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Characterizations of Generalized Paracompactness in Ideal Topological Spaces

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Abstract. The concept of δ-$\beta_{\mathcal{I}}$-paracompactness in ideal topological spaces is introduced as a weaker form of β-paracompactness, which was looked at in [17]. This study examines several characterizations of δ-$\beta_{\mathcal{I}}$-paracompact spaces and its subsets. Furthermore, we investigate the invariants of δ-$\beta_{\mathcal{I}}$-paracompactness through functions.

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1. Introduction

Established considerably later than the two earlier classes, paracompact spaces are considered one of the most important classes of topological spaces, concurrently generalizing both metrizable and compact spaces. Topologists and analysts quickly acknowledged paracompact areas. The concept of a paracompact space in mathematics refers to a topological space in which each open cover possesses an open refinement that is locally finite. This concept of spaces was first developed by Dieudonné [4] in 1944. A Hausdorff space is considered paracompact if and only if it allows partitions of unity that are subordinate to any open cover. All paracompact Hausdorff spaces are normal; see [6]. In literature, different kinds of generalized paracompactness, such as S-paracompactness [1], P_3-paracompactness [3], and β-paracompactness [2], are studied. In 2006, Al-Zoubi [1] used semi-open sets to define S-paracompact spaces, which are a generalization of paracompact spaces, and studied the relationship between the spaces. Li and Song [14]

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constructed a Hausdorff S-paracompact space that is not a paracompact space and studied more characterizations of S-paracompact spaces.

An ideal topological space was proposed by Kuratowski in 1930 [13]. Moreover, Jankovic and Hamlett [11] have examined and described the significant properties of ideal topological spaces. They established the concept of I-open sets and undertook comprehensive investigations into topologies utilizing ideals. Abd El-Monsef et al. [7] performed an advanced examination into the notions of I-open sets. I_g-closed sets were first introduced by Dontchev et al. in 1999 [5]. Abd El-Monsef et al. [8] first introduced the concept of the s-local function, which was later examined by Khan and Noiri [12].

The concept of paracompactness with respect to an ideal was initially introduced by Zahid [19] and later investigated by Hamlett et al. [9]. In addition, Sathiyasundari and Renukadevi [16] explored the concept of I-paracompact and examined its characteristics. They extended certain results derived from paracompact spaces to the notion of I-paracompact spaces. The notion of S-paracompactness in ideal topological spaces was studied by Sanabria et al. [15]. Their work involved the introduction and examination of a new kind of space, namely I-S-paracompact spaces, which are defined on an ideal topological space. This class includes spaces that are S-paracompact and I-paracompact.

In 2013, Demir and Ozbakir [3] introduced a diminished variant of expandable and paracompact spaces, termed β-expandable spaces and β-paracompact spaces, respectively. The proof was presented indicating that any β-paracompact space is essentially a β-expandable space. Yildirim et al. [17] introduced the notion of β-paracompactness inside an ideal topological space and performed a comparison study with existing types of paracompactness. In this paper, the δ-$\beta_{\mathcal{I}}$-paracompact spaces are constructed using δ-$\beta_{\mathcal{I}}$-open sets. The spaces under examination are an extension of the β-paracompact spaces as delineated in reference [17].

2. Preliminaries

Throughout this paper, spaces (X,τ) and (Y,σ) (or simply X and Y), always mean topological spaces on which no separation axiom is assumed. For a subset A of a topological space (X,τ), $CI(A)$ and $Int(A)$ will denote the closure and interior of A in (X,τ), respectively. An ideal $\mathcal{I}$ on a topological space (X,τ) is a nonempty collection of subsets of X which satisfies:

(i) $A\in\mathcal{I}$ and $B\subset A$ implies $B\in\mathcal{I}$,

(ii) $A\in\mathcal{I}$ and $B\in\mathcal{I}$ implies $A\cup B\in\mathcal{I}$.

An ideal topological space $(X,\tau,\mathcal{I})$ is a topological space (X,τ) with an ideal $\mathcal{I}$ on X. The set of all subsets of X is denoted as $P(X)$. A set operator $(.)^*: P(X)\to P(X)$, which is a local function [13], is defined with respect to τ and $\mathcal{I}$: for $A\subset X$, $A^*(\mathcal{I},\tau)=\{x\in X: U\cap A\notin\mathcal{I}$ for every $U\in\tau(x)\}$, where $\tau(x)=\{U\in\tau: x\in U\}$. We simply write A^* instead of $A^*(\mathcal{I},\tau)$. X^* is often a proper subset of X and $X=X^*$ if $\tau\cap\mathcal{I}=\{\emptyset\}$. A topology $\tau^*(\mathcal{I})$, or more simply τ^*, finer than τ, exists for any ideal topological space and is generated by $\beta(\mathcal{I},\tau)=\{U-I: U\in\tau$ and $I\in\mathcal{I}\}$. However, in general, $\beta(\mathcal{I},\tau)$ is not always a topology. Additionally, $Cl^*(A)=A\cup A^*$ defines a Kuratowski closure operator for $\tau^*(\mathcal{I})$.

Lemma 1. [11] *Let A and B be subsets of an ideal topological space $(X, \tau, \mathcal{I})$. Then the following statements are true:*

(i) If $A \subset B$, then $A^* \subset B^*$;

(ii) $G \cap A^* \subset (G \cap A)^*$ for all $G \in \tau$;

(iii) $A^* = Cl(A^*) \subset Cl(A)$.

Definition 1. [10] *Let A be a subset of an ideal topological space $(X, \tau, \mathcal{I})$. A point $x \in X$ is called a $\delta_{\mathcal{I}}$-cluster point of A if $Int(Cl^*(U)) \cap A \neq \emptyset$ for each neighborhood U of x. The set of all $\delta_{\mathcal{I}}$-cluster points of A is called the $\delta_{\mathcal{I}}$-closure of A and will be denoted by $\delta Cl_{\mathcal{I}}(A)$. A is said to be $\delta_{\mathcal{I}}$-closed [18] if $\delta Cl_{\mathcal{I}}(A) = A$. The complement of a $\delta_{\mathcal{I}}$-closed set is called a $\delta_{\mathcal{I}}$-open set. $\delta_{\mathcal{I}}$-interior of A, will be denoted by $\delta Int_{\mathcal{I}}(A)$, is the union of all $\delta_{\mathcal{I}}$-open sets contained in A.*

Lemma 2. [10] *Let A and B be subsets of an ideal topological space $(X, \tau, \mathcal{I})$. Then the following statements are true:*

(i) If $A \subset B$ then $\delta Cl_{\mathcal{I}}(A) \subset \delta Cl_{\mathcal{I}}(B)$;

(ii) If A is an open set, then $\delta Cl_{\mathcal{I}}(A) = A$;

(iii) If A is a closed set, then $\delta Int_{\mathcal{I}}(A) = A$.

