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Further New Operators in Primal Spaces

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Abstract. In this paper, we Introduce new operators depending on the definitions of γ and γ^*-operators that were defined in [7] using the structure of primal topological spaces. We provide some examples to illustrate the relation between these new operators, Additionally, we give more results regarding to γ-diamond operator.

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1. Introduction

Recently, there are several topological structures have emerged and gained prominence in a wide array of research studies. One of these structures was named grill introduced in [9]. Moreover, the concept of ideal [10, 11] and filter [9] are examples of other topological structures. In addition, the primal structure is among these topological structures that have been defined by Acharjee et al. [1]. It is known that the primal structure was studied in the framework of both soft and fuzzy set theory in [6, 8]. Al-Omari and Alqahtani defined closure operators using the concept of primal spaces in [4]. Furthermore, more operators were defined using the structure of primal spaces and soft primal space in [2, 5]. Moreover, regularity and normality in primal spaces was discussed in [3]. Additionally, Alghamdi et al. have defined new operators using the structure of primal spaces in [7]. In this work, we will continue present new results regarding to diamond operators that were defined in [7]. Then, we provide definitions of new operators called λ, $\lambda^{\diamond}$ and $\widetilde{\lambda}$. Finally, we present some examples to clarify the relations between them under different definitions of primal spaces.

We are going now to present some definitions and results that we use through the paper.

Definition 1. *[1] Let $\mathcal{B}$ be a nonempty set. We say that $\mathrm{P} \subseteq \mathcal{P}(\mathcal{B})$, where $\mathcal{P}(\mathcal{B})$ is the power set of $\mathcal{B}$, is a primal on $\mathcal{B}$ if the following conditions hold:*

1. $\mathcal{B} \notin \mathrm{P}$.

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2. If $C \subseteq Q$ and $Q \in \mathrm{P}$, then $C \in \mathrm{P} \Leftrightarrow$ If $C \subseteq Q$ and $C \notin \mathrm{P}$, then $Q \notin \mathrm{P}$.

3. If $C \cap Q \in \mathrm{P}$, then either $C \in \mathrm{P}$ or $Q \in \mathrm{P} \Leftrightarrow$ If $C \notin \mathrm{P}$ and $Q \notin \mathrm{P}$, then $C \cap Q \notin \mathrm{P}$.

We say that $(\mathcal{B}, \Gamma, \mathrm{P})$ is a primal topological space ($\mathbb{PS}$), where Γ is a topology on $\mathcal{B}$.

Definition 2. *[1] Let $(\mathcal{B}, \Gamma, \mathrm{P})$ be a $\mathbb{PS}$. For any set $Q \subseteq \mathcal{B}$, the function $(\cdot)^{\diamond} : \mathcal{P}(\mathcal{B}) \to \mathcal{P}(\mathcal{B})$ is defined as:*

$$Q^{\diamond} = Q^{\diamond}(\mathcal{B}, \Gamma, \mathrm{P}) = \{r \in \mathcal{B} : Q^c \cup O^c \in \mathrm{P} \text{ for all } O \in \Gamma(r)\}.$$

From the definition, it is clear that $\forall\ r \in Q^{\diamond}$, we have $W \cap Q \neq \emptyset$ for every $W \in \Gamma(r)$; hence, $Q^{\diamond} \subseteq \mathsf{CL}(Q)$.

Definition 3. *[7] Let $(\mathcal{B}, \Gamma, \mathrm{P})$ be a $\mathbb{PS}$ and let $O \subseteq \mathcal{B}$ be any set. Then, the function $\gamma : \mathcal{P}(\mathcal{B}) \to \mathcal{P}(\mathcal{B})$ is defined as:*

$$\gamma(\mathcal{O}) = \{r \in \mathcal{B} : \mathcal{O}^c \cup (W^{\diamond})^c \in \mathrm{P} \ \text{ for all } \ W \in \Gamma(r)\}.$$

Definition 4. *[7] Let $(\mathcal{B}, \Gamma, \mathrm{P})$ be a $\mathbb{PS}$. The operator $\gamma^* : \mathcal{P}(\mathcal{B}) \to \mathcal{P}(\mathcal{B})$ is defined as:*

$$\gamma^*(\mathcal{O}) = \{r \in \mathcal{B} : \exists\ W \in \Gamma(r) \text{ such that } (W^{\diamond} - \mathcal{O})^c \notin \mathrm{P}\}$$

for every $\mathcal{O} \subseteq \mathcal{B}$.

Definition 5. *[7] If $(\mathcal{B}, \Gamma, \mathrm{P})$ is a $\mathbb{PS}$, then the topology produced by the operator γ^* is defined as follows:*

$$\Gamma_{\gamma^*} = \{U \subseteq \mathcal{B} \mid U \subseteq \gamma^*(U)\} \text{ and } \mathsf{CL}_{\gamma^*}(S) = S \cup \gamma(S).$$

Definition 6. *[12] If $(\mathcal{B}, \Gamma, \mathrm{P})$ is a $\mathbb{PS}$, then Γ_θ is defined as follows:*

$$\Gamma_\theta = \{W \in \Gamma \mid \forall\ r \in W\ \exists\ H \in \Gamma(r) \text{ such that } r \in H \subseteq \mathsf{CL}(H) \subseteq W\}.$$

Observe that $\Gamma_\theta \subseteq \Gamma$. Moreover, $\mathsf{CL}_\theta(\mathcal{K}) = \{r \in \mathcal{B} \mid \mathcal{K} \cap \mathsf{CL}(\mathcal{O}) \neq \emptyset\ \forall\ \mathcal{O} \in \Gamma(r)\}$ and $Int_\theta(\mathcal{K}) = \{\bigcup_{\alpha \in \Lambda} U_\alpha \text{ such that } U_\alpha \subseteq \mathcal{K} \text{ and } U_\alpha \in \Gamma_\theta\}$.

Theorem 1. *[7] Let $(\mathcal{B}, \Gamma, \mathrm{P})$ be a $\mathbb{PS}$ and let $\mathcal{O}, \mathcal{O}_1, \mathcal{O}_2 \subseteq \mathcal{B}$. The following properties hold:*

(i) $\gamma(\emptyset) = \emptyset$.

(ii) If $\mathcal{O}_1 \subseteq \mathcal{O}_2$, then $\gamma(\mathcal{O}_1) \subseteq \gamma(\mathcal{O}_2)$.

(iii) $\gamma(\mathcal{O})$ is closed.

(iv) $\gamma(\mathcal{O}) \subseteq \mathsf{CL}_\theta(\mathcal{O})$.

(v) If $\mathcal{O} \subseteq \gamma(\mathcal{O})$ and $\gamma(\mathcal{O})$ is open, then $\gamma(\mathcal{O}) = \mathsf{CL}_\theta(\mathcal{O})$.

(vi) *If* $\mathcal{O}^c \notin \mathrm{P}$, *then* $\gamma(\mathcal{O}) = \emptyset$.

(vii) $\gamma(\mathcal{O}_1 \cup \mathcal{O}_2) = \gamma(\mathcal{O}_1) \cup \gamma(\mathcal{O}_2)$.

Theorem 2. *[7] Let* $(\mathcal{B}, \Gamma, \mathrm{P})$ *be a* $\mathbb{PS}$ *and let* $\mathcal{O}, \mathcal{O}_1, \mathcal{O}_2 \subseteq \mathcal{B}$. *Then,*

(i) $\gamma^*(\mathcal{O}) = [\gamma(\mathcal{O}^c)]^c$.

(ii) $\gamma^*(\mathcal{O})$ *is open.*

(iii) *If* $\mathcal{O}_1 \subseteq \mathcal{O}_2$, *then* $\gamma^*(\mathcal{O}_1) \subseteq \gamma^*(\mathcal{O}_2)$.

(iv) $\gamma^*(\mathcal{O}_1 \cap \mathcal{O}_2) = \gamma^*(\mathcal{O}_1) \cap \gamma^*(\mathcal{O}_2)$.

(v) $\gamma^*(\mathcal{O}) = \gamma^*(\gamma^*(\mathcal{O})) \Leftrightarrow \gamma(\mathcal{O}^c) = \gamma(\gamma(\mathcal{O}^c))$.

(vi) *If* $\mathcal{O}^c \notin \mathrm{P}$, *then* $\gamma^*(\mathcal{O}) = [\gamma(\mathcal{B})]^c$.

(vii) *If* $[(\mathcal{O}_1 - \mathcal{O}_2) \cup (\mathcal{O}_2 - \mathcal{O}_1)]^c \notin \mathrm{P}$, *then* $\gamma^*(\mathcal{O}_1) = \gamma^*(\mathcal{O}_2)$.

(viii) *A subset* $\mathcal{O}$ *is a diamond-closed if and only if* $\gamma(\mathcal{O}) \subseteq \mathcal{O}$.

Corollary 1. *[7] Let* $(\mathcal{B}, \Gamma, \mathrm{P})$ *be a* $\mathbb{PS}$ *and* $\mathcal{O}, M \subseteq \mathcal{B}$ *such that* $\mathcal{O}^c \notin \mathrm{P}$. *Then,*

$$\gamma(\mathcal{O} \cup M) = \gamma(\mathcal{O}) = \gamma(M - \mathcal{O}).$$

Theorem 3. *[7] Let* $(\mathcal{B}, \Gamma, \mathrm{P})$ *be a* $\mathbb{PS}$. *Then,*

$$\gamma(\mathcal{B}) = \mathcal{B} \text{ if and only if } \Gamma - \{\mathcal{B}\} \subseteq \mathrm{P} \text{ and } \triangle = \{\mathcal{K} \in \Gamma \mid \mathcal{K}^\diamond = \emptyset\} = \{\emptyset\}.$$

Definition 7. *[7] Let* $(\mathcal{B}, \Gamma, \mathrm{P})$ *be a* $\mathbb{PS}$ *and let* $\mathcal{O} \subseteq \mathcal{B}$. *Then,* Γ *is compatible with* P *if for every* $r \in \mathcal{O}$ *there exists* $W \in \Gamma(r)$ *such that* $(W^\diamond)^c \cup \mathcal{O}^c \notin \mathrm{P}$, *then* $\mathcal{O}^c \notin \mathrm{P}$.

Theorem 4. *[7] Let* $(\mathcal{B}, \Gamma, \mathrm{P})$ *be a* $\mathbb{PS}$ *and let* $\mathcal{O} \subseteq \mathcal{B}$. *Then,*

$$\Gamma \text{ is compatible with } \mathrm{P} \Longleftrightarrow [\mathcal{O} - \gamma(\mathcal{O})]^c \notin \mathrm{P}.$$

Lemma 1. *[7] Let* $(\mathcal{B}, \Gamma, \mathrm{P})$ *be a* $\mathbb{PS}$. *If* $G \in \Gamma_\theta$, *then* $G \in \Gamma_{\gamma^*}$.

Lemma 2. *[2] Let* $(\mathcal{B}, \Gamma, \mathrm{P})$ *be a* $\mathbb{PS}$ *such that* $C(\mathcal{B}) - \{\mathcal{B}\} \subseteq \mathrm{P}$, *where* $C(\mathcal{B}) = \{F : F \subseteq \mathcal{B}$ *is a closed set*$\}$. *Then,* $\mathcal{O} \subseteq \mathcal{O}^\diamond$ *for all* $\mathcal{O} \in \Gamma$.

2. More Results about γ and γ^*

In this section, we present further results pertaining to the operators γ and γ^*.

Theorem 5. *Let* $(\mathcal{B}, \Gamma, \mathrm{P})$ *be a* $\mathbb{PS}$. *Then,* $Int_\theta(\mathcal{O}) \subseteq \gamma^*(\mathcal{O})$ *for any set* $\mathcal{O} \subseteq \mathcal{B}$.

