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Some Generator Subgraphs of the Square of a Cycle

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Abstract. Graphs considered in this paper are finite simple graphs, which have no loops and multiple edges. Let $G = (V(G), E(G))$ be a graph with $E(G) = \{e_1, e_2, \ldots, e_m\}$, for some positive integer m. The *edge space* of G, denoted by $\mathscr{E}(G)$, is a vector space over the field $\mathbb{Z}_2$. The elements of $\mathscr{E}(G)$ are all the subsets of $E(G)$. Vector addition is defined as $X + Y = X \Delta Y$, the symmetric difference of sets X and Y, for $X, Y \in \mathscr{E}(G)$. Scalar multiplication is defined as $1 \cdot X = X$ and $0 \cdot X = \emptyset$ for $X \in \mathscr{E}(G)$. Let H be a subgraph of G. The *uniform set of* H with respect to G, denoted by $E_H(G)$, is the set of all elements of $\mathscr{E}(G)$ that induces a subgraph isomorphic to H. The subspace of $\mathscr{E}(G)$ generated by $E_H(G)$ shall be denoted by $\mathscr{E}_H(G)$. If $E_H(G)$ is a generating set, that is $\mathscr{E}_H(G) = \mathscr{E}(G)$, then H is called a *generator subgraph* of G. This paper provides characterization for the star graph, path graph, $(3, r)-$ tadpole graph, and kite graph $Kt_{r,s}$ so that these classes of graphs are generator subgraphs of the square of a cycle.

2020 Mathematics Subject Classifications: 05C25

Key Words and Phrases: Edge space of a graph, Generator Subgraph, Square of a cycle, Uniform set

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1. Introduction

Many interesting studies in graph theory use algebraic structures to define new classes of graphs. Then, determine the characteristics of the new developed graphs using graph-theoretic properties. For example, to mention some, the set of $k-$ subset of an artibrary set was used in [10]. The notion of group was used in [1]. In [2], the set of all induced subgraphs were utilized to develop new classes of graphs. There are several similar studies that can be found in the literature, although some uses different algebraic structures.

The notion of the generator subgraph of a graph introduced by Gervacio in 2008 links the graph theory with algebra. This notion stems from the theory of the edge space of a graph. In this study, graphs considered are finite simple undirected graphs, which have no loops and multiple edges.

Let G be a graph with $E(G) = \{e_1, e_2, \ldots, e_m\}$, for some positive integer m. The *edge space* of G, denoted by $\mathscr{E}(G)$, is a vector space over the field $\mathbb{Z}_2 = \{0, 1\}$. The elements

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of $\mathscr{E}(G)$ are all the subsets of $E(G)$. Vector addition is defined as $X + Y = X \Delta Y$, the symmetric difference of sets X and Y, for $X, Y \in \mathscr{E}(G)$. Scalar multiplication is defined as $1 \cdot X = X$ and $0 \cdot X = \emptyset$ for $X \in \mathscr{E}(G)$. The set $S \subseteq \mathscr{E}(G)$ is called a generating set if every element of $\mathscr{E}(G)$ is a linear combination of the elements of S.

For a non-empty set $X \subseteq E(G)$, the smallest subgraph of G with edge set X is called the edge-induced subgraph of G, which we denote by $G[X]$. In this paper, when we say induced subgraph, we mean an edge-induced subgraph of a graph. Let H be a subgraph of G.The *uniform set of* H with respect to G, denoted by $E_H(G)$, is the set of all elements of $\mathscr{E}(G)$ that induces a subgraph isomorphic to H. The subspace of $\mathscr{E}(G)$ generated by $E_H(G)$ is denoted by $\mathscr{E}_H(G)$. If $E_H(G)$ is a generating set, that is $\mathscr{E}_H(G) = \mathscr{E}(G)$, then H is called a *generator subgraph* of G.

It can be verified that the set $\mathscr{A} = \{\{e_1\}, \{e_2\}, \ldots, \{e_m\}\}$ forms a basis of $\mathscr{E}(G)$. Hence, $\dim \mathscr{E}(G) = m$, the size of G. The set $\mathscr{A}$ is called the natural basis for the edge space of G, as adopted from [7]. Clearly, $\mathscr{E}_H(G) \subseteq \mathscr{E}(G)$. To show that a subgraph H is a generator subgraph of G, it is sufficient to show that $\mathscr{E}(G) \subseteq \mathscr{E}_H(G)$. That is, the basis $\{\{e_1\}, \{e_2\}, \ldots, \{e_m\}\} \subseteq \mathscr{E}_H(G)$. Equivalently, we have the following useful remark.