Definition 2. [10] *A subset A of an ideal topological space $(X, \tau, \mathcal{I})$ is called δ-$\beta_{\mathcal{I}}$-open if $A \subset Cl(Int(\delta Cl_{\mathcal{I}}(A)))$ and it is called δ-$\beta_{\mathcal{I}}$-closed if $Int(Cl(\delta Int_{\mathcal{I}}(A))) \subset A$.*

Definition 3. [10] *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. The union of all δ-$\beta_{\mathcal{I}}$-open sets contained in A is called the δ-$\beta_{\mathcal{I}}$-interior of A denoted by δ-$\beta Int_{\mathcal{I}}(A)$. The intersection of all δ-$\beta_{\mathcal{I}}$-closed sets containing A is called the δ-$\beta_{\mathcal{I}}$-closure of A denoted by δ-$\beta Cl_{\mathcal{I}}(A)$.*

Lemma 3. *Let A be a subset of an ideal topological space $(X, \tau, \mathcal{I})$. Then,*

(i) δ-$\beta Cl_{\mathcal{I}}(A) \subset Cl(A)$.

(ii) If A is open, then A is δ-$\beta_{\mathcal{I}}$-open.

(iii) If A is closed, then A is δ-$\beta_{\mathcal{I}}$-closed and δ-$\beta Cl_{\mathcal{I}}(A) = A$.

(iv) $x \in \delta$-$\beta Cl_{\mathcal{I}}(A)$ if and only if $A \cap V \neq \emptyset$ for every δ-$\beta_{\mathcal{I}}$-open set V containing x.

Proof. (i) Applying the closure and δ-$\beta_{\mathcal{I}}$-closure definitions in X, we obtain (i).

(ii) If A is open, then $A = Int(A)$. Consequently, $A = \delta Cl_{\mathcal{I}}(A)$ as stated in Lemma 2. Therefore, $A = Int(A) = Int(\delta Cl_{\mathcal{I}}(A)) \subset Cl(Int(\delta Cl_{\mathcal{I}}(A)))$. Thus, A is characterized as a δ-$\beta_{\mathcal{I}}$-open set.

(*iii*) In order to demonstrate (*iii*), we assume that A is closed. This implies that $A = Cl(A)$, and by Lemma 2, $\delta Int_{\mathcal{I}}(A) = A$. Therefore, $Cl(\delta Int_{\mathcal{I}}(A)) = Cl(A) = A$, which implies that $Int(Cl(\delta Int_{\mathcal{I}}(A))) = Int(A) \subset A$. Consequently, A is δ-$\beta_{\mathcal{I}}$-closed. It implies that δ-$\beta Cl_{\mathcal{I}}(A) \subset Cl(A) = A$. We deduce that δ-$\beta Cl_{\mathcal{I}}(A) = A$.

(*iv*) Let $x \in \delta$-$\beta Cl_{\mathcal{I}}(A)$. The point x is then included in all δ-$\beta_{\mathcal{I}}$-closed sets that contain A. Assume that $V \cap A = \emptyset$ for a δ-$\beta_{\mathcal{I}}$-open set V that contains x. It implies that $X - V$ is a δ-$\beta_{\mathcal{I}}$-closed set that contains A but $x \notin X - V$. Consequently, we have a contraction. As a result, $A \cap V \neq \emptyset$ for any δ-$\beta_{\mathcal{I}}$-open set V that contains x. According to Theorem 1 in [10], if $A \cap V \neq \emptyset$ for every δ-$\beta_{\mathcal{I}}$-open sets V containing x, then $x \in \delta$-$\beta Cl_{\mathcal{I}}(A)$.

3. δ-$\beta_{\mathcal{I}}$-paracompactness and Characterizations

This part talks about the idea of δ-$\beta_{\mathcal{I}}$-paracompactness, which is a weaker form of $\mathcal{I}$-β-paracompactness that was studied by Yildirim et al. [17]. We will then look at how to describe it. Let $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda_1\}$ and $\mathcal{V} = \{V_\mu : \mu \in \Lambda_2\}$ be two collections of subsets of a topological space X. The collection $\mathcal{U}$ is called a refinement of the collection $\mathcal{V}$ if for every $\alpha \in \Lambda_1$ there exists $\mu \in \Lambda_2$ such that $U_\alpha \subset V_\mu$. A collection $\mathcal{V}$ of subsets of a topological space (X, τ) is said to be β-locally finite [3] if for each $x \in X$, there exists a β-open set U containing x and U intersects at most finitely many members of $\mathcal{V}$. Yildirim et al. [17] introduced the concept of paracompactness in an ideal topological space as follows: an ideal topological space $(X, \tau, \mathcal{I})$ is said to be $\mathcal{I}$-β-paracompact if every open cover $\mathcal{U}$ of X has a β-locally finite β-open refinement $\mathcal{V}$ such that $X - \cup\{V : V \in \mathcal{V}\} \in \mathcal{I}$.

Definition 4. *A collection $\mathcal{V}$ of subsets of an ideal topological space $(X, \tau, \mathcal{I})$ is said to be δ-$\beta_{\mathcal{I}}$-locally finite if for each $x \in X$, there exists a δ-$\beta_{\mathcal{I}}$-open set U containing x and U intersects at most finitely many members of $\mathcal{V}$.*

Lemma 4. *Let $\mathcal{V}$ be a collection of subsets of an ideal topological space $(X, \tau, \mathcal{I})$. If $\mathcal{V}$ is β-locally finite, then it is δ-$\beta_{\mathcal{I}}$-locally finite.*

Proof. Let $\mathcal{V}$ be β-locally finite. We will verify that $\mathcal{V}$ is δ-$\beta_{\mathcal{I}}$-locally finite. Let $x \in X$. Since $\mathcal{V}$ is β-locally finite, there exists a β-open set G_x containing x, which intersects at most finitely many elements of $\mathcal{V}$. Given that G_x is β-open, it follows that it is δ-$\beta_{\mathcal{I}}$-open [10]. Consequently, $\mathcal{V}$ is δ-$\beta_{\mathcal{I}}$-locally finite.

Definition 5. *An ideal topological space $(X, \tau, \mathcal{I})$ is said to be δ-$\beta_{\mathcal{I}}$-paracompact if every open cover $\mathcal{U}$ of X has a δ-$\beta_{\mathcal{I}}$-locally finite δ-$\beta_{\mathcal{I}}$-open refinement $\mathcal{V}$ (not necessarily a cover) such that $X - \cup\{V : V \in \mathcal{V}\} \in \mathcal{I}$. The collection $\mathcal{V}$ of subsets of X such that $X - \cup\{V : V \in \mathcal{V}\} \in \mathcal{I}$ is called an $\mathcal{I}$-cover. A subset A of an ideal topological space $(X, \tau, \mathcal{I})$ is said to be δ-$\beta_{\mathcal{I}}$-paracompact if for any open cover $\mathcal{U}$ of A has a δ-$\beta_{\mathcal{I}}$-locally finite δ-$\beta_{\mathcal{I}}$-open refinement $\mathcal{V}$ such that $A - \cup\{V : V \in \mathcal{V}\} \in \mathcal{I}$.*

The two theorems that follow arise from the fact that every open set is β-open, every β-open is δ-$\beta_{\mathcal{I}}$-open and $\emptyset$ is in any ideal.

