EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 1, Article Number 5560
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Approximation Theorems for Exponentially Bounded K-Convoluted C-Cosine Functions

Youssef Bajjou[1,*], Abdelkhalek El Amrani[1], Aziz Blali[3]

[1] *Department of Mathematics, Dhar El Mahraz Faculty of Sciences, Sidi Mohamed Ben Abdellah University, Atlas Fez, Morocco*
[2] *Department of Mathematics, Higher Normal School, Sidi Mohamed Ben Abdellah University, B.P. 5206 Bensouda-Fez, Morocco*

Abstract. Let $C : E \to E$ be a bounded linear operator on a complex Banach space E and $K : [0, +\infty[\to \mathbb{C}$ a locally integrable function. The aim of this paper, based on the theory of K-convoluted C-cosine functions, is to study the approximation theorem for K-convoluted C-cosine functions by showing the relation between the convergence of the sequence of C-resolvent and the exponentially bounded sequence of K-convoluted C-cosine functions.

2020 Mathematics Subject Classifications: 46A32, 47D09, 47A58, 60J35

Key Words and Phrases: K-convoluted C-cosine functions, C-resolvent, Approximation

1. Introduction

Throughout this paper E denote a non-trivial complex Banach space, $L(E)$ denotes the Banach algebra of bounded linear operators from E into E, C is an injective element of $L(E)$. For a linear operator A acting on E, $D(A)$, $N(A)$, $R(A)$ and $\rho_C(A)$, denotes its domain (equipped with the graph norm), kernel, range and the C-resolvent set of A, defined by $\rho_C(A) := \{\lambda \in \mathbb{C} \mid R(C) \subseteq R(\lambda I - A) \ and \ \lambda I - A$ is injective in $B(E)\}$ and if $\lambda \in \rho_C(A)$ then we denoted by $R_C(\lambda, A)$ the C-resolvent defined by $R_C(\lambda, A) = (\lambda I - A)^{-1}C$. If $t \in \mathbb{R}$, $\lfloor t \rfloor = \sup\{n \in \mathbb{Z}, n \leq t\}$ denotes the integer part of t. K is a complex-valued locally integrable function in $[0, +\infty[$ (ie $K \in L^1_{loc}([0, +\infty[))$, not identical to zero such that:

- (P): K is Laplace transformable, that is to say there exists $\beta \in \mathbb{R}$ so that $L(K)(\lambda) = \int_0^{+\infty} e^{-\lambda t} K(t)dt < +\infty$ for all $\lambda \in \mathbb{C}$ with $Re(\lambda) > \beta$. Put $abs(K) := \inf\{Re(\lambda) : L(K)(\lambda) < +\infty\}$.
- (Q): For all $\lambda > abs(K)$, $L(K)(\lambda) \neq 0$.

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v18i1.5560

Email addresses: youssefbajjou2017@gmail.com (Y. Bajjou), abdelkhalek.elamrani@usmba.ac.ma (A. El Amrani), aziz.blali@usmba.ac.ma (A. Blali)

- (R): $0 \in supp(K)$ (According to Titchmarsh's theorem [[2]], for every $\varphi \in C([0,+\infty[$, the assumption
  for all $t \in [0,+\infty[,\ \int_0^t K(t-s)\varphi(s)ds = 0$ implies $\varphi \equiv 0$).

For example the following function is a kernel:

$$K(t) := \frac{1}{2\sqrt{2\pi t^3}} e^{\frac{-1}{4t}}\ if\ t > 0\ and\ K(0) = 0$$

see [1]. We can define on $[0,+\infty[$, the absolutely continuous function by

$$for\ all\ t \geq 0,\ \Theta(t) := \int_0^t K(s)ds,$$

then

$$for\ all\ t \geq 0,\ \Theta'(t) = K(t)\ a.e\ t \in [0,+\infty[.$$

We let

$$l^{\infty}(E) = \{(x_k)_{k\in\mathbb{N}} : x_k \in E\ \ and\ \ \sup_{k\in\mathbb{N}} \mid x_k \mid < +\infty\}$$

the Banach space equipped with the norm

$$\| (x_k)_{k\in\mathbb{N}} \| = \sup_{k\in\mathbb{N}} \mid x_k \mid$$

for all sequence $x = (x_k)_{k\in\mathbb{N}} \in l^{\infty}(E)$ and $c(E)$, the closed subspace of $l^{\infty}(E)$, defined by

$$c(E) = \{(x_k)_{k\in\mathbb{N}} :\ x_k \in E\ \ and\ \ \lim_{k\to\infty} x_k\ \ exists\}.$$

See [1] for more details.
In this work we will use the theory of integration in the sense of Bochner.

2. K-convoluted C-cosine function

A strongly continous operator family $(C(t))_{t\geq 0}$ such that:

- For all $t \geq 0\ \ C(t)A \subseteq AC(t)$,
- For all $t \geq 0\ \ C(t)C \subseteq CC(t)$,
- For all $x \in E$ and $t \geq 0$

$$\int_0^t (t-s)C(s)xds \in D(A) \text{ and } A\int_0^t C(s)xds = C(t)x - \Theta(t)Cx,$$

- There exist $M \geq 1,\ there\ exist\ \omega \geq 0 :\ for\ all\ t \geq 0,\ \| C(t) \| \leq Me^{\omega t}$, is called an exponentially bounded $K-$convoluted $C-$cosine function with subgenerator A. We can prove that $CA \subset AC$ see [9]. For example, if $K(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ for some $\alpha \geq 0$ a $K-$convoluted $C-$cosine function on E is called an $\alpha-$times integrated $C-$cosine function on E see [10] and [13] for more details. We say that $(C(t))_{t\geq 0}$ is non-degenerate if additionnally $C(t)x = 0$ for all $t \geq 0$ implies that $x = 0$, since C is injective then each $K-$convoluted C-cosine function is no degenerate (see [11], [3], [6], [12], [8], [7] and [4]). If $(C(t))_{t\geq 0}$ is $K-$convoluted C-cosine function then the following formulae holds:

$$2C(t)C(s)x = \{\int_0^{t+s} - \int_0^{t} - \int_0^{s}\}K(t+s-r)C(r)xdr + \int_{|t-s|}^{t} K(s-t+r)C(r)Cx \quad (1)$$

$$+ \quad \int_{|t-s|}^{s} K(t-s+r)C(r)Cxdr + \int_0^{|t-s|} K(\mid t-s \mid +r)C(r)Cxdr$$