Remark 1 ([8])**.** *Let H be a subgraph of G. Then H is a generator subgraph of G if and only if for every $e \in E(G)$ the singleton $\{e\} \in \mathscr{E}_H(G)$.*

Readers may refer to [8] for an illustration of finding the generator subgraph of a graph using Remark 1.

In [8], the concept of even edge space $\mathscr{E}^*(G)$ of a graph was introduced. If G is a graph with size m, it was shown that $\mathscr{E}^*(G)$ is a maximal subspace of the edge space of G with dimension $m - 1$. The results on the notion of even edge space are useful in this study.

Several studies on this problem focuses on the determination of the generator subgraphs of some common classes of graphs, see [11], [8], [6], [5], [7]. It can be noted that among the classes of graphs being studied, only the generator subgraphs of the complete graph and star graph were completely known. One significant result on this problem was a necessary condition that the size of a subgraph H of G must be odd, [4]. Hence, in finding the generator subgraph of a graph, we consider only those subgraphs with odd sizes. Equivalently, we have the following theorem.

Theorem 1 ([4])**.** *Let H be a subgraph of the graph G. If H is a generator subgraph of G, then $|E(H)|$ is odd.*

By a graph G, we mean an ordered pair $(V(G), E(G))$, where $V(G)$ is a finite non-empty set of elements called *vertices* and $E(G)$ is a set of $2-$ subset of $V(G)$ whose elements are called *edges*. The sets $V(G)$ and $E(G)$ are called the *vertex set* and *edge set* of G, respectively. The *order* of G is the cardinality of $V(G)$, denoted by $|V(G)|$, and the *size* of G is the cardinality of $E(G)$, denoted by $|E(G)|$. If $[x, y] \in E(G)$, we say that x is *adjacent* to y or y is *adjacent* to x. For the two graphs G and H, by $G \simeq H$, we mean G is isomorphic to H. A vertex in a graph with degree 1 is called a *pendant vertex* while an edge of the graph incident to a pendant vertex is called *pendant edge*. We used the usual notations for some special classes of graphs, K_n for complete graph of order n, P_n for path of order n, and S_n for star graph of order $n+1$. Some other classes of graphs, which were identified to be a generator subgraphs of the square of a cycle, are defined in the appropriate section of this paper. For other basic concepts in graph theory, readers may refer to the book written by Chartrand & Zhang [3]. For the algebra concepts,particularly vector spaces and some of its properties, readers may refer to the book written by E.D. Nering [9].

Let $x, y \in V(G)$. The distance between x and y, denoted by $d(x, y)$, is the length of the shortest $x - y$ path. Let C_n be a cycle of length n. The square of the cycle C_n, denoted by C_n^2, is the graph obtained from C_n by adding the edge $[x, y]$ to the cycle C_n if and only if $d(x, y) = 2$.

Examples of square of some cycle graphs C_n are given in Figure 1.

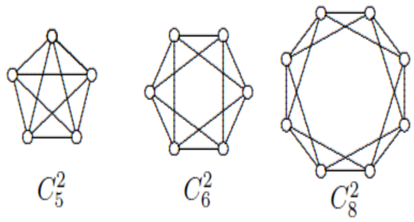

Figure 1: Illustrating the square of different cycle graphs

This study focuses on determining the generator subgraphs of the square of the cycle. At first, we provide the fixed labeling of the square of a cycle and define the edges in terms of its vertices. Then, use some properties of the square of a cycle, such as rotational symmetry to determine its generator subgraphs. Finally, some classes of graphs were found to be generator subgraphs of the square of the cycle.

In determining the dimension of the edge space of the square of a cycle graph, we utilize the following theorem, a well-known theorem in graph theory.

Theorem 2. *If G is a graph of size m, then*

$$\sum_{v \in V(G)} \deg(v) = 2m.$$

1.1. Some other known results on the generator subgraph of a graph

This section provides some other results on the generator subgraph of a graph. These results are useful in proving results of this study.

The remaining theorems can be found in [8]. The first theorem states that the path P_2 is a generator subgraph of a nonempty graph G.