Theorem 1. *If a topological space (X, τ) is paracompact, then $(X, \tau, \mathcal{I})$ is δ-$\beta_{\mathcal{I}}$-paracompact.*

Proof. It is evident, as $\emptyset \in \mathcal{I}$.

Theorem 2. *If $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-β-paracompact then it is δ-$\beta_{\mathcal{I}}$-paracompact.*

Proof. Every β-locally finite collection of subsets of X is δ-$\beta_{\mathcal{I}}$-locally finite, as demonstrated by Lemma 4. Furthermore, each β-open set is δ-$\beta_{\mathcal{I}}$-open [10]. We can continue with the proof by following to the definitions of $\mathcal{I}$-β-paracompactness and δ-$\beta_{\mathcal{I}}$-paracompactness.

Consider the set X consisting of all positive integers. Let $\tau = \{\emptyset\} \cup \{X\} \cup \{\{1, 2, ..., n\} : n \in X\}$ be a topology on X. Define $\mathcal{I} = \{H \subset X : 1 \notin H\}$ as an ideal on the set X. An open cover of X is defined as $\mathcal{W} = \{\{1, 2, 3, ..., n\} : n \in X\}$. However, there is no locally finite open refinement $\mathcal{V}$ that covers X. Thus, (X, τ) is not a paracompact space. But, X is a δ-$\beta_{\mathcal{I}}$-paracompact space, as for each open cover $\mathcal{U}$ of X has a δ-$\beta_{\mathcal{I}}$-locally finite δ-$\beta_{\mathcal{I}}$-open refinement $\mathcal{V} = \{\{1\}\}$ such that $X - \{1\} = \{2, 3, ...\} \in \mathcal{I}$.

Theorem 3. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and let G be a δ-$\beta_{\mathcal{I}}$-open subset of X. Then $G \cap \delta$-$\beta Cl_{\mathcal{I}}(A) = \emptyset$ if and only if $G \cap A = \emptyset$, for all $A \subset X$.*

Proof. It follows from (iv) of Lemma 3 and the fact that $A \subset \delta$-$\beta Cl_{\mathcal{I}}(A)$.

Theorem 4. *Let $\mathcal{V} = \{V_\lambda : \lambda \in \Lambda\}$ be a collection of subsets of a topological space (X, τ). The following statements are true.*

(i) If $\mathcal{V}$ is δ-$\beta_{\mathcal{I}}$-locally finite and $H_\lambda \subset V_\lambda$ for all $\lambda \in \Lambda$, then $\mathcal{H} = \{H_\lambda : \lambda \in \Lambda\}$ is δ-$\beta_{\mathcal{I}}$-locally finite.

(ii) $\mathcal{V}$ is δ-$\beta_{\mathcal{I}}$-locally finite if and only if $\{\delta$-$\beta Cl_{\mathcal{I}}(V_\lambda) : \lambda \in \Lambda\}$ is δ-$\beta_{\mathcal{I}}$-locally finite.

Proof. (i) Let $x \in X$. Since $\mathcal{V}$ is δ-$\beta_{\mathcal{I}}$-locally finite, there exists a δ-$\beta_{\mathcal{I}}$-open set U containing x, which intersects at most finitely many elements of $\mathcal{V}$. As $H_\lambda \subset V_\lambda$ for all $\lambda \in \Lambda$, U intersects with at most finitely many of the elements in $\mathcal{H} = \{H_\lambda : \lambda \in \Lambda\}$. Hence, $\mathcal{H} = \{H_\lambda : \lambda \in \Lambda\}$ is δ-$\beta_{\mathcal{I}}$-locally finite.

(ii) Let $\mathcal{V}$ be δ-$\beta_{\mathcal{I}}$-locally finite and let $x \in X$. Consequently, there exists a δ-$\beta_{\mathcal{I}}$-open set G that contains x and satisfies $G \cap V_\lambda = \emptyset$ for every $\lambda \neq \lambda_1, \lambda_2, \ldots, \lambda_n$. According to Theorem 3, it follows that $G \cap \delta$-$\beta Cl_{\mathcal{I}}(V_\lambda) = \emptyset$ for every $\lambda \neq \lambda_1, \lambda_2, \ldots, \lambda_n$. Thus, $\{\delta$-$\beta Cl_{\mathcal{I}}(V_\lambda) : \lambda \in \Lambda\}$ is δ-$\beta_{\mathcal{I}}$-locally finite. If $\{\delta$-$\beta Cl_{\mathcal{I}}(V_\lambda) : \lambda \in \Lambda\}$ is δ-$\beta_{\mathcal{I}}$-locally finite, then $\mathcal{V}$ is δ-$\beta_{\mathcal{I}}$-locally finite, according to (i). Therefore, (ii) has been demonstrated.

Theorem 5. *If $(X, \tau, \mathcal{I})$ is δ-$\beta_{\mathcal{I}}$-paracompact and $\mathcal{J}$ is an ideal on X with $\mathcal{I} \subset \mathcal{J}$, then $(X, \tau, \mathcal{J})$ is δ-$\beta_{\mathcal{J}}$-paracompact.*

Proof. Let $(X, \tau, \mathcal{I})$ be δ-$\beta_{\mathcal{I}}$-paracompact and $\mathcal{I} \subset \mathcal{J}$, and let $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ be an open cover of X. Since $(X, \tau, \mathcal{I})$ is δ-$\beta_{\mathcal{I}}$-paracompact, $\mathcal{U}$ has a δ-$\beta_{\mathcal{I}}$-locally finite refinement $\mathcal{V}$ of δ-$\beta_{\mathcal{I}}$-open sets such that $X - \cup\{V : V \in \mathcal{V}\} \in \mathcal{I}$. As $\mathcal{I} \subset \mathcal{J}$, $X - \cup\{V : V \in \mathcal{V}\} \in \mathcal{J}$. Consequently, $(X, \tau, \mathcal{J})$ is δ-$\beta_{\mathcal{J}}$-paracompact.

Lemma 5. *If an open cover $\mathcal{U} = \{U_\lambda : \lambda \in \Lambda\}$ of an ideal topological space $(X, \tau, \mathcal{I})$ has a δ-$\beta_\mathcal{I}$-locally finite of δ-$\beta_\mathcal{I}$-open refinement $\mathcal{V}$ such that $X - \cup\{V : V \in \mathcal{V}\} \in \mathcal{I}$, then there exists a precise δ-$\beta_\mathcal{I}$-locally finite δ-$\beta_\mathcal{I}$-open refinement $\mathcal{H} = \{H_\lambda : \lambda \in \Lambda\}$ of $\mathcal{U}$ such that $X - \cup\{H_\lambda : \lambda \in \Lambda\} \in \mathcal{I}$.*

Proof. The proof is comparable to that of Lemma 1.3 in [15].