for all $t, s \geq 0$ *and* $x \in E$, see chapter 2 theorem 2.1.13 of [5].
For a $K-$convoluted $C-$cosine function $(C(t))_{t\geq 0}$, we define its integral generator $\hat{A} : D(\hat{A}) \subset E \to E$ by

$$D(\hat{A}) = \{x \in E : (\exists y_x \in E) : C(t)x - \Theta(t)Cx = \int_0^t (t-s)C(s)y_x ds \ for\ all\ t \geq 0\}$$

and $\hat{A}x = y_x$ for all $x \in D(\hat{A})$. $\hat{A}$ is a closed operator which is an extension of any subgenerator of $(C(t))_{t\geq 0}$, $C^{-1}AC = \hat{A}$ and $(C(t))_{t\geq 0}$ is uniquely determined by one of its subgenerators see [9]. In the rest of this part, let $M > 0$, $\omega \geq \max(0, abs(K))$ and let's put $\omega_1 = \max(\omega, abs(K))$. Suppose that $(A, D(A))$ is closed linear operator and $(C(t))_{t\geq 0}$ is strongly continuous operator family and for all $t \geq 0$ $\parallel C(t) \parallel \leq Me^{\omega t}$ then we have the following useful properties:

(i)

- (i) Assume that A is a subgenerator of an exponentially bounded, $K-$convoluted $C-$cosine function $(C(t))_{t\geq 0}$ then

  $$\{\lambda^2 : \ \Re(\lambda) > \omega_1 \ L(K)(\lambda) \neq 0\} \subset \rho_C(A), \quad (2)$$

  and

  $$\lambda(\lambda^2 - A)Cx = \frac{1}{L(K)(\lambda)} \int_0^{+\infty} e^{-\lambda t}C(t)xdt, \ x \in E, \Re(\lambda) > \omega_1, \ L(K)(\lambda) \neq 0. \quad (3)$$

  For more details see [9] and [5].

- (ii) Suppose that the family $(C(t))_{t\geq 0}$ satisfies the two conditions (2)-(3), then $(C(t))_{t\geq 0}$ is an exponentially bounded, $K-$convoluted $C-$cosine function with subgenerator A. For more details see [9] and [5].

- (iii) Assume that (2)-(3) hold only for real values of λ's, then $(C(t))_{t\geq 0}$ is still an exponentially bounded, $K-$convoluted $C-$cosine function with subgenerator A. For more details see [[9]].

(ii) Put for all $x \in E$ and $\lambda > \omega$, $R_{\lambda^2}x := \frac{1}{\lambda L(K)(\lambda)} \int_0^{+\infty} e^{-\lambda t} C(t)xdt$. Then for all $\lambda, \mu > \omega$ and all $x \in E$, $(\lambda^2 - \mu^2)R_{\lambda^2}R_{\mu^2}x = R_{\mu^2}Cx - R_{\lambda^2}Cx$ if the formula (1) holds for all $x \in E$ and $s \geq 0$. For more details see [13].

Remark 1. *(i) If for all $t \geq 0$, $CC(t) = C(t)C$ then for all $\lambda > \omega$, $CR_{\lambda^2} = R_{\lambda^2}C$. Indeed for all $x \in E$ and all $\lambda > \omega$ we have:*

$$\begin{aligned} R_{\lambda^2}Cx &= \frac{1}{\lambda L(K)(\lambda)} \int_0^{+\infty} e^{-\lambda t} C(t)Cxdt \\ &= \frac{1}{\lambda L(K)(\lambda)} \int_0^{+\infty} Ce^{-\lambda t} C(t)xdt \\ &= C\frac{1}{\lambda L(K)(\lambda)} \int_0^{+\infty} e^{-\lambda t} C(t)xdt \\ &= CR_{\lambda^2}x. \end{aligned}$$

(ii) We assumed that for all $\lambda, \mu > \omega$ and all $x \in E$,

$$(\lambda^2 - \mu^2)R_{\lambda^2}R_{\mu^2} = R_{\mu^2}Cx - R_{\lambda^2}Cx,$$

then for all $\lambda, \mu > \omega$ $R_{\lambda^2}R_{\mu^2} = R_{\mu^2}R_{\lambda^2}$. Indeed for all $x \in E$ and all $\lambda, \mu > \omega$ we have

$$\begin{aligned} 0 &= (R_\lambda^2 Cx - R_\mu^2 Cx) + (R_\mu^2 Cx - R_\lambda^2 Cx) \\ &= (\mu^2 - \lambda^2)R_{\lambda^2}R_{\mu^2}x + (\lambda^2 - \mu^2)R_{\mu^2}R_{\lambda^2} \\ &= (\lambda^2 - \mu^2)(R_{\mu^2}R_{\lambda^2}x - R_{\lambda^2}R_{\mu^2}x) \end{aligned}$$

(iii) We assumed that $CC(.) = C(.)C$ and for all $\lambda, \mu > \omega$ and all $x \in E$,

$$(\lambda^2 - \mu^2)R_{\lambda^2}R_{\mu^2} = R_{\mu^2}Cx - R_{\lambda^2}Cx,$$

then

- *$N(R_{\lambda^2})$ is independent of $\lambda > \omega$. Indeed for all $\lambda > \omega$, all $x \in E$ such that $R_{\lambda^2}x = 0$ and for all $\mu > \omega$ we have*

$$\begin{aligned} 0 &= CR_{\lambda^2}x \\ &= R_{\lambda^2}Cx \\ &= (R_{\lambda^2}Cx - R_{\mu^2}Cx) + R_{\mu^2}Cx \\ &= (\mu^2 - \lambda^2)R_{\lambda^2}R_{\mu^2}x + R_{\mu^2}Cx \\ &= (\mu^2 - \lambda^2)R_{\mu^2}R_{\lambda^2}x + R_{\mu^2}Cx \\ &= 0 + R_{\mu^2}Cx \\ &= R_{\mu^2}Cx \\ &= CR_{\mu^2}x, \end{aligned}$$

hence $R_{\mu^2}x = 0$ since C is injective.