Proof. Let $r \notin \gamma^*(\mathcal{O})$ and let $W \in \Gamma(r)$. Then, $r \in \gamma(\mathcal{O}^c)$; hence $(W^\diamond \cap \mathcal{O}^c)^c \in \mathrm{P}$ which implies that $W^\diamond \cap \mathcal{O}^c \neq \emptyset$. Then, $\mathsf{CL}(W) \cap \mathcal{O}^c \neq \emptyset$. Hence, $\mathsf{CL}(W) \nsubseteq \mathcal{O}$. Then, $r \notin Int_\theta(\mathcal{O})$.

Theorem 6. *Let $(\mathcal{B}, \Gamma, \mathrm{P})$ be a $\mathbb{PS}$. If $\Gamma - \{\mathcal{B}\} \subseteq \mathrm{P}$ and $\triangle = \{\emptyset\}$, then $\gamma^*(\mathcal{K}) \subseteq \gamma(\mathcal{K})$ for every $\mathcal{K} \subseteq \mathcal{B}$.*

Proof. Since $\Gamma - \{\mathcal{B}\} \subseteq \mathrm{P}$ and $\triangle = \{\emptyset\}$, then $\mathcal{B} = \gamma(\mathcal{B})$ by using Theorem 3. Let $r \in \gamma^*(\mathcal{K})$. Then, $[W^\diamond - \mathcal{K}]^c \notin \mathrm{P}$ for some $W \in \Gamma(r)$ which implies that $(W^\diamond)^c \cup \mathcal{K} \notin \mathrm{P}$. Thus, $r \notin \gamma(\mathcal{K}^c)$. Since $r \in \mathcal{B} = \gamma(\mathcal{B}) = \gamma(\mathcal{K} \cup \mathcal{K}^c) = \gamma(\mathcal{K}) \cup \gamma(\mathcal{K}^c)$, then $r \in \gamma(\mathcal{K})$.

Theorem 7. *Let $(\mathcal{B}, \Gamma, \mathrm{P})$ be a $\mathbb{PS}$. Then,*

(i) If $\Gamma - \{\mathcal{B}\} \subseteq \mathrm{P}$ and $\triangle = \{\emptyset\}$, then $\gamma^*(\mathcal{B}) = \gamma(\mathcal{B})$.

(ii) If $\gamma^*(\mathcal{K}) = \gamma(\mathcal{K})$ for any $\mathcal{K} \subseteq \mathcal{B}$, then $\Gamma - \{\mathcal{B}\} \subseteq \mathrm{P}$ and $\triangle = \{\emptyset\}$.

Proof.

(i) $\gamma^*(\mathcal{B}) = [\gamma(\emptyset)]^c = \mathcal{B} = \gamma(\mathcal{B})$ by Theorem 3.

(ii) We know that $\gamma^*(\mathcal{K}) = [\gamma(\mathcal{K}^c)]^c$. Then, $\gamma(\mathcal{B}) = \gamma(\mathcal{K}^c \cup \mathcal{K}) = \gamma(\mathcal{K}^c) \cup \gamma(\mathcal{K}) = \gamma(\mathcal{K}^c) \cup [\gamma(\mathcal{K}^c)]^c = \mathcal{B}$. By using the result from Theorem 3, we get that $\Gamma - \{\mathcal{B}\} \subseteq \mathrm{P}$ and $\triangle = \{\emptyset\}$.

Theorem 8. *Let $(\mathcal{B}, \Gamma, \mathrm{P})$ be a $\mathbb{PS}$. If there exists a set $\mathcal{K} \subseteq \mathcal{B}$ such that $\gamma^*(\mathcal{K}) \neq \gamma(\mathcal{K})$. Then, one of the following holds:*

(i) There exists $r \in \mathcal{B}$ and $G \in \Gamma(r)$ such that $\mathcal{B} - G^\diamond = (G^\diamond)^c \in \Gamma - \{\mathrm{P}\}$.

(ii) There exists $r \in \mathcal{B}$ and $G \in \Gamma(r)$ such that $Int(G^c) \in \mathrm{P}$ for every $G \in \Gamma(r)$.

Proof. We know that $\gamma^*(\mathcal{K}) = [\gamma(\mathcal{K}^c)]^c$. If $\gamma^*(\mathcal{K}) \neq \gamma(\mathcal{K})$, we have two cases:

(i) $\exists\ r \in \gamma^*(\mathcal{K}) - \gamma(\mathcal{K})$ which implies that $r \notin \gamma(\mathcal{K}^c) \cup \gamma(\mathcal{K})$. Hence, there exist $L, \mathcal{O} \in \Gamma(r)$ such that $[L^\diamond \cap \mathcal{K}]^c \notin \mathrm{P}$ and $[\mathcal{O}^\diamond \cap \mathcal{K}^c]^c \notin \mathrm{P}$. Let $G = L \cap \mathcal{O}$. Then, as $[L^\diamond \cap \mathcal{K}]^c \subseteq [G^\diamond \cap \mathcal{K}]^c$ and $[\mathcal{O}^\diamond \cap \mathcal{K}^c]^c \subseteq [G^\diamond \cap \mathcal{K}^c]^c$, we have $[G^\diamond \cap \mathcal{K}]^c \notin \mathrm{P}$ and $[G^\diamond \cap \mathcal{K}^c]^c \notin \mathrm{P}$. Hence,

$$\begin{aligned}(G^\diamond)^c = [G^\diamond \cap \mathcal{B}]^c &= [G^\diamond \cap (\mathcal{K} \cup \mathcal{K}^c)]^c \\ &= [(G^\diamond \cap \mathcal{K}) \cup (G^\diamond \cap \mathcal{K}^c)]^c \\ &= (G^\diamond \cap \mathcal{K})^c \cap (G^\diamond \cap \mathcal{K}^c)^c \notin \mathrm{P}.\end{aligned}$$

Since $G^\diamond$ is a closed set, then $(G^\diamond)^c \in \Gamma$.

(ii) $\exists\ r \in \gamma(\mathcal{K}) - \gamma^*(\mathcal{K})$ which implies that $r \in \gamma(\mathcal{K})$ and $r \in \gamma(\mathcal{K}^c)$. Let $G \in \Gamma(r)$. Then, $(G^\diamond)^c \cup \mathcal{K}^c \in \mathrm{P}$; hence $(\mathsf{CL}(G))^c \subseteq (G^\diamond)^c \cup \mathcal{K}^c \Longrightarrow (\mathsf{CL}(G))^c = Int(G^c) \in \mathrm{P}$.

Theorem 9. *Let $(\mathcal{B}, \Gamma, \mathrm{P})$ be a $\mathbb{PS}$ and $\mathcal{K}, \mathcal{O} \subseteq \mathcal{B}$. Then,*

$$Int_\theta(\mathcal{K}) \cap \gamma(\mathcal{O}) \subseteq \gamma(\mathcal{K} \cap \mathcal{O}).$$

Proof. Let $r \in Int_\theta(\mathcal{K}) \cap \gamma(\mathcal{O})$. Then, there exists $H \in \Gamma(r)$ such that $r \in H \subseteq \mathsf{CL}(H) \subseteq \mathcal{K}$. Since $r \in \gamma(\mathcal{O})$, then for any $D \in \Gamma(r)$, we have $(D^\diamond)^c \cup \mathcal{O}^c \in \mathrm{P}$.

$$\begin{aligned}[D^\diamond \cap (\mathcal{K} \cap \mathcal{O})]^c \subseteq & [D^\diamond \cap (\mathsf{CL}(H) \cap \mathcal{O})]^c \\ \subseteq & [(D^\diamond \cap H^\diamond) \cap \mathcal{O}]^c \text{ using the relation } H^\diamond \subseteq \mathsf{CL}(H). \\ \subseteq & [(D \cap H)^\diamond \cap \mathcal{O}]^c \text{ since } (D \cap H)^\diamond \subseteq D^\diamond \cap H^\diamond \text{ by [1].} \\ = & [(D \cap H)^\diamond \cap \mathcal{O}]^c \in \mathrm{P} \text{ since } r \in \gamma(\mathcal{O}).\end{aligned}$$

Hence, $r \in \gamma(\mathcal{K} \cap \mathcal{O})$.

Theorem 10. *Let* $(\mathcal{B}, \Gamma, \mathrm{P})$ *be a* $\mathbb{PS}$ *and* $\mathcal{K}, \mathcal{O} \subseteq \mathcal{B}$. *Then,*

$$Int_\theta(\mathcal{K}) \cap Int(\gamma(\mathcal{O})) \subseteq Int(\gamma(\mathcal{K} \cap \mathcal{O})).$$

Proof. By using Theorem 9,

$$\begin{aligned}Int_\theta(\mathcal{K}) \cap Int(\gamma(\mathcal{O})) = & Int[Int_\theta(\mathcal{K}) \cap \gamma(\mathcal{O})] \\ \subseteq & Int(\gamma(\mathcal{K} \cap \mathcal{O})).\end{aligned}$$

Definition 8. *Let* $(\mathcal{B}, \Gamma, \mathrm{P})$ *be a* $\mathbb{PS}$. *Then,* $(\mathcal{B}, \Gamma, \mathrm{P})$ *is a* $\diamond$*-extremally disconnected if* $U^\diamond \in \Gamma$ *for every* $U \in \Gamma$.

The following example is an example of $\diamond$-extremally disconnected primal topological space.

Example 1. *Define* $(\mathbb{R}, \Gamma_0, \mathrm{P}_0)$ *as follows:*

$$T \in \mathrm{P}_0 \Longleftrightarrow 0 \notin T \text{ and } U \in \Gamma_0 \Longleftrightarrow U = \emptyset \text{ or } 0 \in U.$$

Let $U \in \Gamma_0$. *Then,*

$$U^\diamond = \begin{cases} \emptyset & \text{if } U = \emptyset \\ \mathbb{R} & \text{if } U \neq \emptyset \end{cases}$$

Hence, $(\mathbb{R}, \Gamma_0, \mathrm{P}_0)$ *is* $\diamond$*-extremally disconnected.*

Theorem 11. *Let* $(\mathcal{B}, \Gamma, \mathrm{P})$ *be* $\diamond$*-extremally disconnected. Then, for any* $C \subseteq \mathcal{B}$,

$$\gamma(\gamma(C)) \subseteq \gamma(C).$$

Proof. Let $r \in \gamma(\gamma(C))$. Then, $[\mathcal{O}^\diamond \cap \gamma(C)]^c \in \mathrm{P}$ for each $\mathcal{O} \in \Gamma(r)$. Hence, $\mathcal{O}^\diamond \cap \gamma(C) \neq \emptyset$. Let $t \in \mathcal{O}^\diamond \cap \gamma(C)$. Therefore, since $\mathcal{O}^\diamond \in \Gamma(t)$ and $t \in \gamma(C)$, then $[C \cap (\mathcal{O}^\diamond)^\diamond]^c \in \mathrm{P}$. As $[C \cap \mathcal{O}^\diamond]^c \subseteq [C \cap (\mathcal{O}^\diamond)^\diamond]^c \in \mathrm{P}$, then $[C \cap \mathcal{O}^\diamond]^c \in \mathrm{P}$ which implies that $r \in \gamma(C)$.

Theorem 12. *Let* $(\mathcal{B}, \Gamma, \mathrm{P})$ *be a* $\diamond$*-extremally disconnected and* $C \subseteq Int(\gamma(C))$. *Then,* $\gamma(C) = \gamma(\gamma(C))$.

Proof. Since $\gamma(\gamma(C)) \subseteq \gamma(C)$ by Theorem 11, then it remains to show that $\gamma(C) \subseteq \gamma(\gamma(C))$. Indeed, since $C \subseteq Int(\gamma(C))$, then $\gamma(C) \subseteq \gamma(Int(\gamma(C)) \subseteq \gamma(\gamma(C))$.

