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Analysis of Some Structures in Ternary Soft Topological Spaces

Mubashir Nawaz[1], Yuxin Liu[2,*], Seethalakshmi Ramaswamy[3], Maha Mohammed Saeed[4], Arif Mehmood[1]

[1] *Department of Mathematics, Institute of Numerical Sciences, Gomal University, Dera Ismail Khan, 29050, KPK, Pakistan*
[2] *School of Science, Dalian Maritime University, Dalian, 116026, P. R. China*
[3] *Department of Mathematics (H and S), Rajalakshmi Institute of Technology, Kuthambakkam, Chennai - 600124, Tamilnadu, India.*
[4] *Department of Mathematics, Faculty of Sciences, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, 15 Saudi, Arabia*

Abstract. The objective of this research is to explore ternary soft sets over three initial universal sets, incorporating a parameter set, also known as the set of decision variables. Fundamental operations such as subset, superset, equality, complement, null set, and absolute set are examined, along with the union and intersection of two ternary soft sets. The study further investigates the difference and symmetric difference between two ternary soft sets, as well as the "AND" and "OR" operations, particularly in relation to the crisp points of the sets. The behavior and properties of ternary soft sets are analyzed, with examples provided for clarification. A novel mathematical structure, ternary soft topological structures, is introduced, focusing on three initial universal sets and a fixed parameter set. Key concepts such as ternary soft open sets, closed sets, closures, interiors, boundaries, and neighborhoods are defined and explored in depth. The relationships between these concepts are examined to provide a comprehensive understanding. Illustrative examples demonstrate the practical applications of these ideas. Additionally, ternary soft semi-separation axioms, along with various properties, such as ternary soft semi-regular, semi-normal, and invariance properties, including the ternary soft topological and hereditary properties, are discussed.

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Key Words and Phrases: Ternary soft set, Ternary soft topology, ternary soft interior, ternary soft closure, ternary soft neighborhood, separation axioms, hereditary properties.

*Corresponding author.
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Email addresses: mubashirmaths@gmail.com (M. Nawaz), mehdaniyal@gmail.com (A. Mehmood), 034005lyx@dlmu.edu.cn (Y. Liu), seethamaths79@gmail.com (S. Ramaswamy), (mmmohammed@kau.edu.sa) (M. M. Saeed)

1. Introduction

Most of our traditional techniques are crisp in nature and are used for formal modeling and reasoning. These techniques are supposed to be precise and concise in nature. However, these techniques are getting badly failed when applied to problems which are complex in nature. These complex problems may be problems in Engineering, Medical Sciences and social sciences etc. These problems cannot be overcome using traditional techniques. To overcome the uncertainties and chaotic situation we have to use few techniques which are probability and fuzzy sets techniques [27], techniques of intuitionistic [5, 6], technique of vague [12], technique of interval Mathematics [6, 13] and finally technique of rough sets [19]. The said techniques can effectively be used to diffuse the complexity that exists in our problems. In this direction a Russian researcher Molodtsov [18], designed a new technique of soft set theory. This technique shoulders up the responsibility of washing-out the uncertainties that are caught by our problems. Different researchers were active in this direction and were coming out with deferent and strange ideas. Pawlak [20], launched the idea of rough set technique which is entirely different notion and used to solve some other kind of problems that contained error. With the passage of time, the concept of soft set technique was arresting the attention of researchers. Considerable attention was given in (see [2]-[28]). In continuation the application of soft set technique was examined in (see [9]-[29]). The applications of soft sets techniques were installed in decision making problems in (see [7]-[22]) and in demand analysis in [11] as well as in clustering analysis in [21]. The study of forecasting analysis with respect to soft set technique was discussed in [25].

1.1. Research Gap

The current study is limited to ternary soft sets, defined on three initial universal sets. There is a gap in extending these structures to n-dimensional soft sets, where $n \geq 3$. This extension could provide a more comprehensive framework for handling complex relationships involving more than three sets, thus broadening the scope of applications in fields such as multi-criteria decision analysis (MCDA) and multi-dimensional optimization problems. In our study, we selected$n = 3$ as the sample size and developed a novel research space that had not previously been explored or addressed by any other researchers.

1.2. Motivation

The research on binary soft topological spaces [8], which focuses on two initial universe sets with a fixed set of parameters, became a source of motivation for the development of ternary soft topology by providing a foundational framework for extending the concepts into more complex spaces. In binary soft topology, key concepts like binary soft open sets, closed sets, closure, interior, boundary, and neighborhoods were introduced and their basic properties were explored. This laid the groundwork for extending these definitions to ternary soft sets, where three universes are involved instead of just two.

1.3. Literature review

Maji et al. [17] has given more depth to soft set theory technique. The authors made the concept of different operations namely, sub-set, intersection, union and complement of soft sets. Mathematics were continuously working over this particular technique to make it more applicable. During this journey of research in general some result in [17] were pointed out to be weak and these results were not true in journal. The attempt was made by yang [26], Ali et al. [3] and Sezgin and Atagun [23]. It is worth nothing that the complement defined in [3] are define in two different ways. One is defining with NOT set of parameters and the other is define without the NOT set of parameter. Journey was continued toward this goal and finally Maji et al. [15] defined the concept of fuzzy soft set which is actually Extension of fuzzy set to a new domain. The concept of another technique which is known as intuitionistic fuzzy soft set was discussed in [16]. Feng et al. [10] made a marriage of fuzzy set with rough sets as tentative approach. M. I. Ali et al. [4] launched new structure known as algebraic structure of soft sets. Shabir et al. [24] for the first time leaked out the concept of soft topology. Acikgoz et al. [1] tried his hand for the first time on binary soft set theory and was beautifully succeeded. Examples were also given regarding this theory. Shivanagappa et al. [8] on the basis of [1] ushered in a new concept of binary soft topological spaces. Mehmood et al. [14] discussed binary soft topological spaces. With respect to generalized open set known as pre-open sets. This paper is structured into nine sections. Section 1 provides an introduction to the study, organized into three subsections: 1.1 Research Gap, 1.2 Motivation, and 1.3 Literature Review. Each subsection addresses a specific aspect of the research, laying the foundation for the study's focus and approach. Section 2 revisits the fundamental concepts relevant to the study. Section 3 is devoted to characterizing ternary soft sets, including their definitions, operations, and properties. Section 4 characterizes some results in terms of operators. Section 5 explores some structures of ternary soft topological spaces. Section 6 is devoted to the characterization of additional results in terms of interior and closures. Section 7 is the most important section, as it highlights the strength of our new work. Section 8 discusses some hereditary, separation axioms and other separation axioms. Section 9 discusses the comparative analysis. Section 9 discusses the merits, while Section 10 addresses the demerits of our new work. The final section, Section 11, is devoted to the conclusion and future work

2. Preliminaries / Basic Concepts

In this section, various operations on binary soft sets, including union, intersection, difference, and logical operations (AND, OR), as well as the definitions of binary null and absolute soft sets and their complements, are discussed, providing a foundational framework for further applications and theoretical developments in soft set theory.

Definition 1. *[1] Let U be a universal set, and let $P(U)$ be the power set of U. If E is a set of parameters and $A \subseteq E$, then a pair (F, A) is said to be a soft set over U, where F is defined as below:*

$$F : A \to P(U).$$

.

Definition 2. *[1] Let U_1 and U_2 be two universal sets, and let $P(U_1)$ and $P(U_2)$ be the power sets of U_1 and U_2 . If E is a set of parameters and $A \subseteq E$, then a pair (F, A) is said to be a binary soft set (BSS) over U_1, U_2 ,where F is defined as below:*

$$F : A \to P(U_1) \times P(U_1),$$

$F(e) = (X, Y)$ for each $e\epsilon A$ such that $X \subseteq U_1, Y \subseteq U_2$.

Definition 3. *[1] . Let (F, A) and (G, B) two binary soft sets over universal sets U_1 and U_2 then $(F, A)$is said to be a binary soft subset of (G, B) if*

(i) $A \subseteq B$

(ii) $X_1 \subseteq X_2$ *and* $Y_1 \subseteq Y_2$ *such that* $F(e) = (X_1, Y_1)$, $G(e) = (X_2, Y_2)$ *for each* $e \in A$.

Symbolically, it is denoted as $(F, A) \subseteq (G, B)$, briefly.

Definition 4. *[1] The complement of the binary soft subset (F, A) is denoted by $(F, A)^c$ and is defined as:*

$$(F, A)^c = (F^c, \rceil A),$$

where $F^c : \rceil A \to P(U_1) \times P(U_2)$ is the mapping given by:

$$F^c(e) = (U_1 - X, U_2 - Y) \quad \text{such that} \quad F(e) = (X, Y).$$

Clearly, $((F, A)^c)^c = (F, A)$.

Definition 5. *[1] A binary soft set (F, A) over U_1, U_2 is called a binary null soft set, denoted by $\tilde{\emptyset}$, if:*

$$F(e) = (\emptyset, \emptyset) \quad \text{for each } e \in A.$$

Definition 6. *[1] A binary soft set (F, A) over U_1, U_2 is called a binary absolute soft set, denoted by $\tilde{A}$, if:*

$$F(e) = (U_1, U_2) \quad \text{for each } e \in A.$$

Definition 7. *[1] The union of two binary soft subsets (F, A) and (G, B) over the common U_1, U_2 is the binary soft set (H, C), where $C = A \cup B$, and for each $e \in C$,*

$$H(e) = \begin{cases} (X_1, Y_1), & e \in A - B, \\ (X_2, Y_2), & e \in B - A, \\ (X_1 \cup X_2, Y_1 \cup Y_2), & e \in A \cap B, \end{cases}$$

such that $F(e) = (X_1, Y_1)$ for each $e \in A$ and $G(e) = (X_2, Y_2)$ for each $e \in B$. We denote the union of two binary soft subsets (F, A) and (G, B) as:

$$(F, A)\tilde{\cup}(G, B) = (H, C).$$

Definition 8. *[1] The intersection of two binary soft subsets (F, A) and (G, B) over the common universes U_1, U_2 is the binary soft set (H, C), where $C = A \cap B$, and*

$$H(e) = (X_1 \cap X_2, Y_1 \cap Y_2) \quad \text{for each } e \in C$$

such that $F(e) = (X_1, Y_1)$ for each $e \in A$ and $G(e) = (X_2, Y_2)$ for each $e \in B$. We denote the intersection of two binary soft subsets (F, A) and (G, B) as:

$$(F, A) \cap (G, B) = (H, C).$$

Definition 9. *[1] The difference of two binary soft sets (F, A) and (G, B) over the common U_1, U_2 is the binary soft set (H, A), where*

$$H(e) = (X_1 - X_2, Y_1 - Y_2) \quad \text{for each } e \in A$$

such that $(F, A) = (X_1, Y_1)$ and $(G, B) = (X_2, Y_2)$.

Definition 10. *[1] If (F, A) and (G, B) are two binary soft subsets, then "(F, A) AND (G, B)" denoted by $(F, A)\tilde{\tilde{\wedge}}(G, B)$ is defined by*

$$(F, A)\tilde{\tilde{\wedge}}(G, B) = (H, A \times B),$$

where

$$H(e, f) = (X_1 \cap X_2, Y_1 \cap Y_2) \quad \text{for each } (e, f) \in A \times B$$

such that $F(e) = (X_1, Y_1)$ and $G(e) = (X_2, Y_2)$.

Definition 11. *[1] If (F, A) and (G, B) are two binary soft subsets, then "(F, A) $OR(G, B)$" denoted by $(F, A)\tilde{\tilde{\vee}}(G, B)$ is defined by*

$$(F, A)\tilde{\tilde{\vee}}(G, B) = (O, A \times B)$$

where

$$O(e, f) = (X_1 \cup X_2, Y_1 \cup Y_2)$$

for each $(e, f) \in A \times B$ such that $F(e) = (X_1, Y_1)$ and $G(e) = (X_2, Y_2)$.

3. Characterizing Ternary Soft Sets: Definitions, Operations, and Properties

In this section, we explore the fundamental definitions related to ternary soft sets. We will introduce key concepts such as ternary soft set, ternary soft subset, ternary soft equal set, ternary soft null set, ternary soft absolute set, ternary soft union, ternary soft intersection, ternary soft laws, and ternary soft difference. Each of these concepts will be explained in detail, accompanied by clear and understandable examples to help solidify their understanding.

Definition 12. *Let U_1, U_2, U_3 be three initial universe sets and E be a set of parameters. Let $P(U_1), P(U_2), P(U_3)$ denote the power set of U_1, U_2, U_3, respectively. Also, let $A, B, \c{C} \subseteq E$.*

Definition 13. *Let U_1, U_2, U_3 be three universal sets, and let $P(U_1), P(U_2), P(U_3)$ be the power sets of U_1, U_2, U_3. If E is a set of parameters and $A \subseteq E$, then a pair (F, A) is said to be a ternary soft set (TSS) over U_1, U_2, U_3, where F is defined as below:*

$$F : A \to P(U_1) \times P(U_2) \times P(U_3), \quad F(e) = (X, Y, \c{Z}) \quad \textit{for each } e \in A$$

such that $X \subseteq U_1, Y \subseteq U_2, \c{Z} \subseteq U_3$.

Example 1. *Consider the following sets:*

$$U_1 = \{p_1, p_2, p_3, p_4, p_5\} \quad \textit{is the set of paints.}$$
$$U_2 = \{d_1, d_2, d_3, d_4, d_5\} \quad \textit{is the set of dresses.}$$
$$U_3 = \{j_1, j_2, j_3, j_4, j_5\} \quad \textit{is the set of jackets.}$$
$$E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}, e_{11}\}$$

E is the set of parameters, where

e_1 : *expensive*, e_2 : *cheap*, e_3 : *sport*, e_4 : *classic*, e_5 : *colorful*, e_6 : *plain*, e_7 : *small*, e_8 : *large*,

e_9 : *attractive*, e_{10} : *dirty*, e_{11} : *expire.*

The ternary soft set (F, A) describes "the special feature of paints, dresses, and jackets" which Mr. Wisal Khattak is going to buy, where $A = \{e_1, e_2, e_3, e_4\} \subseteq E$.

(F, A) is a ternary soft set over U_1, U_2, U_3, defined as follows:

$$F(e_1) = (\{p_1, p_2\}, \{d_1, d_3\}, \{j_1, j_3\})$$
$$F(e_2) = (\{p_3, p_4\}, \{d_2, d_4, d_5\}, \{j_2, j_4, j_5\})$$
$$F(e_3) = (\{p_2, p_3, p_5\}, \{d_1, d_5\}, \{j_1, j_5\})$$
$$F(e_4) = (\{p_1, p_4\}, \{d_2, d_3\}, \{j_2, j_3\})$$

So, we can say the ternary soft set (F, A) is:

$$\textit{expensive paints, dresses, jackets:} \quad F(e_1) = (\{p_1, p_2\}, \{d_1, d_3\}, \{j_1, j_3\})$$
$$\textit{cheap paints, dresses, jackets:} \quad F(e_2) = (\{p_3, p_4\}, \{d_2, d_4, d_5\}, \{j_2, j_4, j_5\})$$
$$\textit{sports paints, dresses, jackets:} \quad F(e_3) = (\{p_2, p_3, p_5\}, \{d_1, d_5\}, \{j_1, j_5\})$$
$$\textit{classic paints, dresses, jackets:} \quad F(e_4) = (\{p_1, p_4\}, \{d_2, d_3\}, \{j_2, j_3\})$$

We denote the ternary soft set (F, A) as follows:

$$(F, A) = \{(e_1, (\{p_1, p_2\}, \{d_1, d_3\}, \{j_1, j_3\})), (e_2, (\{p_3, p_4\}, \{d_2, d_4, d_5\}, \{j_2, j_4, j_5\})),$$
$$(e_3, (\{p_2, p_3, p_5\}, \{d_1, d_5\}, \{j_1, j_5\})), (e_4, (\{p_1, p_4\}, \{d_2, d_3\}, \{j_2, j_3\}))\}$$

In this example, we can see the views of Mrs. Wisal Khattak who wants to buy paints, dresses, and jackets under the same parameters.

Definition 14. *Let (F, A) and (G, B) be two ternary soft sets over the universal sets U_1, U_2, U_3. Then (F, A) is called a ternary soft subset of (G, B) if:*

(i) $A \subseteq B$

(ii) $X_1 \subseteq X_2, Y_1 \subseteq Y_2$ *and* $\mathcal{Z}_1 \subseteq \mathcal{Z}_2$ *such that* $F(e) = (X_1, Y_1, \mathcal{Z}_1)$ *and* $G(e) = (X_2, Y_2, \mathcal{Z}_2)$ *for each* $e \in A$.

Symbolically, it is denoted as:

$$(F, A) \subseteq (G, B)$$

(F, A) is called a ternary soft superset of (G, B) if (G, B) is a ternary soft subset of (F, A). Symbolically, it is denoted as:

$$(F, A) \supseteq (G, B)$$

Example 2. *Let $U_1 = \{p_1, p_2, p_3, p_4, p_5\}, U_2 = \{d_1, d_2, d_3, d_4, d_5\}, U_3 = \{j_1, j_2, j_3, j_4, j_5\}$, and $E = \{e_1, e_2, e_3, e_4, e_5\}$. Let $A = \{e_1, e_2, e_3\} \subseteq E$ and $B = \{e_1, e_2, e_3, e_4\} \subseteq E$.*

(F, A) and (G, B) are two ternary soft subsets over U_1, U_2, U_3, defined as follows:

$$(F, A) = \{(e_1, (\{p_1, p_2\}, \{d_1\}, \{s_1\})), (e_2, (\{p_3\}, \{d_3, d_4\}, \{s_3, s_4\})), (e_3, (\{p_1, p_4\}, \{d_1, d_2\}, \{j_1, j_2\}))\}$$

$$(G, B) = \{(e_1, (\{p_1, p_2, p_3\}, \{d_1\}, \{v_1\})), (e_2, (\{p_1, p_3\}, \{d_3, d_4, d_5\}, \{j_3, j_4, j_5\})), (e_3, (\{p_1, p_3, p_4\}, U_2, U_3)), (e_4, (U_1, U_2, U_3))\}$$

Therefore, $(F, A) \subseteq (G, B)$.

Definition 15. *Let (F, A) and (G, B) be two ternary soft sets over the common universes U_1, U_2, U_3. (F, A) is called a ternary soft equal of (G, B) if (F, A) is a ternary soft subset of (G, B) and (G, B) is a ternary soft subset of (F, A).*

Symbolically, it is denoted as:

$$(F, A) = (G, B)$$

Definition 16. *The complement of ternary soft sets (F, A) is denoted by $(F, A)^c$ and is defined as:*

$$(F, A)^c = (F^c, \rceil A)$$

where $F^c : \rceil A \rightarrow \mathcal{P}(U_1) \times \mathcal{P}(U_2) \times \mathcal{P}(U_3)$ is the mapping given by:

$$F^c(e) = (U_1 - X, U_2 - Y, U_3 - \mathcal{Z})$$

such that $F(e) = (X, Y, \mathcal{Z})$. Clearly,

$$((F, A)^c)^c = (F, A)$$

Example 3. *Consider Example 1. Then*

$$(F,A)^c = \left\{ \begin{matrix} \textit{not expensive paints, dresses, jackets:} & rep.\{\{p_1,p_2\},\{d_1,d_3\},\{j_1,j_3\}\} \\ \textit{not cheap paints, dresses, jackets:} & rep.\{\{p_3,p_4\},\{d_2,d_4,d_5\},\{j_2,j_4,j_5\}\} \\ \textit{not sports paints, dresses, jackets:} & rep.\{\{p_2,p_3,p_5\},\{d_1,d_5\},\{j_1,j_5\}\} \\ \textit{not classic paints, dresses, jackets:} & rep.\{\{p_1,p_5\},\{d_2,d_3\},\{j_2,j_3\}\}. \end{matrix} \right\}.$$

Definition 17. *A ternary soft set (F,A) over U_1,U_2,U_3 is called a ternary null soft set, denoted by $\tilde{\emptyset}$, if $F(e) = (\emptyset,\emptyset,\emptyset)$ for each $e \in A$.*

Example 4. *Consider the following sets:*

$$U_1 = \{\mathrm{j}_1,\mathrm{j}_2,\mathrm{j}_3\} \text{ is the set of jeans,}$$

$$U_2 = \{p_1,p_2,p_3,p_4\} \text{ is the set of paints,}$$

$$U_3 = \{g_1,g_2,g_3,g_4\} \text{ is the set of glasses.}$$

$$(F,A) = \{e_1 = \textit{expensive}, e_2 = \textit{smart}, e_3 = \textit{beautiful}\},$$

where A is the set of parameters. Let (F,A) be a ternary soft set as follows:

$$(F,A) = \{(e_1,(\emptyset,\emptyset,\emptyset)),(e_2,(\emptyset,\emptyset,\emptyset)),(e_3,(\emptyset,\emptyset,\emptyset))\}.$$

Therefore, (F,A) is a ternary null soft set.

Definition 18. *A ternary soft set (F,A) over U_1,U_2,U_3 is called a ternary absolute soft set, denoted by $\tilde{A}$, if $F(e) = (U_1,U_2,U_3)$ for each $e \in A$.*

Example 5. *Let U_1,U_2,U_3 and A be sets as in Example 4. Let (F,A) be a ternary soft set as follows:*

$$(F,A) = \{(e_1,(U_1,U_2,U_3)),(e_2,(U_1,U_2,U_3)),(e_3,(U_1,U_2,U_3))\}.$$

Therefore, (F,A) is a ternary absolute soft set. Clearly, $(\tilde{\tilde{A}})_c = \tilde{\tilde{\emptyset}}$ and $(\tilde{\tilde{\emptyset}})_c = \tilde{\tilde{A}}$.

Definition 19. *The union of two ternary soft sets (F,A) and (G,B) over the common U_1,U_2,U_3 is the ternary soft set (H,C), where $C = A \cup B$, and for each $e \in C$,*

$$H(e) = \left\{ \begin{matrix} (X_1,Y_1,Z_1), e \in A - B \\ (X_2,Y_2,Z_2), e \in B - A \\ (X_1 \cup X_2, Y_1 \cup Y_2, Z_1 \cup Z_2), e \in A \cap B \end{matrix} \right\}$$

such that $F(e) = (X_1,Y_1,Z_1)$ for each $e \in A$ and $G(e) = (X_2,Y_2,Z_2)$ for each $e \in B$. We denote it as

$$(F,A)\tilde{\tilde{\cup}}(G,B) = (H,C).$$

Example 6. *Consider the following sets:*

$$U_1 = \{s_1, s_2, s_3, s_4, s_5, s_6\} \text{ is the set of shoes,}$$

$$U_2 = \{p_1, p_2, p_3, p_4\} \text{ is the set of purses,}$$

$$U_3 = \{l_1, l_2, l_3, l_4\} \text{ is the set of lipsticks,}$$

$E = \{e_1 = expensive, e_2 = cheap, e_3 = black, e_4 = brown, e_5 = leather, e_6 = sport, e_7 = classic, e_8 = smart\}$.