Theorem 3. *Let G be a graph with $|E(G)| = m > 0$. Then the path P_2 is a generator subgraph of G.*

In [8], it was found that the subspace generated by set of all elements of $\mathscr{E}(G)$ with even cardinality has dimension $m - 1$, where m is the size of the graph. This vector space is called even edge space of graph, denoted by $\mathscr{E}^*(G)$. Equivalently, the following theorem is stated below.

Theorem 4 ([8]). *Let G be a graph with $E(G) = \{e_1, e_2, \ldots, e_m\}$. Then $\mathscr{E}^*(G)$ is a subspace of $\mathscr{E}(G)$. Moreover,* $\dim \mathscr{E}^*(G) = m - 1$.

Also, they found a basis for $\mathscr{E}^*(G)$, which is stated below.

Theorem 5 ([8]). *Let G be a graph with $E(G) = \{e_1, e_2, \ldots, e_m\}$ and define $\mathscr{B} = \{X_1, X_2, \ldots, X_{m-1}\}$, where $X_1 = \{e_1, e_2\}, X_2 = \{e_1, e_3\}, \ldots, X_{m-1} = \{e_1, e_m\}$. Then $\mathscr{B}$ forms a basis for $\mathscr{E}^*(G)$.*

Finally, in [8], they determined some properties of graphs wherein a star is one of its generator subgraphs.

Theorem 6. *Let $p > 0$ be an odd integer. If G is a graph such that for every edge $[a, b]$ in G either* $\deg(a) > p$ *or* $\deg(b) > p$, *then star S_p is a generator subgraph of G.*

Below is an immediate consequence of Theorem 6.

Corollary 1. *Let $p > 0$ be odd. If G is k- regular and $k > p$ then star S_p is a generator subgraph of G.*

The converse of Theorem 6 is not true for $p = 1$ since star $S_1 \simeq P_2$ is a generator subgraph of the graph $G = kP_2$, a graph consisting of k vertex-disjoint copies of P_2. If $p \neq 1$, we have the following result.

Theorem 7. *Let $p > 1$ be odd. Then S_p is a generator subgraph of G if and only if for every edge $[a, b]$ in G, either* $\deg(a) > p$ *or* $\deg(b) > p$.

2. Results

The main results of this study are divided into two parts. The first part investigated the edge space of the square of a cycle, its dimension and discusses some preliminary results. The second part provides some special classes of graphs which are generator subgraphs of the square of a cycle. The preliminary results in the first part were utilized in obtaining the generator subgraphs of the square of a cycle.

2.1. Edge Space of the Square of a Cycle

Let C_n^2 denote the square of a cycle of order n. Let $V(C_n^2) = \{1, 2, 3, \ldots, n\}$ where the sequence of vertices $[1, 2, 3, \ldots, n]$ forms the cycle C_n. We shall assume that the vertices $1, 2, 3, \ldots, n-1$ and n are arranged in increasing order in a clockwise direction. Thus, the edges of C_n^2 are of the form $[i, i+1]$ and $[i, i+2]$, where $1 \leq i \leq n$ and $n+1 = 1$ & $n+2 = 2$. Define $e_i = [i, i+1]$ and $s_i = [i, i+2]$. Although $[i, i+1] = [i+1, i]$ and $[i, i+2] = [i+2, i]$, for isomorphism purposes, we shall observe the order of the vertices in the definition of e_i and s_i. Thus, by the mapping of vertices $e_i \mapsto e_j$, we mean the mapping of vertices $i \mapsto j$

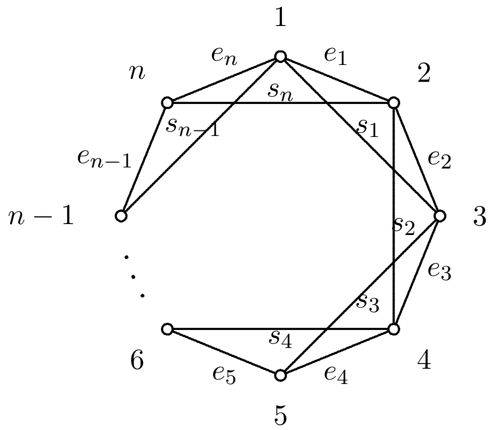

Figure 2: The labeling of C_n^2

and $i+1 \mapsto j+1$. Here, subscripts are taken modulo n. Unless otherwise stated, we shall use this labeling throughout the discussion of this paper and we shall call this the *labeling* of C_n^2. Figure 2 represents the labeling of C_n^2.