3. On λ operator

In this part, a new operator called the λ operator is defined. We provide some results and examples to illustrate the relationship between this operator and others.

Definition 9. *Let $(\mathcal{B}, \Gamma, \mathrm{P})$ be a $\mathbb{PS}$. Then, $\lambda : \mathcal{P}(\mathcal{B}) \to \mathcal{P}(\mathcal{B})$ is defined as:*

$$\lambda(\mathcal{K}) = \gamma^*(\mathcal{K}) - \mathcal{K} \text{ for every } \mathcal{K} \subseteq \mathcal{B}.$$

Remark 1. *Let $(\mathcal{B}, \Gamma, \mathrm{P})$ be a $\mathbb{PS}$ such that $\mathrm{P} = \mathcal{P}(\mathcal{B}) - \{\mathcal{B}\}$. Then, $\gamma(\mathcal{O}) = \mathsf{CL}_\theta(\mathcal{O})$ for all $\mathcal{O} \subseteq \mathcal{B}$.*

Proof. We know from (iv) in Theorem 1 that $\gamma(\mathcal{O}) \subseteq \mathsf{CL}_\theta(\mathcal{O})$. For the converse, let $r \notin \gamma(\mathcal{O})$. Then, $[W^\diamond \cap \mathcal{O}]^c \notin \mathrm{P}$ for some $W \in \Gamma(r)$. Hence, $W^\diamond \cap \mathcal{O} = \emptyset$. Thus, if $t \in \mathcal{O}$, we have that $U \cap W = \emptyset$ for some $U \in \Gamma(t)$. Thus, $t \notin \mathsf{CL}(W)$; hence $\mathcal{O} \cap \mathsf{CL}(W) = \emptyset$ which implies that $r \notin \mathsf{CL}_\theta(\mathcal{O})$. Therefore, $\mathsf{CL}_\theta(\mathcal{O}) = \gamma(\mathcal{O})$.

Remark 2. *Let $(\mathcal{B}, \Gamma, \mathrm{P})$ be a $\mathbb{PS}$. Then,*

1. *If $\mathrm{P} = \emptyset$ and $\mathcal{K}$ is any nonempty proper subset of $\mathcal{B}$, then $\lambda(\mathcal{K}) = \mathcal{K}^c$.*
   To show that, we have $\lambda(\mathcal{K}) = \gamma^*(\mathcal{K}) - \mathcal{K} = [\gamma(\mathcal{K}^c)]^c - \mathcal{K}$. Since $\mathcal{K} \neq \emptyset$, then we have $\gamma(\mathcal{K}^c) = \emptyset$ by (vi) in Theorem 1. Hence, $\lambda(\mathcal{K}) = \mathcal{K}^c$.

2. *If $\mathrm{P} = \mathcal{P}(\mathcal{B}) - \{\mathcal{B}\}$ and $\mathcal{K}$ is any nonempty proper subset of $\mathcal{B}$, then $\lambda(\mathcal{K}) = \emptyset$.*
   Since $\gamma(S) = \mathsf{CL}_\theta(S)$ for every $S \subseteq \mathcal{B}$, then $\lambda(\mathcal{K}) = \gamma^*(\mathcal{K}) - \mathcal{K} = [\gamma(\mathcal{K}^c)]^c - \mathcal{K} = [\mathsf{CL}_\theta(\mathcal{K}^c)]^c - \mathcal{K} = int_\theta(\mathcal{K}) - \mathcal{K} = \emptyset$.

Example 2. *Let $(\mathcal{B} = \mathbb{R}, \tau_{\sqrt{2}}, \mathrm{P})$ be defined as follows:*

$$W \in \tau_{\sqrt{2}} \iff \sqrt{2} \in W \text{ or } W = \emptyset,$$

$$L \in \mathrm{P} \iff L^c \text{ is an infinite subset of } \mathbb{R}.$$

Let $K \subseteq \mathbb{R}$ be any set. Then, since $W = \{\sqrt{2}\}$ is an open neighborhood of $\sqrt{2}$ such that $W^c \notin \mathsf{P}$ because W is finite, we have $\sqrt{2} \notin K^\diamond$. Moreover, if r is any real number different from $\sqrt{2}$, then $W = \{r, \sqrt{2}\}$ is an open neighborhood of r such that $W^c \notin \mathsf{P}$. Hence, $r \notin K^\diamond$ which implies that $K^\diamond = \emptyset$. Thus, $\gamma^*(K) = \mathbb{R}$. Therefore, $\lambda(K) = K^c$.

Theorem 13. *Let $(\mathcal{B}, \Gamma, \mathrm{P})$ be a $\mathbb{PS}$ and let $\mathcal{K}, \mathcal{O} \subseteq \mathcal{B}$. The following properties hold:*

(i) $\lambda(\emptyset) = [\gamma(\mathcal{B})]^c$.

(ii) $\lambda(\mathcal{B}) = \emptyset$.

(iii) *If $\Gamma - \{\mathcal{B}\} \subseteq \mathrm{P}$ with $\triangle = \{\emptyset\}$ and $\mathcal{K}^c \notin \mathrm{P}$, then $\lambda(\mathcal{K}) = \emptyset$.*

(iv) $\lambda(\mathcal{K}) = \mathcal{K}^c - \gamma(\mathcal{K}^c)$.

(v) $\lambda(\mathcal{K}) \cap \lambda(\mathcal{O}) \subseteq \lambda(\mathcal{K} \cup \mathcal{O})$.

(vi) $\lambda(\mathcal{K} \cap \mathcal{O}) = (\lambda(\mathcal{K}) \cap \gamma^*(\mathcal{O})) \cup (\lambda(\mathcal{O}) \cap \gamma^*(\mathcal{K})) \subseteq \lambda(\mathcal{K}) \cup \lambda(\mathcal{O})$.

(vii) $\lambda(\lambda(\mathcal{K})) \subseteq \lambda(\gamma^*(\mathcal{K})) \cup \gamma^*(\gamma^*(\mathcal{K}))$.

(viii) $\gamma(\lambda(\mathcal{K})) \subseteq \gamma(\gamma^*(\mathcal{K}))$.

(ix) $\lambda(\mathcal{K}) \cap \mathcal{K} = \emptyset$*; thus* $\lambda(\mathcal{K}) \subseteq \mathcal{K}^c$.

(x) $\lambda(\mathcal{K}) \cup \lambda(\mathcal{O}) \subseteq (\mathcal{K} \cap \lambda(\mathcal{O})) \cup (\lambda(\mathcal{K}) \cap \mathcal{O}) \cup \lambda(\mathcal{K} \cup \mathcal{O})$.

Proof.

(i) $\lambda(\emptyset) = \gamma^*(\emptyset) - \emptyset = [\gamma(\emptyset^c)]^c = [\gamma(\mathcal{B})]^c$ by using (i) in Theorem 2.

(ii) $\lambda(\mathcal{B}) = \gamma^*(\mathcal{B}) - \mathcal{B} = [\gamma(\mathcal{B}^c)]^c - \mathcal{B} = \emptyset$ since $\gamma(\emptyset) = \emptyset$.

(iii) $\lambda(\mathcal{K}) = \gamma^*(\mathcal{K}) - \mathcal{K} = [\gamma(\mathcal{K}^c)]^c - \mathcal{K}$. Since $\mathcal{K}^c \notin$ P, then $\gamma(\mathcal{K}) = \emptyset$ by (vi) in Theorem 1. By using Theorem 3 and (vii) in Theorem 1, we have:

$$\mathcal{B} = \gamma(\mathcal{B}) = \gamma(\mathcal{K} \cup \mathcal{K}^c) = \gamma(\mathcal{K}) \cup \gamma(\mathcal{K}^c) = \gamma(\mathcal{K}^c).$$

Then, $\lambda(\mathcal{K}) = \emptyset$.

(iv) $\lambda(\mathcal{K}) = \gamma^*(\mathcal{K}) - \mathcal{K} = [\gamma(\mathcal{K}^c)]^c \cap \mathcal{K}^c = \mathcal{K}^c - \gamma(\mathcal{K}^c)$.

(v)

$$\begin{aligned} r \in \lambda(\mathcal{K}) \cap \lambda(\mathcal{O}) &\Longleftrightarrow r \in \lambda(\mathcal{K}) \text{ and } r \in \lambda(\mathcal{O}) \\ &\Longleftrightarrow r \in \gamma^*(\mathcal{K}) \cap \gamma^*(\mathcal{O}) \text{ and } r \notin \mathcal{K} \cup \mathcal{O} \\ &\Longleftrightarrow r \in \gamma^*(\mathcal{K} \cap \mathcal{O}) \text{ and } r \notin \mathcal{K} \cup \mathcal{O} \text{ using (iv) in Theorem 2} \\ &\Longleftrightarrow r \in \gamma^*(\mathcal{K} \cup \mathcal{O}) \text{ and } r \notin \mathcal{K} \cup \mathcal{O} \text{ by (iii) in Theorem 2} \\ &\Longleftrightarrow r \in \gamma^*(\mathcal{K} \cup \mathcal{O}) - (\mathcal{K} \cup \mathcal{O}) = \lambda(\mathcal{K} \cup \mathcal{O}). \end{aligned}$$

(vi) By item (iv) of Theorem 2, we have

$$\begin{aligned} r \in \lambda(\mathcal{K} \cap \mathcal{O}) &\Longleftrightarrow r \in \gamma^*(\mathcal{K} \cap \mathcal{O}) \text{ and } r \notin (\mathcal{K} \cap \mathcal{O}) \\ &\Longleftrightarrow r \in [\gamma^*(\mathcal{K}) \cap \gamma^*(\mathcal{O})] \text{ and } r \notin \mathcal{K} \text{ or } r \notin \mathcal{O}. \end{aligned}$$

Suppose that $r \notin \mathcal{K}$. Then,

$$\begin{aligned} & r \in [\gamma^*(\mathcal{K}) \cap \gamma^*(\mathcal{O})] \text{ and } r \notin \mathcal{K} \\ \Longrightarrow & r \in \lambda(\mathcal{K}) \cap \gamma^*(\mathcal{O}). \end{aligned}$$

Now, suppose that $r \notin \mathcal{O}$. Then,

$$\begin{aligned} & r \in [\gamma^*(\mathcal{K}) \cap \gamma^*(\mathcal{O})] \text{ and } r \notin \mathcal{O} \\ \Longrightarrow & r \in \lambda(\mathcal{O}) \cap \gamma^*(\mathcal{K}) \\ \Longrightarrow & r \in [\lambda(\mathcal{K}) \cap \gamma^*(\mathcal{O})] \cup [\lambda(\mathcal{O}) \cap \gamma^*(\mathcal{K})] \subseteq \lambda(\mathcal{K}) \cup \lambda(\mathcal{O}). \end{aligned}$$

(vii)

$$
\begin{aligned}
\lambda(\lambda(\mathcal{K})) =& \gamma^*(\lambda(\mathcal{K})) - \lambda(\mathcal{K}) \\
=& \gamma^*(\gamma^*(\mathcal{K}) - \mathcal{K}) - [\gamma^*(\mathcal{K}) - \mathcal{K}] \\
=& \gamma^*[\gamma^*(\mathcal{K}) \cap \mathcal{K}^c] - [\gamma^*(\mathcal{K}) \cap \mathcal{K}^c] \\
=& \gamma^*(\gamma^*(\mathcal{K})) \cap \gamma^*(\mathcal{K}^c) \cap [\gamma^*(\mathcal{K}) \cap \mathcal{K}^c]^c \\
=& \gamma^*(\gamma^*(\mathcal{K})) \cap \gamma^*(\mathcal{K}^c) \cap [(\gamma^*(\mathcal{K}))^c \cup \mathcal{K}] \\
\subseteq& \gamma^*(\gamma^*(\mathcal{K})) \cap [(\gamma^*(\mathcal{K}))^c \cup \mathcal{K}] \\
=& [\gamma^*(\gamma^*(\mathcal{K})) \cap (\gamma^*(\mathcal{K}))^c] \cup [\gamma^*(\gamma^*(\mathcal{K})) \cap \mathcal{K}] \\
=& [\gamma^*(\gamma^*(\mathcal{K})) - \gamma^*(\mathcal{K})] \cup [\gamma^*(\gamma^*(\mathcal{K})) \cap \mathcal{K}] \\
\subseteq& \lambda(\gamma^*(\mathcal{K})) \cup \gamma^*(\gamma^*(\mathcal{K})).
\end{aligned}
$$

(viii) $\gamma(\lambda(\mathcal{K})) = \gamma[\gamma^*(\mathcal{K}) - \mathcal{K}] \subseteq \gamma(\gamma^*(\mathcal{K}))$ by using (ii) in Theorem 1.