Definition 6. *An ideal topological space $(X, \tau, \mathcal{I})$ is δ-$\beta_\mathcal{I}$-regular if for any closed subset F of X and $x \notin F$, there exist disjoint δ-$\beta_\mathcal{I}$-open sets U and V such that $x \in U$ and $F - V \in \mathcal{I}$.*

Theorem 6. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Assume that the subsequent assertions are true:*

(i) X is δ-$\beta_\mathcal{I}$-paracompact;

(ii) X is Hausdorff;

(iii) The arbitrary union of δ-$\beta_\mathcal{I}$-closed; sets remains δ-$\beta_\mathcal{I}$-closed.

Then $(X, \tau, \mathcal{I})$ is δ-$\beta_\mathcal{I}$-regular.

Proof. Let F be a closed subset of X, and let $x \notin F$. Utilizing (ii), there exist disjoint open sets V_x and O_x that include x and y, respectively, for any point $y \in F$. It suggests that $y \notin Cl(V_x)$. Now we have that $\mathcal{U} = \{O_x : x \in F\} \cup \{X - F\}$ is an open cover of X. By (i), there exists a δ-$\beta_\mathcal{I}$-locally finite δ-$\beta_\mathcal{I}$-open refinement $\mathcal{H} = \{H_x : x \in F\} \cup \{W\}$ such that $H_x \subset O_x$ for each x, $W \subset X - F$, and $X - (\cup\{H_x : x \in F\} \cup \{W\}) \in \mathcal{I}$. Assume that $V = \cup\{H_x : x \in F\}$ and $U = X - \cup\{\delta\text{-}\beta Cl_\mathcal{I}(H_x) : x \in F\}$. Using (iii), U and V, therefore, are disjoint δ-$\beta_\mathcal{I}$-open sets in which $x \in U$ and $F - V \subset X - F \in \mathcal{I}$. Consequently, $(X, \tau, \mathcal{I})$ is δ-$\beta_\mathcal{I}$-regular.

Theorem 7. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Suppose that the following statements hold:*

(i) X is δ-$\beta_\mathcal{I}$-paracompact;

(ii) X is Hausdorff;

(iii) $\delta\text{-}\beta Cl_\mathcal{I}(\cup\{V_\lambda : \lambda \in \Lambda\}) = \cup\{\delta\text{-}\beta Cl_\mathcal{I}(V_\lambda) : \lambda \in \Lambda\}$, for any δ-$\beta_\mathcal{I}$-locally finite collection $\mathcal{V} = \{V_\lambda : \lambda \in \Lambda\}$ of X.

Then $(X, \tau, \mathcal{I})$ is δ-$\beta_\mathcal{I}$-regular.

Proof. Let F be a closed set and $x \notin F$. Using (ii), for any $y \in F$, there exists an open set G_y containing y such that $x \notin Cl(G_y)$. Then, $\mathcal{G} = \{G_y : y \in F\} \cup \{X - F\}$ is an open cover of X. By (i) and Lemma 5, $\mathcal{G}$ has a precise δ-$\beta_\mathcal{I}$-locally finite δ-$\beta_\mathcal{I}$-open refinement $\mathcal{W} = \{W_y : y \in F\} \cup \{G\}$ such that $W_y \subset G_y$ for each $y \in F$, $G \subset X - F$

and $X - (\cup\{W_y : y \in F\} \cup \{G\}) \in \mathcal{I}$. As $F - \cup\{W_y : y \in F\} = F - (\cup\{W_y : y \in F\} \cup \{G\}) \subset X - (\cup\{W_y : y \in F\} \cup \{G\})$, we get that $F - \cup\{W_y : y \in F\} \in \mathcal{I}$. It follows that $V = \cup\{W_y : y \in F\}$ is a δ-$\beta_{\mathcal{I}}$-open set in X and $F - V \in \mathcal{I}$. Given that $x \notin Cl(G_y)$, it follows that $x \notin Cl(W_y)$, and consequently, $x \notin \delta\text{-}\beta Cl_{\mathcal{I}}(W_y)$. By (iii), the fact that $\mathcal{W}$ is δ-$\beta_{\mathcal{I}}$-locally finite suggests that $\delta\text{-}\beta Cl_{\mathcal{I}}(\cup\{W_y : y \in F\}) = \cup\{\delta\text{-}\beta Cl_{\mathcal{I}}(W_y) : y \in F\}$. We now obtain $U \cap V = \emptyset$ such that $x \in U$ for a δ-$\beta_{\mathcal{I}}$-open set $U = X - \delta\text{-}\beta Cl_{\mathcal{I}}(V)$. Therefore, $(X, \tau, \mathcal{I})$ is δ-$\beta_{\mathcal{I}}$-regular.

Theorem 8. *If an ideal topological space $(X, \tau, \mathcal{I})$ is δ-$\beta_{\mathcal{I}}$-paracompact and regular, then every open cover of X has a δ-$\beta_{\mathcal{I}}$-locally finite $\mathcal{I}$-cover refinement of closed sets.*

Proof. Let $\mathcal{U}$ be an open cover of X. By regularity of X, for each $x \in X$ and $U_x \in \mathcal{U}$ containing x, there exists an open set G_x of x such that $Cl(G_x) \subset U_x$. Thus $\mathcal{U}_1 = \{G_x : x \in X\}$ is an open cover of X. Since X is δ-$\beta_{\mathcal{I}}$-paracompact, $\mathcal{U}_1$ has a δ-$\beta_{\mathcal{I}}$-locally finite refinement $\mathcal{V}_1 = \{V_\lambda : \lambda \in \Lambda\}$ of δ-$\beta_{\mathcal{I}}$-open sets such that $X - \cup\{V_\lambda : \lambda \in \Lambda\} \in \mathcal{I}$. As $V_\lambda \subset \delta\text{-}\beta Cl_{\mathcal{I}}(V_\lambda)$ and $\mathcal{I}$ is an ideal, $X - \cup\{\delta\text{-}\beta Cl_{\mathcal{I}}(V_\lambda) : \lambda \in \Lambda\} \in \mathcal{I}$. By Theorem 4, $\mathcal{V} = \{\delta\text{-}\beta Cl_{\mathcal{I}}(V_\lambda) : V_\lambda \in \mathcal{V}_1\}$ is δ-$\beta_{\mathcal{I}}$-locally finite. Because $\mathcal{V}_1$ refines $\mathcal{U}_1$, for every $\lambda \in \Lambda$, there is some $G_x \in \mathcal{U}_1$ such that $V_\lambda \subset G_x$. Then $\delta\text{-}\beta Cl_{\mathcal{I}}(V_\lambda) \subset Cl(V_\lambda) \subset Cl(G_x)$, and hence $\delta\text{-}\beta Cl_{\mathcal{I}}(V_\lambda) \subset U_x$. Therefore $\mathcal{V} = \{\delta\text{-}\beta Cl_{\mathcal{I}}(V_\lambda) : V_\lambda \in \mathcal{V}_1\}$ refines $\mathcal{U}$, and hence $\mathcal{V}$ is a δ-$\beta_{\mathcal{I}}$-locally finite $\mathcal{I}$-cover refinement of closed sets.