Let $A = \{e_1, e_3, e_5\} \subseteq E$ *and* $B = \{e_3, e_4, e_6, e_8\} \subseteq E$. *Let* (F, A), (G, B) *be two ternary soft sets as follows:*

$(F, A) = \{(e_1, \{s_1, s_2\}, \{p_1\}, \{l_1\}), (e_3, \{s_4, s_5, s_6\}, \{p_1, p_3\}, \{l_1, l_3\}), (e_5, \{s_2, s_4, s_6\}, \{p_2, p_4\}, \{l_2, l_4\})\}$.

$(G, B) = \{(e_3, \{s_4, s_5\}, \{p_1, p_4\}, \{l_1, l_4\}), (e_4, \{s_1\}, \{p_2\}, \{l_2\}), (e_6, \{s_1, s_2\}, \{p_4\}, \{l_4\}), (e_8, \{s_5\}, \{p_1\}, \{l_1\})\}$.

Then $(H, C) = (F, A)\tilde{\tilde{\cup}}(G, B)$ *is the ternary soft set where* $C = A \cup B$.

$$(H, \acute{c}) = \left\{ \begin{array}{l} (e_1, \{s_1, s_2\}, \{p_1\}, \{l_1\}), \\ (e_3, \{s_4, s_5, s_6\}, \{p_1, p_3, p_4\}, \{l_1, l_3, l_4\}), \\ (e_4, \{s_1\}, \{p_2\}, \{l_2\}), \\ (e_5, \{s_2, s_4, s_6\}, \{p_2, p_4\}, \{l_2, l_4\}), \\ (e_6, \{s_1, s_2\}, \{p_4\}, \{l_4\}), \\ (e_8, \{s_5\}, \{p_1\}, \{l_1\}) \end{array} \right\}.$$

Definition 20. *The intersection of two ternary soft sets* (F, A) *and* (G, B) *over the common* U_1, U_2, U_3 *is the ternary soft set* (H, C), *where* $C = A \cap B$, *and*

$$H(e) = (X_1 \cap X_2, Y_1 \cap Y_2, Z_1 \cap Z_2)$$

for each $e \in C$ *such that* $F(e) = (X_1, Y_1, Z_1)$ *for each* $e \in A$ *and* $G(e) = (X_2, Y_2, Z_2)$ *for each* $e \in B$. *Symbolized as*

$$(F, A)\tilde{\tilde{\cap}}(G, B) = (H, C).$$

Example 7. *In the Example 6, the intersection of two ternary soft sets* (F, A) *and* (G, B) *is the ternary soft set* (H, C), *where*

$$C = A \cap B = \{e_3\}$$

and

$$(H, C) = \{(e_3, (\{s_4, s_5\}, \{p_1\}, \{l_1\}))\}.$$

Proposition 1. *Let* (F, A), (G, B), *and* (H, C) *be three ternary soft sets. Then we have the following results:*

(i) $(F, A)\tilde{\tilde{\cup}}(F, A) = (F, A)$.

(ii) $(F, A)\tilde{\tilde{\cup}}(G, B) = (G, B)\tilde{\tilde{\cup}}(F, A)$.

(iii) $(F, A)\tilde{\tilde{\cup}}((G, B)\tilde{\tilde{\cup}}(H, \acute{C})) = ((F, A)\tilde{\tilde{\cup}}(G, B))\tilde{\tilde{\cup}}(H, C)$.

(iv) $(F, A)\tilde{\tilde{\cup}}\tilde{\tilde{\emptyset}} = (F, A)$.

(v) $(F, A)\tilde{\tilde{\cup}}\tilde{\tilde{A}} = \tilde{\tilde{A}}$.

(vi) $(F, A) \subseteq \tilde{\tilde{\cup}}(F, A)\tilde{\tilde{\cup}}(G, B)$ *and* $(G, B) \subseteq \tilde{\tilde{\cup}}(F, A)\tilde{\tilde{\cup}}(G, B)$.

(vii) $(F, A)\tilde{\tilde{\cup}}(G, B) = \tilde{\tilde{\emptyset}}$ *if and only if* $(F, A) = \tilde{\tilde{\emptyset}}$ *and* $(G, B) = \tilde{\tilde{\emptyset}}$.

(viii) $(F, A) \subseteq (G, B)$ *if and only if* $(F, A)\tilde{\tilde{\cup}}(G, B) = (G, B)$.

Proof. *It is obvious.*

Proposition 2. *Let* (F, A) *and* (G, B) *be two ternary soft sets. Then we have the following results:*

(i) $(F, A)\tilde{\tilde{\cup}}(F, A)^c = \tilde{\tilde{A}}$.

(ii) $(F, A)\tilde{\tilde{\cap}}(F, A)^c = \tilde{\tilde{\emptyset}}$.

(iii) $(F, A)\tilde{\tilde{\subseteq}}(G, B)$ *if and only if* $(G, B)^c\tilde{\tilde{\subseteq}}(F, A)^c$.

(iv) $((F, A)\tilde{\tilde{\cup}}(G, B))^c = (F, A)^c\tilde{\tilde{\cup}}(G, B)^c$.

(v) $((F, A)\tilde{\tilde{\cap}}(G, B))^c = (F, A)^c\tilde{\tilde{\cap}}(G, B)^c$.

Proof.

(i) *It is obvious.*

(ii) *It is obvious.*

(iii) *It is obvious.*

(iv) $(F, A)\tilde{\tilde{\cup}}(G, B) = (H, A \cup B)$, *where for each* $e \in A \cup B$*:*

$$H(e) = \left\{ \begin{array}{ll} (X_1, Y_1, Z_1), & e \in A - B \\ (X_2, Y_2, Z_2), & e \in B - A \\ (X_1 \cup X_2, Y_1 \cup Y_2, Z_1 \cup Z_2), & e \in A \cap B. \end{array} \right\}$$

Such that $F(e) = (X_1, Y_1, Z_1)$ *for each* $e \in A$ *and* $G(e) = (X_2, Y_2, Z_2)$ *for each* $e \in B$.

Therefore,

$$((F, A)\tilde{\tilde{\cup}}(G, B))^c = (H, A \cup B)^c = (H^c, \nmid A \cup \nmid B),$$

where

$$H^c(\nmid e) = (U_1 - X, U_2 - Y, U_3 - \mathcal{Z}) \text{ for each } \nmid e \in_{\nmid} A \cup \nmid B,$$

such that $H(e) = (X, Y, \mathcal{Z})$.

Now,

$$H^c(\nmid e) = \left\{ \begin{array}{ll} (U_1 - X_1, U_2 - Y_1, U_3 - \mathcal{Z}_1), & e \in_{\nmid} A - \nmid B \\ (U_1 - X_2, U_2 - Y_2, U_3 - \mathcal{Z}_2), & e \in_{\nmid} B - \nmid A \\ (U_1 - (X_1 \cup X_2), U_2 - (Y_1 \cup Y_2), U_3 - (\mathcal{Z}_1 \cup \mathcal{Z}_2)), & e \in_{\nmid} A \cap \nmid B. \end{array} \right\}$$

Similarly,

$$(F, A)^c \widetilde{\widetilde{\cup}} (G, B)^c = (F^c, \nmid A) \widetilde{\widetilde{\cup}} (G^c, \nmid B) = (K, \nmid A \cup \nmid B),$$

where

$$K(\nmid e) = \left\{ \begin{array}{ll} (U_1 - X_1, U_2 - Y_1, U_3 - \mathcal{Z}_1), & e \in_{\nmid} A - \nmid B \\ (U_1 - X_2, U_2 - Y_2, U_3 - \mathcal{Z}_2), & e \in_{\nmid} B - \nmid A \\ (U_1 - (X_1 \cup X_2), U_2 - (Y_1 \cup Y_2), U_3 - (\mathcal{Z}_1 \cup \mathcal{Z}_2)), & e \in_{\nmid} A \cap \nmid B. \end{array} \right\}$$

Finally, H^c *and* K *are the same. Thus, the proof is completed.*

(v) It is proved in a similar way.

4. Characterization of Some Results in Terms of Operators

In this section few results are characterized in terms of operators. These operators are union, intersection, AND and OR respectively. Examples are generated to understand the applications and logic of these operators.

Proposition 3. *Let* (F, A), (G, B), *and* $(H, \acute{C})$ *be three ternary soft sets. Then we have the following results:*

$$\text{(i) } (F, A) \widetilde{\widetilde{\cup}} \big((G, B) \widetilde{\widetilde{\cap}} (H, \acute{C})\big) = \big((F, A) \widetilde{\widetilde{\cup}} (G, B)\big) \widetilde{\widetilde{\cap}} \big((F, A) \widetilde{\widetilde{\cup}} (H, \acute{C})\big).$$

$$\text{(ii) } (F, A) \widetilde{\widetilde{\cap}} \big((G, B) \widetilde{\widetilde{\cup}} (H, \acute{C})\big) = \big((F, A) \widetilde{\widetilde{\cap}} (G, B)\big) \widetilde{\widetilde{\cup}} \big((F, A) \widetilde{\widetilde{\cap}} (H, \acute{C})\big).$$

Proof. It is obvious.

Definition 21. *The difference of two ternary sets* (F, A) *and* (G, B) *over the common* U_1, U_2, U_3 *is the ternary soft set* (H, A), *where*

$$H(e) = (X_1 - X_2, Y_1 - Y_2, \mathcal{Z}_1 - \mathcal{Z}_2)$$

for each $e \in A$ *such that* $(F, A) = (X_1, Y_1, \mathcal{Z}_1)$ *and* $(G, B) = (X_2, Y_2, \mathcal{Z}_2)$.

Example 8. *Consider the following sets:*

$$U_1 = \{\acute{c}_1, \acute{c}_2, \acute{c}_3, \acute{c}_4, \acute{c}_5\} \text{ is the set of computers,}$$

$$U_2 = \{\acute{m}_1, \acute{m}_2, \acute{m}_3, \acute{m}_4, \acute{m}_5\} \text{ is the set of mobile phones,}$$

$$U_3 = \{h_1, h_2, h_3, h_4, h_5\} \text{ is the set of hand-free,}$$

$$E = \{e_1 = expensive, e_2 = outlook, e_3 = functions\}.$$

Let (F, E), (G, E) be two ternary soft sets as follows:

$$(F, E) = \left\{ \begin{array}{l} (e_1, (\{\acute{c}_1, \acute{c}_3\}, \{\acute{m}_2, \acute{m}_3\}, \{h_2, h_3\})), \\ (e_2, (\{\acute{c}_4\}, \{\acute{m}_1, \acute{m}_5\}, \{h_1, h_5\})), \\ (e_3, (\{c_3, c_4\}, \{m_2\}, \{h_2\})). \end{array} \right\}$$

$$(G, E) = \left\{ \begin{array}{l} (e_1, (\{\acute{c}_1, \acute{c}_4\}, \{\acute{m}_1\}, \{h_1\})), \\ (e_2, (\{\acute{c}_4\}, \{\acute{m}_2, \acute{m}_5\}, \{h_2, h_5\})), \\ (e_3, (\{\acute{c}_4\}, \{\acute{m}_2\}, \{h_2\})). \end{array} \right\}$$

then

$$(H, E) = \left\{ \begin{array}{l} (e_1, (\{\acute{c}_3\}, \{\acute{m}_2, \acute{m}_3\}, \{h_2, h_3\})), \\ (e_2, (\emptyset, \{\acute{m}_1\}, \{h_1\})), \\ (e_3, (\{\acute{c}_3\}, \emptyset, \emptyset)). \end{array} \right\}$$

Definition 22. *The symmetric difference of two ternary soft sets (F, A) and (G, B) over the common U_1, U_2, U_3 is the ternary soft set (H, A) defined as:*

$$(H, A) = ((F, A) - (G, A))\tilde{\tilde{\cup}}((G, A) - (F, A)).$$

We denote it as:

$$(H, A) = (F, A)\Delta(G, A).$$

Example 9. *In the Example 8, the symmetric difference of two $\langle T, S, Ss\rangle$ (F, E) and (G, E) is the ternary soft set (H, E) as follows:*

$$(F, E) = \left\{ \begin{array}{l} (e_1, (\{\acute{c}_1, \acute{c}_3\}, \{\acute{m}_2, \acute{m}_3\}, \{h_2, h_3\})), \\ (e_2, (\{\acute{c}_4\}, \{\acute{m}_1, \acute{m}_5\}, \{h_1, h_5\})), \\ (e_3, (\{\acute{c}_3, \acute{c}_4\}, \{\acute{m}_2\}, \{h_2\})). \end{array} \right\}$$

$$(G, E) = \left\{ \begin{array}{l} (e_1, (\{\acute{c}_1, \acute{c}_4\}, \{\acute{m}_1\}, \{h_1\})), \\ (e_2, (\{\acute{c}_4\}, \{\acute{m}_2, \acute{m}_5\}, \{h_2, h_5\})), \\ (e_3, (\{\acute{c}_4\}, \{\acute{m}_2\}, \{h_2\})). \end{array} \right\}$$

Then:

$$(F, E) - (G, E) = \left\{ \begin{array}{l} (e_1, (\{\acute{c}_3\}, \{\acute{m}_2, \acute{m}_3\}, \{h_2, h_3\})), \\ (e_2, (\varnothing, \{\acute{m}_1\}, \{h_1\})), \\ (e_3, (\{\acute{c}_3\}, \varnothing, \varnothing)). \end{array} \right\}$$

$$(G,E)-(F,E)=\left\{\begin{array}{l}(e_1,(\{\acute{c}_4\},\{\acute{m}_1\},\{h_1\})),\\(e_2,(\varnothing,\{\acute{m}_2\},\{h_2\})),\\(e_3,(\varnothing,\varnothing,\varnothing)).\end{array}\right\}$$

Then:

$$(H,E)=\left(\begin{array}{l}(e_1,(\{\acute{c}_3,\acute{c}_4\},\{\acute{m}_1,\acute{m}_2,\acute{m}_3\},\{h_1,h_2,h_3\})),\\(e_2,(\varnothing,\{\acute{m}_1,\acute{m}_2\},\{h_1,h_2\})),\\(e_3,(\{\acute{c}_3\},\varnothing,\varnothing)).\end{array}\right)$$

Proposition 4. *Let (F,A), (G,A), and (H,A) be three ternary soft sets. Then we have the following results:*

(i) $\widetilde{\widetilde{A}}-\widetilde{\widetilde{\varnothing}}=\widetilde{\widetilde{A}}$ *and* $\widetilde{\widetilde{A}}-\widetilde{\widetilde{A}}=\widetilde{\widetilde{\varnothing}}$.

(ii) $\widetilde{\widetilde{A}}-(F,A)^c=(F,A)$.

(iii) $(F,A)\widetilde{\widetilde{\subseteq}}(G,A)$ *if and only if* $(G,A)^c\widetilde{\widetilde{\subseteq}}(F,A)^c$.

(iv) $(F,A)\widetilde{\widetilde{\cap}}(G,A)$ *if and only if* $(F,A)\widetilde{\widetilde{\subseteq}}(G,A)^c$ *if and only if* $(G,A)\widetilde{\widetilde{\subseteq}}(F,A)^c$.

(v) $(F,A)\widetilde{\widetilde{\cup}}(G,A)=\widetilde{\widetilde{A}}$, $(F,A)\widetilde{\widetilde{\cap}}(G,A)=\widetilde{\widetilde{\varnothing}}$ *if and only if* $(F,A)=(G,A)^c$.

(vi) $((F,A)\widetilde{\widetilde{\cup}}(G,A))^c=(F,A)^c\widetilde{\widetilde{\cap}}(G,A)^c$.

(vii) $((F,A)\widetilde{\widetilde{\cap}}(G,A))^c=(F,A)^c\widetilde{\widetilde{\cup}}(G,A)^c$.

(viii) If $(F,A)\widetilde{\widetilde{\subseteq}}(G,A)$, $(F,A)\widetilde{\widetilde{\cup}}(H,A)\widetilde{\widetilde{\subseteq}}(G,A)\widetilde{\widetilde{\cup}}(H,A)$.

(ix) If $(F,A)\widetilde{\widetilde{\subseteq}}(G,A)$, $(F,A)\widetilde{\widetilde{\cap}}(H,A)\widetilde{\widetilde{\subseteq}}(G,A)\widetilde{\widetilde{\cap}}(H,A)$.

(x) If $(F,A)\widetilde{\widetilde{\subseteq}}(G,A)$ *and* $(F,A)\widetilde{\widetilde{\subseteq}}(H,A)$, $(F,A)\widetilde{\widetilde{\subseteq}}(G,A)\widetilde{\widetilde{\cap}}(H,A)$.

(xi) If $(F,A)\widetilde{\widetilde{\subseteq}}(G,A)$ *and* $(F,A)\widetilde{\widetilde{\subseteq}}(H,A)$, $(F,A)\widetilde{\widetilde{\cup}}(G,A)\widetilde{\widetilde{\subseteq}}(H,A)$.

(xii) $(F,A)-((G,A)-(H,A))=(F,A)-((G,A)\widetilde{\widetilde{\cup}}(H,A))$.

(xiii) $(F,A)-((G,A)\widetilde{\widetilde{\cap}}(H,A))=((F,A)-(G,A))\widetilde{\widetilde{\cup}}((F,A)-(H,A))$.

(xiv) $(F,A)\Delta\widetilde{\widetilde{\varnothing}}=(F,A)$, $(F,A)\Delta(F,A)=\widetilde{\widetilde{\varnothing}}$, $(F,A)\Delta(G,A)=(G,A)\Delta(F,A)$.

(xv) $(F,A)\Delta((G,A)\Delta(H,A))=((F,A)\Delta(G,A))\Delta(H,A)$.

(xvi) $(F,A)\widetilde{\widetilde{\cap}}((G,A)\Delta(H,A))=((F,A)\widetilde{\widetilde{\cap}}(G,A))\Delta((F,A)\widetilde{\widetilde{\cap}}(H,A))$.

Proof. It is obvious.

Definition 23. *If (F, A) and (G, B) are two ternary soft sets, then "(F, A) AND (G, B)," denoted by $(F, A)\widetilde{\widetilde{\wedge}}(G, B)$, as:*

$$(F, A)\widetilde{\widetilde{\wedge}}(G, B) = (H, A \times B)$$

where $H(e, f) = (X_1 \cap X_2, Y_1 \cap Y_2, Z_1 \cap Z_2)$ for each $(e, f) \in A \times B$ such that $F(e) = (X_1, Y_1, Z_1)$ and $G(e) = (X_2, Y_2, Z_2)$.

Example 10.

Consider the following sets:
$U_1 = \{j_1, j_2, j_3, j_4, j_5, j_6\}$ *is the set of shoes,*
$U_2 = \{p_1, p_2, p_3, p_4\}$ *is the set of purses,*
$U_3 = \{l_1, l_2, l_3, l_4\}$ *is the set of lipsticks,*
$E = (e_1 = expensive,\ e_2 = cheap,\ e_3 = black,\ e_4 = brown,\ e_5 = leather,\ e_6 = sport,\ e_7 = classic,\ e_8 = smart)$
$(F, A) = \{(e_1, \{j_1, j_2\}, \{p_2\}, \{l_1\}),\ (e_3, \{j_4, j_5, j_6\}, \{p_1, p_3\}, \{l_1, l_3\}),$
$(e_5, \{j_2, j_4, j_6\}, \{p_2, p_4\}, \{l_2, l_4\})\},$
$(G, B) = \{(e_3, \{j_4, j_5\}, \{p_1, p_4\}, \{l_1, l_4\}),\ (e_4, \{j_1\}, \{p_2\}, \{l_2\}),$
$(e_6, \{j_1, j_2\}, \{p_4\}, \{l_4\}),\ (e_8, \{j_5\}, \{p_1\}, \{l_1\})\},$
$(H, A \times B) = (F, A)\widetilde{\widetilde{\wedge}}(G, B)$ *is the ternary soft set as follows:*
$(H, A \times B) = \{((e_1, e_3), (\emptyset, \emptyset, \{l_1\})),\ ((e_1, e_4), (\{j_1\}, \{p_2\}, \emptyset)),\ ((e_1, e_6), (\{j_1, j_2\}, \emptyset, \emptyset)),$
$((e_1, e_8), (\emptyset, \emptyset, \{l_1\})),\ ((e_3, e_3), (\{j_4, j_5\}, \{p_1\}, \{l_1\})),\ ((e_3, e_4), (\emptyset, \emptyset, \emptyset)),$
$((e_3, e_6), (\emptyset, \emptyset, \emptyset)),\ ((e_3, e_8), (\{j_5\}, \{p_1\}, \{l_1\})),\ ((e_5, e_3), (\{j_4\}, \{p_4\}, \{l_4\})),$
$((e_5, e_4), (\emptyset, \{p_2\}, \{l_2\})),\ ((e_5, e_6), (\{j_2\}, \{p_4\}, \{l_4\})),\ ((e_5, e_8), (\emptyset, \emptyset, \emptyset))\}.$

Definition 24.

If (F, A) and (G, B) are two ternary soft sets, then "(F, A) OR (G, B)" denoted by $(F, A)\widetilde{\widetilde{\vee}}(G, B)$as:

$$(F, A)\widetilde{\widetilde{\vee}}(G, B) = (O, A \times B),$$

$$\text{where } O(e, f) = (X_1 \cup X_2, Y_1 \cup Y_2, Z_1 \cup Z_2) \text{ for each } (e, f) \in A \times B,$$

$$\text{such that } F(e) = (X_1, Y_1, Z_1) \text{ and } G(e) = (X_2, Y_2, Z_2).$$

Example 11.