(ix) Since $\lambda(\mathcal{K}) = \gamma^*(\mathcal{K}) - \mathcal{K}$, then $\lambda(\mathcal{K}) \cap \mathcal{K} = \emptyset$. Hence, $\lambda(\mathcal{K}) \subseteq \mathcal{K}^c$.

(x) Let $r \in \lambda(\mathcal{K}) \cup \lambda(\mathcal{O})$. Then, we have two cases:

Case 1. $r \in \lambda(\mathcal{K}) \Longrightarrow r \in \gamma^*(\mathcal{K})$ and $r \notin \mathcal{K}$.

Subcase 1.1. If $r \notin \mathcal{O}$, then $r \notin \mathcal{K} \cup \mathcal{O}$. Since $r \in \gamma^*(\mathcal{K}) \subseteq \gamma^*(\mathcal{K} \cup \mathcal{O})$, then $r \in \lambda(\mathcal{K} \cup \mathcal{O})$.

Subcase 1.2. If $r \in \mathcal{O}$, then $r \in \lambda(\mathcal{K}) \cap \mathcal{O}$.

Case 2. $r \in \lambda(\mathcal{O}) \Longrightarrow r \in \gamma^*(\mathcal{O})$ and $r \notin \mathcal{O}$.

Subcase 2.1. If $r \notin \mathcal{K}$, then $r \in \lambda(\mathcal{K} \cup \mathcal{O})$.

Subcase 2.2. If $r \in \mathcal{K}$, then $r \in \lambda(\mathcal{O}) \cap \mathcal{K}$.
Hence, $\lambda(\mathcal{K}) \cup \lambda(\mathcal{O}) \subseteq \lambda(\mathcal{K} \cup \mathcal{O}) \cup [\lambda(\mathcal{K}) \cap \mathcal{O}] \cup [\lambda(\mathcal{O}) \cap \mathcal{K}]$.

The equality in the properties (vii) and (viii) of Theorem 13 may not be true in general as shown in example below.

Example 3. *Let $(\mathbb{R}, \Gamma_0, \mathrm{P}_0)$ be defined as in Example 1. Consider the set $\mathbb{W} = \{0, 1, 2, ...\}$. Let $r \in \mathbb{R}$ and let $O \in \Gamma_0(r)$. Then, $O^\diamond = \mathbb{R}$. Hence, $(O^\diamond - \mathbb{W})^c = (\mathbb{R} - \mathbb{W})^c = \mathbb{W} \notin \mathrm{P}_0$; thus, $\gamma^*(\mathbb{W}) = \mathbb{R}$. Therefore, $\lambda(\mathbb{W}) = \mathbb{W}^c$. Now, let A be any set such that $0 \notin A$ and let $r \in \mathbb{R}$. Then, if O is any arbitrary open set with $r \in O$, we have $O^\diamond = \mathbb{R}$. Thus, $(O^\diamond - A)^c \in \mathrm{P}_0$. Therefore, $\gamma^*(A) = \emptyset$ which leads to that $\lambda(A) = \emptyset$. Then, $\lambda(\lambda(\mathbb{W})) = \emptyset$ and $\gamma^*(\gamma^*(\mathbb{W})) = \mathbb{R}$.*
On the other hand, $\gamma(\gamma^*(\mathbb{W})) = \mathbb{R}$. Indeed, let $r \in \mathbb{R}$ and let $O \in \Gamma_0(r)$. Then, since $(O^\diamond)^c \cup U^c \in \mathrm{P}_0$ for all $U \in \Gamma_0(r)$, we have $\gamma(\mathbb{R}) = \mathbb{R}$.
Moreover, $\gamma(\lambda(\mathbb{W})) = \gamma(\mathbb{W}^c) = \emptyset$ because $0 \in \mathbb{W}$ which implies that $\mathbb{W} \notin \mathrm{P}_0$.

Corollary 2. *Let $(\mathcal{B}, \Gamma, \mathrm{P})$ be a $\mathbb{PS}$. Then, $\Gamma - \{\mathcal{B}\} \subseteq \mathrm{P}$ and $\triangle = \{\emptyset\}$ if and only if $\lambda(\emptyset) = \emptyset$.*

Proof. It is obvious by Theorem 3 and property (i) of Theorem 13.

Theorem 14. *Let* $(\mathcal{B}, \Gamma, \mathrm{P})$ *be a* $\mathbb{PS}$*. Then,* $\lambda(\mathcal{O}^c) = \mathcal{O}$ *if and only if* $\mathcal{O} \cap \gamma(\mathcal{O}) = \emptyset$ *for any* $\mathcal{O} \subseteq \mathcal{B}$*.*

Proof. Let $\mathcal{O} \subseteq \mathcal{B}$. Then,

$$\begin{aligned} \lambda(\mathcal{O}^c) = \mathcal{O} &\Longleftrightarrow \gamma^*(\mathcal{O}^c) - \mathcal{O}^c = \gamma^*(\mathcal{O}^c) \cap \mathcal{O} = \mathcal{O} \\ &\Longleftrightarrow \mathcal{O} \subseteq \gamma^*(\mathcal{O}^c) = [\gamma(\mathcal{O})]^c \\ &\Longleftrightarrow \mathcal{O} \cap \gamma(\mathcal{O}) = \emptyset. \end{aligned}$$

Theorem 15. *Let* $(\mathcal{B}, \Gamma, \mathrm{P})$ *be a* $\mathbb{PS}$*. Then,* Γ *is compatible with* P *if and only if* $[\lambda(\mathcal{O})]^c \notin \mathrm{P}$ *for all* $\mathcal{O} \subseteq \mathcal{B}$*.*

Proof.
By Theorem 4, we have:

$$\Gamma \text{ is compatible with } \mathrm{P} \Longleftrightarrow [\mathcal{O} - \gamma(\mathcal{O})]^c \notin \mathrm{P} \text{ for all } \mathcal{O} \subseteq \mathcal{B}.$$

Hence,

$$\begin{aligned} \Gamma \text{ is compatible with } \mathrm{P} &\Longleftrightarrow [\mathcal{O}^c - \gamma(\mathcal{O}^c)]^c \notin \mathrm{P} \\ &\Longleftrightarrow [(\gamma(\mathcal{O}^c))^c - \mathcal{O}]^c \notin \mathrm{P} \\ &\Longleftrightarrow [\gamma^*(\mathcal{O}) - \mathcal{O}]^c \notin \mathrm{P} \\ &\Longleftrightarrow [\lambda(\mathcal{O})]^c \notin \mathrm{P}. \end{aligned}$$

Theorem 16. *Let* $(\mathcal{B}, \Gamma, \mathrm{P})$ *be a* $\mathbb{PS}$ *and* $\mathcal{O}$ *be a diamond-closed subset of* $\mathcal{B}$*. Then,* $\lambda(\mathcal{O}) \subseteq [\gamma(\mathcal{B})]^c$*.*

Proof. If $\mathcal{O}$ is a diamond-closed, then $\gamma(\mathcal{O}) \subseteq \mathcal{O}$ by using (viii) in Theorem 2. Hence,

$$\begin{aligned} \lambda(\mathcal{O}) &= \gamma^*(\mathcal{O}) - \mathcal{O} \\ &\subseteq \gamma^*(\mathcal{O}) - \gamma(\mathcal{O}) \\ &= \gamma^*(\mathcal{O}) \cap [\gamma(\mathcal{O})]^c \\ &= \gamma^*(\mathcal{O}) \cap \gamma^*(\mathcal{O}^c) \\ &= \gamma^*(\mathcal{O} \cap \mathcal{O}^c) \text{ by using (iv) in Theorem 2} \\ &= \gamma^*(\emptyset) = [\gamma(\mathcal{B})]^c. \end{aligned}$$

Corollary 3. *Let* $(\mathcal{B}, \Gamma, \mathrm{P})$ *be a* $\mathbb{PS}$*.*

(i) If $\mathcal{O}$ *is a closed subset of* $\mathcal{B}$*, then* $\lambda(\mathcal{O}) \in \Gamma$*.*

(ii) If $\mathcal{O}$ is a diamond-closed subset of $\mathcal{B}$ and $\Gamma - \{\mathcal{B}\} \subseteq \mathrm{P}$ and $\triangle = \{\emptyset\}$, then $\lambda(\mathcal{O}) = \emptyset$.

Proof.

(i) $\lambda(\mathcal{O}) = \gamma^*(\mathcal{O}) - \mathcal{O}$ is an open set by (ii) in Theorem 2.

(ii) Let $\mathcal{O}$ be a diamond-closed subset of $\mathcal{B}$, $\Gamma - \{\mathcal{B}\} \subseteq \mathrm{P}$ and $\triangle = \{\emptyset\}$. Then, by Theorem 3, we have $\gamma(\mathcal{B}) = \mathcal{B}$. Hence, $\lambda(\mathcal{O}) = \gamma^*(\mathcal{O}) - \mathcal{O} \subseteq \gamma^*(\mathcal{O}) - \gamma(\mathcal{O}) = \gamma^*(\emptyset) = [\gamma(\mathcal{B})]^c = \mathcal{B}^c = \emptyset$.

4. On $\lambda^\diamond$ operator

Definition 10. *Let $(\mathcal{B}, \Gamma, \mathrm{P})$ be a $\mathbb{PS}$. Then, the operator $\lambda^\diamond : \mathcal{P}(\mathcal{B}) \to \mathcal{P}(\mathcal{B})$ is defined as follows:*

$$\lambda^\diamond(O) = \gamma^*(O) - \gamma(O) \text{ for any set } O \subseteq \mathcal{B}.$$

Remark 3. *Let $(\mathcal{B}, \Gamma, \mathrm{P})$ be a $\mathbb{PS}$ and let $O \subseteq \mathcal{B}$ be a nonempty proper subset of $\mathcal{B}$.*

(i) $\lambda^\diamond(O) = [\gamma(\mathcal{B})]^c$.

(ii) If $\mathrm{P} = \emptyset$, then $\lambda^\diamond(O) = \mathcal{B}$.

(iii) If $\mathrm{P} = \mathcal{P}(\mathcal{B}) - \{\mathcal{B}\}$, then $\lambda^\diamond(O) = \emptyset$.

Proof. Let O be any nonempty proper subset of $\mathcal{B}$. Then,

(i)

$$\begin{aligned}\lambda^\diamond(O) =&\gamma^*(O) - \gamma(O)\\ =&\gamma^*(O) \cap [\gamma(O)]^c\\ =&\gamma^*(O) \cap \gamma^*(O^c)\\ =&\gamma^*(O \cap O^c) = \gamma^*(\emptyset) = [\gamma(\mathcal{B})]^c.\end{aligned}$$

(ii) If $\mathrm{P} = \emptyset$, then $\mathcal{O}^c \notin \mathrm{P}$ since it is a proper subset of $\mathcal{B}$; hence $\gamma(\mathcal{O}) = \emptyset$ by (vi) in Theorem 1. Thus, since $\mathcal{O} \neq \emptyset$, $\lambda^\diamond(\mathcal{O}) = \gamma^*(\mathcal{O}) = \mathcal{B}$.