Theorem 9. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. The following statements are equivalent:*

(i) For every closed subset F of X and every $x \notin F$, there exist disjoint δ-$\beta_{\mathcal{I}}$-open sets U and V such that $x \in U$ and $F - V \in \mathcal{I}$.

(ii) For every open subset G of X and every $x \in G$, there exists a δ-$\beta_{\mathcal{I}}$-open set U such that $x \in U$ and $\delta\text{-}\beta Cl_{\mathcal{I}}(U) - G \in \mathcal{I}$.

Proof. $(i) \Rightarrow (ii)$ Let G be open and $x \in G$. Then, $X - G$ remains closed, and x does not belong to $X - G$. By (i), there exist disjoint δ-$\beta_{\mathcal{I}}$-open sets U and V such that $x \in U$ and $(X - G) - V \in \mathcal{I}$. As U and V are disjoint, by Theorem 3, we have $\delta\text{-}\beta Cl_{\mathcal{I}}(U) \subset X - V$. That implies $\delta\text{-}\beta Cl_{\mathcal{I}}(U) \cap (X - G) \subset (X - G) - V$. Hence $\delta\text{-}\beta Cl_{\mathcal{I}}(U) \cap (X - G) = \delta\text{-}\beta Cl_{\mathcal{I}}(U) - G \in \mathcal{I}$.

$(ii) \Rightarrow (i)$ Let F be closed and $x \notin F$. It implies that $X - F$ is open and $x \in X - F$. By (ii), there exists a δ-$\beta_{\mathcal{I}}$-open set U such that $x \in U$ and $\delta\text{-}\beta Cl_{\mathcal{I}}(U) - (X - F) \in \mathcal{I}$. Hence, $V = X - \delta\text{-}\beta Cl_{\mathcal{I}}(U)$ is a δ-$\beta_{\mathcal{I}}$-open set, and $U \cap V = \emptyset$. That is $F - V = F - (X - \delta\text{-}\beta Cl_{\mathcal{I}}(U)) = \delta\text{-}\beta Cl_{\mathcal{I}}(U) - (X - F) \in \mathcal{I}$.

4. δ-$\beta_{\mathcal{I}}$-paracompactness of subsets

In the preceding section, we presented the notion of δ-$\beta_{\mathcal{I}}$-paracompactness for subsets of an ideal topological space. Prior to delineating the characterizations, we note that the union of a finite family of δ-$\beta_{\mathcal{I}}$-locally finite collections of sets inside an ideal topological space remains δ-$\beta_{\mathcal{I}}$-locally finite.

Theorem 10. *Let A and B be subsets of an ideal topological space $(X, \tau, \mathcal{I})$. If A and B are δ-$\beta_{\mathcal{I}}$-paracompact in X, then $A \cup B$ is also δ-$\beta_{\mathcal{I}}$-paracompact.*

Proof. Let $\mathcal{U} = \{U_\lambda : \lambda \in \Lambda\}$ be an open cover of $A \cup B$. Then $\mathcal{U}$ is an open cover of A and B. Hence there are δ-$\beta_{\mathcal{I}}$-locally finite δ-$\beta_{\mathcal{I}}$-open families $\mathcal{V} = \{V_\alpha : \alpha \in \Lambda_1\}$ of A and $\mathcal{W} = \{W_\mu : \mu \in \Lambda_2\}$ of B which refine $\mathcal{U}$ such that $A - \cup\{V_\alpha : \alpha \in \Lambda_1\} \in \mathcal{I}$ and $B - \cup\{W_\mu : \mu \in \Lambda_2\} \in \mathcal{I}$. It implies that $A - \cup\{V_\alpha : \alpha \in \Lambda_1\} = I_1$ and $B - \cup\{W_\mu : \mu \in \Lambda_2\} = I_2$, where $I_1, I_2 \in \mathcal{I}$. Therefore, $A \cup B = \cup\{V_\alpha : \alpha \in \Lambda_1\} \cup \{W_\mu : \mu \in \Lambda_2\} \cup (I_1 \cup I_2)$. It follows that $A \cup B - \cup\{V_\alpha \cup W_\mu : \alpha \in \Lambda_1, \mu \in \Lambda_2\} \in \mathcal{I}$. We see that the collection $\mathcal{H} = \{V_\alpha \cup W_\mu : \alpha \in \Lambda_1, \mu \in \Lambda_2\}$ of δ-$\beta_{\mathcal{I}}$-open sets is a δ-$\beta_{\mathcal{I}}$-locally finite and $\mathcal{H}$ refines $\mathcal{U}$. Consequently, $A \cup B$ is δ-$\beta_{\mathcal{I}}$-paracompact.

Theorem 11. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. If A is δ-$\beta_{\mathcal{I}}$-paracompact and B is closed in X, then $A \cap B$ is δ-$\beta_{\mathcal{I}}$-paracompact.*

Proof. Let $\mathcal{U} = \{U_\lambda : \lambda \in \Lambda\}$ be an open cover of $A \cap B$. As $X - B$ is open in X, $\mathcal{U}_1 = \{U_\lambda : \lambda \in \Lambda\} \cup \{X - B\}$ is an open cover of A. By assumption and Lemma 5, $\mathcal{U}_1$ has a δ-$\beta_{\mathcal{I}}$-locally finite precise δ-$\beta_{\mathcal{I}}$-open refinement $\mathcal{V} = \{V_\lambda : \lambda \in \Lambda\} \cup \{V\}$ such that $V_\lambda \subset U_\lambda$ for every $\lambda \in \Lambda$, and $V \subset X - B$ such that $A - \cup(\{V_\lambda : \lambda \in \Lambda\} \cup \{V\}) \in \mathcal{I}$. Since $A \cap B - \cup\{V_\lambda : \lambda \in \Lambda\} = A \cap B - \cup(\{V_\lambda : \lambda \in \Lambda\} \cup \{V\}) \subset A - \cup(\{V_\lambda : \lambda \in \Lambda\} \cup \{V\})$, we have that $A \cap B - \cup\{V_\lambda : \lambda \in \Lambda\} \in \mathcal{I}$. It is obvious that the collection $\mathcal{V}_1 = \{V_\lambda : \lambda \in \Lambda\}$ of δ-$\beta_{\mathcal{I}}$-open sets is a δ-$\beta_{\mathcal{I}}$-locally finite and refines $\mathcal{U}$. $A \cap B$ is therefore δ-$\beta_{\mathcal{I}}$-paracompact.