Consider the following sets:

$U_1 = \{s_1, s_2, s_3, s_4, s_5, s_6\}$ *is the set of shoes,*

$U_2 = \{p_1, p_2, p_3, p_4\}$ *is the set of purses,*

$U_3 = \{l_1, l_2, l_3, l_4\}$ *is the set of lipsticks,*

$E = \{e_1 = expensive,\ e_2 = cheap,\ e_3 = black,\ e_4 = brown,\ e_5 = leather,\ e_6 = sport,\ e_7 = classic,\ e_8 = smart$

$(F, A) = \{(e_1, \{s_1, s_2\}, \{p_2\}, \{l_1\}), (e_3, \{s_4, s_5, s_6\}, \{p_1, p_3\}, \{l_1, l_3\}), (e_5, \{s_2, s_4, s_6\}, \{p_2, p_4\}, \{l_2, l_4\})\}$.

$(G, B) = \{(e_3, \{s_4, s_5\}, \{p_1, p_4\}, \{l_1, l_4\}), (e_4, \{s_1\}, \{p_2\}, \{l_2\}), (e_6, \{s_1, s_2\}, \{p_4\}, \{l_4\}), (e_8, \{s_5\}, \{p_1\}, \{l_1\})\}$.

$(F, A)\tilde{\tilde{\vee}}(G, B) = (O, A \times B)$ *is a ternary soft set as follows:*

$$(H, A \times B) = \left\{ \begin{array}{l} ((e_1, e_3), (\{s_1, s_2, s_4, s_5\}, \{p_1, p_2, p_4\}, \{l_1, l_4\})), \\ ((e_1, e_4), (\{s_1, s_2\}, \{p_2\}, \{l_1, l_2\})), \\ ((e_1, e_6), (\{s_1, s_2\}, \{p_2, p_4\}, \{l_1, l_4\})), \\ ((e_1, e_8), (\{s_1, s_2, s_5\}, \{p_1, p_2\}, \{l_1\})), \\ ((e_3, e_3), (\{s_4, s_5, s_6\}, \{p_1, p_3, p_4\}, \{l_1, l_3, l_4\})), \\ ((e_3, e_4), (\{s_1, s_4, s_5, s_6\}, \{p_1, p_2, p_3\}, \{l_1, l_2, l_3\})), \\ ((e_3, e_6), (\{s_1, s_2, s_4, s_5, s_6\}, \{p_1, p_3, p_4\}, \{l_1, l_3, l_4\})), \\ ((e_3, e_8), (\{s_4, s_5, s_6\}, \{p_1, p_3\}, \{l_1, l_3\})), \\ ((e_5, e_3), (\{s_2, s_4, s_5, s_6\}, \{p_1, p_2, p_4\}, \{l_1, l_2, l_4\})), \\ ((e_5, e_4), (\{s_1, s_2, s_4, s_6\}, \{p_2, p_4\}, \{l_2, l_4\})), \\ ((e_5, e_6), (\{s_1, s_2, s_4, s_6\}, \{p_2, p_4\}, \{l_2, l_4\})), \\ ((e_5, e_8), (\{s_2, s_4, s_5, s_6\}, \{p_1, p_2, p_4\}, \{l_1, l_2, l_4\})) \end{array} \right\}.$$

Proposition 5.

Let $(F, A), (G, A)$ and (H, A) be three ternary soft sets. Then we have the following results:

(i)
$$((F, A)\tilde{\tilde{\vee}}(G, A))^c = (F, A)^c\tilde{\tilde{\wedge}}(G, A)^c.$$

(ii)
$$((F, A)\tilde{\tilde{\wedge}}(G, A))^c = (F, A)^c\tilde{\tilde{\vee}}(G, A)^c.$$

(iii)
$$(F, A) \vee ((G, B)\tilde{\tilde{\vee}}(H, \acute{C})) = ((F, A)\tilde{\tilde{\vee}}(G, B))\tilde{\tilde{\vee}}(H, \acute{C}).$$

(iv)
$$(F, A)\tilde{\tilde{\wedge}}((G, B)\tilde{\tilde{\wedge}}(H, \acute{C})) = ((F, A)\tilde{\tilde{\wedge}}(G, B))\tilde{\tilde{\wedge}}(H, \acute{C}).$$

(v)
$$(F, A)\tilde{\tilde{\vee}}((G, B)\tilde{\tilde{\wedge}}(H, \acute{C})) = ((F, A)\tilde{\tilde{\vee}}(G, B))\tilde{\tilde{\wedge}}((F, A)\tilde{\tilde{\vee}}(H, \acute{C})).$$

(vi)
$$(F, A)\tilde{\tilde{\wedge}}((G, B)\tilde{\tilde{\vee}}(H, \acute{C})) = ((F, A)\tilde{\tilde{\wedge}}(G, B))\tilde{\tilde{\vee}}((F, A)\tilde{\tilde{\wedge}}(H, \acute{C})).$$

Proof. It is obvious.

5. Exploring the Structure of Ternary Soft Topological Spaces

In this section, the most important space is introduced, known as the ternary soft topological space. Examples are provided, and a few fundamental results related to this space are discussed with respect to soft points. It is demonstrated that the union of two ternary soft topological spaces may not necessarily be a ternary soft topological space. However, the intersection of these spaces works seamlessly to maintain the properties of a ternary soft topological space.

Definition 25. *Let τ_Δ be the collection of ternary soft sets over U_1, U_2, U_3, then τ_Δ is said to be a ternary soft topology (TST) on U_1, U_2, U_3 if:*

(i) $\widetilde{\widetilde{\emptyset}}, \widetilde{\widetilde{X}} \in \tau_\Delta$,

(ii) The union of any number of members of ternary soft sets in τ_Δ belongs to τ_Δ,

(iii) The intersection of any two ternary soft sets in τ_Δ belongs to τ_Δ.

Then $(U_1, U_2, U_3, \tau_\Delta, E)$ is called a ternary soft topological space over U_1, U_2, U_3.

Definition 26. *Let $(U_1, U_2, U_3, \tau_\Delta, E)$ be a ternary soft topological space over U_1, U_2, U_3. Then the members of τ_Δ are said to be ternary soft open sets in U_1, U_2, U_3.*

Definition 27. *Let $(U_1, U_2, U_3, \tau_\Delta, E)$ be a ternary soft topological space over U_1, U_2, U_3. Then the members of τ_Δ are said to be ternary soft closed sets in U_1, U_2, U_3 if their relative complements $(F, E)'$ belong to τ_Δ.*

Definition 28. *Let U_1, U_2, U_3 be three initial universe sets, E be a set of parameters, and $\tau_\Delta = \{\widetilde{\widetilde{\emptyset}}, \widetilde{\widetilde{X}}\}$. Then τ_Δ is called the ternary soft indiscrete topology on U_1, U_2, U_3, and $(U_1, U_2, U_3, \tau_\Delta, E)$ is said to be a ternary soft indiscrete space over U_1, U_2, U_3.*

Definition 29. *Let U_1, U_2, U_3 be three initial universe sets, E be a set of parameters, and let τ_Δ be the collection of all ternary soft sets which can be defined over U_1, U_2, U_3. The τ_Δ is called the ternary soft discrete topology on U_1, U_2, U_3, and $(U_1, U_2, U_3, \tau_\Delta, E)$ is said to be a ternary soft discrete space over U_1, U_2, U_3.*

Example 12. *Consider the following sets:*

$U_1 = \{a_1, a_2, a_3, a_4, a_5\}, \quad U_2 = \{d_1, d_2, d_3, d_4\}, \quad U_3 = \{\acute{c}_1, \acute{c}_2, \acute{c}_3, \acute{c}_4\}, \quad E = \{e_1, e_2, e_3, e_4, e_5\}.$

Let $A = \{e_1, e_2, e_4\}$. Then

$$\tau_\Delta = \{\widetilde{\widetilde{\emptyset}}, \widetilde{\widetilde{X}}, (F_1, A), (F_2, A), (F_3, A), (F_4, A)\},$$

where (F_1, A), (F_2, A), (F_3, A), (F_4, A) are ternary soft sets defined as follows:

$$(F_1, A) = \{(e_1, (\{a_1\}, \{d_1\}, \{\acute{c}_1\})), (e_2, (\{a_2\}, \{d_2\}, \{\acute{c}_2\})), (e_4, (\{a_3\}, \{d_3\}, \{\acute{c}_3\}))\}.$$

$(F_2, A) = \{(e_1, (\{a_4\}, \{d_4\}, \{\acute{c}_4\})), (e_2, (\{a_3\}, \{d_1\}, \{\acute{c}_1\})), (e_4, (\{a_3, a_5\}, \{d_1, d_2\}, \{\acute{c}_1, \acute{c}_2\}))\}.$

$$(F_3, A) = \left\{ \begin{array}{l} \{(e_1, (\{a_1, a_4\}, \{d_1, d_4\}, \{\acute{c}_1, \acute{c}_4\})), \\ (e_2, (\{a_2, a_3\}, \{d_1, d_2\}, \{\acute{c}_1, \acute{c}_2\})), \\ (e_4, (\{a_3, a_5\}, \{d_1, d_2, d_3\}, \{\acute{c}_1, \acute{c}_2, \acute{c}_3\}))\}. \end{array} \right\}$$

$$(F_4, A) = \{(e_4, (\{a_3\}, \{d_3\}, \{\acute{c}_3\}))\}.$$

Clearly, τ_Δ is a ternary soft topology. Then automatically, $\tilde{\tilde{\tilde{\emptyset}}}, \tilde{\tilde{\tilde{X}}}, (F_1, A), (F_2, A), (F_3, A), (F_4, A)$ are ternary soft open sets. Similarly, $\tilde{\tilde{\tilde{\emptyset}}}, \tilde{\tilde{\tilde{X}}}, (F_1, A)', (F_2, A)', (F_3, A)', (F_4, A)'$ are ternary soft closed sets.

Remark 1. *Any collection of ternary soft sets does not necessarily form a ternary soft topology. The following example illustrates this.*

Example 13. *Following is the ternary soft sets collection of sets:*

$$\mathcal{T}_\Delta = \left\{ \begin{array}{l} \tilde{\tilde{\tilde{\emptyset}}},\ \tilde{\tilde{\tilde{X}}},\ \{(e_1, (\{a_2, a_4\}, \{d_2, d_4\}, \{\acute{c}_2, \acute{c}_4\})), (e_2, (\{a_3\}, \{d_4\}, \{\acute{c}_4\}))\}, \\ \{(e_1, (\{a_2, a_3\}, \{d_1, d_4\}, \{\acute{c}_1, \acute{c}_4\})), (e_2, (\{a_2\}, \{d_1\}, \{\acute{c}_1\})), \\ (e_5, (\{a_1, a_3\}, \{d_2\}, \{\acute{c}_2\}))\}, \\ \{(e_1, (\{a_1, a_3\}, \{d_2, d_3\}, \{\acute{c}_2, \acute{c}_3\})), (e_4, (\{a_1\}, \{d_1, d_2\}, \{\acute{c}_1, \acute{c}_2\}))\} \end{array} \right\}$$

We see that this is not a ternary soft topology.

Remark 2. *Let $(U_1, U_2, U_3, \tau_\Delta, E)$ and $(U_1, U_2, U_3, \tau_{\Delta'}, E)$ be ternary soft topological spaces over the same universal sets U_1, U_2, U_3. Then $(U_1, U_2, U_3, \tau_\Delta \tilde{\tilde{\tilde{\cup}}} \tau_{\Delta'}, E)$ may not be a ternary soft topological space over (U_1, U_2, U_3).*

Example 14. *Let τ_Δ be given by*

$$\tau_\Delta = \left\{ \begin{array}{l} \tilde{\tilde{\tilde{\emptyset}}},\ \tilde{\tilde{\tilde{X}}},\ \{(e_1, (\{a_1\}, \{d_1\}, \{\acute{c}_1\})), (e_2, (\{a_2\}, \{d_2\}, \{c_2\})), (e_4, (\{a_3\}, \{d_3\}, \{\acute{c}_3\}))\}, \\ \{(e_1, (\{a_4\}, \{d_4\}, \{\acute{c}_4\})), (e_2, (\{a_3\}, \{d_1\}, \{\acute{c}_1\})), (e_3, (\{a_1, a_2\}, \{d_3\}, \{\acute{c}_3\})), \\ (e_4, (\{a_3, a_5\}, \{d_1, d_2\}, \{\acute{c}_1, \acute{c}_2\}))\}, \\ \{(e_1, (\{a_1, a_4\}, \{d_1, d_4\}, \{\acute{c}_1, \acute{c}_4\})), (e_2, (\{a_2, a_3\}, \{d_1, d_2\}, \{\acute{c}_1, \acute{c}_2\})), \\ (e_3, (\{a_1, a_2\}, \{d_3\}, \{c_3\})), (e_4, (\{a_3, a_5\}, \{d_1, d_2, d_3\}, \{c_1, c_2, c_3\}))\}, \end{array} \right\}$$

and $\tau_{\Delta'}$ be given by

$$\tau_{\Delta'} = \left\{ \begin{array}{l} \tilde{\tilde{\tilde{\emptyset}}},\ \tilde{\tilde{\tilde{X}}},\ \{(e_1, (\{a_2\}, \{d_2\}, \{\acute{c}_2\})), (e_5, (\{a_3, a_4\}, \{d_1, d_3\}, \{\acute{c}_1, \acute{c}_3\})), \\ (e_8, (\{a_1, a_3\}, \{d_2\}, \{\acute{c}_2\}))\}, \\ \{(e_2, (\{a_1, a_2\}, \{d_4\}, \{\acute{c}_2\})), (e_5, (\{a_3, a_5\}, \{d_3\}, \{\acute{c}_3\})), \\ (e_7, (\{a_1\}, \{d_2, d_3\}, \{\acute{c}_2, \acute{c}_3\}))\}, \\ \{(e_1, (\{a_2\}, \{d_2\}, \{\acute{c}_2\})), (e_2, (\{a_1, a_2\}, \{d_4\}, \{\acute{c}_4\})), (e_5, (\{a_3, a_4, a_5\}, \{d_1, d_3\}, \{\acute{c}_1, \acute{c}_3\})), \\ (e_7, (\{a_1\}, \{d_1, d_3\}, \{\acute{c}_1, \acute{c}_3\}))\} \end{array} \right\}$$

Then clearly, τ_Δ and $\tau_{\Delta'}$ are ternary soft topological spaces, but

$$\tau_\Delta \tilde{\tilde{\cup}} \tau_{\Delta'} = \left\{ \begin{array}{l} \tilde{\tilde{\emptyset}},\ \tilde{\tilde{X}},\ \{(e_1,(\{a_1,a_2\},\{d_1,d_2\},\{\acute{c}_1,\acute{c}_2\})),(e_2,(\{a_2\},\{d_2\},\{\acute{c}_2\})),\\ (e_4,(\{a_3\},\{d_3\},\{\acute{c}_3\})),(e_5,(\{a_3,a_4\},\{d_1,d_3\},\{\acute{c}_1,\acute{c}_3\})),(e_8,(\{a_1,a_3\},\{d_2\},\{\acute{c}_2\}))\},\\ \{(e_1,(\{a_1\},\{d_1\},\{\acute{c}_1\})),(e_2,(\{a_1,a_2\},\{d_2,d_4\},\{\acute{c}_2,\acute{c}_4\})),\\ (e_4,(\{a_3\},\{d_3\},\{\acute{c}_3\})),(e_5,(\{a_3,a_5\},\{d_3\},\{\acute{c}_3\})),(e_7,(\{a_1\},\{d_2,d_3\},\{\acute{c}_2,\acute{c}_3\}))\},\\ \{(e_1,(\{a_1,a_2\},\{d_1,d_2\},\{\acute{c}_1,\acute{c}_2\})),(e_2,(\{a_1,a_2\},\{d_2,d_4\},\{\acute{c}_2,\acute{c}_4\})),\\ (e_4,(\{a_3\},\{d_3\},\{\acute{c}_3\})),(e_5,(\{a_3,a_4,a_5\},\{d_1,d_3\},\{\acute{c}_1,\acute{c}_3\})),\\ (e_7,(\{a_1\},\{d_2,d_3\},\{\acute{c}_2,\acute{c}_3\}))\},\\ \{(e_1,(\{a_2,a_4\},\{d_2,d_4\},\{\acute{c}_2,\acute{c}_3\})),(e_2,(\{a_3\},\{d_1\},\{\acute{c}_1\})),\\ (e_3,(\{a_1,a_2\},\{d_3\},\{\acute{c}_3\})),(e_5,(\{a_3,a_4\},\{d_1,d_3\},\{\acute{c}_1,\acute{c}_3\})),\\ (e_8,(\{a_1,a_3\},\{d_2\},\{\acute{c}_2\}))\},\\ \{(e_1,(\{a_4\},\{d_4\},\{\acute{c}_4\})),(e_2,(\{a_1,a_2,a_3\},\{d_1,d_4\},\{\acute{c}_1,\acute{c}_4\})),\\ (e_3,(\{a_1,a_2\},\{d_3\},\{\acute{c}_3\})),(e_5,(\{a_3,a_5\},\{d_3\},\{\acute{c}_3\})),\\ (e_7,(\{a_1\},\{d_2,d_3\},\{\acute{c}_2,\acute{c}_3\}))\},\\ \{(e_1,(\{a_2,a_4\},\{d_2,d_4\},\{\acute{c}_2,\acute{c}_4\})),(e_2,(\{a_1,a_2,a_3\},\{d_1,d_4\},\{\acute{c}_1,\acute{c}_4\})),\\ (e_3,(\{a_1,a_2\},\{d_3\},\{\acute{c}_3\})),(e_5,(\{a_3,a_4,a_5\},\{d_1,d_3\},\{\acute{c}_1,\acute{c}_3\})),\\ (e_7,(\{a_1\},\{d_2,d_3\},\{\acute{c}_2,\acute{c}_3\}))\} \end{array} \right\}$$

Clearly,

$$\{(e_5,(\{a_3,a_4\},\{d_1,d_3\},\{\acute{c}_1,\acute{c}_3\}))\}\tilde{\tilde{\cap}}\{(e_5,(\{a_3,a_5\},\{d_3\},\{\acute{c}_3\}))\} = \{(e_5,(\{a_3\},\{d_3\},\{\acute{c}_3\}))\} \notin \tau_\Delta \tilde{\tilde{\cup}} \tau_{\Delta'}.$$

Thus, $\tau_\Delta \tilde{\tilde{\cup}} \tau_{\Delta'}$ is not a ternary soft topology.

Theorem 1. *Let $(U_1,U_2,U_3,\tau_\Delta,E)$ and $(U_1,U_2,U_3,\tau'_\Delta,E)$ be two ternary soft topological spaces over the common initial sets U_1,U_2,U_3. Then $(U_1,U_2,U_3,\tau_\Delta\tilde{\tilde{\cap}}\tau'_\Delta,E)$ is a ternary soft topological space over U_1,U_2,U_3.*

Proof.

(i) $\tilde{\tilde{\emptyset}}$, $\tilde{\tilde{X}}$, *belongs to*$\tau_\Delta\tilde{\tilde{\cap}}\tau'_\Delta$.

(ii) *Let $\{G_i, E/i \in \hat{I}\}$ be a family of ternary soft sets in $\tau_\Delta\tilde{\tilde{\cap}}\tau'_\Delta$. Then $(G_i,E) \in \tau_\Delta$ and $(G_i,E) \in \tau'_\Delta$ for all $i \in \hat{I}$, so $\tilde{\tilde{\cup}}_{i\in\hat{I}}(G_i,E) \in \tau_\Delta$ and $\tilde{\tilde{\cup}}_{i\in\hat{I}}(G_i,E) \in \tau'_\Delta$. Thus $\tilde{\tilde{\cup}}_{i\in\hat{I}}(G_i,E) \in \tau_\Delta \cap \tau'_\Delta$.*

(iii) *Let the two ternary soft sets (H,E), $(\hat{I},E) \in \tau_\Delta\tilde{\tilde{\cap}}\tau'_\Delta$. Then (H,E), $(\hat{I},E) \in \tau_\Delta$ and (H,E), $(\hat{I},E) \in \tau'_\Delta$. Since $(H,E)\tilde{\tilde{\cap}}(G,E) \in \tau_\Delta$ and $(H,E)\tilde{\tilde{\cap}}(G,E) \in \tau'_\Delta$, so $(H,E)\tilde{\tilde{\cap}}(G,E) \in \tau_\Delta\tilde{\tilde{\cap}}\tau'_\Delta$.*

6. Characterization of More Results in Terms of Interior and Closure

In this section, several additional results are introduced concerning the concepts of the interior and closure of sets within the context of ternary soft topological spaces.

The relationships and interconnections between these results are thoroughly explored to provide a deeper understanding of their significance. To facilitate a clearer comprehension of the concepts, relevant examples are included, demonstrating how these results can be effectively applied in various scenarios and highlighting their practical implications.

Definition 30. *Let $(U_1, U_2, U_3, \tau_\Delta, E)$ be a ternary soft topological space over U_1, U_2, U_3 and (G, E) be the ternary soft set over common universal sets U_1, U_2, U_3. Then the ternary soft closure of (G, E), denoted by $\overline{\overline{(G, E)}}$, is the intersection of all ternary soft closed sets of (G, E). Thus, $\overline{\overline{(G, E)}}$ is the smallest ternary soft closed set over U_1, U_2, U_3 which contains (G, E).*

Theorem 2. *Let $(U_1, U_2, U_3, \tau_\Delta, E)$ be a ternary soft topological space over U_1, U_2, U_3 and let $(H, E), (\hat{I}, E)$ be ternary soft sets over U_1, U_2, U_3. Then:*

(i) $\overline{\overline{\tilde{\tilde{\emptyset}}}} = \tilde{\tilde{\emptyset}}$ *and* $\overline{\overline{\tilde{\tilde{X}}}} = \tilde{\tilde{X}}$.

(ii) $(H, E)\tilde{\tilde{\subseteq}}\overline{\overline{(H, E)}}$ *implies* $\overline{\overline{(H, E)}}$ *is a ternary soft closed set and* $\overline{\overline{(H, E)}}$ *contains* (H, E).

(iii) (H, E) *is a ternary soft closed set if and only if* $(H, E)\tilde{\tilde{\approx}}\overline{\overline{(H, E)}}$.

(iv) $\overline{\overline{\overline{\overline{(H, E)}}}} = (H, E)$.

(v) $(H, E)\tilde{\tilde{\subseteq}}(\hat{I}, E)$ *implies* $\overline{\overline{(H, E)}}\tilde{\tilde{\subseteq}}\overline{\overline{(\hat{I}, E)}}$.