(iii) Suppose that $\mathrm{P} = \mathcal{P}(\mathcal{B}) - \{\mathcal{B}\}$. Then, we have $r \in \gamma(\mathcal{B})$ for every $r \in \mathcal{B}$ since $U^c \in \mathrm{P}$ for each $U \in \Gamma(r)$.

$$\therefore \lambda^\diamond(O) = [\gamma(\mathcal{B})]^c = [\mathcal{B}]^c = \emptyset.$$

Example 4. *Let $(\mathbb{R}, \tau_{\sqrt{2}}, \mathrm{P})$ be defined as in Example 2 and let $\mathcal{K} \subseteq \mathbb{R}$. Then, We have $\lambda^\diamond(\mathcal{K}) = [\gamma(\mathbb{R})]^c = \mathbb{R}$.*

Lemma 3. *Let* $(\mathcal{B}, \Gamma, \mathrm{P})$ *be a* $\mathbb{PS}$ *and* $O \subseteq \mathcal{B}$*. Then,*

(i) $\lambda^{\diamond}(\mathcal{O}) \in \Gamma$.

(ii) $\lambda^{\diamond}(\mathcal{O}) = \emptyset$ *if and only if* $\gamma^{*}(\mathcal{O}) \subseteq \gamma(\mathcal{O})$.

Proof.

(i) Since $\lambda^{\diamond}(\mathcal{O}) = [\gamma(\mathcal{B})]^{c}$, then $\lambda^{\diamond}(\mathcal{O}) \in \Gamma$ by (iii) in Theorem 1.

(ii) It is obvious by the definition of $\lambda^{\diamond}$ operator.

Corollary 4. *Let* $(\mathcal{B}, \Gamma, \mathrm{P})$ *be a* $\mathbb{PS}$ *and* $\mathcal{O} \subseteq \mathcal{B}$*. Then,*

$$\Gamma - \{\mathcal{B}\} \subseteq \mathrm{P} \textit{ and } \triangle = \{\emptyset\} \Leftrightarrow \lambda^{\diamond}(\mathcal{O}) = \emptyset.$$

Proof. Let $\mathcal{O} \subseteq \mathcal{B}$ and $\Gamma - \{\mathcal{B}\} \subseteq \mathrm{P}$ such that $\triangle = \{\emptyset\}$. Then, by Theorem 3, $\gamma(\mathcal{B}) = \mathcal{B}$ which implies that $\lambda^{\diamond}(\mathcal{O}) = \emptyset$.
For the converse, if $\lambda^{\diamond}(\mathcal{O}) = \emptyset = [\gamma(\mathcal{B})]^{c}$, then $\gamma(\mathcal{B}) = \mathcal{B}$ which implies that $\Gamma - \{\mathcal{B}\} \subseteq \mathrm{P}$ and $\triangle = \{\emptyset\}$ by Theorem 3.

Theorem 17. *Let* $(\mathcal{B}, \Gamma, \mathrm{P})$ *be a* $\mathbb{PS}$ *and let* $\emptyset \neq \mathcal{O} \subseteq \mathcal{B}$*. Then, the following properties hold:*

(i) $\lambda^{\diamond}(\mathcal{O}) = \mathcal{B}$ *if and only if for every* $r \in \mathcal{B}$*, there exists* $R \in \Gamma(r)$ *such that* $(R^{\diamond})^{c} \notin \mathrm{P}$.

(ii) If $\lambda^{\diamond}(\mathcal{O}) = \mathcal{O}$*, then* $\gamma(\mathcal{B}) \neq \mathcal{B}$.

(iii) $\lambda^{\diamond}(\mathcal{O}) = \gamma^{*}(\mathcal{O}^{c})$ *if and only if* $[\gamma(\mathcal{O})]^{c} \subseteq \gamma^{*}(\mathcal{O})$.

(iv) If $\mathcal{O}$ *is a diamond-open set, then* $\mathcal{O} - \gamma(\mathcal{O}) \subseteq \lambda^{\diamond}(\mathcal{O})$.

(v) $\lambda^{\diamond}(\mathcal{O}) \subseteq \gamma^{*}(\lambda^{\diamond}(\mathcal{O}))$.

(vi) If $\mathcal{O}^{c} \notin \mathrm{P}$*, then* $\lambda^{\diamond}(\mathcal{O}) = \gamma^{*}(\mathcal{O})$.

Proof.

(i) $\lambda^{\diamond}(\mathcal{O}) = [\gamma(\mathcal{B})]^{c} = \mathcal{B} \Longleftrightarrow \gamma(\mathcal{B}) = \emptyset$. Hence, for every $r \in \mathcal{B}$, there exists $R \in \Gamma(r)$ such that $(R^{\diamond} \cap \mathcal{B})^{c} = (R^{\diamond})^{c} \notin \mathrm{P}$.
Conversely, suppose that for every $r \in \mathcal{B}$, there exists $R \in \Gamma(r)$ such that $(R^{\diamond})^{c} \notin \mathrm{P}$. Then, $\gamma(\mathcal{B}) = \emptyset$; hence, $\lambda^{\diamond}(\mathcal{O}) = \mathcal{B}$ for any $\mathcal{O} \subseteq \mathcal{B}$.

(ii) Suppose that $\lambda^{\diamond}(\mathcal{O}) = \mathcal{O}$. Then, $[\gamma(\mathcal{B})]^{c} = \mathcal{O}$ which implies that $\gamma(\mathcal{B}) = \mathcal{O}^{c} \neq \mathcal{B}$ since $\mathcal{O} \neq \emptyset$.

(iii)

$$\begin{aligned}\lambda^{\diamond}(\mathcal{O}) =&\gamma^{*}(\mathcal{O}) - \gamma(\mathcal{O}) = \gamma^{*}(\mathcal{O}^{c})\\ =&\gamma^{*}(\mathcal{O}) \cap [\gamma(\mathcal{O})]^{c} = \gamma^{*}(\mathcal{O}^{c})\\ =&\gamma^{*}(\mathcal{O}) \cap \gamma^{*}(\mathcal{O}^{c}) = \gamma^{*}(\mathcal{O}^{c})\\ \Leftrightarrow&\gamma^{*}(\mathcal{O}^{c}) = [\gamma(\mathcal{O})]^{c} \subseteq \gamma^{*}(\mathcal{O}).\end{aligned}$$

(iv) Let $r \in \mathcal{O} - \gamma(\mathcal{O})$. Then, $r \in \mathcal{O}$ and $r \notin \gamma(\mathcal{O})$. Since $\mathcal{O}$ is a diamond open, then $\mathcal{O} \subseteq \gamma^{*}(\mathcal{O})$ which implies that $r \in \gamma^{*}(\mathcal{O})$; hence, $r \in \lambda^{\diamond}(\mathcal{O})$.

(v) Let $\mathcal{O} \subseteq \mathcal{B}$. By (iii) in Theorem 2, we have $\gamma^{*}(\emptyset) \subseteq \gamma^{*}(\lambda^{\diamond}(\mathcal{O}))$. Then,

$$\lambda^{\diamond}(\mathcal{O}) = [\gamma(\mathcal{B})]^{c} = \gamma^{*}(\emptyset) \subseteq \gamma^{*}(\lambda^{\diamond}(\mathcal{O})).$$

(vi) Since $\mathcal{O}^{c} \notin \mathrm{P}$, then $\gamma(\mathcal{O}) = \emptyset$ by (vi) in Theorem 1. Hence, $\lambda^{\diamond}(\mathcal{O}) = \gamma^{*}(\mathcal{O})$.

Theorem 18. *Let $(\mathcal{B}, \Gamma, \mathrm{P})$ be a $\mathbb{PS}$ and let $\mathcal{O} \subseteq \mathcal{B}$. Then, the following properties hold:*

(i) $\lambda(\mathcal{O}) \cap \lambda^{\diamond}(\mathcal{O}) = \lambda^{\diamond}(\mathcal{O}) - \mathcal{O}$.

(ii) $\lambda^{\diamond}(\mathcal{O}) - \lambda(\mathcal{O}) = \lambda^{\diamond}(\mathcal{O}) \cap \mathcal{O}$.

(iii) $\lambda(\mathcal{O}) \cup \lambda^{\diamond}(\mathcal{O}) = \gamma^{*}(\mathcal{O}) - (\mathcal{O} \cap \gamma(\mathcal{O}))$.

(iv) $\lambda(\lambda^{\diamond}(\mathcal{O})) = \gamma(\mathcal{B}) - \gamma(\gamma(\mathcal{B}))$.

(v) $\lambda^{\diamond}(\mathcal{O}) \subseteq \gamma^{*}(\lambda(\mathcal{O}))$.

Proof.

(i)

$$\begin{aligned}\lambda(\mathcal{O})\cap\lambda^{\diamond}(\mathcal{O}) =&[\gamma^*(\mathcal{O})-\mathcal{O}]\cap[\gamma^*(\mathcal{O})-\gamma(\mathcal{O})]\\ =&[\gamma^*(\mathcal{O})\cap\mathcal{O}^c]\cap[\gamma^*(\mathcal{O})\cap(\gamma(\mathcal{O}))^c]\\ =&[\gamma^*(\mathcal{O})\cap\mathcal{O}^c]\cap[\gamma^*(\mathcal{O})\cap\gamma^*(\mathcal{O}^c)]\\ =&\gamma^*(\mathcal{O})\cap[\mathcal{O}^c\cap\gamma^*(\mathcal{O}^c)]\\ =&[\gamma^*(\mathcal{O})\cap\gamma^*(\mathcal{O}^c)]\cap\mathcal{O}^c\\ =&\lambda^{\diamond}(\mathcal{O})-\mathcal{O}.\end{aligned}$$

(ii)

$$\begin{aligned}\lambda^{\diamond}(\mathcal{O})-\lambda(\mathcal{O}) =&[\gamma^*(\mathcal{O})-\gamma(\mathcal{O})]-[\gamma^*(\mathcal{O})-\mathcal{O}]\\ =&[\gamma^*(\mathcal{O})\cap\gamma^*(\mathcal{O}^c)]\cap[\gamma^*(\mathcal{O})\cap\mathcal{O}^c]^c\\ =&[\gamma^*(\mathcal{O})\cap\gamma^*(\mathcal{O}^c)]\cap[(\gamma^*(\mathcal{O}))^c\cup\mathcal{O}]\\ =&[\gamma^*(\mathcal{O})\cap\gamma^*(\mathcal{O}^c)\cap(\gamma^*(\mathcal{O}))^c]\cup[\gamma^*(\mathcal{O})\cap\gamma^*(\mathcal{O}^c)\cap\mathcal{O}]\\ =&[\gamma^*(\mathcal{O}^c)\cap\emptyset]\cup[\lambda^{\diamond}(\mathcal{O})\cap\mathcal{O}]\\ =&\lambda^{\diamond}(\mathcal{O})\cap\mathcal{O}.\end{aligned}$$

(iii)

$$\begin{aligned}\lambda(\mathcal{O})\cup\lambda^{\diamond}(\mathcal{O}) =&[\gamma^*(\mathcal{O})-\mathcal{O}]\cup[\gamma^*(\mathcal{O})-\gamma(\mathcal{O})]\\ =&[\gamma^*(\mathcal{O})\cap\mathcal{O}^c]\cup[\gamma^*(\mathcal{O})\cap(\gamma(\mathcal{O}))^c]\\ =&[\gamma^*(\mathcal{O})\cap\mathcal{O}^c]\cup[\gamma^*(\mathcal{O})\cap\gamma^*(\mathcal{O}^c)]\\ =&\gamma^*(\mathcal{O})\cap[\mathcal{O}^c\cup\gamma^*(\mathcal{O}^c)]\\ =&\gamma^*(\mathcal{O})\cap[\mathcal{O}^c\cup(\gamma(\mathcal{O}))^c]\\ =&\gamma^*(\mathcal{O})\cap[\mathcal{O}\cap\gamma(\mathcal{O})]^c\\ =&\gamma^*(\mathcal{O})-[\mathcal{O}\cap\gamma(\mathcal{O})].\end{aligned}$$

(iv) We know that $\lambda(\lambda^{\diamond}(\mathcal{O}))=\lambda\left([\gamma(\mathcal{B})]^c\right)$. Then,

$$\lambda(\lambda^{\diamond}(\mathcal{O}))=\lambda\left([\gamma(\mathcal{B})]^c\right)=\gamma(\mathcal{B})-[\gamma(\gamma(\mathcal{B}))] \text{ by (iv) in Theorem 13.}$$

(v) Clearly, for any $\mathcal{O}\subseteq\mathcal{B}$, we have $\lambda^{\diamond}(\mathcal{O})=\gamma^*(\emptyset)\subseteq\gamma^*(\lambda(\mathcal{O}))$.