Remark 2. *The following statements hold.*

i. $C_3^2 \simeq K_3$, $C_4^2 \simeq K_4$, *and* $C_5^2 \simeq K_5$.

ii. Let n *be a positive integer. If* $n \geq 5$ *then* C_n^2 *is a* $4-$ *regular graph.*

In this study we consider the square of C_n where $n > 5$ since C_n^2 is a complete graph of order n if $n = 3, 4, \&5$ and the generator subgraphs of complete graphs were completely known by Gervacio [4].

Next, we determine the dimension of the edge space of C_n^2.

Theorem 8. *Let* $n \geq 5$ *be an integer. Then* $\dim \mathscr{E}(C_n^2) = 2n$.

Proof. To show that $\dim \mathscr{E}(C_n^2) = 2n$, it is enough to show that the size of C_n^2 is $2n$. Let $V(C_n^2) = \{v_1, v_2, v_3, \ldots, v_n\}$. By Theorem 2, $2|E(C_n^2)| = \sum_{i=1}^{n} \deg(v_i)$. By Remark 2, C_n^2 is a $4-$ regular graph. Hence, $2|E(C_n^2)| = \sum_{i=1}^{n} 4 = 4n$. This implies that $|E(C_n^2)| = 2n$.

The following remark and lemma are simple observations.

Remark 3. *Let* H *be a subgraph of* C_n^2. *If* $\{e_i, s_i\} \in \mathscr{E}(C_n^2)$ *for some integer* i, $1 \leq i \leq n$, *then by rotational symmetry on* C_n^2, $\{e_i, s_i\} \in \mathscr{E}(C_n^2)$ *for all* i. *Similarly, if* $\{e_i, s_{i-1}\} \in \mathscr{E}(C_n^2)$ *for some integer* $i, 1 \leq i \leq n$, *then* $\{e_i, s_{i-1}\} \in \mathscr{E}(C_n^2)$ *for all* i.

Lemma 1. *Let* H *be a subgraph of* C_n^2. *Then* $\{e_i, s_i\} \in \mathscr{E}_H(C_n^2)$ *if and only if* $\{e_i, s_{i-1}\} \in \mathscr{E}_H(C_n^2)$ *where* $1 \leq i \leq n$.

Proof. Let us consider the labeling of C_n^2. Assume that $\{e_i, s_i\} \in \mathscr{E}_H(C_n^2)$. Then $\{e_i, s_i\} = H_1 \Delta H_2 \Delta \cdots \Delta H_k$, where $H_j \in E_H(C_n^2)$, j and k are positive integers, where

$1 \leq j \leq k$. Define the mapping $\phi : H_j \longrightarrow H'_j$ by $\phi(e_i) = e_{n-(i-1)}$ and $\phi(s_i) = s_{n-i}$. Here, subscripts are taken modulo n. It can be verified that ϕ is an isomorphism. Thus, $H'_j \in E_H(C_n^2)$ for all j. Hence, $\{\phi(e_i), \phi(s_i)\} = \{e_{n-(i-1)}, s_{n-i}\} = \{e_{(n-i+1)}, s_{n-i}\} = H'_1 \Delta H'_2 \Delta \cdots \Delta H'_k \in \mathscr{E}_H(C_n^2)$. By Remark 3, $\{e_i, s_{i-1}\} \in \mathscr{E}_H(C_n^2)$. For the converse, the proof is similar.

The next result is an extension of the above lemma.

