{(r)}(u, \lambda)$ defined by

$$\sum_{n=0}^{\infty} T_n^{(r)}(u, \lambda)\frac{t^n}{n!} = \left(\frac{1-u}{\lambda e^{2t} - u}\right)^r, \tag{8}$$

known as **Apostol-Frobenius-Type Tangent numbers of higher order**.

Meanwhile, in [1] we define a sequence of polynomials $\{P_n(x)\}_0^\infty$ satisfying

$$P_n'(x) = nP_{n-1}(x), \ \ n \geq 1, \tag{9}$$

as Appell polynomials. Moreover, [3], [15], and [16] established an important characterization of Appell polynomials in the following equivalent conditions:

(a) $\{P_n(x)\}_0^\infty$ is a sequence of Appell polynomials.

(b) $\{P_n(x)\}_0^\infty$ has a generating function of the form

$$A(t)e^{xt} = \sum_{n=0}^{\infty} P_n(x)\frac{t^n}{n!}, \tag{10}$$

where $A(t)$ is a formal power series independent of x with $A(0) \neq 0$.

(c) There exists a sequence $\{a_n\}_{n=0}^\infty$ with $a_0 \neq 0$ such that

$$P_n(x) = \sum_{k=0}^{n} \binom{n}{k} a_{n-k}x^k. \tag{11}$$

(d) There exists a sequence $\{a_n\}_{n=0}^\infty$ with $a_0 \neq 0$ such that

$$P_n(x) = \left(\sum_{k=0}^{\infty} \frac{a_k}{k!}D^k\right) x^n, \tag{12}$$

where $D = \frac{d}{dx}$.

The next lemma will be useful in some of our results.

Lemma 1. *Let f be a function and $\{f(N)\}_0^\infty$ a sequence and coefficients of the power series*

$$\sum_{N=0}^{\infty} f(N)\frac{(x+y)^N}{N!}.$$

There exists a pair of integers n and m such that

$$\sum_{N=0}^{\infty} f(N)\frac{(x+y)^N}{N!} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} f(m+n)\frac{x^m y^n}{m!n!}, \tag{13}$$

where $N = n + m$.

Proof. For each $N \in \mathbb{N}$, we write

$$\begin{aligned} f(N)\frac{(x+y)^N}{N!} &= \frac{f(N)\sum_{i=0}^{N}\binom{N}{i}x^i y^{N-i}}{N!} \\ &= \sum_{i=0}^{N} \frac{f(N)\binom{N}{i}x^i y^{N-i}}{N!} \\ &= \sum_{i=0}^{N} \frac{f(N)\frac{N!}{(N-i)!i!}x^i y^{N-i}}{N!} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=0}^{N} f(N)\frac{x^i y^{N-i}}{(N-i)!i!} \\
&= \sum_{i=0}^{N} f((N-i)+i)\frac{x^i y^{N-i}}{(N-i)!i!} \\
&= \sum_{N=0}^{\infty}\sum_{i=0}^{N} f((N-i)+i)\frac{x^i y^{N-i}}{(N-i)!i!} \\
\sum_{N=0}^{\infty} f(N)\frac{(x+y)^N}{N!} &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} f(m+n)\frac{x^m y^n}{m!n!}.
\end{aligned}
$$

In this study, the authors are interested to explore some properties of Bell-based Apostol-Frobenius-Type Tangent polynomials of higher order ${}_BT_n^{(r)}(x,y;u,\lambda)$ in terms of the three aforementioned cases (6), (7), and (8).

2. Higher Order Bivariate Bell-Based Apostol-Frobenius-Type Tangent Polynomials

The following theorems contain identities for the bivariate Bell-based ApostolFrobenius-Type Tangent polynomials of higher order expressed in terms of (6), (7), and (8) and the Bell polyomials.

Theorem 1. *The Bell-based Apostol-Frobenius-Type Tangent Polynomials of higher order* ${}_BT_n^{(r)}(x,y;u,\lambda)$ *satisfies the equation*

$$
{}_BT_n^{(r)}(x,y;u,\lambda) = \sum_{k=0}^{n}\binom{n}{k} T_n^{(r)}(x;u,\lambda)B_{n-k}(y) \tag{14}
$$