(vi) $\overline{\overline{(H, E)\tilde{\tilde{\cup}}(\hat{I}, E)}} = \overline{\overline{(H, E)}}\tilde{\tilde{\cup}}\overline{\overline{(\hat{I}, E)}}$.

(vii) $\overline{\overline{(H, E)\tilde{\tilde{\cap}}(\hat{I}, E)}}\tilde{\tilde{\subseteq}}\overline{\overline{(H, E)}}\tilde{\tilde{\cap}}\overline{\overline{(\hat{I}, E)}}$.

Proof.

(i) *This is obvious.*

(ii) *Let $\{(H_i, E) \mid i \in \hat{I}\}$ be the family of all the ternary closed sets containing (H, E). Then, by definition, we know that:*

$$\overline{\overline{(H, E)\tilde{\tilde{\cap}}}} \in \hat{I}(H_i, E) \to (1).$$

Now, since $\{(H_i, E) \mid i \in \hat{I}\}$ is a ternary soft closed set $\forall\ i \in I \Rightarrow \tilde{\tilde{\cap}}_i \in \hat{I}(H_i, E)$ is also a ternary soft closed set. Since an arbitrary intersection of ternary soft closed sets is a ternary soft closed set, $\overline{\overline{(H, E)}}$ is a ternary soft closed set $(from(1))$.

$\Rightarrow$ Thus, $\overline{\overline{(H, E)}}$ is a ternary soft closed set. Now, we prove that $\overline{\overline{(H, E)}}\tilde{\tilde{\supseteq}}(H, E)$. We know that $\forall\ i \in I$, $\{(H_i, E) \mid i \in \hat{I}\}\tilde{\tilde{\supseteq}}(H, E)$.

$\Rightarrow (H,E)\tilde{\tilde{\subseteq}}\ \tilde{\tilde{\cap}}_i \in \hat{I}\ (H_i,E) \implies (H,E)\ \tilde{\tilde{\subseteq}}\ \overline{\overline{(H,E)}}$ $[using(1)]$
$\Rightarrow \overline{\overline{(H,E)}}\tilde{\tilde{\subseteq}}(H,E)$ *Thus* $\overline{\overline{(H,E)}}$ *contains* (H,E). *Hence,* $\overline{\overline{(H,E)}}$ *is a ternary soft closed set and* $\overline{\overline{(H,E)}}$ *contains* (H,E).

(iii) *Let* (H,E) *be a ternary soft set. To prove* $\overline{\overline{(H,E)}} = (H,E)$, *suppose* (H,E) *is a ternary soft closed set. Now, we have* $(H,E)\tilde{\tilde{\supseteq}}(H,E)$, *so* (H,E) *is a* $<< (T,S) >>$ *closed set containing* $(H,E) \to (1)$. *But* $\overline{\overline{(H,E)}}$ *is the smallest ternary soft closed set containing* $(H,E) \to (2)$. *Therefore from* (1) *and* (2), *it follows that* $\overline{\overline{(H,E)}}$ *is smaller then* (H,E) *that is* $\overline{\overline{(H,E)}}\tilde{\tilde{\subseteq}}(H,E)$. *But from from* (ii) *of this theorem, we have* $(H,E)\tilde{\tilde{\subseteq}}\overline{\overline{(H,E)}}$ *is always true. Therefore we have* $\overline{\overline{(H,E)}}\tilde{\tilde{\subseteq}}(H,E)$. *Thus* $(H,E) = \overline{\overline{(H,E)}}$. *Consequently , if* (H,E) *is a ternary soft closed set then* $\overline{\overline{(H,E)}} = (H,E)$.

(iv) *Since* $\overline{\overline{(H,E)}}$ *is a ternary soft closed set, therefore by (iii), we have* $\overline{\overline{\overline{\overline{(H,E}}}} = (H,E)$.

(v) *If* $(H,E)\tilde{\tilde{\subseteq}}(\hat{I},E)$, *then* $\overline{\overline{(H,E)}}\tilde{\tilde{\subseteq}}\overline{\overline{(\hat{I},E)}}$. *Suppose* $(H,E)\tilde{\tilde{\subseteq}}(\hat{I},E)$. *We know that* $(\hat{I},E)\tilde{\tilde{\subseteq}}\overline{\overline{(\hat{I},E)}}$, *and we have*

$$(H,E)\tilde{\tilde{\subseteq}}(\hat{I},E)\tilde{\tilde{\subseteq}}\overline{\overline{(\hat{I},E)}}.$$

Therefore, $(H,E)\tilde{\tilde{\subseteq}}\overline{\overline{(\hat{I},E)}}$. *Therefore,* $\overline{\overline{(\hat{I},E)}}$ *is a ternary soft closed set containing* $(\hat{I},E) \to (1)$. *But* $\overline{\overline{(H,E)}}$ *is the smallest ternary soft closed set containing* $(H,E) \to$ (2) *it follows that* $\overline{\overline{(H,E)}}$ *is smaller than* $\overline{\overline{(\hat{I},E)}}$, *that is* $\overline{\overline{(H,E)}}\tilde{\tilde{\subseteq}}\overline{\overline{(\hat{I},E)}}$. *Thus if* $(H,E)\tilde{\tilde{\subseteq}}(\hat{I},E)$. *Then* $\overline{\overline{(H,E)}}\tilde{\tilde{\subseteq}}\overline{\overline{(\hat{I},E)}}$.

(vi) *We know* $(H,E)\tilde{\tilde{\subseteq}}(H,E)\tilde{\tilde{\cup}}(\hat{I},E)$ *and* $(\hat{I},E)\tilde{\tilde{\subseteq}}(H,E)\tilde{\tilde{\cup}}(\hat{I},E)$. *Therefore:*

$$\overline{\overline{(H,E)}}\tilde{\tilde{\subseteq}}(H,E)\tilde{\tilde{\cup}}(\hat{I},E) \quad and \quad (\hat{I},E)\tilde{\tilde{\subseteq}}(H,E)\tilde{\tilde{\cup}}(\hat{I},E).$$

Since $(H,E)\tilde{\tilde{\subseteq}}(\hat{I},E)$ *implies* $\overline{\overline{(H,E)}}\tilde{\tilde{\subseteq}}\overline{\overline{(\hat{I},E)}}$

$$\Rightarrow \overline{\overline{\{(H,E)\tilde{\tilde{\cup}}(\hat{I},E)\}}}\tilde{\tilde{\subseteq}}\overline{\overline{\{(H,E)\tilde{\tilde{\cup}}(\hat{I},E)\}}} \qquad \tilde{\tilde{\cup}}\{\overline{\overline{(H,E)}}\tilde{\tilde{\cup}}\overline{\overline{(\hat{I},E)}}\}.$$

$(H,E)\tilde{\tilde{\cup}}(\hat{I},E)\tilde{\tilde{\subseteq}}(H,E)\tilde{\tilde{\cup}}(\hat{I},E) \to (1)$.

Also, from the ternary soft closure property, we have $(H,E)\tilde{\tilde{\subseteq}}\overline{\overline{(H,E)}}$ *and* $(\hat{I},E)\tilde{\tilde{\subseteq}}\overline{\overline{(\hat{I},E)}}$.
Thus$(H,E)\tilde{\tilde{\cup}}(\hat{I},E)\tilde{\tilde{\subseteq}}\overline{\overline{(H,E)}}\tilde{\tilde{\cup}}\overline{\overline{(\hat{I},E)}} \implies \overline{\overline{(H,E)}}\tilde{\tilde{\cup}}\overline{\overline{(\hat{I},E)}}$
is the ternary soft closed set containing $(H,E)\tilde{\tilde{\cup}}(\hat{I},E)$. *But* $(H,E)\tilde{\tilde{\cup}}(\hat{I},E)$ *is the smallest ternary soft closed set containing* $(H,E)\tilde{\tilde{\cup}}(\hat{I},E) \to (2)$

Comparing (1) and (2), we have $\overline{\overline{(H,E)\widetilde{\widetilde{\cup}}(\hat{I},E)}}$ *is smaller than* $\overline{\overline{(H,E)}}\widetilde{\widetilde{\cup}}\overline{\overline{(\hat{I},E)}}$*. Thus, from (1) and (2), we have* $\overline{\overline{(H,E)\widetilde{\widetilde{\cup}}(\hat{I},E)}} = \overline{\overline{(H,E)}}\widetilde{\widetilde{\cup}}\overline{\overline{(\hat{I},E)}}$.

(vii) Since $(H,E)\widetilde{\widetilde{\cap}}(\hat{I},E)\widetilde{\widetilde{\subseteq}}(H,E)$*, so by part (v),* $\overline{\overline{(H,E)\widetilde{\widetilde{\cap}}(\hat{I},E)}}\widetilde{\widetilde{\subseteq}}\overline{\overline{(H,E)}}$ *and* $\overline{\overline{(H,E)\widetilde{\widetilde{\cap}}(\hat{I},E)}}\widetilde{\widetilde{\subseteq}}\overline{\overline{(\hat{I},E)}}$*.*
Thus, $\overline{\overline{(H,E)\widetilde{\widetilde{\cap}}(\hat{I},E)}}\widetilde{\widetilde{\subseteq}}\overline{\overline{(H,E)}}\widetilde{\widetilde{\cap}}\overline{\overline{(\hat{I},E)}}$*.*

Definition 31. *Let* $(U_1, U_2, U_3, \tau_\Delta, A)$ *be a ternary soft topological space over* U_1, U_2, U_3 *and* (H, A) *be a ternary soft set. Then we associate point wise ternary soft closure of* (F, E) *over* U_1, U_2, U_3*, which is denoted by* $(\overline{\overline{H}}, A)$ *and defined as* $(\overline{\overline{H}}, A)_{\widetilde{\alpha}} = \overline{\overline{(H,A)}}_{\widetilde{\alpha}}$ *where* $\overline{\overline{(H,A)}}_{\widetilde{\alpha}}$ *is the ternary soft closure of* $(H,A)_{\widetilde{\alpha}}$ *in* $(U_1, U_2, U_3, \tau_\Delta, A)$ *for each* $\widetilde{\alpha} \in A$*.*

Theorem 3. *Let* $(U_1, U_2, U_3, \tau_\Delta, A)$ *be a ternary soft topological space over* U_1, U_2, U_3 *and* (H, A) *be a ternary soft set. Then* $(\overline{\overline{H}}, A)\widetilde{\widetilde{\subseteq}}\overline{\overline{(H,A)}}$*.*

Proof. *For any parameter* $\widetilde{\alpha} \in E$*,* $\overline{\overline{(H,A)}}_{\widetilde{\alpha}}$ *is the smallest ternary soft closed set in* $(U_1, U_2, U_3, \tau_\Delta, A)$ *which contains* $(H,A)_{\widetilde{\alpha}}$*. Moreover, if* $(\overline{\overline{H}}, A)_{\widetilde{\alpha}} = (Ł, A)$*, then* $(Ł, A)$ *is also a ternary soft closed set in* $(U_1, U_2, U_3, \tau_\Delta, A)$ *containing* $(H,A)_{\widetilde{\alpha}}$*. This implies that* $(\overline{\overline{H}}, A)_{\widetilde{\alpha}} =, \overline{\overline{(H,A)}}_{\widetilde{\alpha}}\widetilde{\widetilde{\subseteq}}(Ł, A)$*. Thus,* $\overline{\overline{(H,A)}}\widetilde{\widetilde{\subseteq}}\overline{\overline{(H,A)}}$*.* □

Theorem 4. *Let* $(U_1, U_2, U_3, \tau_\Delta, A)$ *be the ternary soft topological space and* (F, A) *be the ternary soft set over* (U_1, U_2, U_3)*. Then,* $(\overline{\overline{F}}, A)\widetilde{\widetilde{\subseteq}}\overline{\overline{(F,A)}}$*.*

Proof. *Let* $(U_1, U_2, U_3, \tau_\Delta, A)$ *be the ternary soft topological space over* U_1, U_2, U_3*. If* $(\overline{\overline{F}}, A)\widetilde{\widetilde{\subseteq}}\overline{\overline{(F,A)}}$*, then* $(\overline{\overline{F}}, A)$ *is a ternary soft closed set and so* $(\overline{\overline{F}}, A)^{\acute{c}} \in \tau_\Delta$*.*

Conversely, if $(\overline{\overline{F}}, A)^{\acute{c}} \in \tau_\Delta$*, then ternary soft closed set containing* (F, A)*. By the above theorem,* $(\overline{\overline{F}}, A)\widetilde{\widetilde{\subseteq}}\overline{\overline{(F,A)}}$*, and by the definition of the ternary soft closure of* (F, A)*, any ternary closed set over* U_1, U_2, U_3 *that contains* (F, A) *will contain* $(\overline{\overline{F}}, A)$*. Thus,* $\overline{\overline{(F,A)}}\widetilde{\widetilde{\subseteq}}(\overline{\overline{F}}, A)$*, hence* $(\overline{\overline{F}}, A) = \overline{\overline{(F,A)}}$*.* □

Definition 32. *Let* $(U_1, U_2, U_3, \tau_\Delta, A)$ *be a ternary soft topological space over* U_1, U_2, U_3 *and* (H, A) *be a ternary soft set. Let* $e_x \in E$*. Then* e_x *is said to be a ternary soft interior point of* (H, A) *if there exists a ternary soft open set* $(\mathcal{K}, A)$ *such that* $e_x \in (H, A)\widetilde{\widetilde{\subseteq}}(\mathcal{K}, A)$*.*

Definition 33. *Let* $(U_1, U_2, U_3, \tau_\Delta, A)$ *be a ternary soft topological space over* U_1, U_2, U_3 *and* (F, A) *be a ternary soft set. Let* $e_x \in A$*. Then* (F, A) *is said to be a ternary soft neighborhood of* e_x *if there exists a ternary soft open set* $(\mathcal{K}, A)$ *such that* $e_x \in (F, A)\widetilde{\widetilde{\subseteq}}(\mathcal{K}, A)$*.*

Theorem 5. *Let* $(U_1, U_2, U_3, \tau_\Delta, A)$ *be a ternary soft topological space over* U_1, U_2, U_3*. Let* (F, E) *be a ternary soft set over* U_1, U_2, U_3*, and let* $e_x \in E$*. If* e_x *is a ternary soft interior point of* (F, E)*, then* e_x *is a ternary soft interior point of* $(F, E)_{\widetilde{\alpha}}$ *in* $(U_1, U_2, U_3, \tau_\Delta, A)$ *for each* $\widetilde{\alpha} \in E$*.*

The above theorem is not true in general.

Theorem 6. *Let $(U_1, U_2, U_3, \tau_\Delta, A)$ be a ternary soft topological space over the initial parameter (U_1, U_2, U_3). Then:*

(i) Each $e_x \in E$ has a ternary soft neighborhood.

(ii) The intersection of any two ternary soft neighborhoods of a ternary soft point e_x is again a ternary soft neighborhood.

(iii) Every ternary soft superset of a ternary soft neighborhood of a point e_x is again a ternary soft neighborhood of the point e_x.

Proof. ***(i)****. For any $e_x \in \widetilde{\widetilde{X}}$, we have $e_x \in \widetilde{\widetilde{X}} \mathrel{\widetilde{\widetilde{\subseteq}}} \widetilde{\widetilde{X}}$. Thus, $\widetilde{\widetilde{X}}$ is a ternary soft neighborhood of e_x.*

(ii)*. Let $(U_1, U_2, U_3, \tau_\Delta, A)$ be a ternary soft topological space, and let $e_x \in E$ be any ternary soft point. Let (F, E) and (G, E) be any two ternary soft neighborhoods of e_x. To prove that $(F, E) \mathbin{\widetilde{\widetilde{\cap}}} (G, E)$ is also a ternary soft neighborhood of e_x, we note that (F, E) being a ternary soft neighborhood of e_x implies there exists a ternary soft open set $(\mathcal{K}, E)$ such that $e_x \in (\mathcal{K}, E) \mathrel{\widetilde{\widetilde{\subseteq}}} (F, E)$. Similarly, (G, E) is a ternary soft neighborhood of e_x, implying there exists a ternary soft open set $(Ł, E)$ such that $e_x \in (Ł, E) \mathrel{\widetilde{\widetilde{\subseteq}}} (G, E)$.*

Now, $(\mathcal{K}, E) \mathbin{\widetilde{\widetilde{\cap}}} (Ł, E)$ is a ternary soft open set, and from the previous conditions, we have $e_x \in [(\mathcal{K}, E) \mathbin{\widetilde{\widetilde{\cap}}} (Ł, E)] \mathrel{\widetilde{\widetilde{\subseteq}}} [(F, E) \mathbin{\widetilde{\widetilde{\cap}}} (G, E)]$. Thus, there exists a ternary soft open set $[(\mathcal{K}, E) \mathbin{\widetilde{\widetilde{\cap}}} (Ł, E)]$ such that $e_x \in [(\mathcal{K}, E) \mathbin{\widetilde{\widetilde{\cap}}} (Ł, E)] \mathrel{\widetilde{\widetilde{\subseteq}}} [(F, E) \mathbin{\widetilde{\widetilde{\cap}}} (G, E)]$. From the definition of a ternary soft neighborhood, it follows that $(F, E) \mathbin{\widetilde{\widetilde{\cap}}} (G, E)$ is a ternary soft neighborhood of e_x. Hence, the intersection of any two ternary soft neighborhoods is again a ternary soft neighborhood.

(iii)*. Let $(U_1, U_2, U_3, \tau_\Delta, E)$ be a ternary soft topological space, and let $e_x \in E$ be any ternary soft point. Let (F, E) be a ternary soft neighborhood of e_x, and let (G, E) be any ternary soft superset of (F, E). Since (G, E) is also a ternary soft neighborhood of e_x, there exists a ternary soft open set (H, E) such that $e_x \in (H, E) \mathrel{\widetilde{\widetilde{\subseteq}}} (F, E)$.*

Now, (F, E) being a ternary soft subset of (G, E) implies $(G, E) \mathrel{\widetilde{\widetilde{\supseteq}}} (F, E)$, which gives $(F, E) \mathrel{\widetilde{\widetilde{\subseteq}}} (F, E)$. From the previous results, we have $e_x \in (H, E) \mathrel{\widetilde{\widetilde{\subseteq}}} (F, E) \mathrel{\widetilde{\widetilde{\subseteq}}} (G, E)$, which implies $e_x \in (H, E) \mathrel{\widetilde{\widetilde{\subseteq}}} (G, E)$. Therefore, there exists a ternary soft open set (H, E) such that $e_x \in (H, E) \mathrel{\widetilde{\widetilde{\subseteq}}} (G, E)$. Hence, (G, E) is a ternary soft neighborhood of e_x. Thus, every ternary soft superset of a ternary soft neighborhood is again a ternary soft neighborhood of that point.

Theorem 7. *Let $(U_1, U_2, U_3, \tau_\Delta, E)$ be a ternary soft topological space. Let (F, E) be any ternary soft subset over U_1, U_2, U_3. Then the following hold true:*

(i) $(F, E)^\circ$ is a ternary soft open set contained in (F, E), i.e. $(F, E)^\circ$ is a ternary soft open set and $(F, E)^\circ \mathrel{\widetilde{\widetilde{\subseteq}}} (F, E)$.

(ii) $(F, E)^\circ$ is the largest ternary soft open set contained in (F, E).

(iii) (F,E) *is ternary soft open if and only if* $(F,E)\tilde{\tilde{\cong}}(F,E)^{\circ}$.

Proof. *By the definition of ternary soft interior, we have* $(F,E)^{\circ} = \widetilde{\widetilde{\cup_{\lambda\in A}}}(H,E)_{\lambda}$, *where* $\{(H,E)_{\lambda}\} : \lambda \in A$ *is the family of all ternary soft open sets contained in* (F,E). $(H,E)_{\lambda}\tilde{\tilde{\subseteq}}(F,E)\ \forall\ \lambda \in \Lambda$ *implies the union of all ternary soft open sets implies that open sets by the definition of ternary soft topological space. Also, we have* $(H,E)_{\lambda}\tilde{\tilde{\subseteq}}(F,E)\ \forall\ \lambda \in \Lambda \Rightarrow \widetilde{\widetilde{\cup_{\lambda\in A}}}(H,E)_{\lambda}\tilde{\tilde{\subseteq}}(F,E)$
$\Rightarrow\quad (F,E)^{\circ}\tilde{\tilde{\subseteq}}(F,E)$. *Hence,* $(F,E)^{\circ}$ *is a ternary soft open set and* $(F,E)^{\circ}\tilde{\tilde{\subseteq}}(F,E)$.

(ii) from (i), we have that $(F,E)^{\circ}$ *is a ternary soft open set contained in* (F,E). *Let* (H,E) *be any ternary soft open set contained in* (F,E). *Let* (H,E) *be any ternary soft open set contained in* (F,E). *This implies that the family* $\{(H,E)_{\lambda} : \lambda \in \Lambda\} =$ *, the family of all ternary soft open sets contained in* (F,E) *implies* $(H,E)\tilde{\tilde{\subseteq}}\widetilde{\widetilde{\cup_{\lambda\in A}}}(H,E)_{\lambda} \Rightarrow (H,E)\tilde{\tilde{\subseteq}}(F,E)^{\circ}$. $\Rightarrow (F,E)^{\circ}\tilde{\tilde{\supseteq}}(H,E)$. $\Rightarrow (F,E)^{\circ}$ *is larger than every ternary soft open set contained in* (F,E). *Thus,* $(F,E)^{\circ}$ *is the largest ternary soft open set contained in* (F,E).

(ii)*. Suppose* (F,E) *is ternary soft open. Therefore,* (F,E) *is a ternary soft open set contained in* (F,E), *(i.e.* $(F,E)\tilde{\tilde{\subseteq}}(F,E)) \to (1)$. *But* $(F,E)^{\circ}$ *is the largest ternary soft open set contained in* $(F,E) \to (2)$. *Therefore, from* (1) *and* (2), *it follows that* $(F,E)^{\circ}$ *must be larger than* (F,E), *that is,* $(F,E)^{\circ}\tilde{\tilde{\supseteq}}(F,E)$ *or* (F,E) *is smaller than* $(F,E)^{\circ}$, *that is* $(F,E)\tilde{\tilde{\subseteq}}(F,E)^{\circ} \to (3)$. *But* $(F,E)^{\circ}\tilde{\tilde{\supseteq}}(F,E)$ *is always true* $\to (4)$. *From* (3) *and* (4), *we have* $(F,E)\tilde{\tilde{\cong}}(F,E)^{\circ}$. *Note that the right-hand side result, i.e.,* $(F,E)^{\circ}$ *is a ternary soft open set, implies that the left-hand side, i.e.,* (F,E), *must also be a ternary soft open set. Consequently,* (F,E) *is a ternary soft open set. If* $(F,E)\tilde{\tilde{\cong}}(F,E)^{\circ}$, *then* (F,E) *is a ternary soft open set. Hence,* (F,E) *is ternary soft open if and only if* $(F,E)\tilde{\tilde{\cong}}(F,E)^{\circ}$.