Lemma 2. *Let H be a subgraph of C_n^2. If $\{e_i, s_i\} \in \mathscr{E}_H(C_n^2)$ for some integer i, where $1 \leq i \leq n$, then $\mathscr{E}^*(C_n^2) \subseteq \mathscr{E}_H(C_n^2)$.*

Proof. Consider the labeling of C_n^2, let $\mathscr{B} = \{\{e_1, e_2\}, \{e_1, e_3\}, \ldots, \{e_1, e_n\}, \{e_1, s_1\}, \{e_1, s_2\}, \ldots, \{e_1, s_n\}\}$. By Theorem 5, $\mathscr{B}$ forms a basis for $\mathscr{E}^*(C_n^2)$ so it is enough to show that $\mathscr{B} \subseteq \mathscr{E}_H(C_n^2)$. Let $X \in \mathscr{B}$. Then $X = \{e_1, e_i\}$ or $X = \{e_1, s_i\}$ for some i, where $1 \leq i \leq n$. First, we show that $\{e_1, e_i\} \in \mathscr{E}_H(C_n^2)$. Since $\{e_i, s_i\} \in \mathscr{E}_H(C_n^2)$ for some i, by Remark 3, $\{e_i, s_i\} \in \mathscr{E}_H(C_n^2)$ for all i. By Lemma 1, $\{e_{i+1}, s_i\} \in \mathscr{E}_H(C_n^2)$ for all i. Now, $\{e_i, e_{i+1}\} = \{e_i, s_i\}\Delta\{e_{i+1}, s_i\} \in \mathscr{E}_H(C_n^2)$ for all i. In particular, $\{e_1, e_2\} \in \mathscr{E}_H(C_n^2)$. Thus, for $3 \leq i \leq n$, $X = \{e_1, e_i\} = \{e_1, e_2\}\Delta\{e_2, e_3\}\Delta \cdots \Delta\{e_{i-1}, e_i\} \in \mathscr{E}_H(C_n^2)$. Next, we show that $\{e_1, s_i\} \in \mathscr{E}_H(C_n^2)$. Clearly, $\{e_1, s_1\} \in \mathscr{E}_H(C_n^2)$. Now, for $2 \leq i \leq n$, we have $\{e_1, s_i\} = \{e_1, e_i\}\Delta\{e_i, s_i\} \in \mathscr{E}_H(C_n^2)$. Thus, $X \in \mathscr{E}_H(C_n^2)$. Therefore, $\mathscr{B} \subseteq \mathscr{E}_H(C_n^2)$.

Now, we give necessary and sufficient conditions for a subgraph H to be a generator subgraph of C_n^2.

Lemma 3. *Let H be a subgraph of C_n^2 where $|E(H)|$ is odd. Then H is a generator subgraph of C_n^2 if and only if $\{e_i, s_i\} \in \mathscr{E}_H(C_n^2)$ for some integer i, where $1 \leq i \leq n$.*

Proof. Let us consider the labeling of C_n^2. Assume that H is a generator subgraph of C_n^2. Then $\{e_i, s_i\} \in \mathscr{E}_H(C_n^2)$ for all i. Conversely, let H be a subgraph of C_n^2. We show $\{e_i\}, \{s_i\} \in \mathscr{E}_H(C_n^2)$ for all i, where $1 \leq i \leq n$. Since $\{e_i, s_i\} \in \mathscr{E}_H(C_n^2)$ for some integer i, there exists $A \in E_H(C_n^2)$, such that $e_i \in A$. Observe that $|A|$ is odd since $|E(H)|$ is odd. Define $B = A\backslash\{e_i\}$. Then $|B|$ is even so $B \in \mathscr{E}^*(C_n^2)$. By Lemma 2, $B \in \mathscr{E}_H(C_n^2)$. Thus, $\{e_i\} = A\Delta B \in \mathscr{E}_H(C_n^2)$. By rotational symmetry on C_n^2, $\{e_i\} \in \mathscr{E}_H(C_n^2)$ for all i. In similar argument, we can show that $\{s_i\} \in \mathscr{E}_H(C_n^2)$ for all i. By Remark 1, H is a generator subgraph of C_n^2.

2.2. Generator Subgraphs of the Square of a Cycle

First, we determine the necessary and sufficient conditions for star graph S_q to be a generator subgraph of C_n^2.

Theorem 9. *Let q and n be positive integers. Then the star Sq is a generator subgraph of C_n^2 if and only if $q = 1$ or $q = 3$.*

Proof. Assume that S_q is a generator subgraph of C_n^2. We show that $q = 1$ or $q = 3$. Suppose on the contrary, $q \neq 1$ and $q \neq 3$. Note that C_n^2 is 4-regular. Then by Theorem 1

, $q < 4$. This implies that $q = 2$. Thus $|E(S_q)| = 2$, which is even. This is a contradiction in view of Theorem 1. Therefore, $q = 1$ or $q = 3$. Conversely, suppose $q = 1$ or $q = 3$. We show that S_q is a generator subgraph of C_n^2. Case 1, $q = 1$. Then S_q is isomorphic to P_2. By Theorem 3, S_q is a generator subgraph of C_n^2. For case 2, $q = 3$, since C_n^2 is a 4-regular graph and $q < 4$ then the star S_q is a generator subgraph of C_n^2 in view of Corollary 1.