where $B_n(y)$ *is the Bell Polynomial defined by the generating function*

$$
\sum_{n=0}^{\infty} B_n(y)\frac{t^n}{n!} = e^{y\left(e^t-1\right)}.
$$

Proof. We write

$$
\begin{aligned}
\sum_{n=0}^{\infty} {}_BT_n^{(r)}(x,y;u,\lambda)\frac{t^n}{n!} &= \left(\frac{(1-u)}{\lambda e^{2t}-u}\right)^r e^{xt+y(e^t-1)} \\
&= \left\{\left(\frac{(1-u)}{\lambda e^{2t}-u}\right)^r e^{xt}\right\} e^{y(e^t-1)} \\
&= \left(\sum_{n=0}^{\infty} T_n^{(r)}(x;u,\lambda)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty} B_n(y)\frac{t^n}{n!}\right)
\end{aligned}
$$

$$= \sum_{n=0}^{\infty}\left\{\sum_{k=0}^{n}\binom{n}{k}T_k^{(r)}(x;u,\lambda)B_{n-k}(y)\right\}\frac{t^n}{n!}.$$

Comparing coefficients, we obtain the desired result

$${}_BT_n^{(r)}(x,y;u,\lambda) = \sum_{k=0}^{n}\binom{n}{k}T_k^{(r)}(x;u,\lambda)B_{n-k}(y).$$

Theorem 2. *The function* ${}_BT_n^{(r)}(x,y;u,\lambda)$ *satisfies the equation*

$${}_BT_n^{(r)}(x,y;u,\lambda) = \sum_{k=0}^{n}\binom{n}{k}T_n^{(r)}(u,\lambda)B_{n-k}(x,y) \tag{15}$$

where $B_n(x,y)$ *is the Bivariate Bell Polynomial defined by the generating function*

$$\sum_{n=0}^{\infty}B_n(x,y)\frac{t^n}{n!} = e^{xt+y\left(e^t-1\right)}.$$

Proof. We start with

$$\begin{aligned}\sum_{n=0}^{\infty}{}_BT_n^{(r)}(x,y;u,\lambda)\frac{t^n}{n!} &= \left(\frac{(1-u)}{\lambda e^{2t}-u}\right)^r e^{xt+y(e^t-1)} \\ &= \left(\sum_{n=0}^{\infty}T_n^{(r)}(u,\lambda)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}B_n(x,y)\frac{t^n}{n!}\right) \\ &= \sum_{n=0}^{\infty}\left\{\sum_{k=0}^{n}\binom{n}{k}T_k^{(r)}(u,\lambda)B_{n-k}(x,y)\right\}\frac{t^n}{n!}.\end{aligned}$$

Comparing coefficients,

$${}_BT_n^{(r)}(x,y;u,\lambda) = \sum_{k=0}^{n}\binom{n}{k}T_k^{(r)}(u,\lambda)B_{n-k}(x,y).$$

as desired.

In view of (5), we can express the bivariate Bell Polynomial $B_n(x,y)$ as

$$B_n(x,y) = {}_BT_n^{(0)}(x,y;u,\lambda), \tag{16}$$

which allows as to write the result from Theorem 1.2 into

$${}_BT_n^{(r)}(x,y;u,\lambda) = \sum_{k=0}^{n}\binom{n}{k}T_k^{(r)}(u,\lambda){}_BT_{n-k}^{(0)}(x,y;u,\lambda).$$

Theorem 3. *The function* ${}_BT_n^{(r)}(x,y;u,\lambda)$ *satisfies the equation*

$$ {}_BT_n^{(r)}(x,y;u,\lambda) = \sum_{k=0}^{n}\binom{n}{k}{}_BT_{n-k}^{(r)}(y,u,\lambda)x^k. \tag{17}$$

Proof. Writing

$$\begin{aligned}\sum_{n=0}^{\infty}{}_BT_n^{(r)}(x,y;u,\lambda)\frac{t^n}{n!} &= \left(\frac{(1-u)}{\lambda e^{2t}-u}\right)^r e^{y(e^t-1)}e^{xt}\\ &= \left(\sum_{n=0}^{\infty}{}_BT_n^{(r)}(y;u,\lambda)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\frac{(xt)^n}{n!}\right)\\ &= \sum_{n=0}^{\infty}\left\{\sum_{k=0}^{n}\binom{n}{k}{}_BT_k^{(r)}(y;u,\lambda)x^{n-k}\right\}\frac{t^n}{n!}.\end{aligned}$$

Comparing coefficients,

$$\begin{aligned}{}_BT_n^{(r)}(x,y;u,\lambda) &= \sum_{k=0}^{n}\binom{n}{k}{}_BT_k^{(r)}(y;u,\lambda)x^{n-k}\\ &= \sum_{k=0}^{n}\binom{n}{k}{}_BT_{n-k}^{(r)}(y;u,\lambda)x^{k}\end{aligned}$$

as desired.