Corollary 1. *If A is a closed subset of ideal topological space $(X, \tau, \mathcal{I})$ which is δ-$\beta_{\mathcal{I}}$-paracompact, then A is δ-$\beta_{\mathcal{I}}$-paracompact.*

Corollary 2. *If A is δ-$\beta_{\mathcal{I}}$-paracompact in X and B is an open contained A, then $A - B$ is δ-$\beta_{\mathcal{I}}$-paracompact.*

Lemma 6. [9] *Let $\mathcal{I}$ be an ideal on a topological space X. If Y is a subset of X, then $\mathcal{I}_Y = \{I \cap Y : I \in \mathcal{I}\}$ is an ideal on Y.*

Theorem 12. *Let A and B be subsets of an ideal topological space $(X, \tau, \mathcal{I})$ such that $A \subset B$. If A is δ-$\beta_{\mathcal{I}_{\mathcal{B}}}$-paracompact in B and B is δ-$\beta_{\mathcal{I}}$-open in X, then A is δ-$\beta_{\mathcal{I}}$-paracompact in X.*

Proof. Let $\mathcal{U} = \{U_\lambda : \lambda \in \Lambda\}$ be an open cover of A in X. Then $\mathcal{U}_{\mathcal{A}} = \{U_\lambda \cap B : \lambda \in \Lambda\}$ is an open cover of A in B. As A is δ-$\beta_{\mathcal{I}_{\mathcal{B}}}$-paracompact in B, $\mathcal{U}_{\mathcal{A}}$ has a δ-$\beta_{\mathcal{I}_{\mathcal{B}}}$-locally finite precise δ-$\beta_{\mathcal{I}_{\mathcal{B}}}$-open refinement $\mathcal{V}_{\mathcal{A}} = \{V_\lambda \cap B : \lambda \in \Lambda\}$ such that $V_\lambda \subset U_\lambda$ for all $\lambda \in \Lambda$, and $A - \cup\{V_\lambda \cap B : \lambda \in \Lambda\} \in \mathcal{I}_{\mathcal{B}}$. Since V_λ is δ-$\beta_{\mathcal{I}}$-open in X, the collection $\mathcal{V} = \{V_\lambda : \lambda \in \Lambda\}$ of δ-$\beta_{\mathcal{I}}$-open sets in X is δ-$\beta_{\mathcal{I}}$-locally finite that refines $\mathcal{U}$, and $A - \cup\{V_\lambda : \lambda \in \Lambda\} \subset A - \cup\{V_\lambda \cap B : \lambda \in \Lambda\} \in \mathcal{I}_{\mathcal{B}} \subset \mathcal{I}$. Subsequently, A is δ-$\beta_{\mathcal{I}}$-paracompact in X.

5. Preserving paracompactness

This section will illustrate the preservation of δ-$\beta_{\mathcal{J}}$-paracompactness under specific situations. We start by delineating the subsequent definitions.

Definition 7. *Let $(X, \tau, \mathcal{I})$, and $(Y, \tau', \mathcal{J})$ be ideal topological spaces, and $f : X \to Y$ be a function.*

(i) f is called δ-$\beta_{\mathcal{I}}$-open if $f(G)$ is a δ-$\beta_{\mathcal{J}}$-open set in Y for every δ-$\beta_{\mathcal{I}}$-open set G in X.

(ii) f is called δ-$\beta_{\mathcal{I}}$-closed if $f(F)$ is a δ-$\beta_{\mathcal{J}}$-closed set in Y for every δ-$\beta_{\mathcal{I}}$-closed set F in X.

(iii) f is called δ-$\beta_{\mathcal{I}}$-irresolute if $f^{-1}(V)$ is a δ-$\beta_{\mathcal{I}}$-open set in X for every δ-$\beta_{\mathcal{J}}$-open set V in Y.

Definition 8. *An ideal topological space $(X, \tau, \mathcal{I})$ is said to be δ-$\beta_{\mathcal{I}}$-compact if every cover $\mathcal{V}$ of δ-$\beta_{\mathcal{I}}$-open subsets of X has $V_1, V_2, ..., V_n \in \mathcal{V}$ such that $X \subset V_1 \cup V_2 \cup \cdots \cup V_n$*

Note that $f^{-1}(\mathcal{J})$ is an ideal on X if $f : X \to Y$ is a function, (X, τ) is a topological space, and (Y, τ') is a topological space with an ideal $\mathcal{J}$. Furthermore, given that f is surjective and X possesses an ideal $\mathcal{I}$, $f(\mathcal{I})$ is an ideal on Y. Before establishing Theorem 13, we will first present the following lemma.

Lemma 7. *Let $(X, \tau, \mathcal{I})$ and $(Y, \tau', \mathcal{J})$ be ideal topological spaces, and $f : X \to Y$ be surjective. Then f is δ-$\beta_{\mathcal{I}}$-closed if and only if for every $y \in Y$ and a δ-$\beta_{\mathcal{I}}$-open set U in X containing $\{f^{-1}(y)\}$, there exists a δ-$\beta_{\mathcal{J}}$-open set V containing y such that $f^{-1}(V) \subset U$.*

Proof. Let $y \in Y$ and U be a δ-$\beta_{\mathcal{I}}$-open set in X such that $\{f^{-1}(y)\} \subset U$. We have that $V = Y - f(X - U)$ is a δ-$\beta_{\mathcal{J}}$-closed set such that $y \in V$ and $f^{-1}(V) \subset U$. Subsequently, the necessity is verified. We next demonstrate sufficiency. Let F be a δ-$\beta_{\mathcal{I}}$-closed subset in X and $y \in Y - f(F)$. Thus, $\{f^{-1}(y)\} \subset X - F$. By hypothesis, there is a δ-$\beta_{\mathcal{J}}$-open set V_y such that $f^{-1}(V_y) \subset X - F$, and hence, $y \in V_y \subset Y - f(F)$. Therefore $Y - f(F) = \cup\{V_y : y \in Y\}$ is a δ-$\beta_{\mathcal{J}}$-open set in Y. Consequently, $f(F)$ is a δ-$\beta_{\mathcal{J}}$-closed set.

Theorem 13. *Let $(X, \tau, \mathcal{I})$ and $(Y, \tau', \mathcal{J})$ be ideal topological spaces, and $f : X \to Y$ be continuous, δ-$\beta_{\mathcal{I}}$-open, δ-$\beta_{\mathcal{I}}$-closed, surjective with $\{f^{-1}(y)\}$ being δ-$\beta_{\mathcal{I}}$-compact for every $y \in Y$ and $f(\mathcal{I}) \subset \mathcal{J}$. If X is δ-$\beta_{\mathcal{I}}$-paracompact, then Y is δ-$\beta_{\mathcal{J}}$-paracompact.*