The theorem below determines the necessary and sufficient conditions for the path P_k to be a generator subgraph of C_n^2.

Theorem 10. *Let k and n are positive integers. Then the path P_k is a generator subgraph of C_n^2 if and only if k is even and $2 \le k \le n$.*

Proof. Assume that P_k is a generator subgraph of C_n^2. Then by Theorem 1, $|E(P_k)|$ must be odd. This implies that k is even. Next, we claim that $2 \le k \le n$. Suppose not, then either $k = 1$ or $k > n$. If $k = 1$, then P_k has no edge so $E_{P_k}(C_n^2) = \emptyset$. If $k > n$, then P_k is not a subgraph of C_n^2, so $E_{P_k}(C_n^2) = \emptyset$ also. In either case $\mathscr{E}_{P_k}(C_n^2) = \emptyset \neq \mathscr{E}(C_n^2)$. This is a contradiction to the assumption that P_k is a generator subgraph of C_n^2. Conversely, since k is even, $|E(P_k)|$ is odd. If $k = 2$, then $P_k \simeq P_2$. By theorem 3, P_k is a generator subgraph of C_n^2.Let us assume that $2 < k \le n$. Consider the labeling of C_n^2. Let $A = \{e_1, e_2, e_3, \ldots, e_k\}$ and define $B = A\backslash\{e_2\} \cup \{s_1\}$. It can be verified that $A, B \in E_{P_k}(C_n^2)$, as shown in Figure 3.

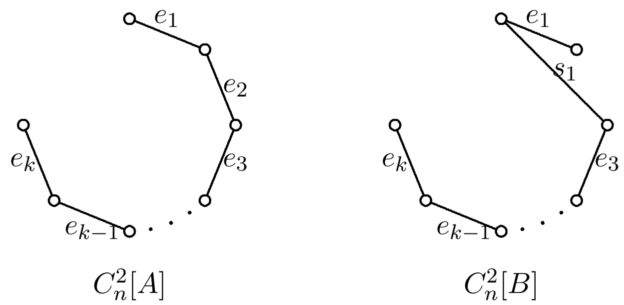

Figure 3: Illustrating the subgraphs $C_n^2[A]$ and $C_n^2[B]$

Now, $A\Delta B = \{e_2, s_1\} \in \mathscr{E}_{P_k}(C_n^2)$. By Lemma 2, $\{e_1, s_1\} \in \mathscr{E}_{P_k}(C_n^2)$. By Lemma 3.4.3 , P_k is a generator subgraph of C_n^2.

We consider another class of subgraphs of C_n^2, the tadpole graph. The (k, r)- tadpole graph, denoted by $T_{k,r}$, is the graph obtained by joining a cycle graph C_k to a path graph P_r with an edge $[a, b]$ where $a \in V(C_k)$ and $b \in V(P_r)$, deg(b) in P_r is either 0 or 1.

For instance the graphs $T_{6,2}$ and $T_{8,1}$ are shown in Figure 4.

First, we investigated the tadpole graph $T_{3,2}$. The result is stated below.

Theorem 11. *The tadpole graph $T_{3,2}$ is a generator subgraph of C_n^2.*

Proof. Consider the labeling of C_n^2. Let $A = \{e_1, e_2, e_3, e_4, s_1\}$ and define $B = A\backslash\{e_3\}\cup \{s_3\}$. As shown in Figure 5, it can be observed that $C_n^2[A] \simeq T_{3,2}$ and $C_n^2[B] \simeq T_{3,2}$. Thus

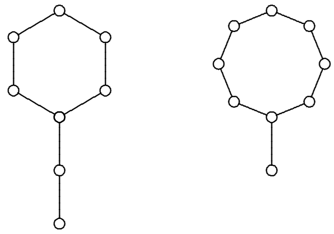

Figure 4: Illustrating the graphs $T_{6,2}$ and $T_{8,1}$

Figure 5: Illustrating the subgraphs $C_n^2[A]$ and $C_n^2[B]$

$A, B \in E_{T_{3,2}}(C_n^2)$. Now, $A\Delta B = \{e_3, s_3\} \in \mathscr{E}_{T_{3,2}}(C_n^2)$. By Lemma 3, $T_{3,2}$ is a generator subgraph of C_n^2.