The next theorem contains the addition formula for bivariate Bell-based Apostol-Frobenius-Type Tangent polynomials of higher order.

Theorem 4. *The Bell-based Apostol-Frobenius-Type Tangent Polynomials of higher order* ${}_BT_n^{(r)}(x,y;u,\lambda)$ *satisfies the equation*

$$ {}_BT_n^{(r)}(x+y,z;u,\lambda) = \sum_{k=0}^{n}\binom{n}{k}T_k^{(r)}(x;u,\lambda)B_{n-k}(y,z) \tag{18}$$

Proof. We start by writing

$$\begin{aligned}\sum_{n=0}^{\infty}{}_BT_n^{(r)}(x+y,z;u,\lambda)\frac{t^n}{n!} &= \left(\frac{(1-u)}{\lambda e^{2t}-u}\right)^r e^{(x+y)t+z(e^t-1)}\\ &= \left\{\left(\frac{(1-u)}{\lambda e^{2t}-u}\right)^r e^{xt}\right\}e^{yt+z(e^t-1)}\\ &= \left(\sum_{n=0}^{\infty}T_n^{(r)}(x;u,\lambda)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}B_n(y,z)\frac{t^n}{n!}\right)\\ &= \sum_{n=0}^{\infty}\left\{\sum_{k=0}^{n}\binom{n}{k}T_k^{(r)}(x;u,\lambda)B_{n-k}(y,z)\right\}\frac{t^n}{n!}.\end{aligned}$$

Comparing coefficients, we obtain the desired result

$$ {}_BT_n^{(r)}(x+y,z;u,\lambda) = \sum_{k=0}^{n} \binom{n}{k} T_k^{(r)}(x;u,\lambda) B_{n-k}(y,z). $$

Implicit Summation Formula

Within this section, we will derive different summation formulas for ${}_BT_n^{(r)}(x+y,z;u,\lambda)$, establishing implicit connections among the variables by considering them as arguments. The subsequent theorem shows a particular expression of these summation formulas.

Theorem 5. *The bivariate Bell-based Apostol-Frobenius-Type Tangent polynomials of higher order ${}_BT_n^{(r)}(x,y;u,\lambda)$ satisfy the summation formula:*

$$ {}_BT_n^{(r_1+r_2)}(x_1+x_2,y_2+y_2;u,\lambda) = \sum_{k=0}^{n} \binom{n}{k} {}_BT_k^{(r_1)}(x_1,y_1;u,\lambda) {}_BG_{n-k}^{(r_2)}(x_2,y_2;u,\lambda) \tag{19} $$

Proof. we can express the right hand side of (5) as follows:

$$ \begin{aligned} &\left(\frac{(1-u)}{\lambda e^{2t}-u}\right)^{r_1+r_2} e^{(x_1+x_2)t+(y_1+y_2)(e^t-1)} \\ &\quad = \left\{\left(\frac{(1-u)}{\lambda e^{2t}-u}\right)^{r_1} e^{x_1t+y_1(e^t-1)}\right\}\left\{\left(\frac{(1-u)}{\lambda e^{2t}-u}\right)^{r_2} e^{x_2t+y_2(e^t-1)}\right\} \\ &\sum_{n=0}^{\infty} {}_BT_n^{(r_1+r_2)}(x_1+x_2,y_2+y_2;u,\lambda)\frac{t^n}{n!} = \left(\sum_{n=0}^{\infty} {}_BT_n^{(r_1)}(x_1,y_1;u,\lambda)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty} {}_BT_n^{(r_2)}(x_2,y_2;u,\lambda)\frac{t^n}{n!}\right) \\ &\qquad = \sum_{n=0}^{\infty}\sum_{k=0}^{n} {}_BT_n^{(r_1)}(x_1,y_1;u,\lambda) {}_BT_{n-k}^{(r_2)}(x_2,y_2;u,\lambda)\binom{n}{k}. \end{aligned} $$

Comparing coefficients, we obtain the desired result

$$ {}_BT_n^{(r_1+r_2)}(x_1+x_2,y_2+y_2;u,\lambda) = \sum_{k=0}^{n} \binom{n}{k} {}_BT_k^{(r_1)}(x_1,y_1;u,\lambda) {}_BT_{n-k}^{(r_2)}(x_2,y_2;u,\lambda). $$