Proof. Let $\mathcal{U} = \{U_\lambda : \lambda \in \Lambda\}$ be an open cover of Y. As f is continuous, $\mathcal{H} = \{f^{-1}(U_\lambda) : \lambda \in \Lambda\}$ is an open cover of X. Given that X is δ-$\beta_{\mathcal{I}}$-paracompact, $\mathcal{H}$ has a precise δ-$\beta_{\mathcal{I}}$-locally finite refinement $\mathcal{V} = \{V_\lambda : \lambda \in \Lambda\}$ of δ-$\beta_{\mathcal{I}}$-open subsets such that $X - \cup\{V_\lambda : \lambda \in \Lambda\} \in \mathcal{I}$. Since f is δ-$\beta_{\mathcal{I}}$-open, $f(\mathcal{V}) = \{f(V_\lambda) : \lambda \in \Lambda\}$ is a precise δ-$\beta_{\mathcal{J}}$-open refinement of $\mathcal{U}$, and $Y - \cup\{f(V_\lambda) : \lambda \in \Lambda\} \in \mathcal{J}$. Next, we shall verify that $f(\mathcal{V})$ is δ-$\beta_{\mathcal{J}}$-locally finite. Let $y \in Y$. As $\mathcal{V}$ is δ-$\beta_{\mathcal{I}}$-locally finite, for $x \in \{f^{-1}(y)\}$, there exists a δ-$\beta_{\mathcal{I}}$-open set G_x containing x such that G_x intersects at most finitely many members of $\mathcal{V}$. Because $\{f^{-1}(y)\}$ is δ-$\beta_{\mathcal{I}}$-compact and $\{G_x : f(x) = y\}$ is a δ-$\beta_{\mathcal{I}}$-open cover of $\{f^{-1}(y)\}$, there exists a finite subcollection H_y, such that $\{f^{-1}(y)\} \subset \cup H_y$, and $\cup H_y$ intersects at most finitely many members of $\mathcal{V}$. As f is δ-$\beta_{\mathcal{I}}$-closed, using Lemma 7, there exists a δ-$\beta_{\mathcal{J}}$-open set W_y containing y such that $f^{-1}(W_y) \subset \cup H_y$. Hence, $f^{-1}(W_y)$ intersects at most finitely many members of $\mathcal{V}$. This implies that W_y intersects at most finitely many members of $f(\mathcal{V})$. Hence, $f(\mathcal{V})$ is a δ-$\beta_{\mathcal{J}}$-locally finite in Y. Therefore, $(Y, \tau', \mathcal{J})$ is δ-$\beta_{\mathcal{J}}$-paracompact.

Theorem 14. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and (Y, τ') be a topological space. Let $f : X \to Y$ be δ-$\beta_{\mathcal{I}}$-irresolute, continuous, δ-$\beta_{\mathcal{I}}$-open, surjective, and $f(\mathcal{V})$ be a δ-$\beta_{f(\mathcal{I})}$-locally finite in Y for every δ-$\beta_{\mathcal{I}}$-locally finite $\mathcal{V}$ in X. If $(X, \tau, \mathcal{I})$ is δ-$\beta_{\mathcal{I}}$-paracompact, then $(Y, \tau', f(\mathcal{I}))$ is δ-$\beta_{f(\mathcal{I})}$-paracompact.*

Proof. Let $\mathcal{U} = \{U_\lambda : \lambda \in \Lambda\}$ represent an open cover of Y. Consequently, we have that $\mathcal{H} = \{f^{-1}(U_\lambda) : \lambda \in \Lambda\}$ creates an open cover of X. Since X is δ-$\beta_{\mathcal{I}}$-paracompact, $\mathcal{H}$ has a δ-$\beta_{\mathcal{I}}$-locally finite precise δ-$\beta_{\mathcal{I}}$-open refinement $\mathcal{V} = \{V_\lambda : \lambda \in \Lambda\}$ such that $X - \cup\{V_\lambda : \lambda \in \Lambda\} \in \mathcal{I}$. As $Y - \cup\{f(V_\lambda) : \lambda \in \Lambda\}) \subset f(X - \cup\{V_\lambda : \lambda \in \Lambda\})$ and $f(X - \cup\{V_\lambda : \lambda \in \Lambda\}) \in f(\mathcal{I})$, we have $Y - \cup\{f(V_\lambda) : \lambda \in \Lambda\}) \in f(\mathcal{I})$. Given that f is surjective, $f(\mathcal{I})$ is an ideal of Y. By assumption, we have that $f(\mathcal{V}) = \{f(V_\lambda) : \lambda \in \Lambda\}$ is a precise δ-$\beta_{f(\mathcal{I})}$-locally finite of δ-$\beta_{f(\mathcal{I})}$-open subsets in Y. Next, we will confirm that $f(\mathcal{V})$ refines $\mathcal{U}$. Let $f(V_\lambda) \in f(\mathcal{V})$. Thus, $V_\lambda \subset f^{-1}(U_\lambda)$ for some $U_\lambda \in \mathcal{H}$, as $\mathcal{V}$ refines $\mathcal{H}$. This indicates that $f(V_\lambda) \subset f(f^{-1}(U_\lambda)) \subset U_\lambda$. Subsequently, $(Y, \tau', f(\mathcal{I}))$ is δ-$\beta_{f(\mathcal{I})}$-paracompact.

Theorem 15. *Let (X, τ) be a topological space and $(Y, \tau', \mathcal{J})$ be an ideal topological space. Let $f : X \to Y$ be open, δ-$\beta_{f^{-1}(\mathcal{J})}$-irresolute, and bijective. If $(Y, \tau', \mathcal{J})$ is δ-$\beta_{\mathcal{J}}$-paracompact, then $(X, \tau, f^{-1}(\mathcal{J}))$ is δ-$\beta_{f^{-1}(\mathcal{J})}$-paracompact.*

Proof. Let $\mathcal{U} = \{U_\lambda : \lambda \in \Lambda\}$ be an open cover of X. As f is open, $f(\mathcal{U}) = \{f(U_\lambda) : \lambda \in \Lambda\}$ is an open cover of Y. By hypothesis, $f(\mathcal{U})$ has a δ-$\beta_{\mathcal{J}}$-locally finite precise δ-$\beta_{\mathcal{J}}$-open refinement $\mathcal{H} = \{V_\lambda : \lambda \in \Lambda\}$ such that $Y - \cup\{V_\lambda : \lambda \in \Lambda\} \in \mathcal{J}$. It implies that $Y - \cup\{V_\lambda : \lambda \in \Lambda\} = J$ for some $J \in \mathcal{J}$, which follows that $f^{-1}(Y) - \cup\{f^{-1}(V_\lambda) : \lambda \in \Lambda\} = f^{-1}(J)$. Then, $X - \cup\{f^{-1}(V_\lambda) : \lambda \in \Lambda\} \in f^{-1}(\mathcal{J})$. As f is δ-$\beta_{f^{-1}(\mathcal{J})}$-irresolute, $\mathcal{V} = \{f^{-1}(V_\lambda) : \lambda \in \Lambda\}$ is a δ-$\beta_{f^{-1}(\mathcal{J})}$-locally finite δ-$\beta_{f^{-1}(\mathcal{J})}$-open collection. The refinement of $\mathcal{U}$ by $\mathcal{V}$ will be asserted. Let $f^{-1}(V_\lambda) \in \mathcal{V}$. Hence, $V_\lambda \in \mathcal{H}$, and there exists $U_\lambda \in \mathcal{U}$ such that $V_\lambda \subset f(U_\lambda)$ as $\mathcal{H}$ refines $f(\mathcal{U})$. Therefore $f^{-1}(V_\lambda) \subset f^{-1}(f(U_\lambda)) = U_\lambda \in \mathcal{U}$. Accordingly, X is δ-$\beta_{f^{-1}(\mathcal{J})}$-paracompact.