The next result gives the characterization for the tadpole $T_{3,r}$ so that it is a generator subgraph of C_n^2.

Theorem 12. *Let n and r be positive integers. Then the tadpole graph $T_{3,r}$ is a generator subgraph of C_n^2 if and only if r is even and $2 \leq r \leq n-3$.*

Proof. Assume that the tadpole graph $T_{3,r}$ is a generator subgraph of C_n^2. Then the size of $T_{3,r}$ must be odd in view of Theorem 1. It follows that r is even. We claim that $2 \leq r \leq n-3$. Suppose, on the contrary, $r = 1$ or $r > n-3$. If $r = 1$ then $T_{3,r}$ consists of four edges, which are even. This is a contradiction by Theorem 1. If $r > n-3$, then the order of $T_{3,r}$ is greater than n. Meaning, $T_{3,r}$ is not a subgraph of C_n^2. Again, a contradiction to the assumption that $T_{3,r}$ is a generator subgraph of C_n^2. Conversely, assume that r is even and $2 \leq r \leq n-3$. We show that $T_{3,r}$ is a generator subgraph of C_n^2. Since r is even, then the size of the $T_{3,r}$ is odd. If $r = 2$, then $T_{3,r}$ is a generator subgraph of C_n^2 by Theorem 11. Let us assume that $2 < r \leq n-3$. Consider the labeling of C_n^2. Let $A = \{e_{n-1}, s_{n-1}, s_n, e_1, e_2, \ldots, e_r\}$and define $B = A \backslash \{e_1\} \cup \{s_n\}$. It can be verified that $A, B \in E_{T_{3,r}}(C_n^2)$, as shown in Figure 6. Now, $A\Delta B = \{e_1, s_n\} \in \mathscr{E}_{T_{3,r}}(C_n^2)$. By Lemma 1, $\{e_1, s_n\} \in \mathscr{E}(C_n^2)$. By Lemma 3, $T_{3,r}$ is a generator subgraph of C_n^2.

By a *kite graph*, denoted by $Kt_{r,s}$, we mean a graph formed by joining a path graph P_r, a path graph P_s and a cycle C_3 with two edges. One edge joins one vertex of C_3 to a

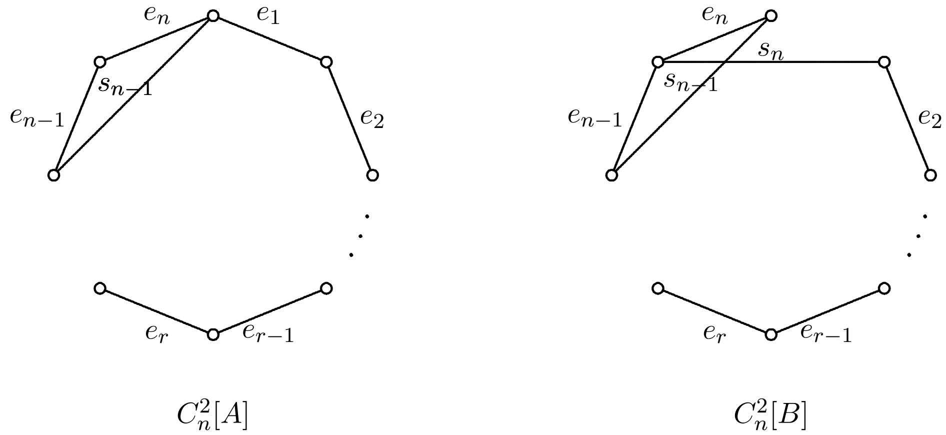

Figure 6: Illustrating the subgraphs $C_n^2[A]$ and $C_n^2[B]$

vertex of P_r whose degree in P_r is either 0 or 1. The second edge joins another vertex of C_3 to a vertex of P_s whose degree in P_s is either 0 or 1.

Some examples of kites are shown in Figure 7.