Remark 1. *When $r_1 = r$, $r_2 = 0$, $x_1 = x$, $x_2 = 1$,$y_1 = y$, $y_2 = 0$, the summation formula in (19) reduces to*

$$ \begin{aligned} {}_BT_n^{(r)}(x+1,y;u,\lambda) &= \sum_{k=0}^{n} \binom{n}{k} {}_BT_k^{(r)}(x,y;u,\lambda) B_{n-k}(1,0) \\ &= \sum_{k=0}^{n} \binom{n}{k} {}_BT_k^{(r)}(x,y;u,\lambda). \end{aligned} \tag{20} $$

On the other hand, when $y = 1$, (18) gives

$$ {}_BT_n^{(r)}(x+1, z; u, \lambda) \;=\; \sum_{k=0}^{n} \binom{n}{k} T_k^{(r)}(x; u, \lambda) B_{n-k}(1, z). \tag{21} $$

Replacing z with y in (21) and compare it to (20) yields

$$ \sum_{k=0}^{n} \binom{n}{k} T_k^{(r)}(x; u, \lambda) B_{n-k}(1, y) \;=\; \sum_{k=0}^{n} \binom{n}{k} {}_BT_k^{(r)}(x, y; u, \lambda). $$

In view of (5), notice that we can write

$$ \sum_{n=0}^{\infty} {}_BT_n^{(r)}(x, y; u, \lambda) \frac{(t+v)^n}{n!} \;=\; \left(\frac{1-u}{\lambda e^{2(t+v)} - u} \right)^r e^{x(t+v) + y(e^{t+v} - 1)} $$

which allows us to express

$$ \left(\frac{1-u}{\lambda e^{2(t+v)} - u} \right)^r e^{x(t+v)} e^{y(e^{t+v} - 1)} \;=\; \sum_{n=0}^{\infty} {}_BT_n^{(r)}(x, y; u, \lambda) \frac{(t+v)^n}{n!}, $$

consequently

$$ \left(\frac{1-u}{\lambda e^{2(t+v)} - u} \right)^r e^{y(e^{t+v} - 1)} \;=\; e^{-x(t+v)} \sum_{n=0}^{\infty} {}_BT_n^{(r)}(x, y; u, \lambda) \frac{(t+v)^n}{n!}. \tag{22} $$

Applying (13), we obtain

$$ \left(\frac{1-u}{\lambda e^{2(t+v)} - u} \right)^r e^{y(e^{t+v} - 1)} \;=\; e^{-x(t+v)} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} {}_BT_{k+l}^{(r)}(x, y; u, \lambda) \frac{t^k}{k!} \frac{v^l}{l!}. \tag{23} $$

Replacing x with z, equation (23) becomes

$$ \begin{aligned} \left(\frac{(1-u)}{\lambda e^{2(t+v)} - u} \right)^r e^{y(e^{t+v} - 1)} &= e^{-z(t+v)} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} {}_BT_{k+l}^{(r)}(z, y; u, \lambda) \frac{t^k}{k!} \frac{v^l}{l!} \\ \left(\frac{(1-u)}{\lambda e^{2(t+v)} - u} \right)^r e^{y(e^{t+v} - 1)} e^{x(t+v)} &= e^{x(t+v)} e^{-z(t+v)} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} {}_BT_{k+l}^{(r)}(z, y; u, \lambda) \frac{t^k}{k!} \frac{v^l}{l!} \\ \left(\frac{(1-u)}{\lambda e^{2(t+v)} - u} \right)^r e^{x(t+v) + y(e^{t+v} - 1)} &= e^{(x-z)(t+v)} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} {}_BT_{k+l}^{(r)}(z, y; u, \lambda) \frac{t^k}{k!} \frac{v^l}{l!} \end{aligned} $$