Theorem 8. *Let* $(U_1, U_2, U_3, \tau_{\Delta}, E)$ *be the ternary soft topological space. Let* (F,E) *and* (G,E) *be any two ternary soft subsets over* U_1, U_2, U_3. *Then the following properties hold true:*

(i) $\widetilde{\widetilde{X^{\circ}}}\tilde{\tilde{\cong}}\tilde{\tilde{X}}$.

(ii) $\tilde{\tilde{\emptyset}}^{\circ} = \tilde{\tilde{\emptyset}}$.

(iii) If $(F,E)\tilde{\tilde{\subseteq}}(G,E)$, *then* $(F,E)^{\circ}\tilde{\tilde{\subseteq}}(G,E)^{\circ}$.

(iv) $[(F,E)\tilde{\tilde{\cap}}(G,E)]^{\circ}\tilde{\tilde{\cong}}(F,E)^{\circ}\tilde{\tilde{\cap}}(G,E)^{\circ}$.

(v) $[(F,E)^{\circ}]^{\circ} = (F,E)^{\circ}$.

(vi) $(F,E)^{\circ}\tilde{\tilde{\cup}}(G,E)^{\circ}\tilde{\tilde{\subseteq}}[(F,E)\tilde{\tilde{\cup}}(G,E)]^{\circ}$.

Proof.

(i) *We know that* $\overset{\approx}{X}$ *is a ternary soft open set. This implies* $\overset{\approx}{X}^{\circ}\overset{\approx}{=}\overset{\approx}{X}$*. (Since* (F,E) *is open if and only if* $(F,E)\overset{\approx}{=}(F,E)^{\circ}$*). Therefore,* $\overset{\approx}{X}^{\circ}\overset{\approx}{=}\overset{\approx}{X}$*.*

(ii) *The result follows from (i).*

(iii) *Suppose* $(F,E)\overset{\approx}{\subseteq}(G,E)$*. Then we know that* $(F,E)^{\circ}\overset{\approx}{\subseteq}(F,E)$ *and* $(F,E)\overset{\approx}{\subseteq}(G,E)$*, therefore* $(F,E)^{\circ}\overset{\approx}{\subseteq}(G,E)$*. Therefore,* $(F,E)^{\circ}$ *is a* $\ll (T,S) \gg$ *ternary soft open set contained in* $(G,E) \to (1)$*. But* $(G,E)^{\circ}$ *is the largest ternary soft open set contained in* $(G,E) \to (2)$*. From* (1) *and* (2)*, we have that* $(G,E)^{\circ}$ *is larger than* $(F,E)^{\circ}$*, which implies* $(F,E)^{\circ}\overset{\approx}{\subseteq}(G,E)^{\circ}$*. Thus,* $(F,E)\overset{\approx}{\subseteq}(G,E)$ *implies* $(F,E)^{\circ}\overset{\approx}{\subseteq}(G,E)^{\circ}$*.*

(iv) *Let* $(U_1,U_2,U_3,\tau_{\Delta},E)$ *be the ternary soft topological space. To prove*

$$[(F,E)\overset{\approx}{\cap}(G,E)]^{\circ}\overset{\approx}{=}(F,E)^{\circ}\overset{\approx}{\cap}(G,E)^{\circ},$$

we know that $(F,E)\overset{\approx}{\cap}(G,E)\overset{\approx}{\subseteq}(F,E)$ *and* $(F,E)\overset{\approx}{\cap}(G,E)\overset{\approx}{\subseteq}(G,E)$*, which implies*

$$[(F,E)\overset{\approx}{\cap}(G,E)]^{\circ}\overset{\approx}{\subseteq}(F,E)^{\circ} \quad \textit{and} \quad [(F,E)\overset{\approx}{\cap}(G,E)]^{\circ}\overset{\approx}{\subseteq}(G,E)^{\circ}(by(iii)).$$

This implies

$$[(F,E)\overset{\approx}{\cap}(G,E)]^{\circ}\overset{\approx}{\subseteq}(F,E)^{\circ}\overset{\approx}{\cap}(G,E)^{\circ}.$$

Also, we have $(F,E)^{\circ}\overset{\approx}{\subseteq}(F,E)$ *and* $(G,E)^{\circ}\overset{\approx}{\subseteq}(G,E)$*, which implies*

$$(F,E)^{\circ}\overset{\approx}{\cap}(G,E)^{\circ}\overset{\approx}{\subseteq}(F,E)\overset{\approx}{\cap}(G,E),$$

which implies that $(F,E)^{\circ}\overset{\approx}{\cap}(G,E)^{\circ}$ *is a ternary soft open set contained in* $(F,E)\overset{\approx}{\cap}(G,E)$ $\Rightarrow (2)$*. But* $[(F,E)\overset{\approx}{\cap}(G,E)]^{\circ}$ *is the largest ternary soft open set contained in* $(F,E)\overset{\approx}{\cap}(G,E) \Rightarrow (3)$*. Therefore, from* (2) *and* (3)*, it follows that* $[(F,E)\overset{\approx}{\cap}(G,E)]^{\circ}$ *is larger than* $(F,E)^{\circ}\overset{\approx}{\cap}(G,E)^{\circ}$*, that is* $(F,E)^{\circ}\overset{\approx}{\cap}(G,E)^{\circ}$ *is smaller than* $[(F,E)\overset{\approx}{\cap}(G,E)]^{\circ}$ *this leads to* $(F,E)^{\circ}\overset{\approx}{\cap}(G,E)^{\circ}\overset{\approx}{\subseteq} [(F,E)\overset{\approx}{\cap}(G,E)]^{\circ} \to (4)$*. From* (1) *and* (4) *it follows that* $[(F,E)\overset{\approx}{\cap}(G,E)]^{\circ} \overset{\approx}{=}(F,E)^{\circ}\overset{\approx}{\cap}(G,E)^{\circ}$*.*

(v) *We know that* $[(F,E)^{\circ}]^{\circ}$ *is a ternary soft open set. Let us assume* $(F,E)^{\circ}\overset{\approx}{=}(H,E)$*. Therefore,* (H,E) *is a ternary soft open set, which implies* $(H,E)\overset{\approx}{=}(H,E)^{\circ}$*. Therefore,* $(F,E)^{\circ}\overset{\approx}{=}[(F,E)^{\circ}]^{\circ}$*, hence the result.*

(vi) *Since* $(F,E)\overset{\approx}{\subseteq}(F,E)\overset{\approx}{\cup}(G,E)$ *and* $(G,E)\overset{\approx}{\subseteq}(F,E)\overset{\approx}{\cup}(G,E)$*, by (iii), we have*

$$(F,E)^{\circ}\overset{\approx}{\subseteq}[(F,E)\overset{\approx}{\cup}(G,E)]^{\circ} \quad \textit{and} \quad (F,E)^{\circ}\overset{\approx}{\subseteq}[(F,E)\overset{\approx}{\cup}(G,E)]^{\circ}.$$

So that

$$(F,E)^{\circ}\overset{\approx}{\cup}(G,E)^{\circ}\overset{\approx}{\subseteq}[(F,E)\overset{\approx}{\cup}(G,E)]^{\circ},$$

since $(F,E)^{\circ}\overset{\approx}{\cup}(G,E)^{\circ}$ *is a ternary soft open set.*

Remark 3. *Let $(U_1, U_2, U_3, \tau_\Delta, E)$ be the ternary soft topological space over U_1, U_2, U_3, and let (F, E) be any ternary soft open set. Then we always have*

$$(H,E)^\circ \tilde{\tilde{\subseteq}} (F,E) \tilde{\tilde{\subseteq}} \overline{\overline{(F,E)}}.$$

Remark 4. $\underline{\underline{(H,E)}} \tilde{\tilde{=}} [\underline{\underline{\tilde{\tilde{X}} - (H,E)}}]$ *and* $bd\ (H,E) = bd\ [\tilde{\tilde{X}} - (H,E)]$.
To prove;

$$bd(H,E) = bd\overline{\overline{(H,E)}}\ \tilde{\tilde{\cap}}\ \overline{\overline{[\tilde{\tilde{X}} - (H,E)]}} \quad \Rightarrow \quad (1)$$

$$bd\ [\tilde{\tilde{X}} - (H,E)] = \overline{\overline{[\tilde{\tilde{X}} - (H,E)]}} \tilde{\tilde{\cap}}\ \overline{\overline{[\tilde{\tilde{\mathcal{X}}} - (\tilde{\tilde{X}} - (H,E))]}}.$$

Hence,

$$\overline{\overline{[\tilde{\tilde{X}} - (H,E)]}} \tilde{\tilde{\cap}}\ \overline{\overline{(H,E)}} \quad \Rightarrow \quad \overline{\overline{(H,E)}} \tilde{\tilde{\cap}}\ \overline{\overline{[\tilde{\tilde{X}} - (H,E)]}} \tilde{\tilde{\cap}}\ = bd(H,E).$$

Thus,

$$bd\ (H,E) = bd\ [\tilde{\tilde{X}} - (H,E)].$$

Theorem 9. *Let $(U_1, U_2, U_3, \tau_\Delta, E)$ be the ternary soft topological space. Let (H,E) be any ternary soft subset of $\tilde{\tilde{X}}$. Then the following properties are true:*

(i) $bd(H,E) = \overline{\overline{(H,E)}} - (H,E)^\circ$.

(ii) $(H,E)^\circ = (H,E) - bd(H,E)$.

(iii) $\tilde{\tilde{X}} \tilde{\tilde{=}} (H,E)^\circ \tilde{\tilde{\cap}} bd(H,E) \tilde{\tilde{\cup}} [\tilde{\tilde{X}} - (H,E)^\circ]$.

Proof.

(i) We know that $bd(H,E) = \underline{\underline{(H.E)}} \tilde{\tilde{=}} \overline{\overline{(H,E)}}\ \tilde{\tilde{\cap}}\ \overline{\overline{(H,E)'}}$

$$\Rightarrow \quad \overline{\overline{(H,E)}} = \overline{\overline{(H,E)}} - [\tilde{\tilde{X}} - \overline{\overline{(\tilde{\tilde{X}} - (H,E))}}]^\circ = \overline{\overline{(H,E)}} - [\tilde{\tilde{X}}(\tilde{\tilde{X}} - (H,E))]^\circ$$

.
Thus, $= \overline{\overline{(H,E)}} - (H,E)^\circ = bd\ (H,E) = \overline{\overline{(H,E)}} - (H,E)^\circ$.

(ii) Consider

$$(H,E) - bd\ (H,E) = (H,E) - [\overline{\overline{(H,E)}} - \overline{\overline{(\tilde{\tilde{X}} - (H,E))}}]$$

$$= (H,E)\ \tilde{\tilde{\cap}}\ [\tilde{\tilde{X}} - \overline{\overline{(H,E)}}\ \tilde{\tilde{\cap}}\ \overline{\overline{(\tilde{\tilde{X}} - (H,E))}}]$$

$$= \ (H,E)\ \tilde{\tilde{\cap}}\ [(\tilde{\tilde{X}} - \overline{\overline{(H,E)}})\ \tilde{\tilde{\cup}}\ (\tilde{\tilde{X}} - \overline{\overline{[\tilde{\tilde{X}} - (H,E)]}})]$$

.
This simplifies to

$$\Rightarrow \ (H,E) \mathbin{\tilde{\tilde{\cap}}} [(\tilde{\tilde{X}} - \overline{\overline{(H,E)}}) \mathbin{\tilde{\tilde{\cup}}} (\tilde{\tilde{X}} - [(H,E)^{\circ}])]$$

$$= \ (H,E) \mathbin{\tilde{\tilde{\cap}}} [(\tilde{\tilde{X}} - \overline{\overline{(H,E)}}) \mathbin{\tilde{\tilde{\cup}}} [(H,E) \mathbin{\tilde{\tilde{\cap}}} (H,E)^{\circ}]]$$

. *Therefore,* $(H,E)^{\circ} = (H,E) - bd\ (H,E)$.

(iii) Consider the right-hand side

$(H,E)^{\circ} \tilde{\tilde{\cup}} bd\ (H,E) \tilde{\tilde{\cup}}\ [\tilde{\tilde{X}} - (H,E)^{\circ}]$

$$= (H,E)^{\circ} \mathbin{\tilde{\tilde{\cup}}} bd\ (H,E) \mathbin{\tilde{\tilde{\cup}}} (\tilde{\tilde{X}} - \overline{\overline{(H,E)}})$$

$$= (H,E) \mathbin{\tilde{\tilde{\cup}}} \overline{\overline{(\tilde{\tilde{X}} - (H,E))}} = \tilde{\tilde{X}}.$$

Further, we know that $bd\ (H,E) = \overline{\overline{(H,E)}} - (H,E)^{\circ}$, *which implies that* $bd(H,E)$ *and* $(H,E)^{\circ}$ *are disjoint* $\rightarrow$ (1). *Replacing* (H,E) *with* $(\tilde{\tilde{X}} - (H,E)^{\circ})$, *are disjoint sets. Hence,* $\tilde{\tilde{X}} \ = \ \overline{\overline{(H,E)}} \mathbin{\tilde{\tilde{\cup}}} \overline{\overline{(\tilde{\tilde{X}} - (H,E))}}$, *which is a ternary soft disjoint union.*

Theorem 10. *Let* $(U_1, U_2, U_3, \tau_{\Delta}, E)$ *be the ternary soft topological space. Let* (F,E) *be any ternary soft subset of* U_1, U_2, U_3. *Then* (F,E) *is ternary soft closed if and only if* $(F,E) \supseteq bd\ (F,E)$.

Proof. *Suppose* (F,E) *is a ternary soft closed set. Then,*

$$bd\ (H,E) = \overline{\overline{(F,E)}} \mathbin{\tilde{\tilde{\cap}}} \overline{\overline{(\tilde{\tilde{X}} - (F,E))}} \mathbin{\tilde{\tilde{\subseteq}}} (F,E) \mathbin{\tilde{\tilde{\cap}}} \overline{\overline{(\tilde{\tilde{X}} - (F,E))}} \mathbin{\tilde{\tilde{\subseteq}}} (F,E).$$

Therefore, if $(F,E) \supseteq bd\ (F,E)$. *Hence,* (F,E) *is ternary soft closed if and only if* $(F,E) \supseteq bd(F,E)$. $\quad \Rightarrow (1)$

Conversely, suppose $(F,E) \supseteq bd(F,E)$, *i.e.,* $bd\ (F,E) \mathbin{\tilde{\tilde{\subseteq}}} (F,E)$, *which implies*

$$(F,E) \mathbin{\tilde{\tilde{\cup}}} bd\ (F,E) \mathbin{\tilde{\tilde{\subseteq}}} \overline{\overline{(F,E)}} = (F,E).$$

Therefore, (F,E) *is ternary soft closed. Thus,* $(F,E) \supseteq bd\ (F,E)$ *implies* (F,E) *is ternary soft closed.* $\quad \Rightarrow (2)$

From (1) *and* (2), *it is clear that* (F,E) *is ternary soft closed if and only if* $(F,E) \supseteq bd\ (F,E)$.

Theorem 11. *Let* $(U_1, U_2, U_3, \tau_{\Delta}, E)$ *be the ternary soft topological space. Let* (F,E) *be any ternary soft subset of* U_1, U_2, U_3. *Then* (F,E) *is ternary soft open if and only if*

$$(F,E) \mathbin{\tilde{\tilde{\cap}}} bd\ (F,E) \mathrel{\tilde{\tilde{=}}} \tilde{\tilde{\emptyset}}$$

Proof. *Suppose* (F,E) *is ternary soft open, which implies that* $[\tilde{\tilde{X}}-(F,E)]$ *is ternary soft closed. Then,*

$$\overline{\overline{[\tilde{\tilde{X}}-(F,E)]}}\ \tilde{\tilde{=}}\ [\tilde{\tilde{X}}-(F,E)].$$

Now consider $(F,E)\tilde{\tilde{\cap}}bd\ (F,E)$*. We have*

$$(F,E)\tilde{\tilde{\cap}}\ \left[\overline{\overline{(F,E)}}\ \tilde{\tilde{\cap}}\ \overline{\overline{(\tilde{\tilde{X}}-(F,E))}}\right]=(F,E)\ \tilde{\tilde{\cap}}\ \left[(\tilde{\tilde{X}}-(F,E))\ \tilde{\tilde{\cap}}\ \overline{\overline{(F,E)}}\right]=\tilde{\tilde{\emptyset}}.$$

Therefore, $(F,E)\tilde{\tilde{\cap}}bd\,(F,E)\tilde{\tilde{=}}\tilde{\tilde{\emptyset}}$*. Hence, if* (F,E) *is open, it implies that* $(F,E)\tilde{\tilde{\cap}}bd\,(F,E)\tilde{\tilde{=}}\tilde{\tilde{\emptyset}}$*.* $\Rightarrow (1)$

Conversely, suppose $(F,E)\tilde{\tilde{\cap}}bd\ (F,E)\tilde{\tilde{=}}\ \tilde{\tilde{\emptyset}}$*. Then,*

$$(F,E)\ \tilde{\tilde{\cap}}\ \left[\overline{\overline{(F,E)}}\ \tilde{\tilde{\cap}}\ \overline{\overline{(\tilde{\tilde{X}}-(F,E))}}\right]\ \tilde{\tilde{=}}\ \tilde{\tilde{\emptyset}},$$

which implies

$$(F,E)\ \tilde{\tilde{\cap}}\ \overline{\overline{(F,E)}}\tilde{\tilde{\cap}}\ \overline{\overline{[\tilde{\tilde{X}}-(F,E)]}}\ \tilde{\tilde{=}}\ \tilde{\tilde{\emptyset}}$$

Thus, we have

$$(F,E)\tilde{\tilde{\cap}}\ \overline{\overline{[\tilde{\tilde{X}}-(F,E)]}}\ \tilde{\tilde{=}}\ \ \tilde{\tilde{\emptyset}},$$

which implies

$$(F,E)\ \tilde{\tilde{\subseteq}}\ \tilde{\tilde{X}}\ -\ \overline{\overline{[\tilde{\tilde{X}}-(F,E)]}}\ \tilde{\tilde{=}}\ \tilde{\tilde{\emptyset}}.$$

This leads to

$$(F,E)\tilde{\tilde{\subseteq}}\tilde{\tilde{X}}-[(\tilde{\tilde{X}}-(F,E))^{\circ}],$$

which implies $(F,E)\tilde{\tilde{\subseteq}}\ (F,E)^{\circ}$*. Since* $(F,E)^{\circ}\ \tilde{\tilde{\subseteq}}\ (F,E)$ *is always true, it follows that* $(F,E)\ =\ (F,E)^{\circ}$*. Therefore,* $(F,E)\tilde{\tilde{\cap}}bd(F,E)\tilde{\tilde{=}}\tilde{\tilde{\emptyset}}$ *implies that* (F,E) *is ternary soft open. From* (i) *and* (ii)*, we conclude that* (F,E) *is ternary soft open if and only if* $(F,E)\ \tilde{\tilde{\cap}}\ bd\ (F,E)\tilde{\tilde{=}}\ \tilde{\tilde{\emptyset}}$*.*

7. Comparative Analysis

The following Table 1 provides a detailed comparative analysis of the proposed methods, contrasting them with the established techniques discussed in [8]. This comparison highlights the strengths and weaknesses of each approach, offering insights into how the proposed methods perform relative to the established techniques across various key factors:

Factor	**Binary Soft Sets and Binary Soft Topological Spaces (published work) [8]**	**Ternary Soft Sets and Ternary Soft Topological Spaces (proposed method)**
Core Concept	Focuses on binary soft sets, which are defined over two universal sets and a parameter set (set of decision variables). The core idea is to work with two sets and a parameter set to represent uncertainty or imprecision.	Explores the extension of soft sets to ternary soft sets, which are defined over three universal sets and a parameter set. The primary goal is to extend the binary soft set theory to handle more complex systems involving three sets.
Mathematical Structure	Introduces binary soft topological structures based on two sets. This includes the concept of binary soft open sets, binary soft closed sets, and other basic topological constructs like binary soft neighborhoods. These concepts help define the relationships and operations between binary sets.	Extends the topological framework to ternary soft sets; introducing ternary soft topological structures. This includes the extension of binary topological concepts such as ternary soft open sets, ternary soft closed sets, ternary soft closure, ternary soft boundary, and ternary soft neighborhoods. These structures offer a more complex mathematical framework for analyzing sets with three components.
Operations	Defines various operations on binary soft sets, such as subset, superset, complement, union, intersection, difference between, and symmetric difference two binary soft sets. These operations are basic but essential for working with soft sets.	Defines subset, superset, complement, union, intersection, difference, but applied to ternary soft sets. These operations become more intricate due to the involvement of three sets, expanding the scope of operations to handle more complex interactions.
Logical Operations	Defines AND and OR operations between two binary soft sets. These logical operations are used to combine or intersect the information represented by the two sets. The operations work within the binary framework, allowing for basic logical interactions.	Expands on the logical operations by defining AND and OR operations between ternary soft sets. The extension involves handling three sets, providing a more nuanced logical framework that accounts for three sets' relationships and their interactions.