5. On $\widetilde{\lambda}$ operator

In this section, we introduce an operator, denoted as the $\widetilde{\lambda}$ operator. We will present a definition of this operator and explore its properties. Additionally, we will discuss various results that connecting this operator with other operators introduced in this paper.

Definition 11. *Let $(\mathcal{B}, \Gamma, \mathrm{P})$ be a $\mathbb{PS}$. Then, we define the operator $\widetilde{\lambda} : \mathcal{P}(\mathcal{B}) \to \mathcal{P}(\mathcal{B})$ as follows:*

$$\widetilde{\lambda}(\mathcal{O}) = \mathcal{O} - \gamma(\mathcal{O}) \text{ for every } \mathcal{O} \subseteq \mathcal{B}.$$

Example 5. *Let $(\mathbb{R}, \tau_{\sqrt{2}}, \mathrm{P})$ be defined as in Example 2 and let $\mathcal{K} \subseteq \mathbb{R}$. Then, We have $\widetilde{\lambda}(\mathcal{K}) = \mathcal{K}$.*

Lemma 4. *Let $(\mathcal{B}, \Gamma, \mathrm{P})$ be a $\mathbb{PS}$ and $O \subseteq \mathcal{B}$. Then, we have:*

(i) $\widetilde{\lambda}(\mathcal{O}) = \lambda(\mathcal{O}^c)$.

(ii) $\lambda(\mathcal{O}) \cap \widetilde{\lambda}(\mathcal{O}) = \emptyset$.

(iii) $\lambda(\mathcal{O}) \cap \lambda(\mathcal{O}^c) = \emptyset$.

(iv) If $\mathcal{O} \in \Gamma$, then $\widetilde{\lambda}(\mathcal{O}) \in \Gamma$.

Proof. Let $O \subseteq \mathcal{B}$. Then,

(i) $\lambda(\mathcal{O}^c) = \gamma^*(\mathcal{O}^c) - \mathcal{O}^c = [\gamma(\mathcal{O})]^c \cap \mathcal{O} = \mathcal{O} - \gamma(\mathcal{O}) = \widetilde{\lambda}(\mathcal{O})$.

(ii) $\lambda(\mathcal{O}) \cap \widetilde{\lambda}(\mathcal{O}) = [\gamma^*(\mathcal{O}) \cap \mathcal{O}^c] \cap [\mathcal{O} \cap [\gamma(\mathcal{O})]^c] = \emptyset$.

(iii) By (i) and (ii), we have $\lambda(\mathcal{O}) \cap \lambda(\mathcal{O}^c) = \lambda(\mathcal{O}) \cap \widetilde{\lambda}(\mathcal{O}) = \emptyset$.

(iv) Let $r \in \widetilde{\lambda}(\mathcal{O}) = \mathcal{O} - \gamma(\mathcal{O})$. Then, $r \in \mathcal{O}$ and $r \notin \gamma(\mathcal{O})$. Hence, $r \in [\gamma(\mathcal{O})]^c$. Since $\gamma(\mathcal{O})$ is a closed set by (iii) in Theorem 1, then there exists $H \in \Gamma$ such that $r \in H \subseteq [\gamma(\mathcal{O})]^c$. Thus, $r \in H \cap \mathcal{O} \subseteq [\gamma(\mathcal{O})]^c \cap \mathcal{O} = \mathcal{O} - \gamma(\mathcal{O})$. Then, $\widetilde{\lambda}(\mathcal{O}) \in \Gamma$.

Corollary 5. *Let $(\mathcal{B}, \Gamma, \mathrm{P})$ be a $\mathbb{PS}$ and let $O \subseteq \mathcal{B}$. Then, $\widetilde{\lambda}(\mathcal{O}) = \mathcal{O}$ if and only if $\mathcal{O} \cap \gamma(\mathcal{O}) = \emptyset$.*

Remark 4. *Let $(\mathcal{B}, \Gamma, \mathrm{P})$ be a $\mathbb{PS}$ and let $\mathcal{O}$ be a proper subset of $\mathcal{B}$. Then,*

(i) If $\mathrm{P} = \emptyset$, then $\widetilde{\lambda}(\mathcal{O}) = \mathcal{O}$.

(ii) If $\mathrm{P} = \mathcal{P}(\mathcal{B}) - \{\mathcal{B}\}$, then $\widetilde{\lambda}(\mathcal{O}) = \emptyset$.

Proof.

(i) Since $\mathcal{O}$ is a proper subset of $\mathcal{B}$, then $\mathcal{O}^c \neq \emptyset$ which implies that $\mathcal{O}^c \notin \mathrm{P}$; hence, $\gamma(\mathcal{O}) = \emptyset$ by (vi) in Theorem 1. Thus, $\widetilde{\lambda}(\mathcal{O}) = \mathcal{O}$.

(ii) Since $\mathrm{P} = \mathcal{P}(\mathcal{B}) - \{\mathcal{B}\}$, then $\widetilde{\lambda}(\mathcal{O}) = \mathcal{O} - \gamma(\mathcal{O}) = \mathcal{O} - \mathsf{CL}_\theta(\mathcal{O}) = \emptyset$.

Theorem 19. *Let $(\mathcal{B}, \Gamma, \mathrm{P})$ be a $\mathbb{PS}$ and $O, K \subseteq \mathcal{B}$. Then, we have:*

(i) $\widetilde{\lambda}(\emptyset) = \emptyset$.

(ii) $\widetilde{\lambda}(\widetilde{\lambda}(\mathcal{K})) \subseteq \widetilde{\lambda}(\mathcal{K}) \subseteq \mathcal{K}$.

(iii) $\widetilde{\lambda}(\mathcal{K}) \cap \gamma(\mathcal{K}) = \emptyset$.

(iv) if $\mathcal{K}^c \notin \mathrm{P}$, *then* $\widetilde{\lambda}(\mathcal{K}) = \mathcal{K}$.

(v) $\widetilde{\lambda}(\mathcal{K} \cup \mathcal{O}) = [\widetilde{\lambda}(\mathcal{K}) - \gamma(\mathcal{O})] \cup [\widetilde{\lambda}(\mathcal{O}) - \gamma(\mathcal{K})]$.

(vi) $\widetilde{\lambda}(\mathcal{K}) \cap \widetilde{\lambda}(\mathcal{O}) = (\mathcal{K} \cap \mathcal{O}) - \gamma(\mathcal{K} \cup \mathcal{O})$.

Proof. Let $\mathcal{K}, \mathcal{O} \subseteq \mathcal{B}$. Then,

(i) $\widetilde{\lambda}(\emptyset) = \emptyset - \gamma(\emptyset) = \emptyset$.

(ii) $\widetilde{\lambda}(\widetilde{\lambda}(\mathcal{K})) = \widetilde{\lambda}(\mathcal{K}) - (\gamma(\widetilde{\lambda}(\mathcal{K})) \subseteq \widetilde{\lambda}(\mathcal{K}) \subseteq \mathcal{K}$.

(iii) $\widetilde{\lambda}(\mathcal{K}) \cap \gamma(\mathcal{K}) = [\mathcal{K} - \gamma(\mathcal{K})] \cap \gamma(\mathcal{K}) = \emptyset$.

(iv) If $\mathcal{K}^c \notin \mathrm{P}$, then by (vi) of Theorem 1, we have $\gamma(\mathcal{K}) = \emptyset$. Then, $\widetilde{\lambda}(\mathcal{K}) = \mathcal{K}$.

(v)

$$\begin{aligned}
\widetilde{\lambda}(\mathcal{K} \cup \mathcal{O}) =& (\mathcal{K} \cup \mathcal{O}) - \gamma(\mathcal{K} \cup \mathcal{O}) \\
=& (\mathcal{K} \cup \mathcal{O}) - [\gamma(\mathcal{K}) \cup \gamma(\mathcal{O})] \\
=& (\mathcal{K} \cup \mathcal{O}) \cap [(\gamma(\mathcal{K}))^c \cap (\gamma(\mathcal{O}))^c] \\
=& [\mathcal{K} \cap (\gamma(\mathcal{K}))^c \cap (\gamma(\mathcal{O}))^c] \cup [\mathcal{O} \cap (\gamma(\mathcal{K}))^c \cap (\gamma(\mathcal{O}))^c] \\
=& [\widetilde{\lambda}(\mathcal{K}) \cap (\gamma(\mathcal{O}))^c] \cup [\widetilde{\lambda}(\mathcal{O}) \cap (\gamma(\mathcal{K}))^c] \\
=& [\widetilde{\lambda}(\mathcal{K}) - \gamma(\mathcal{O})] \cup [\widetilde{\lambda}(\mathcal{O}) - \gamma(\mathcal{K})].
\end{aligned}$$

(vi)

$$\begin{aligned}
\widetilde{\lambda}(\mathcal{K}) \cap \widetilde{\lambda}(\mathcal{O}) =& (\mathcal{K} - \gamma(\mathcal{K})) \cap (\mathcal{O} - \gamma(\mathcal{O})) \\
=& (\mathcal{K} \cap \mathcal{O}) \cap [(\gamma(\mathcal{K}))^c \cap (\gamma(\mathcal{O}))^c] \\
=& (\mathcal{K} \cap \mathcal{O}) \cap [(\gamma(\mathcal{K}) \cup \gamma(\mathcal{O}))]^c \\
=& (\mathcal{K} \cap \mathcal{O}) \cap [\gamma(\mathcal{K} \cup \mathcal{O})]^c \\
=& (\mathcal{K} \cap \mathcal{O}) - \gamma(\mathcal{K} \cup \mathcal{O}).
\end{aligned}$$

Lemma 5. *Let* $(\mathcal{B}, \Gamma, \mathrm{P})$ *be a* $\mathbb{PS}$ *and let* $\mathcal{O} \subseteq \mathcal{B}$. *If* $C(\mathcal{B}) - \{\mathcal{B}\} \subseteq \mathrm{P}$ *and* $r \in \widetilde{\lambda}(\mathcal{O})$, *then* $(\{r\})^c \notin \mathrm{P}$.

Proof. Since $r \in \widetilde{\lambda}(\mathcal{O})$, then $r \in \mathcal{O}$ and $r \notin \gamma(\mathcal{O})$ which implies that there exists $Q \in \Gamma(r)$ such that $[Q^\diamond \cap \mathcal{O}]^c \notin \mathrm{P}$. By Lemma 2, we know that $Q \subseteq Q^\diamond$; hence, $[Q^\diamond \cap \mathcal{O}]^c \subseteq [Q \cap \mathcal{O}]^c$. Thus, $[Q \cap \mathcal{O}]^c \notin \mathrm{P}$ and since $r \notin [Q \cap \mathcal{O}]^c$, then $(\{r\})^c \notin \mathrm{P}$.