Thus, using (13) again, we have

$$\begin{aligned}\sum_{k,l\geq 0} {}_BT^{(r)}_{k+l}(x,y;u,\lambda)\frac{t^k}{k!}\frac{v^l}{l!} &= e^{(x-z)(t+v)}\sum_{k,l\geq 0} {}_BT^{(r)}_{k+l}(z,y;u,\lambda)\frac{t^k}{k!}\frac{v^l}{l!}\\ &= \left(\sum_{N=0}^{\infty}(x-z)^N\frac{(t+v)^N}{N!}\right)\left(\sum_{k,l\geq 0} {}_BT^{(r)}_{k+l}(z,y;u,\lambda)\frac{t^k}{k!}\frac{v^l}{l!}\right)\\ &= \left(\sum_{n,m\geq 0}(x-z)^{n+m}\frac{t^n}{n!}\frac{v^m}{m!}\right)\left(\sum_{k,l\geq 0} {}_BT^{(r)}_{k+l}(z,y;u,\lambda)\frac{t^k}{k!}\frac{v^l}{l!}\right)\\ &= \sum_{k,l\geq 0}\left\{\sum_{k,m=0}^{k,l}\binom{k}{n}\binom{l}{m}(x-z)^{n+m}{}_BT^{(r)}_{k+l}(z,y;u,\lambda)\right\}\frac{t^k}{k!}\frac{v^l}{l!}.\end{aligned}$$

Comparing coefficients, we then have

$$\sum_{k,l\geq 0} {}_BT^{(r)}_{k+l}(x,y;u,\lambda) = \sum_{n,m=0}^{k,l}\binom{k}{n}\binom{l}{m}(x-z)^{n+m}{}_BT_{k+l-n-m}(z,y;u,\lambda),$$

which proves our next theorem.

Theorem 6. *The bivariate Bell-based Apostol-Frobenius-Type Tangent polynomials of higher order* ${}_BT^{(r)}_n(x,y;u,\lambda)$ *satisfy the summation formula*

$$\sum_{k,l\geq 0} {}_BT^{(r)}_{k+l}(x,y;u,\lambda) = \sum_{n,m=0}^{k,l}\binom{k}{n}\binom{l}{m}(x-z)^{n+m}{}_BT_{k+l-n-m}(z,y;u,\lambda). \tag{24}$$

The next result provides the difference when x in ${}_BT^{(r)}_n(x,y;u,\lambda)$ is shifted by 1.

Theorem 7. *For* $n\geq 1$*, the difference* ${}_BT^{(r)}_n(x+1,y;u,\lambda) -{}_B T^{(r)}_n(x,y;u,\lambda)$ *is given by the difference formula*

$${}_BT^{(r)}_n(x+1,y;u,\lambda) - {}_BT^{(r)}_n(x,y;u,\lambda) = \sum_{k=0}^{n-1}\binom{n}{k}{}_BT^{(r)}_k(x,y;u,\lambda). \tag{25}$$

Proof.

$$\begin{aligned}\sum_{n=0}^{\infty} {}_BT^{(r)}_n(x+1,y;u,\lambda)\frac{t^n}{n!} &- \sum_{n=0}^{\infty} {}_BT^{(r)}_n(x,y;u,\lambda)\frac{t^n}{n!}\\ &= \left(\frac{(1-u)}{\lambda e^{2t}-u}\right)^r e^{(x+1)t+y(e^t-1)} - \left(\frac{(1-u)}{\lambda e^{2t}-u}\right)^r e^{xt+y(e^t-1)}\end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{(1-u)}{\lambda e^{2t}-u}\right)^r e^{xt+y(e^t-1)}(e^t-1) \\
&= \left(\sum_{n=0}^{\infty} {}_BT_n^{(r)}(x,y;u,\lambda)\frac{t^n}{n!}\right)\left(\sum_{n\geq 0}\frac{t^{n+1}}{(n+1)!}\right) \\
&= \sum_{n=0}^{\infty}\left\{\sum_{k=0}^{n}\binom{n+1}{k} {}_BT_k^{(r)}(x,y;u,\lambda)\right\}\frac{t^{n+1}}{(n+1)!} \\
&= \sum_{n=1}^{\infty}\left\{\sum_{k=0}^{n-1}\binom{n}{k} {}_BT_k^{(r)}(x,y;u,\lambda)\right\}\frac{t^n}{n!}.
\end{aligned}
$$

Comparing coefficients,

$$
{}_BT_n^{(r)}(x+1,y;u,\lambda) - {}_BT_k^{(r)}(x,y;u,\lambda) = \sum_{k=0}^{n-1}\binom{n}{k} {}_BT_k^{(r)}(x,y;u,\lambda)
$$

as desired.

Stirling Number of Second Kind and Bivariate Bell Polynomials

In this subsection, we derive some formulas displaying relationship of ${}_BT_n^{(r)}(x,y;u,\lambda)$ with the Stirling numbers of second kind and bivariate Bell polynomials.