Factor	**Binary Soft Sets and Binary Soft Topological Spaces (published work) [8]**	**Ternary Soft Sets and Ternary Soft Topological Spaces (proposed method)**
Topological Concepts	Focuses on binary soft open sets, binary soft closed sets, binary soft closure, binary soft boundary, and binary soft neighborhoods. These concepts help define the structure and properties of soft sets in the context of two universal sets, establishing basic principles of binary soft topology.	Introduces ternary soft topological concepts such as ternary soft open sets, ternary soft closed sets, ternary soft closure, ternary soft boundary, and ternary soft neighborhoods. These concepts extend the foundational work on binary soft topology to more complex structures involving three sets, creating a more intricate topological framework.
Scope of Study	The scope is limited to binary soft set operations and their interactions within a two-set framework. It provides foundational concepts and operations, and serves as an introduction to the study of soft sets in a simple binary context.	Broadens the scope significantly, focusing on ternary soft sets and the corresponding operations and structures. This expansion to a three-set framework introduces new complexities and opens the door for deeper exploration in more sophisticated and multidimensional decision-making systems.
Mathematical Complexity	The mathematical complexity in this work is relatively simpler, as it deals with binary soft sets and their interactions in a two-set context. This makes the paper easier to follow for those familiar with basic set theory and operations.	Introduces a higher level of complexity because it involves ternary soft sets, which requires handling additional sets and more intricate relationships between them. The extension to three sets naturally introduces more challenging mathematical structures and operations.
Research Objective	The primary objective is to define, explore, and establish properties of binary soft sets, focusing on their operations and topological structures. The paper aims to lay the groundwork for future research in binary soft set theory and applications.	The goal of the work is to extend soft set theory by introducing ternary soft sets and exploring their properties and operations. The paper's contribution is to develop new mathematical structures, such as ternary soft topological spaces, and to study their behavior with respect to more complex decision variables.

Factor	**Binary Soft Sets and Binary Soft Topological Spaces (published work) [8]**	**Ternary Soft Sets and Ternary Soft Topological Spaces (proposed method)**
Applications	Its applications are generally simpler and focused on decision-making in binary contexts. It can be used in scenarios where decisions are based on two sets, such as binary classification or basic set operations in uncertainty modeling.	The applications are more complex, involving three decision variables. This extension is useful in more advanced decision-making models where three sets are needed to describe or analyze the system, making it applicable to more sophisticated systems such as multi-criteria decision analysis or complex uncertainty modeling.
Examples and Engagements	Provides various examples of binary soft set operations and properties to illustrate how these concepts work in practice. These examples help establish the foundational ideas of binary soft sets and their applications.	Provides detailed examples of ternary soft set operations and explores the relationships among ternary soft concepts. It uses these examples to highlight the practical applications and theoretical implications of extending soft set theory to three sets.
Contribution to Soft Set Theory	Contributes foundational knowledge to binary soft set theory, including the introduction of basic operations and the concept of binary soft topology. It is a starting point for future research in binary contexts.	Significantly advances soft set theory by introducing ternary soft sets and expanding the topological framework to handle more complex interactions. It offers new mathematical structures and extends soft set theory to address more multidimensional problems.

Table 1: Comparative analysis

8. Some Hereditary Properties, Separation Axioms, and Other Related Axioms

In this section, hereditary properties, separation axioms, and other related axioms are discussed.

Definition 34. *Let (F, A) be any ternary soft subset of a ternary soft topological space $(U_1, U_2, U_3, \tau_\Delta, E)$. Then (F, A) is called:*

(i) (F, A) *Ternary soft s-open set of* $(U_1, U_2, U_3, \tau_\Delta, E)$ *if* $(F, A) \subseteq cl(int((F, A)))$.

(ii) (F, A) *Ternary soft s-closed set of* $(U_1, U_2, U_3, \tau_\Delta, E)$ *if* $(F, A) \supseteq int(cl((F, A)))$.

Proposition 6. *Let $(U_1, U_2, U_3, \tau_\Delta, E)$ be a ternary soft topological space on $\widetilde{X}$ over $(U_1 \times U_2 \times U_3)$, and $\widetilde{Y}$ be a non-empty ternary soft subset of $\widetilde{\widetilde{X}}$. Then $(U_1, U_2, U_3, \tau_{\Delta_Y}, \alpha)$ is a subspace of $(U_1, U_2, U_3, \tau_{\Delta_Y}, E)$ for each $\alpha \widetilde{\widetilde{\in}} \tilde{E}$.*

Proof. *Let $(U_1, U_2, U_3, \tau_{\Delta_Y}, \alpha)$ be a ternary soft topological space for each $\alpha \in E$. Now, by definition, for any $\alpha \in E$:*

$$\tau_{\Delta_Y} = \{Y \quad_{F(\alpha)/(F,E)} \text{ is ternary soft s-open set}\}$$

$$= \{\widetilde{\widetilde{Y}} \cap F(\alpha)/(F, E) \text{ is ternary soft s-open set}\}$$

$$= \{\widetilde{\widetilde{Y}} \cap F(\alpha)/F(\alpha) \in \tau_{\Delta_\alpha}\}.$$

Thus, $(U_1, U_2, U_3, \tau_{\Delta_Y}, \alpha)$ is a subspace of $(U_1, U_2, U_3, \tau_\Delta, \alpha)$.

Proposition 7. *Let $(U_1, U_2, U_3, \tau_{\Delta_Y}, E)$ be a ternary soft subspace of a ternary soft topological space $(U_1, U_2, U_3, \tau_\Delta, E)$ and (G, E) be a ternary soft s-open set in $\widetilde{Y}$. If $\widetilde{\widetilde{Y}} \in \tau_\Delta$, then $(G, E) \in \tau_\Delta$.*

Proof. *Let (G, E) be a ternary soft s-open set in $\widetilde{\widetilde{Y}}$, then there exists a ternary soft s-open set (H, E) in $\widetilde{\widetilde{X}}$ over $(U_1 \times U_2 \times U_3)$ such that $(G, E) = \widetilde{\widetilde{Y}} \cap (H, E)$. Now, if $\widetilde{\widetilde{Y}} \in \tau_\Delta$, then $\widetilde{Y} \cap (H, E) \in \tau_\Delta$ by the third axiom of the definition of a ternary soft topological space, and hence $(G, E) \in \tau_\Delta$.*

Proposition 8. *Let $(U_1, U_2, U_3, \tau_{\Delta_Y}, E)$ be a ternary soft subspace of a ternary soft topological space $(U_1, U_2, U_3, \tau_\Delta, E)$ and (G, E) be a ternary soft s-open set of $\widetilde{\widetilde{X}}$ over $(U_1 \times U_2 \times U_3)$, then:*

(i) (G, E) is ternary soft s-open in $\widetilde{\widetilde{Y}}$ if and only if $(G, E) = \widetilde{\widetilde{Y}} \widetilde{\widetilde{\cap}} (H, E)$ for some $(H, E) \in \tau_\Delta$.

(ii) (G, E) is ternary soft s-closed in $\widetilde{\widetilde{Y}}$ if and only if $(G, E) = \widetilde{\widetilde{Y}} \widetilde{\widetilde{\cap}} (H, E)$ for some ternary soft s-closed set (H, E) in $\widetilde{\widetilde{X}}$ over $(U_1 \times U_2 \times U_3)$.

Proof.

(i) Follows from the definition of a ternary soft subspace.

(ii) If (G, E) is ternary soft s-closed in $\widetilde{\widetilde{Y}}$, then we have $(G, E) = \widetilde{\widetilde{Y}} - (H, E)$, for some ternary soft s-open $(H, E) \in \tau_{\Delta_Y}$. Now $(H, E) = \widetilde{\widetilde{Y}} \widetilde{\widetilde{\cap}} (H, E)$ for some ternary soft s-open $(K, E) \in \tau_\Delta$. For any $\beta \in E$,

$$G(\beta) = \widetilde{\widetilde{Y}}(\beta) - H(\beta) = \widetilde{\widetilde{Y}} - H(\beta) = \widetilde{\widetilde{Y}} - [\widetilde{\widetilde{Y}}(\beta) \cap K(\beta)]$$

$$= \widetilde{\widetilde{Y}} - [\widetilde{\widetilde{Y}} \cap K(\beta)] = \widetilde{\widetilde{Y}} - K(\beta)$$

$$= \tilde{\tilde{\tilde{Y}}} \tilde{\tilde{\cap}} (\tilde{\tilde{\tilde{X}}} - K(\beta)) = \tilde{\tilde{\tilde{Y}}} \tilde{\tilde{\cap}} [K(\beta)^{\mathcal{C}}].$$

Thus, $(G, E) = \tilde{\tilde{\tilde{Y}}} \tilde{\tilde{\cap}} [K(\beta)^{\mathcal{C}}]$, *where* $(K, E)^{\mathcal{C}}$ *is a ternary soft s-closed set in* $\tilde{\tilde{\tilde{X}}}$ *over* $(U_1 \times U_2 \times U_3)$ *as* $(K, E) \in \tau_\Delta$.

Conversely, assume that $(G, E) = \tilde{\tilde{\tilde{Y}}} \cap (H, E)$ *for some ternary soft s-closed set* (H, E) *in* $\tilde{\tilde{\tilde{X}}}$ *over* $(U_1 \times U_2 \times U_3)$, *which means that* $(H, E) \in \tau_\Delta$. *Now, if* $(H, E) = \tilde{\tilde{\tilde{X}}} - (K, E)$ *where* $(K, E) \tilde{\tilde{\in}} \tau_\Delta$, *then for any* $\beta \in E$:

$$G(\beta) = \tilde{\tilde{\tilde{Y}}}(\beta) \tilde{\tilde{\cap}} H(\beta) = \tilde{\tilde{\tilde{Y}}} \tilde{\tilde{\cap}} H(\beta) = \tilde{\tilde{\tilde{Y}}} \tilde{\tilde{\cap}} (\tilde{\tilde{\tilde{X}}} - K(\beta)) = \tilde{\tilde{\tilde{Y}}} - [\tilde{\tilde{\tilde{Y}}} \tilde{\tilde{\cap}} K(\beta)] = \tilde{Y}(\beta) - [\tilde{\tilde{\tilde{Y}}}(\beta) \tilde{\tilde{\cap}} K(\beta)].$$

Thus, $\tilde{\tilde{\tilde{Y}}} - [\tilde{\tilde{\tilde{Y}}} \tilde{\tilde{\cap}} (K, E)] \in \tau_{\Delta_Y}$, *and hence* (G, E) *is a ternary soft s-closed set in* $\tilde{\tilde{\tilde{Y}}}$.

Let $(U_1, U_2, U_3, \tau_\Delta, E)$ *be a ternary soft topological space. Let* $(U_1, U_2, U_3, \tau_{\Delta_Y}, E)$ *be a ternary soft subspace of* $(U_1, U_2, U_3, \tau_\Delta, E)$. *Let* $(F, E) \tilde{\tilde{\subseteq}} \tilde{\tilde{\tilde{Y}}}$ *be a ternary soft subset of* $\tilde{\tilde{\tilde{Y}}}$. *Then we can find the ternary soft s-closure of* (F, E) *in the space* $(U_1, U_2, U_3, \tau_{\Delta_Y}, E)$. *The ternary soft s-closure of* (F, E) *in* $(U_1, U_2, U_3, \tau_{\Delta_Y}, E)$ *is denoted by* $\overline{\overline{\overline{(F, E)^Y}}}$.

Proposition 9. *Let* $(U_1, U_2, U_3, \tau_{\Delta_Y}, E)$ *be a ternary soft subspace of a ternary soft topological space* $(U_1, U_2, U_3, \tau_\Delta, E)$. *Let* $(F, E) \tilde{\tilde{\subseteq}} \tilde{\tilde{\tilde{Y}}}$ *be a ternary soft subset of* $\tilde{\tilde{\tilde{Y}}}$. *Then we have the following results:*

(i) $\overline{\overline{\overline{(F, E)^Y}}} = \tilde{\tilde{\tilde{Y}}} \tilde{\tilde{\cap}} \overline{\overline{\overline{(F, E)}}}$.

(ii) $(F, E)^{*Y} = \tilde{\tilde{\tilde{Y}}} \tilde{\tilde{\cap}} (F, E)^*$.

(iii) $\underline{\underline{\underline{(F, E)_Y}}} \tilde{\tilde{\subseteq}} \tilde{\tilde{\tilde{Y}}} \tilde{\tilde{\cap}} \underline{\underline{\underline{(F, E)}}}$.

Proof.

(i) To prove, let $\overline{\overline{\overline{(F, E)^Y}}} = \tilde{\tilde{\tilde{Y}}} \tilde{\tilde{\cap}} \overline{\overline{\overline{(F, E)}}}$. *We have:*

$\overline{\overline{\overline{(F, E)^Y}}}$ = *the ternary soft intersection of all the ternary soft s-closed sets containing* (F, E)

$$= \tilde{\tilde{\cap}} \{(G, E)_Y : (G, E)_Y \text{ is } \tau_{\Delta_Y}\text{-ternary soft s-closed set and } (G, E)_Y \tilde{\tilde{\supseteq}} (F, E)\}$$

$$= \tilde{\tilde{\cap}} \{\tilde{\tilde{\tilde{Y}}} \tilde{\tilde{\cap}} (G, E) : (G, E) \text{ is } \tau_\Delta\text{-ternary soft s-closed set and } \tilde{\tilde{\tilde{Y}}} \tilde{\tilde{\cap}} (G, E) \tilde{\tilde{\supseteq}} (F, E)\}$$

$$= \tilde{\tilde{\tilde{Y}}} \tilde{\tilde{\cap}} \{(G, E) : (G, E) \text{ is } \tau_\Delta\text{-ternary soft s-closed set and } (G, E) \tilde{\tilde{\supseteq}} (F, E)\} = \tilde{\tilde{\tilde{Y}}} \tilde{\tilde{\cap}} \overline{\overline{\overline{(F, E)}}}.$$

Thus $\overline{\overline{\overline{(F, E)^Y}}} = \tilde{\tilde{\tilde{Y}}} \tilde{\tilde{\cap}} \overline{\overline{\overline{(F, E)}}}$.

(ii) *To prove that* $(F,E)^Y = \widetilde{\widetilde{\widetilde{Y}}}\widetilde{\widetilde{\cap}}(F,E)^*$, *we know that* $(F,E)^{e\ Y}$ = *the ternary soft union of all the* τ_{Δ_Y}*-ternary soft s-open sets contained in* (F,E)*:*

$$= \widetilde{\widetilde{\cup}}\{(H,E) : (H,E) \textit{ is } \tau_{\Delta_Y}\textit{-ternary soft s-open and } (H,E)\widetilde{\widetilde{\subseteq}}(F,E)\}$$

$$= \widetilde{\widetilde{\cup}}\{(H,E) = \widetilde{\widetilde{\widetilde{Y}}}\widetilde{\widetilde{\cap}}(K,E) : (K,E) \textit{ is } \tau_{\Delta}\textit{-ternary soft s-open set and } \widetilde{\widetilde{\widetilde{Y}}}\widetilde{\widetilde{\cap}}(K,E)\widetilde{\widetilde{\subseteq}}(F,E)\}.$$

Also, we know that $(F,E)^e = \widetilde{\widetilde{\widetilde{Y}}}\widetilde{\widetilde{\cap}}\widetilde{\widetilde{\cup}}\{(L,E) : (L,E)$ *is* τ_Δ*-ternary soft s-open set and* $(L,E)_\gamma\widetilde{\widetilde{\subseteq}}(F,E)\}$. *Now, let* $(M,E)\widetilde{\widetilde{\in}}\widetilde{\widetilde{\widetilde{Y}}}\widetilde{\widetilde{\cap}}(F,E)^*$, *which implies* $(M,E)\widetilde{\widetilde{\in}}\widetilde{\widetilde{\widetilde{Y}}}$ *and* $(M,E)\widetilde{\widetilde{\in}}(F,E)^*$*:*

$(M,E)\widetilde{\widetilde{\in}}\widetilde{\widetilde{\widetilde{Y}}}$ *and* $(M,E)\widetilde{\widetilde{\in}}\widetilde{\widetilde{\cup}}\{(L,E)_\gamma : (L,E)_\gamma$ *is* τ_Δ*-ternary soft s-open set and* $(L,E)_\gamma\widetilde{\widetilde{\subseteq}}(F,E)\}$.

Hence $(M,E)\widetilde{\widetilde{\in}}\widetilde{\widetilde{\widetilde{Y}}}\widetilde{\widetilde{\cap}}(L,E)_{\gamma i}$ *for some* $(L,E)_{\gamma i}$, *where* $(L,E)_{\gamma i}$ *is* τ_Δ*-ternary soft s-open and* $(L,E)_{\gamma i}\widetilde{\widetilde{\subseteq}}(F,E)$. *Therefore,* $(M,E)\widetilde{\widetilde{\in}}(F,E)^{*\ Y}$. *Thus* $(M,E)\widetilde{\widetilde{\in}}\widetilde{\widetilde{\widetilde{Y}}}\widetilde{\widetilde{\cap}}(F,E)^*$ *implies* $(M,E)\widetilde{\widetilde{\in}}(F,E)^*$. *Hence,* $\widetilde{\widetilde{\widetilde{Y}}}\widetilde{\widetilde{\cap}}(F,E)^*\widetilde{\widetilde{\subseteq}}(F,E)^*_Y$.

(iii) *To prove* $\underline{\underline{(F,E)_Y}}\widetilde{\widetilde{\subseteq}}\widetilde{\widetilde{\widetilde{Y}}}\widetilde{\widetilde{\cap}}\underline{\underline{(F,E)}}$. *Now Consider:*

$$\underline{\underline{(F,E)_Y}} = \overline{\overline{(F,E)^Y}}\widetilde{\widetilde{\cap}}\overline{\overline{\widetilde{\widetilde{\widetilde{Y}}} - (F,E)^Y}} \Rightarrow [\widetilde{\widetilde{\widetilde{Y}}}\widetilde{\widetilde{\cap}}\overline{\overline{(F,E)}}]\widetilde{\widetilde{\cap}}\widetilde{\widetilde{\widetilde{Y}}}\widetilde{\widetilde{\cap}}[\overline{\overline{\widetilde{\widetilde{\widetilde{Y}}} - (F,E)}}].$$

Since using result (i), $[\widetilde{\widetilde{\widetilde{Y}}}\widetilde{\widetilde{\cap}}\overline{\overline{(F,E)}}]\widetilde{\widetilde{\cap}}\widetilde{\widetilde{\widetilde{Y}}}\widetilde{\widetilde{\cap}}[\overline{\overline{\widetilde{\widetilde{\widetilde{Y}}} - (F,E)}}]$. *Since*$(\widetilde{\widetilde{\subseteq}}\widetilde{\widetilde{\widetilde{Y}}}\widetilde{\widetilde{\cap}}\underline{\underline{(F,E)}}$. *Thus* $\underline{\underline{(F,E)_Y}}\widetilde{\widetilde{\subseteq}}\widetilde{\widetilde{\widetilde{Y}}}\widetilde{\widetilde{\cap}}\underline{\underline{(F,E)}})$.

Definition 35. *Let* $(U_1, U_2, U_3, \tau_\Delta, A)$ *be a ternary soft topological space of* $\widetilde{X}$ *over* $(U_1 \times U_2 \times U_3)$ *and* $F_e, G_e\widetilde{\widetilde{\in}}\widetilde{\widetilde{\widetilde{X_A}}}$ *such that* $F_e\widetilde{\widetilde{\neq}}G_e$. *Then the ternary soft topological space is said to be a ternary soft s-*τ_o *space, denoted as s-*$T_{\Delta 0}$*, if there exists at least one ternary soft s-open set* (F_1, A) *or* (F_2, A) *such that* $F_e\widetilde{\widetilde{\in}}(F_1, A)$, $G_e\widetilde{\widetilde{\in}}(F_1, A)$ *or* $F_e\widetilde{\widetilde{\in}}(F_2, A)$, $G_e\widetilde{\widetilde{\in}}(F_2, A)$.

Definition 36. *Let* $(U_1, U_2, U_3, \tau_\Delta, A)$ *be a ternary soft topological space of* $\widetilde{X}$ *over* $(U_1 \times U_2 \times U_3)$ *and* $F_e, G_e\widetilde{\widetilde{\in}}\widetilde{\widetilde{\widetilde{X_A}}}$ *such that* $F_e\widetilde{\widetilde{\neq}}G_e$. *Then the ternary soft topological space is said to be a ternary soft s-*τ_1 *space, denoted as s-*$T_{\Delta\ 1}$*, if there exists at least one ternary soft s-open set* (F_1, A) *or* (F_2, A) *such that* $F_e\widetilde{\widetilde{\in}}(F_1, A)$, $G_e\widetilde{\widetilde{\notin}}(F_1, A)$ *or* $F_e\widetilde{\widetilde{\in}}(F_2, A)$, $G_e\widetilde{\widetilde{\notin}}(F_2, A)$.

Definition 37. *Let* $(U_1, U_2, U_3, \tau_\Delta, A)$ *be a ternary soft topological space of* $\widetilde{\widetilde{\widetilde{X}}}$ *over* $(U_1 \times U_2 \times U_3)$ *and* $F_e, G_e\widetilde{\widetilde{\in}}\widetilde{\widetilde{\widetilde{X_A}}}$ *such that* $F_e\widetilde{\widetilde{\neq}}G_e$. *Then the ternary soft topological space is said to be a ternary soft s-*τ_2 *space, denoted as s-*$T_{\Delta 2}$*, if there exists at least one ternary soft s-open set such that* $F_e\widetilde{\widetilde{\in}}(F_1, A)$, $H_e\widetilde{\widetilde{\in}}(F_2, A)$ *and* $(F_1, E)\widetilde{\widetilde{\cap}}(F_2, E) = \widetilde{\widetilde{\widetilde{\varnothing}}}_A$.