- *If $R(R_{\lambda^2}) \subset R(C)$ then $R(R_{\lambda^2})$ is independent of $\lambda > \omega$. Indeed for all $\lambda > \omega$, all $y \in R(R_{\lambda^2})$ such that $y = R_{\lambda^2}x$ and for all $\mu > \omega$ we have*

$$\begin{aligned} CR_{\mu^2}(x + (\mu^2 - \lambda^2)C^{-1}y) &= CR_{\mu^2}x + (\mu^2 - \lambda^2)CR_{\mu^2}C^{-1}y \\ &= R_{\mu}^2 Cx + (\mu^2 - \lambda^2)R_{\mu}^2 CC^{-1}y \\ &= R_{\mu^2}Cx + (\mu^2 - \lambda^2)R_{\mu^2}y \\ &= R_{\mu^2}Cx + (\mu^2 - \lambda^2)R_{\mu^2}R_{\lambda^2}x \\ &= R_{\mu^2}Cx + (R_{\lambda^2}Cx - R_{\mu^2}Cx) \\ &= R_{\lambda^2}Cx \\ &= CR_{\lambda^2}x \\ &= Cy, \end{aligned}$$

hence $y = R_{\mu^2}(x + (\mu^2 - \lambda^2)C^{-1}y) \in R(R_{\mu^2})$ since C is injective.

- *If $R(R_{\lambda^2}) \subset R(C)$ and there exists $\mu > \omega$ such that $N(R_{\mu^2}) = \{0\}$ then there is a linear operator $(A, D(A))$ such that $R_C(\lambda, A) = R_{\lambda^2}$. Indeed*

$for\ all\ \lambda, \mu > \omega,\ for\ all\ y \in R(R_{\lambda^2}),\ there\ is\ a\ unique\ x_{\lambda^2}, x_{\mu^2}) \in E^2 :\ y = R_{\lambda^2}x_{\lambda^2} = R_{\mu^2}x_{\mu^2}.$

On the other hand, if we put

$$W = R_{\lambda^2}R_{\mu^2}((\mu^2 y - Cx_{\mu^2}) - (\lambda^2 y - Cx_{\lambda^2})),$$

then we have :

$$\begin{aligned} W &= R_{\lambda^2}R_{\mu^2}((\mu^2 - \lambda^2)y - C(x_{\mu^2} - x_{\lambda^2})) \\ &= (\mu^2 - \lambda^2)R_{\lambda^2}R_{\mu^2}y - CR_{\lambda^2}R_{\mu^2}(x_{\mu^2} - x_{\lambda^2}) \\ &= (R_{\lambda^2}Cy - R_{\mu^2}Cy) - (CR_{\lambda^2}y - CR_{\mu^2}y) \\ &= (R_{\lambda^2}Cy - R_{\mu^2}Cy) - (R_{\lambda^2}Cy - R_{\mu^2}Cy) \\ &= 0 \end{aligned}$$

It is $(\mu^2 y - Cx_{\mu^2}) = (\lambda^2 y - Cx_{\lambda^2})$ since $R_{\lambda^2}R_{\mu^2}$ is injective, so we cane defined the operator $(A, D(A))$ by $D(A) = R(R_{\mu^2})$ and for all $y \in R(R_{\lambda^2})$, $Ay = \lambda^2 y - CR_{\lambda^2}^{-1}y$, since $R_{\lambda^2} \in B(E)$, moreover for all $y \in R(R_{\lambda^2})$ we have

$$\begin{aligned} CR_{\lambda^2}^{-1}(y) &= \lambda^2 y - Ay \\ &= (\lambda^2 I - A)y \end{aligned}$$

as result $R_{\lambda^2} = (\lambda^2 I - A)^{-1}C = R_C(\lambda^2, A)$.

(iv) *If $(C(t))_{t\geq 0}$ is K-convoluted $C-$cosine function with subgenerator $(A, \mathcal{D}(A))$ such that for all $t \geq 0$, $\| C(t) \| \leq Me^{\omega t}$, then for all $\lambda > \omega$, $R_{\lambda^2} = R_C(\lambda^2, A)$.*

3. Main results

Theorem 1. *Let $(C(t))_{t\geq 0}$ be a $K-$convoluted $C-$cosine functions with subgenerator A. For each $n \in \mathbb{N}$, let $(C_n(t))_{t\geq 0}$ a $K-$convoluted $C-$cosine function with subgenerator $(A_n, \mathcal{D}(A_n))$ and suppose that there exist $\omega \geq 0$ and $M > 0$ such that for all $t \geq 0$ and all $x \in E$, $\parallel C(t)x \parallel \leq Me^{\omega t}$ and for all $n \in \mathbb{N}$ $\parallel C_n(t)x \parallel \leq Me^{\omega t}$. If we put $\omega_1 = \max(\omega + 1, abs(K))$, then the following statements are equivalent:*

(i) *There exist $\lambda_0 > \omega_1)$: $for\ all\ x \in E$ $\lim\limits_{n\to+\infty} R_C(\lambda_0^2, A_n)x = R_C(\lambda_0^2, A)x$ and $(C_n(.)x)_{n\in\mathbb{N}}$ is equicontinuous.*

(ii) *There exist $\lambda_0 > \omega_1$: $for\ all\ y \in R(C))$ $\lim\limits_{n\to+\infty}(\lambda^2 I - A_n)^{-1}y = (\lambda^2 I - A)^{-1}y$ and for all $x \in E$, $(C_n(.)x)_{n\in\mathbb{N}}$ is equicontinuous.*

(iii) *For all $\lambda > \omega_1$, $for\ all\ y \in R(C))$, $\lim\limits_{n\to+\infty}(\lambda^2 I - A_n)^{-1}y = (\lambda^2 I - A)^{-1}y$ and for all $x \in E$, $(C_n(.)x)_{n\in\mathbb{N}}$ is equicontinuous.*

(iv) *For all $\lambda > \omega_1$, $for\ all, x \in E$, $\lim\limits_{n\to+\infty} R_C(\lambda^2, A_n)x = R_C(\lambda^2, A)x$ and $(C_n(.)x)_{n\in\mathbb{N}}$ is equicontinuous.*

(v) *For all $t \geq 0$, $for\ allx \in E$, $\lim\limits_{n\to+\infty} C_n(t)x = C(t)x$, the convergence is uniform on any compact of $[0, +\infty[$.*