Theorem 8. *The bivariate Bell-based Apostol-Frobenius-Type Tangent polynomials of higher order ${}_BT_n^{(r)}(x,y;u,\lambda)$ satisfy the summation formula*

$$
{}_BT_n^{(r)}(x,y;u,\lambda) = \sum_{k=0}^{n}\sum_{j=0}^{k}\binom{n}{k}(x)_j S(k,j) {}_BT_{n-k}^{(r)}(y;u,\lambda). \tag{26}
$$

Proof. Using (5), we write

$$
\begin{aligned}
\sum_{n=0}^{\infty} {}_BT_n^{(r)}(x,y;u,\lambda)\frac{t^n}{n!} &= \left(\frac{(1-u)}{\lambda e^{2t}-u}\right)^r e^{xt+y(e^t-1)} \\
&= \left(\frac{(1-u)}{\lambda e^{2t}-u}\right)^r e^{y(e^t-1)}e^{xt} \\
&= \left(\frac{(1-u)}{\lambda e^{2t}-u}\right)^r e^{y(e^t-1)}(1+e^t-1)^x \\
&= \left(\sum_{n=0}^{\infty} {}_BT_n^{(r)}(y;u,\lambda)\frac{t^n}{n!}\right)\left(\sum_{j=0}^{\infty}\binom{x}{j}(e^t-1)^j\right)
\end{aligned}
$$

$$\begin{aligned}
&= \left(\sum_{n=0}^{\infty} {}_{B}T_n^{(r)}(y;u,\lambda)\frac{t^n}{n!}\right)\left(\sum_{j=0}^{\infty}\frac{x!}{(x-j)!}\frac{(e^t-1)^j}{j!}\right)\\
&= \left(\sum_{n=0}^{\infty} {}_{B}T_n^{(r)}(y;u,\lambda)\frac{t^n}{n!}\right)\left(\sum_{j=0}^{\infty}(x)_j\frac{(e^t-1)^j}{j!}\right)\\
&= \left(\sum_{n=0}^{\infty} {}_{B}T_n^{(r)}(y;u,\lambda)\frac{t^n}{n!}\right)\left(\sum_{j=0}^{\infty}(x)_j\sum_{n=0}^{\infty}S(n,j)\frac{t^n}{n!}\right)\\
&= \left(\sum_{n=0}^{\infty} {}_{B}T_n^{(r)}(y;u,\lambda)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\left\{\sum_{j=0}^{\infty}(x)_jS(n,j)\right\}\frac{t^n}{n!}\right)\\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\left\{\sum_{j=0}^{\infty}(x)_jS(k,j)\,{}_{B}T_{n-k}^{(r)}(y;u,\lambda)\right\}\frac{t^n}{n!}\\
&= \sum_{n=0}^{\infty}\left\{\sum_{k=0}^{n}\binom{n}{k}\sum_{j=0}^{\infty}(x)_jS(k,j)\,{}_{B}T_{n-k}^{(r)}(y;u,\lambda)\right\}\frac{t^n}{n!}.
\end{aligned}$$

Comparing coefficients of $\frac{t^n}{n!}$ we obtain,

$$\begin{aligned}
{}_{B}T_n^{(r)}(x,y;u,\lambda) &= \sum_{k=0}^{n}\sum_{j=0}^{\infty}\binom{n}{k}(x)_jS(k,j)\,{}_{B}T_{n-k}^{(r)}(y;u,\lambda)\\
&= \sum_{k=0}^{n}\sum_{j=0}^{k}\binom{n}{k}(x)_jS(k,j)\,{}_{B}T_{n-k}^{(r)}(y;u,\lambda),
\end{aligned}$$

which proves the theorem.

The next result expresses the bivariate Bell polynomials in terms of bivariate Bell-based Apostol-Frobenius-Type Tangent polynomials.

Theorem 9. *The bivariate Bell polynomials follows the relation*

$$B_n(x,y) = \frac{\lambda\,{}_{B}T_n(x+2,y;u,\lambda) - u\,{}_{B}T_n(x,y;u,\lambda)}{(1-u)}. \tag{27}$$