Theorem 16. *Let $(X, \tau, \mathcal{I})$ and $(Y, \tau', \mathcal{J})$ be ideal topological spaces, and let $f : (X, \tau, \mathcal{I}) \to (Y, \tau', \mathcal{J})$ be open, δ-$\beta_{\mathcal{I}}$-irresolute, bijective, and $f(\mathcal{I}) = \mathcal{J}$. If $A \subset Y$ is δ-$\beta_{\mathcal{J}}$-paracompact in Y, then $f^{-1}(A) \subset X$ is δ-$\beta_{\mathcal{I}}$-paracompact.*

Proof. Let $\mathcal{U} = \{U_\lambda : \lambda \in \Lambda\}$ be an open cover of $f^{-1}(A)$. Given that f is an open mapping, $f(\mathcal{U}) = \{f(U_\lambda) : \lambda \in \Lambda\}$ forms an open cover of A. By hypothesis, $f(\mathcal{U})$

possesses a δ-$\beta_{\mathcal{J}}$-locally finite precise δ-$\beta_{\mathcal{J}}$-open refinement $\mathcal{H} = \{V_\lambda : \lambda \in \Lambda\}$ such that $A - \cup\{V_\lambda : \lambda \in \Lambda\} \in \mathcal{J}$. Then, $f^{-1}(A) - \cup\{f^{-1}(V_\lambda) : \lambda \in \Lambda\} \in f^{-1}(\mathcal{J}) = \mathcal{I}$. As f is δ-$\beta_{\mathcal{I}}$-irresolute, $\mathcal{V} = \{f^{-1}(V_\lambda) : \lambda \in \Lambda\}$ is a δ-$\beta_{\mathcal{I}}$-locally finite δ-$\beta_{\mathcal{I}}$-open collection. Let $f^{-1}(V_\lambda) \in \mathcal{V}$. There exists $f(U_\lambda) \in f(\mathcal{U})$ such that $V_\lambda \subset f(U_\lambda)$, since $\mathcal{H}$ refines $f(\mathcal{U})$. It follows that $f^{-1}(V_\lambda) \subset f^{-1}(f(U_\lambda)) = U_\lambda$, thereby indicating that $\mathcal{H}$ refines $\mathcal{U}$. Therefore $f^{-1}(A)$ is δ-$\beta_{\mathcal{I}}$-paracompact in X.

6. Conclusion

This paper looks at different ways to describe the δ-$\beta_{\mathcal{I}}$-paracompactness of an ideal topological space as a weaker form of β-paracompactness compared to an ideal $\mathcal{I}$ (or $\mathcal{I}$-β-paracompactness). We found that every $\mathcal{I}$-β-paracompact space is a δ-$\beta_{\mathcal{I}}$-paracompact space, and every Hausdorff δ-$\beta_{\mathcal{I}}$-paracompact space under some conditions is δ-$\beta_{\mathcal{I}}$-regular. The union of two δ-$\beta_{\mathcal{I}}$-paracompact subsets is a δ-$\beta_{\mathcal{I}}$-paracompact subset, and the intersection of a δ-$\beta_{\mathcal{I}}$-paracompact subset and a δ-$\beta_{\mathcal{I}}$-closed set is δ-$\beta_{\mathcal{I}}$-paracompact. In addition, we illustrate that δ-$\beta_{\mathcal{I}}$-paracompactness is preserved under certain conditions. If $f : X \to Y$ is δ-$\beta_{\mathcal{I}}$-irresolute, continuous, δ-$\beta_{\mathcal{I}}$-open, and surjective, and X is δ-$\beta_{\mathcal{I}}$-paracompact, then Y is δ-$\beta_{f(\mathcal{I})}$-paracompact. Additionally, provided that $f : X \to Y$ is open, δ-$\beta_{\mathcal{I}}$-irresolute, bijective, and Y is δ-$\beta_{f(\mathcal{I})}$-paracompact, then X is δ-$\beta_{\mathcal{I}}$-paracompact.

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References

[1] K. Al-Zoubi. S-paracompact spaces. *Acta Mathematica Hungarica*, 110:203–212, 2006.

[2] K. Al-Zoubi and S. Al-Ghour. On P_3-paracompact spaces. *International Journal of Mathematics and Mathematical Sciences*, 2007:1–16, 2007.

[3] I. Demir and O.B. Ozbakir. On β-paracompact spaces. *Filomat*, 27(6):971–976, 2013.

[4] J. Dieudonné. Une généralisation des espaces compacts. *Journal de Mathématiques Pures et Appliquées*, 23(9):65–76, 1944.

[5] J. Dontchev, M. Ganster, and T. Noiri. Unified operation approach of generalized closed sets via topological ideals. *Mathematica Japonica*, 49:395–402, 1999.

[6] J. Dugundji. *Topology.* Allyn and Bacon, Boston, 1966.

[7] M. E. Abd El-Monsef, E. F. Lashien, and A. A. Nasef. On I-open sets and I-continuous functions. *Kyungpook Mathematical Journal*, 32(2):21–30, 1992.

[8] M. E. Abd El-Monsef, E. F. Lashien, and A. A. Nasef. Some topological operators via ideals. *Kyungpook Mathematical Journal*, 32(2):273284, 1992.

[9] T. R. Hamlett, D. Rose, and D. Jankovic. Paracompactness with respect to an ideal. *International Journal of Mathematics and Mathematical Sciences*, 20:433–442, 1997.

[10] E. Hatir. On decompositions of continuity and complete continuity in ideal topological spaces. *European Journal of Pure and Applied Mathematics*, 6(3):352–362, 2013.

[11] D. Jankovic and T. R. Hamlett. New topologies from old via ideals. *The American Mathematical Monthly*, 97(4):295–310, 1990.

[12] M. Khan and T. Noiri. Semi-local functions in ideal topological spaces. *Journal of Advanced Research in Pure Mathematics*, 2(1):36–42, 2010.

[13] K. Kuratowski. *Topology I.* Panstwowe Wydawnictwo Naukowe, Warszawa, 1933.

[14] P. Y. Li and Y. K. Song. Some remarks on S-paracompact spaces. *Acta Mathematica Hungarica*, 118:345–355, 2008.

[15] J. Sanabria, E. Rosas, C. Carpintero, M. Salas-Brown, and O. García. S-paracompactness in ideal topological spaces. *Matematički Vesnik*, 68(3):192–203, 2016.

[16] N. Sathiyasundari and V. Renukadevi. Paracompactness with respect to an ideal. *Filomat*, 20(2):333–339, 2013.

[17] E. D. Yildirim, O. B. Ozbakir, and A. C. Guler. β-paracompactness in ideal topological space. *European Journal of Pure and Applied Mathematics*, 12(2):270–278, 2019.

[18] S. Yuksel, A. Acikgoz, and T. Noiri. On δ-I-continuous functions. *Acta Mathematica Hungarica*, 29:39–51, 2005.

[19] M. I. Zahid. *Para H-closed Spaces, Locally Para H-closed Spaces and Their Minimal Topologies.* PhD thesis, University of Pittsburgh, 1981.