Proposition 10. *Let $(U_1, U_2, U_3, \tau_\Delta, E)$ be a ternary soft topological space on $\widetilde{\widetilde{\widetilde{X}}}$ over $(U_1 \times U_2 \times U_3)$. Then each ternary soft point is ternary soft s-closed if and only if $(U_1, U_2, U_3, \tau_\Delta, E)$ is a ternary soft s-T_{Δ_1} space.*

Proof. *Let $(U_1, U_2, U_3, \tau_\Delta, E)$ be a ternary soft topological space on $\widetilde{\widetilde{\widetilde{X}}}$ over $(U_1 \times U_2 \times U_3)$. Now to prove, let $(U_1, U_2, U_3, \tau_\Delta, E)$ be a ternary soft s-T_{Δ_1} space. Suppose ternary soft points $F_{e_1} \widetilde{\widetilde{\widetilde{\equiv}}} (F, E)$ and $G_{e_1} \widetilde{\widetilde{\widetilde{\equiv}}} (G, E)$ are ternary soft s-closed and $F_{e_1} \neq G_{e_1}$. Then $(F, E)^c$ and $(G, E)^c$ are ternary soft s-open in $(U_1, U_2, U_3, \tau_\Delta, E)$. By definition, $(F, E)^c = (F^c, E)$ where $F^c(e_1) = \widetilde{\widetilde{\widetilde{X}}} - F(e_1)$ and $(G, E)^c = (G^c, E)$ where $G^c(e_1) = \widetilde{\widetilde{\widetilde{X}}} - G(e_1)$. Since $F(e_1) \widetilde{\widetilde{\widetilde{\cap}}} G(e_1) = \widetilde{\widetilde{\widetilde{\varnothing}}}$, this implies $F(e_1) = \widetilde{\widetilde{\widetilde{X}}} - G(e_1) = G^c(e_1) \forall e$. Thus $F(e_1) = (F, E) \widetilde{\widetilde{\widetilde{\in}}} (G, E)^c$. Similarly, $G(e_1) = (G, E) \widetilde{\widetilde{\widetilde{\in}}} (F, E)^c$. Hence, $e_1 \widetilde{\widetilde{\widetilde{\in}}} (G, E)^c$, $G(e_1) \widetilde{\widetilde{\widetilde{\notin}}} (G, E)^c$, and $F(e_1) \widetilde{\widetilde{\widetilde{\notin}}} (F, E)^c$, $G(e_1) \widetilde{\widetilde{\widetilde{\in}}} (F, E)^c$. This proves that $(U_1, U_2, U_3, \tau_\Delta, E)$ is a ternary soft s-T_1 space.*

Conversely, let $(U_1, U_2, U_3, \tau_\Delta, E)$ be a ternary soft s-T_{Δ_1} space. To prove that $F(e_1) = (F, E) \widetilde{\widetilde{\widetilde{\in}}} \widetilde{\widetilde{\widetilde{X}}}$ is ternary soft s-closed, we show that $(F, E)^c$ is ternary soft s-open in $(U_1, U_2, U_3, \tau_\Delta, E)$. Let $G_{e_1} = (G, E) \widetilde{\widetilde{\widetilde{\in}}} (F, E)^c$ be ternary soft s-closed. Then $F_{e_1} \neq G_{e_1}$. Since $(U_1, U_2, U_3, \tau_\Delta, E)$ is a ternary soft s-T_{Δ_1} space, there exists a ternary soft s-open set (L, E) such that $G(e_1) \widetilde{\widetilde{\widetilde{\in}}} (L, E) \widetilde{\widetilde{\widetilde{\subseteq}}} (F, E)^c$. Hence $G_{e_1} \widetilde{\widetilde{\widetilde{\in}}} \widetilde{\widetilde{\widetilde{\cup}}} \{(L, E) : G_{e_1} \widetilde{\widetilde{\widetilde{\in}}} (F, E)^c\}$. This proves that $(F, E)^c$ is ternary soft s-open in $(U_1, U_2, U_3, \tau_\Delta, E)$. Therefore, $F_{e_1} = (F, E)$ is ternary soft s-closed in $(U_1, U_2, U_3, \tau_\Delta, E)$.

Proposition 11. *Let $(U_1, U_2, U_3, \tau_\Delta, E)$ be a ternary soft topological space of $\varnothing$ over $(U_1 \times U_2 \times U_3)$ and $F_e, G_e \widetilde{\widetilde{\widetilde{\in}}} \varnothing$ such that $F_e \neq G_e$. If there exist ternary soft s-open sets (F_1, E) and (F_2, E) such that $F_e \widetilde{\widetilde{\widetilde{\in}}} (F_1, E)$ and $G_e \widetilde{\widetilde{\widetilde{\in}}} (F_2, E)^c$, then $(U_1, U_2, U_3, \tau_\Delta, E)$ is a ternary soft s-T_{Δ_0} space and $(U_1, U_2, U_3, \tau_\Delta, E)$ is a ternary soft s-T_{Δ_0} space for each $e \widetilde{\widetilde{\widetilde{\in}}} E$.*

***Proof.**. Clearly, $G_{e_1} \widetilde{\widetilde{\widetilde{\in}}} (F_1, E)^c = (F_1^c, E)$ implies $G_{e_1} \widetilde{\widetilde{\widetilde{\notin}}} (F_2, E)$. Similarly, $F_e \widetilde{\widetilde{\widetilde{\in}}} (F_2, E)^c = (F_2^c, E)$ implies $F_e \widetilde{\widetilde{\widetilde{\notin}}} (F_2, E)$. Thus, we have $F_e \widetilde{\widetilde{\widetilde{\in}}} (F_1, E)$, $G_e \widetilde{\widetilde{\widetilde{\notin}}} (F_1, E)$ or $G_e \widetilde{\widetilde{\widetilde{\in}}} (F_2, E)$, $F_e \widetilde{\widetilde{\widetilde{\notin}}} (F_2, E)$. This proves $(U_1, U_2, U_3, \tau_\Delta, E)$ is a ternary soft s-T_{Δ_0} space. Now for any $e \widetilde{\widetilde{\widetilde{\in}}} E$, $(U_1, U_2, U_3, \tau_\Delta, E)$ is a ternary soft topological space and $F_e \widetilde{\widetilde{\widetilde{\in}}} (F_1, E)$ and $G_e \widetilde{\widetilde{\widetilde{\in}}} (F_1, E)^c$ or $G_e \widetilde{\widetilde{\widetilde{\in}}} (F_2, E)$ and $F_e \widetilde{\widetilde{\widetilde{\notin}}} (F_2, E)^c$. So that $F_e \widetilde{\widetilde{\widetilde{\in}}} F_1(e)$, $G_e \widetilde{\widetilde{\widetilde{\notin}}} F_1(e)$, $G_e \widetilde{\widetilde{\widetilde{\in}}} F_2(e)$, $G_e \widetilde{\widetilde{\widetilde{\notin}}} F_2(e)$. Thus, $(U_1, U_2, U_3, \tau_\Delta, E)$ is a ternary soft s-T_{Δ_0} space.*

Proposition 12. *Let $(U_1, U_2, U_3, \tau_\Delta, E)$ be a ternary soft topological space of $\widetilde{\widetilde{\widetilde{X}}}$ over $(U_1 \times U_2 \times U_3)$ and $F_e, G_e \widetilde{\widetilde{\widetilde{\in}}} \widetilde{\widetilde{\widetilde{X}}}$ such that $F_e \neq G_e$. If there exist ternary soft s-open sets $(F_1, E), (F_2, E)$ such that $F_e \widetilde{\widetilde{\widetilde{\in}}} (F_1, E)$ and $G_e \widetilde{\widetilde{\widetilde{\in}}} (F_1, E)^c$, or $F_e \widetilde{\widetilde{\widetilde{\in}}} (F_2, E)$ and $G_e \widetilde{\widetilde{\widetilde{\in}}} (F_2, E)$. Then $(U_1, U_2, U_3, \tau_\Delta, E)$ is a ternary soft s-T_{Δ_o} space. Additionally, $(U_1, U_2, U_3, \tau_\Delta, E)$ is a ternary soft s-T_{Δ_o} space for each $e \widetilde{\widetilde{\widetilde{\in}}} E$.*

Proof. *Clearly, $G_e \widetilde{\widetilde{\widetilde{\in}}} (F_1, E)^c = (F_1^c, E)$ implies $G_e \widetilde{\widetilde{\widetilde{\notin}}} (F_2, E)$. Similarly, $F_e \widetilde{\widetilde{\widetilde{\in}}} (F_2, E)^c =$*

(F_2^c, E) implies $F_e \tilde{\tilde{\notin}} (F_2, E)$. Thus, we have $F_e \tilde{\tilde{\in}} (F_1, E)$, $G_e \tilde{\tilde{\notin}} (F_1, E)$, or $G_e \tilde{\tilde{\in}} (F_2, E)$, $F_e \tilde{\tilde{\notin}} (F_2, E)$. This proves $(U_1, U_2, U_3, \tau_\Delta, E)$ is a ternary soft s-T_{Δ_o} space. Now, for any $e \tilde{\tilde{\in}} E$, $(U_1, U_2, U_3, \tau_\Delta, E)$ is a ternary soft topological space. Since $F_e \tilde{\tilde{\in}} (F_1, E)$ and $G_e \tilde{\tilde{\notin}} (F_1, E)^c$ or $G_e \tilde{\tilde{\in}} (F_2, E)$ and $F_e \tilde{\tilde{\notin}} (F_2, E)^c$, it follows that $F_e \tilde{\tilde{\in}} F_1(e)$, $G_e \tilde{\tilde{\notin}} F_1(e)$, or $G_e \tilde{\tilde{\in}} F_2(e)$, $F_e \tilde{\tilde{\notin}} F_2(e)$. Thus, $(U_1, U_2, U_3, \tau_\Delta, E)$ is a ternary soft s-T_{Δ_o} space.

Proposition 13. *Let $(U_1, U_2, U_3, \tau_\Delta, E)$ be a ternary soft topological space of $\tilde{\tilde{X}}$ over $(U_1 \times U_2 \times U_3)$ and $F_e, G_e \tilde{\tilde{\in}} \tilde{\tilde{X}}$ such that $F_e \neq G_e$. If there exist ternary soft s-open sets $(F_1, E), (F_2, E)$ such that:*

$$F_e \tilde{\tilde{\in}} (F_1, E) \quad \text{and} \quad G_e \tilde{\tilde{\in}} (F_1, E), \qquad \text{or} \;\; G_e \tilde{\tilde{\in}} (F_2, E) \quad \text{and} \quad F_e \tilde{\tilde{\in}} (F_2, E),$$

then $(U_1, U_2, U_3, \tau_\Delta, E)$ is a ternary soft s-T_{Δ_o} space, and $(U_1, U_2, U_3, \tau_\Delta, E)$ is a ternary soft s-T_{Δ_1} space for each $e \tilde{\tilde{\in}} E$.
Proof. *Obvious.*

Proposition 14. *Let $(U_1, U_2, U_3, \tau_\Delta, E)$ be a ternary soft topological space of $\tilde{\tilde{X}}$ over $(U_1 \times U_2 \times U_3)$ and $\tilde{\tilde{Y}} \tilde{\tilde{\subseteq}} \tilde{\tilde{X}}$. Then, if $(U_1, U_2, U_3, \tau_\Delta, E)$ is a ternary soft s-T_{Δ_o} space, then $(U_1, U_2, U_3, \tau_{\Delta_y}, E)$ is a ternary soft s-T_{Δ_o} space.*

Proof. *Let $F_e, G_e \tilde{\tilde{\in}} \tilde{\tilde{Y}}$ such that $F_e \neq G_e$. Then, $F_e, G_e \tilde{\tilde{\in}} \tilde{\tilde{X}}$. Since $(U_1, U_2, U_3, \tau_\Delta, E)$ is a ternary soft s-T_{Δ_o} space, there exist ternary soft s-open sets (F, E) and (G, E) in $(U_1, U_2, \tau_\Delta, E)$ such that:*

$$F_e \tilde{\tilde{\in}} (F, E), \quad G_e \tilde{\tilde{\notin}} (F, E), \quad \text{or} \;\; G_e \tilde{\tilde{\in}} (G, E), \quad F_e \tilde{\tilde{\notin}} (G, E).$$

Therefore, $F_e \tilde{\tilde{\in}} \tilde{\tilde{Y}} \tilde{\tilde{\cap}} (F, E) = \tilde{\tilde{Y}}(F, E)$. Similarly, it can be shown that $G_e \tilde{\tilde{\in}} \tilde{\tilde{Y}}(G, E)$ and $F_e \tilde{\tilde{\notin}} \tilde{\tilde{Y}}(G, E)$. Thus, $(U_1, U_2, U_3, \tau_{\Delta_y}, E)$ is a ternary soft s-T_{Δ_o} space.

Proposition 15. *Let $(U_1, U_2, U_3, \tau_\Delta, E)$ be a ternary soft topological space of $\tilde{\tilde{X}}$ over $(U_1 \times U_2 \times U_3)$ and $\tilde{\tilde{Y}} \tilde{\tilde{\subseteq}} \tilde{\tilde{X}}$. Then, if $(U_1, U_2, U_3, \tau_\Delta, E)$ is a ternary soft s-T_{Δ_1} space, then $(U_1, U_2, U_3, \tau_{\Delta_y}, E)$ is a ternary soft s-T_{Δ_1} space.*
Proof. *Obvious.*

Proposition 16. *Let $(U_1, U_2, U_3, \tau_\Delta, E)$ be a ternary soft topological space on $\tilde{\tilde{X}}$ over $(U_1 \times U_2 \times U_3)$. If $(U_1, U_2, U_3, \tau_\Delta, E)$ is a ternary soft s-T_{Δ_2} space on $\tilde{\tilde{X}}$ over $(U_1 \times U_2 \times U_3)$, then $(U_1, U_2, \tau_{\Delta_e}, E)$ is a ternary soft s-T_{Δ_2} space for each $e \tilde{\tilde{\in}} E$.*

Proof. *Let $(U_1, U_2, U_3, \tau_{\Delta_y}, E)$ be a ternary soft topological space on $\tilde{\tilde{X}}$ over $(U_1 \times U_2 \times U_3)$. For any $e \tilde{\tilde{\in}} E$, $\tau_{\Delta_e} = \{F(e) : (F, E) \tilde{\tilde{\in}} \tau_\Delta\}$ is a ternary soft topology on $\tilde{\tilde{X}}$ over $(U_1 \times U_2 \times U_3)$. Let $x, y \tilde{\tilde{\in}} \tilde{\tilde{X}}$ such that $x \neq y$. Since $(U_1, U_2, U_3, \tau_\Delta, E)$ is a ternary soft*

s-T_{Δ_2} space, there exist ternary soft points $F_e, G_e \widetilde{\widetilde{\in}} \widetilde{\widetilde{X}}$ such that $F_e \neq G_e$ and $x \widetilde{\widetilde{\in}} F(e)$, $y \widetilde{\widetilde{\in}} G(e)$. There exist ternary soft s-open sets (F_1, E) and (F_2, E) such that:

$$F_e \widetilde{\widetilde{\in}} (F_1, E), \quad G_e \widetilde{\widetilde{\in}} (F_2, E), \quad \text{and} \;\; (F_1, E) \widetilde{\widetilde{\cap}} (F_2, E) = \widetilde{\widetilde{\emptyset}}.$$

This implies that $x \widetilde{\widetilde{\in}} F(e) \widetilde{\widetilde{\subseteq}} F_1(e)$, $y \widetilde{\widetilde{\in}} G(e) \widetilde{\widetilde{\subseteq}} F_2(e)$, and $F_1(e) \widetilde{\widetilde{\cap}} F_2(e) = \widetilde{\widetilde{\emptyset}}$. This proves that $(U_1, U_2, U_3, \tau_{\Delta_e}, E)$ is a ternary soft s-T_{Δ_2} space.

Proposition 17. *Let $(U_1, U_2, U_3, \tau_\Delta, E)$ be a ternary soft topological space on $\widetilde{\widetilde{X}}$ over $(U_1 \times U_2 \times U_3)$ and $\widetilde{\widetilde{Y}} \widetilde{\widetilde{\subset}} \widetilde{\widetilde{X}}$. Then, if $(U_1, U_2, U_3, \tau_\Delta, E)$ is a ternary soft $s - \tau_{\Delta_2}$ space, then $(U_1, U_2, U_3, \tau_{\Delta_y}, E)$ is a ternary soft $s - T_{\Delta_2}$ space, and $(U_1, U_2, U_3, \tau_{\Delta_e}, E)$ is a ternary soft $s - T_{\Delta_2}$ space for each $e \widetilde{\widetilde{\in}} E$.*

Proof. *Let $F_e, G_e \widetilde{\widetilde{\in}} \widetilde{\widetilde{Y}}$ such that $F_e \neq G_e$. Then $F_e, G_e \widetilde{\widetilde{\in}} \widetilde{\widetilde{X}}$. Since $(U_1, U_2, U_3, \tau_\Delta, E)$ is a ternary soft $s - T_{\Delta_2}$ space, there exist ternary soft s-open sets (F_1, E) and (F_2, E) such that $F_e \widetilde{\widetilde{\in}} (F_1, E)$ and $G_e \widetilde{\widetilde{\in}} (F_2, E)$ and $(F_1, E) \widetilde{\widetilde{\cap}} (F_2, E) = \widetilde{\widetilde{\varnothing}}$. Thus, $F_e \widetilde{\widetilde{\in}} \widetilde{\widetilde{Y}} \widetilde{\widetilde{\cap}} =^Y (F_2, E)$ and $^Y(F_2, E) \widetilde{\widetilde{\cap}}^Y (F_2, E) = \widetilde{\widetilde{\varnothing}}$. Therefore, it proves that $(U_1, U_2, U_3, \tau_{\Delta_y}, E)$ is a ternary soft $s - T_{\Delta_2}$ space.*

Proposition 18. *Let $(U_1, U_2, U_3, \tau_\Delta, E)$ be a ternary soft topological space on $\widetilde{\widetilde{X}}$ over $(U_1 \times U_2 \times U_3)$. If $(U_1, U_2, U_3, \tau_\Delta, E)$ is a ternary soft $s - T_{\Delta_2}$ space and for any two ternary soft points $F_e, G_e \widetilde{\widetilde{\in}} \widetilde{\widetilde{X}}$ such that $F_e \neq G_e$, then there exist ternary soft s-closed sets (F_1, E) and (F_2, E) such that $F_e \widetilde{\widetilde{\in}} (F_1, E)$ and $G_e \widetilde{\widetilde{\notin}} (F_1, E)$ or $G_e \widetilde{\widetilde{\in}} (F_2, E)$ and $(F_1, E) \widetilde{\widetilde{\cup}} (F_2, E) = \widetilde{\widetilde{X}}$.*

Proof. *Let $(U_1, U_2, U_3, \tau_\Delta, E)$ be a ternary soft topological space on $\widetilde{\widetilde{X}}$ over $(U_1 \times U_2 \times U_3)$. Since $(U_1, U_2, U_3, \tau_\Delta, E)$ is a ternary soft $s - T_{\Delta_2}$ space and $F_e, G_e \widetilde{\widetilde{\in}} \widetilde{\widetilde{X}}$ such that $F_e \neq G_e$, there exist ternary soft s-open sets (H, E) and (L, E) such that $F_e \widetilde{\widetilde{\in}} (H, E)$ and $G_e \widetilde{\widetilde{\in}} (L, E)$ and $(H, E) \widetilde{\widetilde{\cap}} (L, E) = \widetilde{\widetilde{\varnothing}}$. Clearly, $(H, E) \widetilde{\widetilde{\subset}} (L, E)^c$ and $(L, E) \widetilde{\widetilde{\subset}} (H, E)^c$. Hence, $F_e \widetilde{\widetilde{\in}} (L, E)^c$, and we set $(L, E)^c = (F_1, E)$, which gives $F_e \widetilde{\widetilde{\in}} (F_1, E)$ and $G_e \widetilde{\widetilde{\notin}} (F_1, E)$. Also, $G_e \widetilde{\widetilde{\in}} (F_1, E)^c$, so we set $(H, E)^c = (F_2, E)$. Therefore, $F_e \widetilde{\widetilde{\in}} (F_1, E)$ and $G_e \widetilde{\widetilde{\in}} (F_2, E)$. Moreover, $(F_1, E) \widetilde{\widetilde{\cup}} (F_2, E) = (L, E)^c \widetilde{\widetilde{\cup}} (H, E)^c = \widetilde{\widetilde{X}}$.*

Definition 38. *Let $(U_1, U_2, U_3, \tau_\Delta, E)$ be a ternary soft topological space on $\widetilde{\widetilde{X}}$ over $(U_1 \times U_2 \times U_3)$. Let (F, E) be a ternary soft s-closed set in $(U_1, U_2, U_3, \tau_\Delta, E)$, and $F_e \widetilde{\widetilde{\notin}} (F, E)$. If there exist ternary soft s-open sets (G, E) and (H, E) such that $F_e \widetilde{\widetilde{\in}} (G, E)$, $(F, E) \widetilde{\widetilde{\subset}} (H, E)$, and $(F, E) \widetilde{\widetilde{\cap}} (H, E) = \widetilde{\widetilde{\varnothing}}$, then $(U_1, U_2, U_3, \tau_\Delta, E)$ is called a ternary soft s-regular space.*

Proposition 19. *Let $(U_1, U_2, U_3, \tau_\Delta, E)$ be a ternary soft topological space of $\widetilde{\widetilde{X}}$ over $(U_1 \times U_2 \times U_3)$. Then the following statements are equivalent:*

(i) $(U_1, U_2, U_3, \tau_\Delta, E)$ *is ternary soft s-regular.*

(ii) *For any ternary soft s-open set* (F, E) *in* $(U_1, U_2, U_3, \tau_\Delta, E)$ *and* $G_e \tilde{\tilde{\in}} (F, E)$*, there is a ternary soft s-open set* (G, E) *containing* G_e *such that* $G_e \tilde{\tilde{\in}} \overline{\overline{(G, E)}} \tilde{\tilde{\subseteq}} (F, E)$*.*

(iii) *Each ternary soft point in* $(U_1, U_2, U_3, \tau_\Delta, E)$ *has a ternary soft neighbourhood base consisting of ternary soft s-closed sets.*

Proof. ***(i)*** $\Rightarrow$ ***(ii):*** *Let* (F, E) *be a ternary soft s-open set in* $(U_1, U_2, U_3, \tau_\Delta, E)$ *and* $G_e \tilde{\tilde{\in}} (F, E)$*. Then* $(F, E)^c$ *is a ternary soft s-closed set such that* $G_e \tilde{\tilde{\notin}} (F, E)^c$*. By the ternary soft regularity of* $(U_1, U_2, U_3, \tau_\Delta, E)$*, there are ternary soft s-open sets* (F_1, E)*,* (F_2, E) *such that* $G_e \tilde{\tilde{\notin}} (F_1, E)$*,* $(F, E)^c \tilde{\tilde{\subseteq}} (F_2, E)$ *and* $(F_1, E) \tilde{\tilde{\cap}} (F_2, E) = \tilde{\tilde{\emptyset}}$*. Clearly,* $(F_2, E)^c$ *is a ternary soft set contained in* (F, E)*. Thus,* $(F_1, E) \tilde{\tilde{\subseteq}} (F_2, E)^c \tilde{\tilde{\subseteq}} (F, E)$*. This gives* $\overline{\overline{(F_1, E)}} \tilde{\tilde{\subseteq}} (F_2, E)^c \tilde{\tilde{\subseteq}} (F, E)$*, and we set* $(F_1, E) = (G, E)$*. Consequently,* $G_e \tilde{\tilde{\in}} (G, E)$ *and* $\overline{\overline{(G, E)}} \tilde{\tilde{\subseteq}} (F, E)$*. This proves (ii).*