Proof. From (16), we can write

$$\begin{aligned}
\sum_{n=0}^{\infty}B_n(x,y)\frac{t^n}{n!} &= e^{xt+y(e^t-1)}\\
&= \left(\frac{\lambda e^{2t}-u}{(1-u)}\right)\left(\frac{(1-u)}{\lambda e^{2t}-u}\,e^{xt+y(e^t-1)}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{(1-u)}\left(\lambda\left(\frac{(1-u)}{\lambda e^{2t}-u}e^{(x+2)t+y(e^t-1)}\right)-u\left(\frac{(1-u)}{\lambda e^{2t}-u}e^{xt+y(e^t-1)}\right)\right)\\
&= \frac{1}{(1-u)}\left(\lambda\sum_{n=0}^{\infty}{}_BT_n(x+2,y;u,\lambda)\frac{t^n}{n!}-u\sum_{n=0}^{\infty}{}_BT_n(x,y;u,\lambda)\frac{t^n}{n!}\right)\\
&= \frac{\lambda}{1-u}\sum_{n=-1}^{\infty}{}_BT_n(x+2,y;u,\lambda)\frac{t^n}{n!}-\frac{u}{1-u}\sum_{n=-1}^{\infty}{}_BT_n(x,y;u,\lambda)\frac{t^n}{n!}.
\end{aligned}
$$

Comparing coefficients,

$$
B_n(x,y) = \frac{\lambda_B T_n(x+2,y;u,\lambda)-u_B T_n(x,y;u,\lambda)}{(1-u)}.
$$

Derivative Formulas

Derivative formulas for special polynomials are fundamental tools in mathematics and its applications to physics, engineering, and other scientific fields. These formulas facilitate the analysis of the behavior and properties of special polynomials by quantifying their rates of change, a central aspect in calculus and mathematical analysis for understanding function dynamics. Moreover, they are instrumental in the manipulation of generating functions, which encode sequences of polynomial coefficients. Generating functions, in turn, play a crucial role in combinatorics, number theory, and discrete mathematics, particularly for problems involving counting and enumeration.

The next theorem contains the derivative formula for ${}_BT_n^{(r)}(x,y;u,\lambda)$ with respect to the variable x.

Theorem 10. *The following derivative formula holds*

$$
\frac{\partial}{\partial x}{}_BT_n^{(r)}(x,y;u,\lambda) = n\,{}_BT_{n-1}^{(r)}(x,y;u,\lambda). \tag{28}
$$

Proof. Applying $\frac{\partial}{\partial y}$ to (5),

$$
\begin{aligned}
\frac{\partial}{\partial x}\sum_{n=0}^{\infty}{}_BT_n^{(r)}(x,y;u,\lambda)\frac{t^n}{n!} &= \frac{\partial}{\partial x}\left(\frac{(1-u)}{\lambda e^{2t}-u}\right)^r e^{xt+y(e^t-1)}\\
\sum_{n=0}^{\infty}\frac{\partial}{\partial x}{}_BT_n^{(r)}(x,y;u,\lambda)\frac{t^n}{n!} &= \left(\frac{(1-u)}{\lambda e^{2t}-u}\right)^r e^{xt+y(e^t-1)}\ t\\
&= t\sum_{n=0}^{\infty}{}_BT_n^{(r)}(x,y;u,\lambda)\frac{t^n}{n!}\\
&= \sum_{n=0}^{\infty}{}_BT_n^{(r)}(x,y;u,\lambda)\frac{t^{n+1}}{n!}
\end{aligned}
$$

$$= \sum_{n=1}^{\infty} n {}_BT_{n-1}^{(r)}(x,y;u,\lambda)\frac{t^n}{n!}.$$

Consequently,

$$\frac{\partial}{\partial x} {}_BT_n^{(r)}(x,y;u,\lambda) = n {}_BT_{n-1}^{(r)}(x,y;u,\lambda).$$

Remark 2. *The relation in (28) shows that the sequence of polynomials ${}_BT_n^{(r)}(x,y;u,\lambda)$ satisfy (9), thus ${}_BT_n^{(r)}(x,y;u,\lambda)$ is a sequence of Appell polynomials. The polynomials ${}_BT_n^{(r)}(x,y;u,\lambda)$ are anticipated to exhibit the following characteristics:*

(1) Equation (5) reflects (10), that is,

$$\left(\frac{1-u}{\lambda e^{2t}-u}\right)^r e^{y(e^t-1)}e^{xt} = \sum_{n=0}^{\infty} {}_BT_n^{(r)}(x,y;u,\lambda)\frac{t^n}{n!}$$

where $A(t) = \left(\frac{1-u}{\lambda e^{2t}-u}\right)^r e^{y(e^t-1)}$ is independent of x with $A(0) \neq 0$.