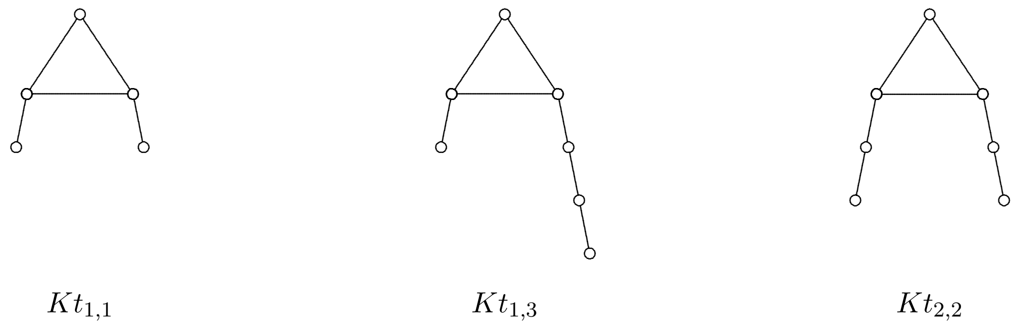

Figure 7: Illustrating the graphs $Kt_{2,2}$, $Kt_{1,3}$ and $Kt_{1,1}$

The following remark can be easily observed.

Remark 4. *The size and order of the graph $Kt_{r,s}$ is $r+s+3$. Thus, $Kt_{r,s}$ is a subgraph of C_n^2 if and only if $r+s \leq n-3$.*

First we show that $Kt_{1,1}$ is a generator subgraph of C_n^2

Theorem 13. *The Kite graph $Kt_{1,1}$ is a generator subgraph of C_n^2.*

Proof. Consider the labeling of C_n^2. Let $A = \{e_{n-1}, s_{n-1}, e_n, e_{n-2}, e_1\}$. Define $B = A\backslash\{e_1\} \cup \{s_1\}$. It can be verified that $A, B \in E_{Kt_{1,1}}(C_n^2)$. Thus, $A + B = \{e_1, s_1\} \in \mathscr{E}_{Kt_{1,1}}(C_n^2)$. Clearly, the size of $Kt_{1,1}$ is 5, which is odd. By Lemma 3, $Kt_{1,1}$ is a generator subgraph of C_n^2.

The next result gives the criteria for the subgraph $Kt_{r,s}$ to be a generator subgraph of C_n^2.

Theorem 14. *Let r and s be positive integers. Then the kite graph $Kt_{r,s}$ is a generator subgraph of C_n^2 if and only if $r+s$ is even and $r+s \leq n-3$.*

Proof. Assume that the kite $Kt_{r,s}$ is a generator subgraph of C_n^2. By Theorem 1, the size of $Kt_{r,s}$ is odd. It follows that that $r+s$ is even in view of Remark 4. We claim that $r+s \leq n-3$. Suppose $r+s > n-3$, then by Remark 4, $Kt_{r,s}$ is not a subgraph of C_n^2. This is a contradiction. Conversely, since $r+s$ is even and $r+s \leq n-3$, the size of $Kt_{r,s}$ is odd and $Kt_{r,s}$ is a subgraph of C_n^2. So the uniform set $E_{Kt_{r,s}}(C_n^2)$ is not empty. If $r = s = 1$ then $Kt_{r,s}$ is a generator subgraph of C_n^2 in view of Theorem 13. Let us assume that $r, s > 1$. Consider the labeling of C_n^2, let $A = \{e_n, e_1, s_n, e_2, e_3, \ldots, e_{r+1}, e_{n-1}, e_{n-2}, \ldots, e_{n-s}\}$. Define $B = A\backslash\{e_2\} \cup \{s_1\}$. It can be verified that $A, B \in E_{Kt_{r,s}}(C_n^2)$. Thus, $A + B = \{e_2, s_1\} \in \mathscr{E}_{Kt_{r,s}}(C_n^2)$. By Lemma 1, $\{e_1, s_1\} \in \mathscr{E}_{Kt_{r,s}}(C_n^2)$. By Lemma 3, $Kt_{r,s}$ is a generator subgraph of C_n^2.

3. Summary and Conclusions

Some classes of generator subgraphs of C_n^2 were found, such as the star graphs, path graphs,$(3, r)-$ tadpole graphs and kite graphs. Characterization for the generator subgraphs of the square of a cycle is still open.

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