Proof. $\underline{1 \Rightarrow 2}$ |
Like $R_C(\lambda_0^2, A_n) = (\lambda_0^2 I - A_n)^{-1}C$ and $R_C(\lambda_0^2, A) = (\lambda^2 I - A)^{-1}C$, then the proof is obvious.
$\underline{2 \Rightarrow 3}$ |
Let's pose

$$U = \{\lambda > \omega_1 : L(K)(\lambda) \neq 0\}\ (=]\omega_1, +\infty[)$$

and

$$V = \{\lambda \in U :\ for\ all\ y \in R(C),\ \lim_{n\to+\infty}(\lambda^2 I - A_n)^{-1}y = (\lambda^2 I - A)^{-1}y\}.$$

According to the statements of (2), V is a nonempty set.
Let be $\lambda \in V$ fixed and $n \in \mathbb{N}$, then for μ in the open set O_λ, where

$$O_\lambda := \{\mu \in U :\parallel (\mu^2 - \lambda^2)(\lambda^2 I - A_n)^{-1} \parallel < \frac{1}{4}\},$$

we have $\parallel (\mu^2 - \lambda^2)(\lambda^2 I - A_n)^{-1} \parallel < \frac{1}{4} < 1$ so $I - (\mu^2 - \lambda^2)(\lambda^2 I - A_n)^{-1}$ is invertible, the series $\sum\limits_{k\geq 0} \left((\mu^2 - \lambda^2)(\lambda^2 I - A_n)^{-1}\right)^k$ is uniformly convergent and

$$(I - (\mu^2 - \lambda^2)(\lambda^2 I - A_n)^{-1})^{-1} = \sum_{k=0}^{+\infty}\{(\mu^2 - \lambda^2)(\lambda^2 I - A_n)^{-1}\}^k.$$

But

$$\begin{aligned}\mu^2 I - A_n &= (\mu^2-\lambda^2)I + (\lambda^2 I - A_n)\\ &= \left((\mu^2-\lambda^2)(\lambda^2 I - A_n)^{-1} + I\right)(\lambda^2 I - A_n)\\ &= \left((I - (\mu^2-\lambda^2)(\lambda^2 I - A_n)^{-1}\right)(\lambda^2 I - A_n),\end{aligned}$$

so

$$\begin{aligned}(\mu^2 I - A_n)^{-1} &= (\lambda^2 I - A_n)^{-1}\left((I - (\mu^2-\lambda^2))(\lambda^2 I - A_n)^{-1}\right)^{-1}\\ &= (\lambda^2 I - A_n)^{-1}\sum_{k=0}^{+\infty}\left((\mu^2-\lambda^2)(\lambda^2 I - A_n)^{-1}\right)^k\\ &= \sum_{k=0}^{+\infty}(\mu^2-\lambda^2)^k\left((\lambda^2 I - A_n)^{-1}\right)^{k+1},\end{aligned}$$

and for all $y \in R(C)$

$$\begin{aligned}\lim_{n\to+\infty}(\mu^2 I - A_n)^{-1}y &= \lim_{n\to+\infty}\sum_{k=0}^{+\infty}(\mu^2-\lambda^2)^k\left((\lambda^2 I - A_n)^{-1}\right)^{k+1}y\\ &= \sum_{k=0}^{+\infty}\lim_{n\to+\infty}(\mu^2-\lambda^2)^k\left((\lambda^2 I - A_n)^{-1}\right)^{k+1}y\\ &= \sum_{k=0}^{+\infty}(\mu^2-\lambda^2)^k\left((\lambda^2 I - A)^{-1}\right)^{k+1}y\\ &= (\lambda^2 I - A)^{-1}\sum_{k=0}^{+\infty}(\mu^2-\lambda^2)\left((\lambda^2 I - A)^{-1}\right)^k y\\ &= (\mu^2 I - A)^{-1}y\\ &\quad (because \lim_{n\to+\infty}\parallel(\mu^2-\lambda^2)(\lambda^2 I - A_n)^{-1}\parallel < 1).\end{aligned}$$

So we can conclude that for all $\lambda \in V$ there exists an open set O_λ such that $O_\lambda \subset V$; therefore V is an open set.
Let be $(\lambda_k)_{k\in\mathbb{N}}$ a sequence in V such that $\lim\limits_{k\to+\infty}\lambda_k = \lambda$ and $\lambda \in U$, let's show that $\lambda \in V$. Like for $n \in \mathbb{N}$, the open set $O'_\lambda := \{\mu \in U : \parallel (\mu^2-\lambda^2)(\mu^2 I - A_n)^{-1} \parallel < \frac{1}{4}\}$ contains λ therefore there exists $\lambda_{k_0} \in U$ such that $\lambda_{k_0} \in O'_\lambda$; but $(\lambda^2 I - A_n)^{-1} = \sum\limits_{k=0}^{+\infty}\left((\lambda^2-\lambda_{k_0}^2)^k(\lambda_{k_0}^2 I - A_n)^{-1}\right)^{k+1}$ whose series converges uniformly on O'_λ and as the function $\beta \longmapsto (\beta^2 I - A_n)^{-1}$ is continuous from $]\omega_1, +\infty[$ to $B(E)$ then

$$\lim_{n\to+\infty}(\lambda^2 I - A_n)^{-1} = \lim_{n\to+\infty}\sum_{k=0}^{+\infty}\left((\lambda^2-\lambda_{k_0}^2)^k(\lambda_{k_0}^2 I - A_n)^{-1}\right)^{k+1}$$

$$\begin{aligned}
&= \sum_{k=0}^{+\infty} \lim_{n\to+\infty} (\lambda^2-\lambda_{k_0}^2)^k \left((\lambda_{k_0}^2 I - A_n)^{-1}\right)^{k+1} \\
&= \sum_{k=0}^{+\infty} (\lambda^2-\lambda_{k_0}^2)^k \left((\lambda_{k_0}^2 I - A)^{-1}\right)^{k+1} \\
&= (\lambda^2 I - A)^{-1},
\end{aligned}$$

the last equality is due to the following inequality

$$\lim_{n\to+\infty} \parallel (\lambda^2-\lambda_{k_0}^2)(\lambda_{k_0}^2 I - A_n)^{-1} \parallel \le \frac{1}{4} < 1,$$

so $\lambda \in V$. Therefore V is relatively closed from U.
Finally the set V is both an open and a closed of the connected set U, whence $V = U$.
$\underline{3 \Rightarrow 4}$ |
Obvious.
$\underline{4 \Rightarrow 5}$ |
Suppose that the conditions of statement 4 are satisfied.
Let $x \in E$ be fixed. We define, for each $n \in \mathbb{N}$, the following functions:
$f_n : \mathbb{R}^+ \to E,\ t \mapsto C_n(t)x.$
$f : \mathbb{R}^+ \to l^\infty(E),\ t \mapsto (f_n(t))_{n\in\mathbb{N}}.$
$F_n :]\omega_1, +\infty[\to E,\ \lambda \mapsto \lambda L(K)(\lambda) R_C(\lambda^2, A_n)x.$
$F :]\omega_1, +\infty[\to l^\infty(E),\ \lambda \mapsto (F_n(\lambda)_{n\in\mathbb{N}}).$
$g_n : \mathbb{R}^+ \to E,\ t \mapsto \int_0^t (t-s) f_n(s) ds.$
$g : \mathbb{R}^+ \to E,\ t \mapsto (g_n(t))_{n\in\mathbb{N}}.$