Theorem 20. *Let* $(\mathcal{B}, \Gamma, \mathrm{P})$ *be a* $\mathbb{PS}$ *and* $C(\mathcal{B}) - \{\mathcal{B}\} \subseteq \mathrm{P}$. *Then,* $r \in \widetilde{\lambda}(\{r\})$ *if and only if* $(\{r\})^c \notin \mathrm{P}$.

Proof. ($\Rightarrow$): It is obvious by Lemma 5.

($\Leftarrow$): Suppose that $(\{r\})^c \notin \mathrm{P}$. We want to show that $r \in \widetilde{\lambda}(\{r\})$ which is equivalent to show that $r \notin \gamma(\{r\})$. Since $\mathcal{B} \in \Gamma(r)$ and $\mathcal{B}^c \cup (\{r\})^c = (\{r\})^c \notin \mathrm{P}$, we get the desired result.

Theorem 21. *Let* $(\mathcal{B}, \Gamma, \mathrm{P})$ *be a* $\mathbb{PS}$ *and let* $\mathcal{O} \subseteq \mathcal{B}$. *Then,*

(i) $\lambda^\diamond(\mathcal{O}) \cap \widetilde{\lambda}(\mathcal{O}) = \gamma^*(\mathcal{O}) \cap \widetilde{\lambda}(\mathcal{O})$.

(ii) $\lambda^\diamond(\mathcal{O}) - \widetilde{\lambda}(\mathcal{O}) = \lambda^\diamond(\mathcal{O}) - \mathcal{O}$.

(iii) $\widetilde{\lambda}(\mathcal{O}) - \lambda^\diamond(\mathcal{O}) = \widetilde{\lambda}(\mathcal{O}) - \gamma^*(\mathcal{O})$.

(iv) $\lambda^\diamond(\mathcal{O}) \cup \widetilde{\lambda}(\mathcal{O}) = [\gamma^*(\mathcal{O}) \cup \mathcal{O}] - \gamma(\mathcal{O})$.

(v) $\widetilde{\lambda}(\lambda^\diamond(\mathcal{O})) = [\gamma(\mathcal{B})]^c$.

(vi) $\widetilde{\lambda}(\lambda(\mathcal{O})) = \lambda(\mathcal{O})$.

Proof. Let $\mathcal{O} \subseteq \mathcal{B}$. Then,

(i)

$$\begin{aligned}\lambda^\diamond(\mathcal{O}) \cap \widetilde{\lambda}(\mathcal{O}) =& [\gamma^*(\mathcal{O}) \cap (\gamma(\mathcal{O}))^c] \cap [\mathcal{O} \cap (\gamma(\mathcal{O}))^c] \\ =& [\gamma(\mathcal{O})]^c \cap [\mathcal{O} \cap \gamma^*(\mathcal{O})] \\ =& \gamma^*(\mathcal{O}) \cap [\mathcal{O} \cap (\gamma(\mathcal{O}))^c)] \\ =& \gamma^*(\mathcal{O}) \cap \widetilde{\lambda}(\mathcal{O}).\end{aligned}$$

(ii)

$$\begin{aligned}\lambda^\diamond(\mathcal{O}) - \widetilde{\lambda}(\mathcal{O}) =& \lambda^\diamond(\mathcal{O}) \cap [\widetilde{\lambda}(\mathcal{O})]^c \\ =& \lambda^\diamond(\mathcal{O}) \cap [\mathcal{O} - (\gamma(\mathcal{O}))]^c \\ =& \lambda^\diamond(\mathcal{O}) \cap [\mathcal{O} \cap (\gamma(\mathcal{O}))^c]^c \\ =& \lambda^\diamond(\mathcal{O}) \cap [\mathcal{O}^c \cup \gamma(\mathcal{O})] \\ =& [\lambda^\diamond(\mathcal{O}) \cap \mathcal{O}^c] \cup [\lambda^\diamond(\mathcal{O}) \cap \gamma(\mathcal{O})] \\ =& [\lambda^\diamond(\mathcal{O}) - \mathcal{O}] \cup [(\gamma^*(\mathcal{O}) - \gamma(\mathcal{O})) \cap \gamma(\mathcal{O})] \\ =& \lambda^\diamond(\mathcal{O}) - \mathcal{O}.\end{aligned}$$

(iii)

$$\begin{aligned}\widetilde{\lambda}(\mathcal{O})-\lambda^{\diamond}(\mathcal{O})=&\widetilde{\lambda}(\mathcal{O})\cap[\lambda^{\diamond}(\mathcal{O})]^{c}\\=&\widetilde{\lambda}(\mathcal{O})\cap[\gamma^{*}(\mathcal{O})\cap\gamma^{*}(\mathcal{O}^{c})]^{c}\\=&\widetilde{\lambda}(\mathcal{O})\cap[(\gamma^{*}(\mathcal{O}))^{c}\cup(\gamma^{*}(\mathcal{O}^{c}))^{c}]\\=&[\widetilde{\lambda}(\mathcal{O})\cap(\gamma^{*}(\mathcal{O}))^{c}]\cup[\widetilde{\lambda}(\mathcal{O})\cap(\gamma^{*}(\mathcal{O}^{c}))^{c}]\\=&[\widetilde{\lambda}(\mathcal{O})-\gamma^{*}(\mathcal{O})]\cup[\mathcal{O}\cap(\gamma(\mathcal{O}))^{c}\cap(\gamma^{*}(\mathcal{O}^{c}))^{c}]\\=&[\widetilde{\lambda}(\mathcal{O})-\gamma^{*}(\mathcal{O})]\cup[\mathcal{O}\cap\gamma^{*}(\mathcal{O}^{c})\cap(\gamma^{*}(\mathcal{O}^{c}))^{c}]\\=&\widetilde{\lambda}(\mathcal{O})-\gamma^{*}(\mathcal{O}).\end{aligned}$$

(iv)

$$\begin{aligned}\lambda^{\diamond}(\mathcal{O})\cup\widetilde{\lambda}(\mathcal{O})=&[\gamma^{*}(\mathcal{O})-\gamma(\mathcal{O})]\cup[\mathcal{O}-\gamma(\mathcal{O})]\\=&[\gamma^{*}(\mathcal{O})\cap(\gamma(\mathcal{O}))^{c}]\cup[\mathcal{O}\cap(\gamma(\mathcal{O}))^{c}]\\=&[\gamma^{*}(\mathcal{O})\cup\mathcal{O}]\cap(\gamma(\mathcal{O}))^{c}\\=&[\gamma^{*}(\mathcal{O})\cup\mathcal{O}]-\gamma(\mathcal{O}).\end{aligned}$$

(v)

$$\begin{aligned}\widetilde{\lambda}(\lambda^{\diamond}(\mathcal{O}))=&\widetilde{\lambda}[(\gamma(\mathcal{B}))^{c}]\\=&(\gamma(\mathcal{B}))^{c}-\gamma((\gamma(\mathcal{B}))^{c})\\=&(\gamma(\mathcal{B}))^{c}\cap[\gamma(\gamma(\mathcal{B}))^{c}]]^{c}\\=&[\gamma(\mathcal{B})\cup\gamma(\gamma(\mathcal{B}))^{c}]^{c}.\end{aligned}$$

Now, by (vii) in Theorem 1, we have $[\gamma(\mathcal{B})\cup\gamma(\gamma(\mathcal{B}))^{c}]^{c}=[\gamma(\mathcal{B}\cup(\gamma(\mathcal{B}))^{c})]^{c}=[\gamma(\mathcal{B})]^{c}$. Hence, $\widetilde{\lambda}(\lambda^{\diamond}(\mathcal{O}))=[\gamma(\mathcal{B})]^{c}$.

(vi)

$$\begin{aligned}\widetilde{\lambda}(\lambda(\mathcal{O}))=&\lambda(\mathcal{O})-\gamma(\lambda(\mathcal{O}))\\=&[\gamma^{*}(\mathcal{O})-\mathcal{O}]-\gamma[\gamma^{*}(\mathcal{O})-\mathcal{O}]\\=&[\gamma^{*}(\mathcal{O})\cap\mathcal{O}^{c}]\cap[\gamma[\gamma^{*}(\mathcal{O})\cap\mathcal{O}^{c}]]^{c}\\=&\mathcal{O}^{c}\cap[\gamma^{*}(\mathcal{O})\cap[\gamma[\gamma^{*}(\mathcal{O})\cap\mathcal{O}^{c}]]^{c}]\\=&\mathcal{O}^{c}\cap[(\gamma(\mathcal{O}^{c}))^{c}\cap[\gamma[(\gamma(\mathcal{O}^{c}))^{c}\cap\mathcal{O}^{c}]]^{c}]\\=&\mathcal{O}^{c}\cap[\gamma(\mathcal{O}^{c})\cup\gamma[(\gamma(\mathcal{O}^{c}))^{c}\cap\mathcal{O}^{c}]]^{c}\\=&\mathcal{O}^{c}\cap[\gamma(\mathcal{O}^{c}\cup\gamma^{*}(\mathcal{O})\cap\mathcal{O}^{c})]^{c}\\=&\mathcal{O}^{c}\cap[\gamma(\mathcal{O}^{c})]^{c}\\=&\gamma^{*}(\mathcal{O})-\mathcal{O}=\lambda(\mathcal{O}).\end{aligned}$$

Corollary 6. *Let* $(\mathcal{B}, \Gamma, \mathrm{P})$ *be a* $\mathbb{PS}$. *Then,*

$$\Gamma \text{ is a compatible with } \mathrm{P} \text{ if and only if } [\widetilde{\lambda}(\mathcal{O})]^c \notin \mathrm{P} \text{ for every } \mathcal{O} \subseteq \mathcal{B}.$$

Theorem 22. *Let* $(\mathcal{B}, \Gamma, \mathrm{P})$ *be a* $\mathbb{PS}$ *and* $\mathcal{O} \subseteq \mathcal{B}$. *Then, the following holds:*

(i) $\widetilde{\lambda}(\lambda^{\diamond}(\lambda(\mathcal{O}))) = [\gamma(\mathcal{B})]^c$.

(ii) $\widetilde{\lambda}(\lambda(\lambda^{\diamond}(\mathcal{O}))) = \gamma(\mathcal{B}) - \gamma(\gamma(\mathcal{B}))$.

Proof.

(i) Let $\mathcal{O} \subseteq \mathcal{B}$. Then, $\widetilde{\lambda}(\lambda^{\diamond}(\lambda(\mathcal{O}))) = \widetilde{\lambda}[\gamma(\mathcal{B})]^c$. Moreover, the property (v) in Theorem 21 leads to $\widetilde{\lambda}(\lambda^{\diamond}(\lambda(\mathcal{O}))) = [\gamma(\mathcal{B})]^c$.

(ii) By the property (vi) in Theorem 21, we know that $\widetilde{\lambda}(\lambda(\lambda^{\diamond}(\mathcal{O}))) = \lambda(\lambda^{\diamond}(\mathcal{O}))$. Additionally, we use property (iv) in Theorem 18 to get $\lambda(\lambda^{\diamond}(\mathcal{O})) = \gamma(\mathcal{B}) - \gamma(\gamma(\mathcal{B}))$.

Corollary 7. *Let* $(\mathcal{B}, \Gamma, \mathrm{P})$ *be a* $\mathbb{PS}$ *and* $\mathcal{O} \subseteq \mathcal{B}$ *such that* $\Gamma - \{\mathcal{B}\} \subseteq \mathrm{P}$ *and* $\triangle = \{\emptyset\}$. *Then, we have:*

(i) $\lambda^{\diamond}(\lambda(\lambda^{\diamond}(\mathcal{O}))) = \emptyset$.