(2) Result in (17) demonstrates (11) and (12),

$$\begin{aligned} {}_BT_n^{(r)}(x,y;u,\lambda) &= \sum_{j=0}^{n} \binom{n}{j} c_j x^{n-j} \\ {}_BT_n^{(r)}(x,y;u,\lambda) &= \left(\sum_{j=0}^{n} \frac{c_j}{j!} D^j\right) x^n \end{aligned}$$

where $c_j = {}_BT_j^{(r)}(y;u,\lambda)$ and $D = \frac{d}{dx}$.

The last result shows the derivative of ${}_BT_n^{(r)}(x,y;u,\lambda)$ with respect to y.

Theorem 11. *The derivative formula given by*

$$\frac{\partial}{\partial y} {}_BT_n^{(r)}(x,y;u,\lambda) = {}_BT_n^{(r)}(x+1,y;u,\lambda) - {}_BT_n^{(r)}(x,y;u,\lambda) \tag{29}$$

holds for ${}_BT_n^{(r)}(x,y;u,\lambda)$.

Proof. Applying $\frac{\partial}{\partial x}$ to both sides of (5)

$$\sum_{n=0}^{\infty} \frac{\partial}{\partial y} {}_BT_n^{(r)}(x,y;u,\lambda)\frac{t^n}{n!} = \left(\frac{(1-u)}{\lambda e^{2t}-u}\right)^r e^{xt+y(e^t-1)}(e^t-1)$$

$$
\begin{aligned}
&= \left(\frac{(1-u)}{\lambda e^{2t}-u}\right)^r e^{(x+1)t+y(e^t-1)} - \left(\frac{(1-u)}{\lambda e^{2t}-u}\right)^r e^{xt+y(e^t-1)} \\
&= \sum_{n=0}^{\infty} {}_BT_n^{(r)}(x+1,y;u,\lambda)\frac{t^n}{n!} - \sum_{n=0}^{\infty} {}_BT_n^{(r)}(x,y;u,\lambda)\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \left\{ {}_BT_n^{(r)}(x+1,y;u,\lambda) - {}_BT_n^{(r)}(x,y;u,\lambda)\right\} \frac{t^n}{n!} \\
\frac{\partial}{\partial y} {}_BT_n^{(r)}(x,y;u,\lambda) &= {}_BT_n^{(r)}(x+1,y;u,\lambda) - {}_BT_n^{(r)}(x,y;u,\lambda).
\end{aligned}
$$

Remark 3. *Combining the results from (25) and (29), we obtain the equation*

$$
\frac{\partial}{\partial y} {}_BT_n^{(r)}(x,y;u,\lambda) = \sum_{k=0}^{n-1} \binom{n}{k} {}_BT_k^{(r)}(x,y;u,\lambda). \tag{30}
$$

To see this, consider the example below. We use (5) to get the following polynomials

$$
\begin{aligned}
{}_BT_0^{(r)}(x,y;u,\lambda) &= \left(\frac{u-1}{u-\lambda}\right)^r, \\
{}_BT_1^{(r)}(x,y;u,\lambda) &= \frac{\left(\frac{u-1}{u-\lambda}\right)^r (\lambda(2r-x-y)+u(x+y))}{u-\lambda}, \\
{}_BT_2^{(r)}(x,y;u,\lambda) &= (x+y)^2\left(\frac{1-u}{\lambda-u}\right)^r + y\left(\frac{1-u}{\lambda-u}\right)^r \\
&\quad + \frac{8\lambda^2 r(1-u)\left(\frac{1-u}{\lambda-u}\right)^{r-1}}{(\lambda-u)^3} + \frac{4\lambda^2(r-1)r(1-u)^2\left(\frac{1-u}{\lambda-u}\right)^{r-2}}{(\lambda-u)^4} \\
&\quad - \frac{2\lambda r(1-u)(x+y)\left(\frac{1-u}{\lambda-u}\right)^{r-1}}{(\lambda-u)^2} - \frac{2\lambda r(1-u)(x+y+2)\left(\frac{1-u}{\lambda-u}\right)^{r-1}}{(\lambda-u)^2}.
\end{aligned}
$$

One can verify using the above polynomials that

$$
\frac{\partial}{\partial y} {}_BT_2^{(r)}(x,y;u,\lambda) = \binom{2}{0} {}_BT_0^{(r)}(x,y;u,\lambda) + \binom{2}{1} {}_BT_1^{(r)}(x,y;u,\lambda).
$$

Acknowledgements

This research has been funded by Cebu Normal University (CNU) through its Center for Research and Development (CRD).

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