(i)
- a) $\underline{f\ is\ well\ defined}$.
  Let t be a positive real.
  We have
  $$for\ all\ n \in \mathbb{N},\ \parallel f_n(t) \parallel \le M \parallel x \parallel e^{\omega_1 t},$$
  so for all $t \ge 0$
  $$\parallel (f_n(t))_{n\in\mathbb{N}} \parallel_\infty \le M \parallel x \parallel e^{\omega_1 t} < +\infty,$$
  therfore the function f is well defined, and since the sequence $(f_n)_{n\in\mathbb{N}}$ is equicontinuous, the function f is continuous.
- b) $\underline{F\ has\ value\ in\ c(E)}$.
  Let λ in $]\omega_1, +\infty[$.
  We now have Theorem 1, for all $n \in \mathbb{N}$,
  $$\begin{aligned}
  F_n(\lambda) &= \lambda L(K)(\lambda) R_C(\lambda^2, A_n)x \\
  &= \int_0^{+\infty} e^{-\lambda t} C(t)x dt \\
  &= \int_0^{+\infty} e^{-\lambda t} f_n(t) dt,
  \end{aligned}$$

so *For all* $n \in \mathbb{N}$, $\| F_n(\lambda) \| \leq \frac{M\|x\|}{\lambda-\omega}$, therefore

$$\begin{aligned}\| F(\lambda) \|_\infty &= \| (F_n(\lambda))_{n\in\mathbb{N}} \|_\infty \\ &\leq \frac{M \| x \|}{\lambda - \omega_1} \\ &< +\infty,\end{aligned}$$

and by hypothesis

$$\lim_{n\to+\infty} F_n(\lambda) = \lim_{n\to+\infty} \lambda L(K)(\lambda) R_C(\lambda^2, A_n)x = \lambda L(K)(\lambda) R_C(\lambda^2, A)x$$

which give $F(\lambda) \in c(E)$.

- c) $F \in C^\infty(]\omega_1, +\infty[, l^\infty(E))$ *and for all* $k \in \mathbb{N}$ *for all* $\lambda > \omega_1$ $F^{(k)}(\lambda) \in c(E)$.

Let t and h be a positive reals. For all $n \in \mathbb{N}$ we have

$$\begin{aligned}\| g_n(t) \| &\leq \int_0^t (t-s) \| f_n(s) \| \, ds \\ &\leq \frac{M \| x \|}{\omega} t e^{\omega t} \\ &\leq \frac{M \| x \|}{\omega} e^{(\omega+1)t} \\ &\leq \frac{M \| x \|}{\omega} e^{\omega_1 t}.\end{aligned}$$

So $\| g(t) \|_\infty = \| (g_n(t))_{n\in\mathbb{N}} \|_\infty \leq \frac{M\|x\|}{\omega} e^{\omega_1 t} < +\infty$
which gives that $g(t) \in l^\infty(E)$. And for all $n \in \mathbb{N}$:

$$\begin{aligned}\| g_n(t+h) - g_n(t) \| &= \| \int_t^{t+h} (t-s) f_n(s) ds + h \int_0^{t+h} f_n(s) ds \| \\ &\leq h\{ \int_t^{t+h} \| f_n(s) \| \, ds + \int_0^{t+h} \| f_n(s) \| \, ds\} \\ &\leq \frac{2hM \| x \|}{\omega_1} e^{\omega_1 (t+h)}.\end{aligned}$$

So $\| g(t+h) - g(t) \| \leq \frac{2hM\|x\|}{\omega} e^{\omega(t+h)}$, from where g is continuous at t. So the function g is well defined and continuous.
On the other hand, the function $I_d : \mathbb{R}^+ \to \mathbb{R}^+,\ t \mapsto t$ is continuous and for all $\lambda > 0$ we have

$$L(I_d)(\lambda) = \int_0^{+\infty} e^{-\lambda s} s ds = \int_0^{+\infty} s e^{-\lambda s} ds = L(I_d)(\lambda) = \frac{1}{\lambda^2},$$

then the proposition 1.6.4 from [1] give $L(I_d * f)(\lambda)$ exists for all $\lambda > \omega_1$ and

$$L(I_d * f)(\lambda) = L(I_d)(\lambda) L(f)(\lambda)$$

$$\begin{aligned}
&= \frac{1}{\lambda^2}L(f)(\lambda)\\
&= \frac{1}{\lambda^2}\int_0^{+\infty} e^{-\lambda t}f(t)dt\\
&= \frac{1}{\lambda^2}\int_0^{+\infty} (e^{-\lambda t}f_n(t))_{n\in\mathbb{N}}dt\\
&= \frac{1}{\lambda^2}(\int_0^{+\infty} e^{-\lambda t}f_n(t)dt)_{n\in\mathbb{N}}\\
&= \frac{1}{\lambda^2}(F_n(\lambda)_{n\in\mathbb{N}}\\
&= \frac{1}{\lambda^2}F(\lambda),
\end{aligned}$$

but

$$\begin{aligned}
L(I_d * f)(\lambda) &= \int_0^{+\infty} e^{-\lambda t}(I_d * f)(t)dt\\
&= \int_0^{+\infty} e^{-\lambda t}\int_0^t (t-s)f(s)dsdt\\
&= \int_0^{+\infty} e^{-\lambda t}g(t)dt\\
&= L(g)(\lambda)
\end{aligned}$$

from which follows the equality $F(\lambda) = \lambda^2 L(g)(\lambda)$, according to Theorem 1.5.1 of [1], $L(g)$ (so F) is infinitely differentiable on $]\omega_1, +\infty[$ and since $c(E)$ is closed of $l^\infty(E)$ then

$$For\ all\ k \in \mathbb{N},\ for\ all\ \lambda \in]\omega_1, +\infty[,\quad F^{(k)}(\lambda) \in c(E).$$

- d) $\lim\limits_{n\to+\infty} C_n(t)xdt = C(t)x$.