(ii) $\lambda^{\diamond}(\lambda(\mathcal{O})) = \emptyset$.

(iii) $\lambda^{\diamond}(\widetilde{\lambda}(\lambda(\mathcal{O}))) = \emptyset$.

(iv) $\lambda^{\diamond}(\lambda(\widetilde{\lambda}(\mathcal{O}))) = \emptyset$.

(v) $\widetilde{\lambda}(\lambda^{\diamond}(\lambda(\mathcal{O}))) = \emptyset$.

(vi) $\widetilde{\lambda}[(\gamma(\mathcal{B}))^c] = \emptyset$.

(vii) $\lambda^{\diamond}(\mathcal{O}) = \emptyset$.

(viii) $\widetilde{\lambda}(\lambda(\lambda^{\diamond}(\mathcal{O}))) = \emptyset$.

(ix) $\lambda(\lambda^{\diamond}(\widetilde{\lambda}(\mathcal{O}))) = \emptyset$.

(x) $\lambda(\widetilde{\lambda}(\lambda^{\diamond}(\mathcal{O}))) = \emptyset$.

(xi) $\gamma(\mathcal{B}) - \gamma(\gamma(\mathcal{B})) = \emptyset$.

6. Examples

We provide illustrative examples that demonstrate the relationships between the $\widetilde{\lambda}$ operator and other operators.

Example 6. *Let* $\mathcal{B} = \{e, r, t\}$. *Define* $\Gamma = \{\emptyset, \mathcal{B}, \{e\}, \{r\}, \{e, r\}\}$ *and* $\mathrm{P} = \{\emptyset, \{e\}, \{r\}, \{e, r\}\}$. *Then,* $(\mathcal{B}, \Gamma, \mathrm{P})$ *is a* $\mathbb{PS}$. *If* $\mathcal{K} \subseteq \mathcal{B}$, *then we have the following table.*

$\mathcal{K}$	$\mathcal{K}^\diamond$	$\gamma(\mathcal{K})$	$\gamma^\star(\mathcal{K})$	$\lambda(\mathcal{K})$	$\lambda^\diamond(\mathcal{K})$	$\widetilde{\lambda}(\mathcal{K})$
$\emptyset$	$\emptyset$	$\emptyset$	$\{e, r\}$	$\{e, r\}$	$\{e, r\}$	$\emptyset$
$\mathcal{B}$	$\{t\}$	$\{t\}$	$\mathcal{B}$	$\emptyset$	$\{e, r\}$	$\{e, r\}$
$\{e\}$	$\emptyset$	$\emptyset$	$\{e, r\}$	$\{r\}$	$\{e, r\}$	$\{e\}$
$\{r\}$	$\emptyset$	$\emptyset$	$\{e, r\}$	$\{e\}$	$\{e, r\}$	$\{r\}$
$\{t\}$	$\{t\}$	$\{t\}$	$\mathcal{B}$	$\{e, r\}$	$\{e, r\}$	$\emptyset$
$\{e, r\}$	$\emptyset$	$\emptyset$	$\{e, r\}$	$\emptyset$	$\{e, r\}$	$\{e, r\}$
$\{e, t\}$	$\{t\}$	$\{t\}$	$\mathcal{B}$	$\{r\}$	$\{e, r\}$	$\{e\}$
$\{r, t\}$	$\{t\}$	$\{t\}$	$\mathcal{B}$	$\{e\}$	$\{e, r\}$	$\{r\}$

Example 7. *Let* $\mathcal{B} = \{e, r, t\}$. *Define* $\Gamma = \{\emptyset, \mathcal{B}, \{e\}, \{r\}, \{e, r\}\}$ *and* $\mathrm{P} = \emptyset$. *If* $\mathcal{K} \subseteq \mathcal{B}$, *then we have the following results.*

$\mathcal{K}$	$\mathcal{K}^\diamond$	$\gamma(\mathcal{K})$	$\gamma^\star(\mathcal{K})$	$\lambda(\mathcal{K})$	$\lambda^\diamond(\mathcal{K})$	$\widetilde{\lambda}(\mathcal{K})$
$\emptyset$	$\emptyset$	$\emptyset$	$\mathcal{B}$	$\mathcal{B}$	$\mathcal{B}$	$\emptyset$
$\mathcal{B}$	$\emptyset$	$\emptyset$	$\mathcal{B}$	$\emptyset$	$\mathcal{B}$	$\mathcal{B}$
$\{e\}$	$\emptyset$	$\emptyset$	$\mathcal{B}$	$\{r, t\}$	$\mathcal{B}$	$\{e\}$
$\{r\}$	$\emptyset$	$\emptyset$	$\mathcal{B}$	$\{e, t\}$	$\mathcal{B}$	$\{r\}$
$\{t\}$	$\emptyset$	$\emptyset$	$\mathcal{B}$	$\{e, r\}$	$\mathcal{B}$	$\{t\}$
$\{e, r\}$	$\emptyset$	$\emptyset$	$\mathcal{B}$	$\{c\}$	$\mathcal{B}$	$\{e, r\}$
$\{e, t\}$	$\emptyset$	$\emptyset$	$\mathcal{B}$	$\{r\}$	$\mathcal{B}$	$\{e, t\}$
$\{r, t\}$	$\emptyset$	$\emptyset$	$\mathcal{B}$	$\{e\}$	$\mathcal{B}$	$\{r, t\}$

Form Example 7, we conclude that $\lambda(\mathcal{K}) = \mathcal{K}^c$, $\lambda^\diamond(\mathcal{K}) = \mathcal{B}$ and $\widetilde{\lambda}(\mathcal{K}) = \mathcal{K}$ since $\gamma(\mathcal{K}) = \emptyset$ for every $\mathcal{K} \subseteq \mathcal{B}$. Observe that $\mathrm{P} = \emptyset$.

Example 8. *Let* $\mathcal{B} = \{e, r, t\}$ *with topology* $\tau = \{\emptyset, \mathcal{B}, \{e\}, \{t\}, \{r, t\}, \{e, t\}\}$ *and the primal* $\mathrm{P} = P(\mathcal{B}) - \{\mathcal{B}\}$*.If* $\mathcal{K} \subseteq \mathcal{B}$*, then:*

$\mathcal{K}$	$\mathcal{K}^\diamond$	$\gamma(\mathcal{K})$	$\gamma^*(\mathcal{K})$	$\lambda(\mathcal{K})$	$\lambda^\diamond(\mathcal{K})$	$\widetilde{\lambda}(\mathcal{K})$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\mathcal{B}$	$\mathcal{B}$	$\mathcal{B}$	$\mathcal{B}$	$\emptyset$	$\emptyset$	$\emptyset$
$\{e\}$	$\{e\}$	$\{e\}$	$\{e\}$	$\emptyset$	$\emptyset$	$\emptyset$
$\{r\}$	$\{r\}$	$\{r\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\{t\}$	$\{r, t\}$	$\{r, t\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\{e, r\}$	$\{e, r\}$	$\mathcal{B}$	$\{e\}$	$\emptyset$	$\emptyset$	$\emptyset$
$\{e, t\}$	$\{e, t\}$	$\mathcal{B}$	$\{e, t\}$	$\emptyset$	$\emptyset$	$\emptyset$
$\{r, t\}$	$\{r, t\}$	$\{r, t\}$	$\{r, t\}$	$\emptyset$	$\emptyset$	$\emptyset$

Form Example 8, we conclude that $\lambda(\mathcal{K}) = \lambda^\diamond(\mathcal{K}) = \widetilde{\lambda}(\mathcal{K}) = \emptyset$ since $\mathcal{K} \subseteq \gamma(\mathcal{K})$ for every $\mathcal{K} \subseteq \mathcal{B}$. Observe that $\mathrm{P} = P(\mathcal{B}) - \{\mathcal{B}\}$.

Example 9. *Let* $(\mathbb{R}, \mathcal{F}, \mathrm{P})$ *be defined as follows:*

$$O \in \mathcal{F} \text{ if and only if } 0 \in O^c \text{ or } O^c \text{ is a finite subset of } \mathbb{R}.$$

Moreover,

$$L \in \mathrm{P} \text{ if and only if } L^c \text{is an infinite subset of } \mathbb{R}.$$

Then, if $D \subseteq \mathbb{R}$*, we have two cases:*

Case 1. D *is a finite subset of* $\mathbb{R}$*. Then, as* $D^c \notin \mathrm{P}$*, we have* $D^\diamond = \gamma(D) = \emptyset$*.*

Case 2. D *is an infinite subset of* $\mathbb{R}$*. Then, let* $r \in \mathbb{R}$*. We have two cases:*

Subcase 2.1. $r \neq 0$*. Then,* $W = \{r\} \in \mathcal{F}(r)$ *and since* $W^c \notin \mathrm{P}$*,* $r \notin D^\diamond$*.*

Subcase 2.2. $r = 0$*. Let* $W \in \mathcal{F}(0)$*. Then,* W^c *is finite; hence* $D \cap W$ *is an infinite subset of* $\mathbb{R}$ *which implies that* $(D \cap W)^c \in \mathrm{P}$*. Thus,* $D^\diamond = \{0\}$*.*

Since $W^\diamond$ *is a finite subset, then* $\gamma(D) = \emptyset$ *for all* $D \subseteq \mathbb{R}$*. Therefore,* $\gamma^*(D) = \mathbb{R}$*. Consequently, for any* $D \subseteq \mathbb{R}$*, we have:*

(i) $\lambda(D) = D^c$*.*

(ii) $\lambda^\diamond(D) = \mathbb{R}$*.*

(iii) $\widetilde{\lambda}(D) = D$*.*

Example 10. *Let* $(\mathbb{R}, \Gamma_0, \mathrm{P}_0)$ *be defined as in Example 1. Then, if* $D \subseteq \mathbb{R}$*, we have two cases:*

Case 1. $0 \notin D$*. Then,* $D^c \notin \mathrm{P}_0$ *since* $0 \in D^c$*. Therefore,* $D^\diamond = \gamma(D) = \emptyset$*.*

Case 2. $0 \in D$*. Let* $r \in \mathbb{R}$ *and let* $W \in \Gamma_0(r)$*. Then,* $0 \in W$ *which implies that* $0 \notin W^c \cup D^c$*. Hence,* $D^\diamond = \mathbb{R}$*. Since we have* $W^\diamond = \mathbb{R}$ *for every* $W \in \Gamma_0(r)$*, we get that* $\gamma(D) = \mathbb{R}$*.*

Note that:

$$\gamma^*(D) = \begin{cases} \emptyset & \text{if } 0 \notin D \\ \mathbb{R} & \text{if } 0 \in D \end{cases}$$

Consequently, for any $O \subseteq \mathbb{R}$*, we have:*

(i)

$$\lambda(D) = \begin{cases} \emptyset & \text{if } 0 \notin D \\ D^c & \text{if } 0 \in D \end{cases}$$

(ii) $\lambda^\diamond(D) = \emptyset$.

(iii)

$$\widetilde{\lambda}(D) = \begin{cases} D & \text{if } 0 \notin D \\ \emptyset & \text{if } 0 \in D \end{cases}$$

7. Conclusion

Our work is a continuation of the paper [7] which discussed γ and γ^*-operators. In this paper, we provide more results haven't been discussed in [7]. Moreover, we define an operator called λ-operator. After that, we define another operator named $\lambda^\diamond$-operator and we show that $\lambda^\diamond(O) = [\gamma(\mathcal{B})]^c$ for every $O \subseteq \mathcal{B}$ in a primal topological space $(\mathcal{B}, \Gamma, \mathrm{P})$. Additionally, we define an operator called $\widetilde{\lambda}$-operator. Finally, we discuss some examples illustrating the differences between these operators.

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