(ii) $\Rightarrow$ ***(iii):*** *Let* $G_e \tilde{\tilde{\in}} \tilde{\tilde{X}}$*. For the ternary soft s-open set* (F, E) *in* $(U_1, U_2, U_3, \tau_\Delta, E)$*, there is a ternary soft s-open set* (G, E) *containing* G_e *such that* $G_e \tilde{\tilde{\in}} \overline{\overline{(G, E)}}$*,* $(G, E) \tilde{\tilde{\subseteq}} (F, E)$*. Thus, for each* $G_e \tilde{\tilde{\in}} \tilde{\tilde{X}}$*, the sets* $\overline{\overline{(G, E)}}$ *form a ternary soft neighborhood base consisting of ternary soft s-closed sets of* $(U_1, U_2, U_3, \tau_\Delta, E)$*, which proves (iii).*

(iii) $\Rightarrow$ ***(i):*** *Let* (F, E) *be a ternary soft s-closed set such that* $G_e \tilde{\tilde{\notin}} (F, E)$*. Then* $(F, E)^c$ *is a ternary soft open neighborhood of* G_e*. By (iii), there is a ternary soft s-closed set* (F_1, E) *which contains* G_e *and is a ternary soft neighborhood of* G_e *with* $(F_1, E) \tilde{\tilde{\subseteq}} (F_1, E)^c$*. Then* $G_e \tilde{\tilde{\notin}} (F, E)^c$*,* $(F, E) \tilde{\tilde{\subseteq}} (F_1, E)^c = (F_2, E)$*, and* $(F_1, E) \tilde{\tilde{\cap}} (F_2, E) = \tilde{\tilde{\emptyset}}$*. Therefore,* $(U_1, U_2, U_3, \tau_\Delta, E)$ *is ternary soft s-regular.*

Proposition 20. *Let* $(U_1, U_2, U_3, \tau_\Delta, E)$ *be a ternary soft s-regular space on* $\tilde{\tilde{X}}$ *over* $(U_1 \times U_2 \times U_3)$*. Then every ternary soft subspace of* $(U_1, U_2, U_3, \tau_\Delta, E)$ *is ternary soft s-regular.*

Proof. *Let* $(U_1, U_2, U_3, \tau_\Delta, E)$ *be a ternary soft subspace of a ternary soft s-regular space* $(U_1, U_2, U_3, \tau_\Delta, E)$*. Suppose* (F, E) *is a ternary soft s-closed set in* $(U_1, U_2, U_3, \tau_\Delta, E)$ *and* $F_e \tilde{\tilde{\in}} \tilde{\tilde{Y}}$*. Then* $(F, E) = (G, E) \tilde{\tilde{\cap}} \tilde{\tilde{Y}}$*, where* (G, E) *is a ternary soft s-closed set in* $(U_1, U_2, U_3, \tau_\Delta, E)$*. Then* $F_e \tilde{\tilde{\notin}} (F, E)$ *since* $(U_1, U_2, U_3, \tau_\Delta, E)$ *is a ternary soft subspace of a ternary soft s-regular space. There exist soft disjoint ternary s-open sets* (F_1, E)*,* (F_2, E) *in* $(U_1, U_2, U_3, \tau_\Delta, E)$*. Then* $F_e \tilde{\tilde{\notin}} (G, E)$*. Since* $(U_1, U_2, U_3, \tau_\Delta, E)$ *is ternary soft s-regular, there exist ternary soft disjoint ternary s-open sets* (F_1, E)*,* (F_2, E) *in* $(U_1, U_2, U_3, \tau_\Delta, E)$ *such that* $F_e \tilde{\tilde{\in}} (F_1, E)$ *and* $(G, E) \tilde{\tilde{\in}} (F_2, E)$*. Clearly,* $F_e \tilde{\tilde{\in}} (F_1, E) \tilde{\tilde{\cap}} \tilde{\tilde{Y}} =^Y (F_2, E)$ *and* $(F, E) \tilde{\tilde{\subseteq}} (F_2, E) \tilde{\tilde{\cap}} \tilde{\tilde{Y}} =^Y (F_2, E)$ *such that* ${}^Y(F_2, E) \overset{Y}{\tilde{\tilde{\cap}}} (F_2, E) = \tilde{\tilde{\varnothing}}$*. This proves that* $(U_1, U_2, \tau_\Delta, E)$ *is a ternary soft s-regular subspace of* $(U_1, U_2, U_3, \tau_\Delta, E)$*.*

Proposition 21. *Let $(U_1, U_2, U_3, \tau_\Delta, E)$ be a ternary soft regular space on $\tilde{\tilde{X}}$ over $(U_1 \times U_2 \times U_3)$. A ternary space $(U_1, U_2, U_3, \tau_\Delta, E)$ is ternary soft s-regular if and only if for each $F_e \tilde{\tilde{\in}} \tilde{\tilde{X}}$ and a ternary soft s-closed set (F, E) in $(U_1, U_2, U_3, \tau_\Delta, E)$ such that $F_e \tilde{\tilde{\notin}} (F, E)$, there exist ternary soft s-open sets (F_1, E), (F_2, E) in $(U_1, U_2, U_3, \tau_\Delta, E)$ such that $F_e \tilde{\tilde{\in}} (F_1, E)$, $(F_1, E) \tilde{\tilde{\subseteq}} (F_2, E)$ and $\overline{\overline{(F_1, E)}} \tilde{\tilde{\cap}} \overline{\overline{(F_2, E)}} = \tilde{\tilde{\varnothing}}$.*

Proof. *For each $F_e \tilde{\tilde{\in}} \tilde{\tilde{X}}$ and a ternary soft s-closed set (G, E) such that $F_e \tilde{\tilde{\notin}} (F, E)$, there is a ternary soft s-open set (G, E) such that $F_e \tilde{\tilde{\in}} (G, E)$, $\overline{\overline{(G, E)}} \tilde{\tilde{\subseteq}} (F_1, E)^c$. Again, there is a ternary soft s-open set (F_1, E) containing F_e such that $\overline{\overline{(F_1, E)}} \tilde{\tilde{\subseteq}} (G, E)$. Let $(F_2, E) = (\overline{\overline{(G, E)}})^c$, then $\overline{\overline{(F_1, E)}} \tilde{\tilde{\subseteq}} (G, E) \tilde{\tilde{\subseteq}} \overline{\overline{(G, E)}} \tilde{\tilde{\subseteq}} (F, E)^c$ implies $\overline{\overline{(F_1, E)}} \tilde{\tilde{\subseteq}} (F_2, E) \tilde{\tilde{\subseteq}} (\overline{\overline{(G, E)}})^c$ or $(F, E) \tilde{\tilde{\subseteq}} (F_2, E)$. Also,*

$$\overline{\overline{(F_1, E)}} \tilde{\tilde{\cap}} \overline{\overline{(F_2, E)}} = (F_1, E) \tilde{\tilde{\cap}} (\overline{\overline{(G, E)}})^c \tilde{\tilde{\subseteq}} (G, E) \tilde{\tilde{\cap}} (\overline{\overline{(G, E)}})^c \tilde{\tilde{\subseteq}} \overline{\overline{(G, E) \tilde{\tilde{\cap}} (\overline{\overline{(G, E)}})^c}} = \overline{\overline{\tilde{\tilde{\varnothing}}}}$$

. Thus, (F_1, E), (F_2, E) are the required ternary soft s-open sets in $(U_1, U_2, U_3, \tau_\Delta, E)$. This proves the necessity. The sufficiency is immediate.

Definition 39. *Let $(U_1, U_2, U_3, \tau_\Delta, E)$ be a ternary soft regular space on $\tilde{\tilde{X}}$ over $(U_1 \times U_2 \times U_3)$. (F, E), (G, E) are ternary soft s-closed sets over $(U_1 \times U_2 \times U_3)$ such that $(F, E) \tilde{\tilde{\cap}} (G, E) = \tilde{\tilde{\varnothing}}$. If there exist ternary soft s-open sets (F_1, E), (F_2, E) such that $(F, E) \tilde{\tilde{\subseteq}} (F_1, E)$, $(G, E) \tilde{\tilde{\subseteq}} (F_2, E)$ and $(F_1, E) \tilde{\tilde{\cap}} (F_2, E) = \tilde{\tilde{\varnothing}}$, then $(U_1, U_2, U_3, \tau_\Delta, E)$ is called a ternary soft s-normal space.*

Definition 40. *Let $(U_1, U_2, U_3, \tau_\Delta, E)$ be a ternary soft regular space on $\tilde{\tilde{X}}$ over $(U_1 \times U_2 \times U_3)$. Then $(U_1, U_2, U_3, \tau_\Delta, E)$ is said to be a ternary soft s-τ_Δ^3 space if it is ternary soft s-regular and a ternary soft s-τ_{Δ_1} space.*

Proposition 22. *Let $(U_1, U_2, U_3, \tau_\Delta, E)$ be a ternary soft regular space on $\tilde{\tilde{X}}$ over $(U_1 \times U_2 \times U_3)$ and $\tilde{\tilde{Y}} \tilde{\tilde{\subseteq}} \tilde{\tilde{X}}$. If $(U_1, U_2, U_3, \tau_\Delta, E)$ is a ternary soft s-τ_{Δ_3} space, then $(U_1, U_2, U_3, \tau_{\Delta_Y}, E)$ is a ternary soft s-τ_{Δ_3} space.*

Definition 41. *A ternary soft topological space $(U_1, U_2, U_3, \tau_\Delta, E)$ on $\tilde{\tilde{X}}$ over $(U_1 \times U_2 \times U_3)$ is said to be a ternary soft s-τ_{Δ_4} space if it is ternary soft s-normal and ternary soft s-τ_{Δ_1} space and τ_{Δ_3}-space.*

Proposition 23. *A ternary soft topological space $(U_1, U_2, U_3, \tau_\Delta, E)$ is ternary soft s-normal if and only if for a soft s-closed set (F, E) and a ternary soft s-open set (G, E), such that $(F, E) \tilde{\tilde{\subseteq}} (G, E)$, there exists at least one ternary soft s-open set (H, E) containing (F, E) such that:*

$$(F, E) \tilde{\tilde{\subseteq}} (H, E) \tilde{\tilde{\subseteq}} \overline{\overline{(H, E)}} \tilde{\tilde{\subseteq}} (G, E).$$

Proof. *Let us suppose that $(U_1, U_2, U_3, \tau_\Delta, E)$ is a ternary soft normal space and (F, E) is any ternary soft s-closed subset of $(U_1, U_2, U_3, \tau_\Delta, E)$ and (G, E) is a ternary soft s-open* *set such that $(F,E)\tilde{\tilde{\subseteq}}(G,E)$. Then $(G,E)^c$ is ternary soft s-closed and $(F,E)\tilde{\tilde{\cap}}(G,E)^c = \tilde{\tilde{\varnothing}}$. So by supposition, there are ternary soft s-open sets (H,E) and (K,E) such that $(F,E)\tilde{\tilde{\subseteq}}(H,E)$, $(G,E)^c\tilde{\tilde{\subseteq}}(K,E)$, and $(H,E)\tilde{\tilde{\cap}}(K,E) = \tilde{\tilde{\varnothing}}$. Since $(H,E)\tilde{\tilde{\cap}}(K,E) = \tilde{\tilde{\varnothing}}$, $(H,E)\tilde{\tilde{\subseteq}}(K,E)^c$. But $(K,E)^c$ is ternary soft s-closed, so that:*

$$(F,E)\tilde{\tilde{\subseteq}}(H,E)\tilde{\tilde{\subseteq}}\overline{\overline{(H,E)}}\tilde{\tilde{\subseteq}}(K,E)^c\tilde{\tilde{\subseteq}}(G,E)$$

. Hence,

$$(F,E)\tilde{\tilde{\subseteq}}(H,E)\tilde{\tilde{\subseteq}}\overline{\overline{(H,E)}}\tilde{\tilde{\subseteq}}(K,E)^c\tilde{\tilde{\subseteq}}(G,E)$$

.

Conversely, suppose that for every ternary soft s-closed set (F,E) and a ternary soft s-open set (G,E) such that $(F,E)\tilde{\tilde{\subseteq}}(H,E)$, there is a ternary soft s-open set (H,E) such that:

$$(F,E)\tilde{\tilde{\subseteq}}(H,E)\tilde{\tilde{\subseteq}}\overline{\overline{(H,E)}}\tilde{\tilde{\subseteq}}(G,E)$$

. Let (F_1,E), (F_2,E) be any two soft disjoint s-closed sets, then $(F_1,E)\tilde{\tilde{\subseteq}}(F_2,E)^c$ where $(F_2,E)^c$ is ternary soft s-open. Hence, there is a ternary soft s-open set (H,E) such that:

$$(F,E)\tilde{\tilde{\subseteq}}(H,E)\tilde{\tilde{\subseteq}}\overline{\overline{(H,E)}}\tilde{\tilde{\subseteq}}(F_2,E)^c$$

. But then $(F_2,E)\tilde{\tilde{\subseteq}}(\overline{\overline{(H,E)}})^c$ and $(H,E)\tilde{\tilde{\cap}}(\overline{\overline{(H,E)}})^c \neq \varnothing$. Hence, $(F_1,E)\tilde{\tilde{\subseteq}}(H,E)$ and $(F_2,E)\tilde{\tilde{\subseteq}}(\overline{\overline{(H,E)}})^c$ with $(H,E)\tilde{\tilde{\cap}}(\overline{\overline{(H,E)}})^c = \varnothing$. Hence, $(U_1,U_2,U_3,\tau_\Delta,E)$ is a ternary soft s-normal space.

9. Comparative Analysis

The following Table 2 provides a detailed comparative analysis of the proposed methods, contrasting them with the established techniques discussed in reference [14]. This comparison highlights the strengths and weaknesses of each approach, offering insights into how the proposed methods perform relative to the established techniques across various key factors:

Aspect	**Binary Soft Axioms (published work)[14]**	**Ternary Soft Semi-Separation Axioms (Proposed Method)**
1. Focus	Study of binary soft sets and their application in binary soft topological spaces.	Study of ternary soft sets and their application in ternary soft topological spaces.
2. Main Objective	Define separation axioms, explore their properties, and establish relationships.	Introduce ternary soft semi-separation axioms and explore related properties in ternary soft topologies.

Aspect	**Binary Soft Axioms (published work)[14]**	**Ternary Soft Semi-Separation Axioms (Proposed Method)**
3. Type of Axioms Introduced	Binary soft separation axioms (e.g., τ_{Δ_i}) and their variants like pre-regular, pre-normal.	Ternary soft semi-separation axioms (e.g., τ_{Δ_i}, semi-regular, semi-normal, τ_{Δ_i}).
4. Properties Studied	Properties of binary soft topological spaces, including regularity, normality, and invariance properties.	Properties of ternary soft topological spaces, such as ternary soft semi-regular, ternary soft semi-normal, and other invariance properties.
5. Relationship with General Topology	Explores the relationship between binary soft topology and general topology.	No direct mention of relationship with general topology; focuses on ternary soft topologies.
6. Invariance Properties	Discusses binary soft invariance properties like topological and hereditary properties.	Discusses ternary soft invariance properties, such as ternary soft topological property and ternary soft hereditary property.
7. Scope of Study	General study of binary soft topological spaces and their separation axioms.	Focus on ternary soft semi-separation axioms and their implications in ternary soft spaces.
8. Applications	Theoretical with hints at practical applications in the future.	Likely intended for more applied studies with an emphasis on solving practical problems using ternary soft topological concepts.
9. Additional Concepts	Focus on binary soft topologies with regularity, normality, and invariance properties.	Introduces semi-separation axioms for ternary spaces, including ternary semi-regular, semi-normal, and other related properties.

Table 2: Comparative analysis

10. Advantages

(i) **Comprehensive Understanding of Ternary Soft Sets:** The research provides a detailed exploration of the basic operations of ternary soft sets, such as subset, superset, complement, union, and intersection. These operations help in the fundamental understanding and manipulation of ternary soft sets, which can be useful for decision-making processes where uncertainty or vagueness exists.

(ii) **New Mathematical Structures:** The introduction of the concept of ternary soft topological structures based on three initial universal sets and decision variables is an innovative approach. This extension opens up new avenues for research in soft set theory and its applications in fields like computer science, decision analysis, and fuzzy logic.

(iii) **Theoretical Contributions to Topology:** By introducing concepts like ternary soft open sets, closed sets, closures, interiors, boundary, and neighborhoods, the study enhances the theoretical foundation of soft set theory, particularly in the context of topology. These new concepts extend existing topological structures and may lead to new insights into how soft sets behave in more complex environments.

(iv) **Practical Applications:** The research includes examples to demonstrate how these concepts can be applied in real-world scenarios. This makes the study highly relevant to practical applications such as decision-making, multi-criteria optimization problems, and uncertainty modeling.

(v) **Interdisciplinary Relevance:** Ternary soft sets and the associated topological structures are versatile and can potentially be applied across a wide range of disciplines, including economics, computer science, engineering, and social sciences. The research provides a foundation for interdisciplinary studies that incorporate uncertainty and vagueness in decision processes.

(vi) **Clarity of Presentation:** The research clearly defines operations and structures with well-illustrated examples, facilitating a better understanding of how these concepts can be used in theoretical and practical contexts.

11. Limitations

(i) **Complexity of Definitions and Operations:** The study delves into advanced mathematical operations that may be difficult for readers unfamiliar with the underlying concepts of soft set theory and topology. For non-experts, understanding the full implications of ternary soft sets and their operations (like difference, symmetric difference, AND/OR operations) may be challenging.

(ii) **Limited Real-World Applications:** While the theoretical aspects are well explored, the actual implementation or application of ternary soft sets in practical systems might be limited or not fully developed. Many of the examples presented might remain theoretical, and there is a need for further research to translate these concepts into real-world solutions.

(iii) **Ambiguity in Parameters:** The concept of a fixed set parameter (decision variables) might introduce ambiguity if not fully defined or if the method to choose these parameters is not clear. The application of decision variables in ternary soft sets can be context-dependent, and without a concrete framework, this flexibility could lead to inconsistencies in practical applications.

(iv) **Scalability Issues:** While ternary soft sets provide a powerful tool for modeling uncertainty with three initial universal sets, the scalability of these structures to handle large datasets or complex systems might be limited. This could restrict their use in big data analytics or large-scale optimization problems unless further advancements are made to improve computational efficiency.

(v) **Dependence on Crisp Points:** The research mentions that the "AND" and "OR" operations are performed with respect to crisp points. However, this might limit the flexibility of the model in handling more complex types of uncertainty, especially when working with vague, fuzzy, or probabilistic data.

(vi) **Lack of Experimental Validation:** Although the research introduces new concepts and mathematical structures, there might be a lack of extensive experimental validation or empirical data to support the practical applicability of these concepts. More real-world case studies or computational experiments would be needed to confirm the effectiveness of ternary soft topological structures.

(vii) **Generalization of Results:** The results are based on three initial universal sets, which might limit the generalization of the findings. It is unclear how these results could be extended to higher dimensions or more complex systems involving more than three sets.

Conclusion and Future Work

Soft set theory is a hybrid theory because of its combination of soft set and crisp set theory. This theory is used for the reduction of errors that exist in data. It is used to decide the character of mathematical structures and employs unconventional definitions of union, intersection, complementation, and subsets criteria. In this work, a few operations are defined on soft sets, which are explained with suitable examples. Furthermore, three operations7intersection, complement, and difference of soft sets7are redefined, leaving other operations unchanged. Upon examining other operations based on these redefined operations, acceptable results were produced, with better examples provided for understanding. In addition, a new structure is defined on soft sets. This structure, an extension of soft sets, uses two soft sets for its generation and is named the binary soft set. This is a super strong structure as it handles two initial universes of discourse. Based on the binary soft set, the basic operators such as binary soft subsets, binary soft absolute set, binary soft union, binary soft intersection, and binary soft difference are defined. Two more operators, AND and OR, which play important roles in the generation of some results, are also defined. All these operators are explained with intelligible examples. Soft set theory is used in both applied and pure mathematics. It is extensively used in engineering and decision-making problems, and in pure mathematics, it is applied in topology, group theory, real analysis, fractional calculus, and operation theory. This particular work extends binary soft set theory into ternary soft set theory, which uses three universal sets and three possible power sets. Basic operations for ternary soft sets are developed, and a few basic theorems and propositions are also studied. Examples are generated for clarification of these operations, results, and propositions. In continuation, one of the most interesting and important structures is discussed based on this newly defined theory: the ternary soft topological space, defined on a ternary soft set.

Soft open sets, soft closed sets, soft interior, soft closure, and the interplay between these concepts have been addressed. Examples are provided for a better understanding of these results. Investigating the generalization of ternary soft sets and ternary soft topological structures to higher dimensions is an interesting avenue for future research. Expanding this approach to n-dimensional soft sets (where $n > 3$) could offer more flexibility and better manage more intricate linkages between various universes and decision factors. The current study focuses on ternary soft sets based on three initial universal sets. Furthermore, ternary soft sets may be better able to represent uncertainty if their operations include fuzzy or probabilistic components. The handling of more complex data, which is typical in real-world situations, would be made possible by permitting "AND" and "OR" operations to operate with fuzzy or probabilistic decision variables. The creation of algorithms for effective computing and optimization is another crucial avenue. The development of computing techniques for ternary soft set operations, such as union, intersection, complement, difference, and symmetric difference, could help the subject and make it more useful for decision support systems. Furthermore, multi-criteria decision analysis, resource allocation, and complicated decision-making problems could be addressed by optimization methods customized for ternary soft sets. Testing in a variety of domains, including engineering, social sciences, and medical decision-making, is necessary to investigate the practicality of ternary soft sets. In these domains, the many levels of uncertainty that frequently define complicated decision-making can be modeled using ternary soft sets. They could assist in representing ambiguous information from patient data and treatment options, for instance, in the medical field. They could also be used in engineering to solve resource management issues or systems with several unknown parameters. Another area of study would be hybrid soft set models, which combine ternary soft sets with other mathematical tools such as interval-valued fuzzy sets, fuzzy sets, or rough sets. By combining the best features of several strategies, these hybrid models may provide more reliable and flexible answers for making decisions in the face of uncertainty. Furthermore, combining ternary soft sets with machine learning methods may create new prospects in fields where complexity and uncertainty are crucial, such as data mining and pattern identification.

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