We have

$$For\ all\ t > 0,\ there\ exist\ k_t \in \mathbb{N}:\ for\ all\ k \geq k_t,\ (-1)^k\frac{1}{k!}(\frac{k}{t})^{k+1}F^{(k)}(\frac{k}{t}) \in c(E),$$

it is that for all $t > 0$ there is $k_t \in \mathbb{N}$ such that

$$((-1)^k\frac{1}{k!}(\frac{k}{t})^{k+1}F^{(k)}(\frac{k}{t}))_{k\geq k_t}$$

is a sequence of elements of $c(E)$. f is continuous on $\mathbb{R}^+$, so each $t > 0$ is a Lebesgue point of f, the Post-Widder theorem (see theorem 1.7.7 of [1]) give for $t > 0$
$f(t) = \lim\limits_{k\to+\infty}(-1)^k\frac{1}{k!}(\frac{k}{t})^{k+1}\hat{f}^{(k)}(\frac{k}{t}) = \lim\limits_{k\to+\infty}(-1)^k\frac{1}{k!}(\frac{k}{t})^{k+1}F^{(k)}(\frac{k}{t})$. But $c(E)$ is closed then $f(t) = (f_n(t))_{n\in\mathbb{N}} \in c(E)$ therefore $\lim\limits_{n\to+\infty} f_n(t)$ exist and this

for all $t > 0$ but $f_n(0) = C_n(0)) = 0$, then if we noted by h the function $h : \mathbb{R}^+ \to E, t \mapsto h(t) = \begin{cases} \lim\limits_{k\to+\infty} f_n(t), & t > 0; \\ 0, & \text{t=0.} \end{cases}$ Then $(f_n)_{n\in\mathbb{N}}$ is a sequence of equicontinuous functions which converges pointwise to h, then h is continuous in $\mathbb{R}^+$. The convergence dominate theorem give that

$$\lim_{n\to+\infty} \int_0^{+\infty} e^{-\lambda t} C_n(t)xdt = \int_0^{+\infty} e^{-\lambda t} h(t)dt,$$

but

$$\begin{aligned}\lim_{n\to+\infty} \int_0^{+\infty} e^{-\lambda t} C_n(t)xdt &= \lim_{n\to+\infty} \lambda L(K)(\lambda) R_C(\lambda^2, A_n)x \\ &= \lambda L(K)(\lambda) R_C(\lambda^2, A)x,\end{aligned}$$

then

$$\lambda L(K)(\lambda) R_C(\lambda^2, A)x = \int_0^{+\infty} e^{-\lambda t} h(t)dt,$$

but $\{\lambda^2 : \ \lambda > \omega_1 \ \textit{and} \ L(K)(\lambda) \neq 0\} \subset \rho_C(A)$, then by the properties 1.(b) and 1.(c), we can deduce that h is K-convoluted C-cosine function generated by A, and like $(C(t))_{t\geq 0}$ is uniquely determined by one of its subgenerators, then $h(.) = C(.)x$, it is $\lim\limits_{n\to+\infty} C_n(t)x = C(t)x$, and this for all $t \geq 0$.

(ii) Let H be a compact of $[0, +\infty[$ and $x \in E$.
Like $H \subset [0, \sup(H)]$ then it suffices to prove that the convergence is uniform on the compact $[0, \sup(H)]$. For that let $\varepsilon > 0$, $(C_n(.)x)_{n\in\mathbb{N}}$ is equicontinuous at all $t \in [0, +\infty[$ so $(C_n(.)x)_{n\in\mathbb{N}}$ is equicontinuous in $[0, \sup(H)]$ which is compact, then $(C_n(.)x)_{n\in\mathbb{N}}$ is uniformly equicontinuous in $[0, \sup(H)]$ which implies the existence of $\eta > 0$ such that

$$(\forall s, t \geq 0) \ \mid t - s \mid < \eta \Longrightarrow (\forall n \in \mathbb{N}) \ \parallel C_n(t)x - C_n(s)x \parallel < \frac{\varepsilon}{3}. \tag{4}$$

For $n_0 = \lfloor \frac{\sup(H)}{\eta} \rfloor + 1 \in \mathbb{N}^*$, we have $\frac{\sup(H)}{n_0} < \eta$ (is therefore $for\ all$ $n \geq n_0$, $\frac{\sup(H)}{n} \leq \frac{\sup(H)}{n_0} < \eta$). For all $i \in \{0, ..., n_0\}$, $t_i = \frac{i}{n_0}\sup(H) \in [0, \sup(H)]$. So for each $t \in [0, \sup(H)]$ *there is* $i \in \{0, ..., n_0 - 1\}$ such that $t_i \leq t \leq t_{i+1}$. For all $i \in \{1, ..., n_0\}$, $(C_n(t_i)x)_{n\in\mathbb{N}}$ is a Cauchy sequence since it is convergent, therefore there exist $m_i \in \mathbb{N}^*$ such that for all $n, m \geq m_i$ $\parallel C_n(t_i)x - C_m(t_i)x \parallel \leq \frac{\varepsilon}{3}$.
If we posed $N_0 = \max\limits_{0\leq i\leq n_0} \{m_i\}$, then for all $m, n \geq N_0$ and all $t \in [0, \sup(H)]$, there exist $i_0 \in \{0, ..., n_0 - 1\}$ such that $t_{i_0} \leq t \leq t_{i_0+1}$, and then if $A_{n,m} = \parallel C_n(t)x - C_m(t)x \parallel$ we have

$$A_{n,m} = \parallel C_n(t)x - C_n(t_{i_0})x + C_n(t_{i_0})x - C_m(t_{i_0})x + C_m(t_{i_0})x - C_m(t)x \parallel$$

$$\begin{aligned}
&\leq \ \| C_n(t)x - C_n(t_{i_0})x \| + \| C_n(t_{i_0})x - C_m(t_{i_0})x \| + \\
&\quad \| C_m(t_{i_0})x - C_m(t)x \| \\
&\leq \ \frac{\varepsilon}{3} + \| C_n(t_{i_0})x - C_m(t_{i_0})x \| + \| C_m(t_{i_0})x - C_m(t)x \| \\
&\quad (because \ \ | t - t_{i_0} | < \eta) \\
&\leq \ \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \| C_m(t_{i_0})x - C_m(t)x \| \ \ (because\ m \geq N_0 \geq m_i) \\
&\leq \ \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \ (because \ \ | t - t_{i_0} | < \eta) \\
&\leq \ \varepsilon.
\end{aligned}$$

So the uniform Cauchy criterion implies that $(C_n(.)x)_{n\in\mathbb{N}}$ converge uniformly on H towards $C(.)x)$.

$\underline{5 \Rightarrow 1}$ |

Let $x \in E$; like $(C_n(.)x)_{n\in\mathbb{N}}$ converge uniformly of any compact in $[0, +\infty[$ then $(C_n(.)x)_{n\in\mathbb{N}}$ is equicontinuous in $[0, +\infty[$. Let $\lambda > \omega_1$, then for all $n \in \mathbb{N}$, we have $\lambda L(K)(\lambda)R_C(\lambda^2, A_n)x = \int_0^{+\infty} e^{-\lambda t} C_n(t)dt$. The convergence dominate theorem give that

$$\begin{aligned}
\lim_{n\to+\infty} \lambda L(K)(\lambda)(\lambda)R_C(\lambda^2, A_n)x &= \lim_{n\to+\infty} \int_0^{+\infty} e^{-\lambda t} C_n(t)xdt \\
&= \int_0^{+\infty} e^{-\lambda t} C(t)xdt \\
&= \lambda L(K)(\lambda)R_C(\lambda^2, A)x.
\end{aligned}$$

Corollary 1. *Let $(C(t))_{t\geq 0}$ be an $\alpha-$times integrated cosine function (for some $\alpha \geq 0$) with generator A, and for each $n \in \mathbb{N}$, let $(C_n(t))_{t\geq 0}$ an $\alpha-$times integrated cosine function with generator $(A_n, \mathcal{D}(A_n))$ such that:*

$$there\ exist\ \omega \geq 0,\ there\ exist\ M > 0:\ for\ all\ t \geq 0,\ for\ all\ x \in E,\ \| C(t)x) \| \leq Me^{\omega t}$$

$$and\ for\ all\ n \in \mathbb{N})\ \ \| C_n(t)x) \| \leq Me^{\omega t}.$$

Then if we put $\omega_1 = \max(\omega + 1, abs(K))$, the following statements are equivalent:

(i) There exist $\lambda_0 > \omega_1$ such that for all $x \in E$,

$$\lim_{n\to+\infty} R(\lambda_0^2, A_n)x = R(\lambda_0^2, A)x$$

and $(C_n(.)x)_{n\in\mathbb{N}}$ is equicontinuous.

(ii) There exist $\lambda_0 > \omega_1$ such that for all $y \in R(C)$,

$$\lim_{n\to+\infty} (\lambda^2 I - A_n)^{-1} y = (\lambda^2 I - A)^{-1} y$$

and for all $x \in E$, $(C_n(.)x)_{n\in\mathbb{N}}$ is equicontinuous.

(iii) For all $t \geq 0$ and $x \in E$, $\lim_{n\to+\infty} C_n(t)x = C(t)x$, the convergence is uniform on any compact of $[0, +\infty[$.

Proof. Let $\alpha \geq 0$. Then if $K(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ and $C = I$, then the $\alpha-$times integrated cosine function is a K-convoluted C-cosine function on E, thus Theorem 1 gives the results.

Corollary 2. *Let $(C(t))_{t\geq 0}$ be a $K-$convoluted $C-$cosine function with subgenerator A and for each $n \in \mathbb{N}$ let $(C_n(t))_{t\geq 0}$ a $K-$convoluted $C-$cosine function with subgenerators $(A_n, \mathcal{D}(A_n))$ such that there exist*

$$\omega \geq 0,\ there\ exist\ M > 0 :\ for\ all\ t, h \geq 0,\ \| C(t + h) - C(t) \| \leq Mhe^{\omega(t+h)}$$

$$and\ for\ all\ n \in \mathbb{N},\quad \| C_n(t + h) - C_n(t) \| \leq Mhe^{\omega(t+h)}.$$

Then if we put $\omega_1 = \max(\omega + 1, abs(K))$, the following statements are equivalent:

(i) There exist $\lambda_0 > \omega_1$ such that $L(K)(\lambda_0) \neq 0$ and for all $x \in E$,

$$\lim_{n\to+\infty} R_C(\lambda_0^2, A_n)x = R_C(\lambda_0^2, A)x.$$

(ii) There exist $\lambda_0 > \omega_1$ such that $L(K)(\lambda_0) \neq 0$ and for all $y \in R(C)$,

$$\lim_{n\to+\infty} (\lambda^2 I - A_n)^{-1} y = (\lambda^2 I - A)^{-1} y.$$

(iii) For all $\lambda > \omega_1$ such that $L(K)(\lambda) \neq 0)$, for all $y \in R(C)$,

$$\lim_{n\to+\infty} (\lambda^2 I - A_n)^{-1} y = (\lambda^2 I - A)^{-1} y.$$

(iv) For all $\lambda > \omega_1$ such that $K(\lambda) \neq 0$, for all $x \in E$,

$$\lim_{n\to+\infty} R_C(\lambda^2, A_n)x = R_C(\lambda^2, A)x.$$

(v) For all $t \geq 0$ and all $x \in E$, $\lim_{n\to+\infty} C_n(t)x = C(t)x$, the convergence is uniform on any compact of $[0, +\infty[$.

Proof. The condition

$$There\ exist\ (\omega, M) \in \mathbb{R}^+ \times \mathbb{R}^+_*,\ for\ all\ n \in \mathbb{N},\ for\ all\ t, h \geq 0\ \| C_n(t+h) - C_n(t) \| \leq Mhe^{\omega(t+h)}$$

implie that for $t = 0$ and $h \geq 0$,

$$\| C_n(h)) \| \leq Mhe^{\omega h} \leq Me^{(\omega+1)h} \leq Me^{\omega_1 h}$$

and for all $x \in E$, $(C_n(.)x)_{n\in\mathbb{N}}$ is equicontinuous, then Theorem 1 gives the result.

Theorem 2. *Suppose that* $(C(t))_{t\geq 0}$ *is a strongly continuous operators family such that for all* $t \geq 0$, $\| C(t) \|\leq Me^{\omega t}$ *and* $CC(.) = C(.)C$. *For all* $x \in E$ *and* $\lambda > \omega$, *put* $R_{\lambda^2}x := \frac{1}{\lambda L(K)(\lambda)} \int_0^{+\infty} e^{-\lambda t}C(t)xdt$.
For each $n \in \mathbb{N}$, *note* $(A_n, \mathcal{D}(A_n))$ *the subgenerators of some* $K-$*convoluted* $C-$*cosine function* $(C_n(t))_{t\geq 0}$, *such that*

- *i) There exist* $\omega \geq 0$ *there exist* $M > 0:$ *for all* $n \in \mathbb{N}$ $\| C_n(t)) \|\leq Me^{\omega t}$.
- *ii) For all* $x \in E$) $(C_n(.)x)_{n\in\mathbb{N}}$ *is equicontinuous.*
- *iii) There exist* $\lambda > \omega$ *such that* $\lim\limits_{n\to+\infty} R_C(\lambda^2, A_n)x = R_{\lambda^2}x$, $R(R_{\lambda^2}) \subset R(C)$ *and* $N(R_{\lambda^2}) = \{0\}$.

Then there is a linear operator A *which is subgenerator of a* $K-$*convoluted* $C-$ *cosine function* $(C(t))_{t\geq 0}$ *such that for all* $t \geqslant 0$ *and all* $x \in E$,

$$\lim_{n\to+\infty} C_n(t)x = C(t)x,$$

the convergence is uniform on any compact of $[0, +\infty[$.

Proof. As $\lim\limits_{n\to+\infty} R(\lambda^2, A_n)x = R_{\lambda^2}x$ then by Theorem 1 and Remark 1, we have for all $\lambda, \mu > \omega$ and all $n \in \mathbb{N}$,

$$(\lambda^2 - \mu^2)R(\lambda^2, A_n)R(\mu^2, A_n) = R(\lambda^2, A_n)Cx - R(\lambda^2, A_n)Cx,$$

then passing to the limit as n tends to $+\infty$, we get for all $\lambda, \mu > \omega$

$$(\lambda^2 - \mu^2)R_{\lambda^2}R_{\mu^2} = R_{\mu^2}Cx - R_{\lambda^2}Cx.$$

The remark 1 implies that there is a linear operator A ($D(A) = R(R_{\lambda^2})$), such that $R_{\lambda^2}x = (\lambda^2 - A)^{-1}Cx = R_C(\lambda^2, A)$. By definition we know that

$$\lambda L(K)(\lambda)R_C(\lambda^2, A_n)x = \int_0^{+\infty} e^{-\lambda t}C_n(t)xdt,$$

but

$$\begin{aligned}\lim_{n\to+\infty} \lambda L(K)(\lambda)R_C(\lambda^2, A_n)x &= \lambda L(K)(\lambda)R_{\lambda^2}x \\ &= \lambda L(K)(\lambda)R_C(\lambda^2, A)x,\end{aligned}$$

by the proof of Theorem 1, we obtain that $\lim\limits_{n\to+\infty} C_n(t)x = C(t)x$, hence $\lambda L(K)(\lambda)R_C(\lambda^2, A)x = \int_0^{+\infty} e^{-\lambda t}C(t)xdt$, then A is subgenerator of $K-$convoluted $C-$cosine function $(C(t))_{t\geq 0}$, such that $\lim\limits_{n\to+\infty} C_n(t)x = C(t)x$ for all $x \in E$, and the convergence is uniform on any compact of $[0, +\infty[$.

4. Conclusion

Among the things that math people like is finding necessary an sufficient conditions so that the limit of a sequence of mathematical objects having specific properties has same properties. Thi is exactly what we did in this article, it is to find necessary and sufficient conditions so that the limit of equicontinuous sequence K-convoluted C-cosine functions (respectively the C-resolvent operators) is also equicontinuous K-convoluted C-cosine functions (respectively C-resolvent operators) and treated the equivalence between the convergence of equicontinuous sequence K-convoluted C-cosine functions and the convergence of the associated $C-$resolvent sequence.

References

[1] W. Arendt, C. J. K. Batty, M. Hieber, and F. Neubrander. *Vector-valued Laplace Transforms and Cauchy Problems*, volume Monographs in Mathematics 96. Springer Basel AG, 2001.

[2] V. Keyantuo, C. Lizama, and P. J. Miana. Algebra homomorphisms defined via convoluted semigroups and cosine functions. *Journal of Functional Analysis*, 257:3454–3487, 2009.

[3] M. Kostić. Convoluted c-cosine functions and convoluted c-semigroups. *Bulletin T. CXXVII de l'Academie Serbe des Sciences et des Arts*, No 28:75–92, 2003.

[4] M. Kostić. Notes on analytic convoluted c-semigroups. *Publications De L'institut Mathematique Nouvelle Serie*, 88(102):67–76, 2010.

[5] M. Kostić. Generalized semigroups and cosine functions. *Mathematical Institute Belgrade*, No 28, 2011.

[6] M. Kostić. Hille-yosida theorems for local convoluted c-semigroups and cosine functions. *Filomat*, 25:4:177–190, 2011.

[7] M. Kostić. Degenerate k-convoluted c-semigroups and degenerate k-convoluted c-cosine functions in locally convex spaces. *Chelyab. Fiz. Mat. Zh*, 3:90–110, 2018.

[8] M. Kostić and S. Pilipović. Convoluted c-operator families and abstract cauchy problems. *Kragujevac Journal of Mathematics*, 30:201–210, 2007.

[9] M. Kostić and S. Pilipović. Convoluted c-cosine functions and semigroups relations with ultradistribution and hyperfunction sines. *University of Novi Sad, Serbia*, 2008.

[10] C.-C. Kuo. On α-times integrated c-cosine functions and abstract cauchy problem i. *Journal of Mathematical Analysis and Applications*, 313:142–162, 2006.

[11] C.-C. Kuo. Local k-convoluted c-semigroups and abstract cauchy problems. *Taiwanese Journal of Mathematics*, 19:1227–1245, 2015.

[12] C.-C. Kuo. Local k-convoluted c-semigroups and complete second order abstract cauchy problems. *Filomat*, 32:19:6789–6797, 2018.

[13] C.-C. Kuo and S. Y. Shaw. On α-times integrated c-semigroups and the abstract cauchy problem. *Studia Mathematica*, 142:201–217, 2000.