**EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS**
2025, Vol. 18, Issue 1, Article Number 5572
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

# Some Algebraic Structures of Complex Fermatean Fuzzy Subgroups

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**Abstract.** A complex Fermatean fuzzy set provides a detailed framework for representing a specific type of information and has been effectively applied to decision-making problems. This study introduces complex Fermatean fuzzy subgroups (CFFSGs), an extension of Fermatean fuzzy subgroups, and complex Pythagorean fuzzy subgroups. The key innovation of CFFSGs lies in their capacity to represent two variables within their algebraic structure, surpassing the capabilities of traditional Fermatean fuzzy subgroups. The research establishes the formal definition and properties of CFFSGs, adapting them to a complex framework that incorporates amplitude and phase components. Additionally, the concepts of complex Fermatean fuzzy cosets and complex Fermatean fuzzy normal subgroups are introduced. The study also investigates and examines the characteristics of homomorphisms between complex Fermatean fuzzy subgroups.

**2020 Mathematics Subject Classifications**: 08A35, 08A72, 20N25

**Key Words and Phrases**: Complex fermatean fuzzy subgroup, complex fermatean fuzzy normal subgroup, homomorphism of fermatean fuzzy subgroup

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## 1. Introduction

Zadeh [40] introduced the fuzzy set (FS) as a novel concept to address uncertainty and vagueness in decision-making (DM) problems. This foundational work sparked the publication of hundreds of studies that extended and applied FS concepts across various fields. A significant advancement was made by Atanassov in 1986 [12], who introduced the intuitionistic fuzzy set (IFS). Later, Yager [38] expanded this framework in 2013 by defining the Pythagorean fuzzy set (PFS), positioning FS as a subset of IFS, which in turn is a subset of PFS. In 2017, Yager further generalized these concepts by introducing q-rung orthopair fuzzy sets [39].

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DOI: https://doi.org/10.29020/nybg.ejpam.v18i1.5572

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This generalization was followed by specific cases, such as the Fermatean fuzzy set (FFS) defined by Senapati et al. in 2020 [33]. These sets differ in their constraints; for IFS, the sum of membership and non-membership degrees must lie between zero and one; for PFS, the sum of their squares must meet this condition; for FFS, it is the sum of their cubes; and for q-rung orthopair fuzzy sets, the sum of the $K^{th}$ powers of membership and non-membership degrees must satisfy this constraint.

Researchers have explored the properties and operations of these sets, developed new measures, and applied them to DM problems, as demonstrated in studies like [18], [23], and [27]. More recently, in 2022, M. Akram et al. [4] proposed a DM approach integrating the attributes of the traditional VIKOR method within the framework of a multidimensional complex Fermatean fuzzy N-soft set.

Real-world problems often involve multiple variables, necessitating more advanced conceptual frameworks. This need led to the intellectual introduction of the complex fuzzy set (CFS) by Ramot et al. (2002) [30], which incorporates two key variables: amplitude and phase. Building on this concept, Alkouri and Salleh (2012) [8] introduced the complex intuitionistic fuzzy set (CIFS), enhancing the properties of CFS. Subsequent works have extended and refined these ideas, including contributions from Alkouri (2013) [9] and Ramot (2003) [29].

In 2019, Ullah et al. [36] developed the complex Pythagorean fuzzy set (CPFS) concept, introducing various associated measures. This was followed in 2020 by Liu et al. [25], who proposed complex q-Rung orthopair fuzzy sets, a generalization encompassing CIFS, CPFS, CFFS, and orthopair fuzzy sets. Further advancements were made in 2021 when Chinnadurai et al. [17] defined the complex Fermatean fuzzy set (CFFS) and explored its applications in decision-making (DM) problems.

The application of complex fuzzy theories in DM has since expanded, as demonstrated by recent works such as Chen (2023) [16] and Wang (2023) [37]. Other notable generalizations include the use of complex neutrosophic graphs for hospital infrastructure design by Alqahtani et al. (2024) [10] and the application of complex hesitant fuzzy graphs by AbuHijleh et al. (2023) [2] and Alkouri (2023) [6]. Additionally, Alqaraleh et al. (2022) [11] introduced bipolar complex fuzzy soft sets with practical applications.

For further exploration of generalizations and applications within this domain, refer to recent works by Al-Masarwah et al. (2023) [5], Fallat et al. (2022) [19], and Hazaymeh et al. (2024–2025) [21, 22].

In parallel with the advancements in fuzzy set theory, significant progress has also been made in fuzzy group theory. Rosenfeld (1971) [32] introduced the concept of a fuzzy subgroup (FSG) as a generalization of the classical group. This foundational work inspired numerous mathematicians to explore group theory through fuzzy sets. In 1989, Biswas [15] expanded on this idea by defining the intuitionistic fuzzy subgroup (IFSG) and analyzing its algebraic properties.

In 2020, Bhunia et al. [14] introduced the Pythagorean fuzzy subgroup (PFSG), further examining its algebraic structure. The same year, manuscripts on the complex intuitionistic fuzzy subgroup (CIFSG) [20] and the complex fuzzy subgroup (CFSG) [3] were published, adding new dimensions to the field. Most recently, a study on the complex

Pythagorean fuzzy subgroup (CPFSG) was published for publication [7].

Building on these developments, Silambarasan (2021) [35] introduced the Fermatean fuzzy subgroup (FFSG), analyzing its properties and its relationships with IFSG and PFSG. This work has inspired further research into FFSG, including studies by Balamurugan (2022) [13], Kalaichelvan (2022) [24], Nagarajan (2021) [26], Onasanya (2022) [28], and Muhammad (2022) [31].

Onasanya et al. (2022) [28] introduced Fermatean fuzzy subgroups within the q-rung orthopair fuzzy sets framework in group theory, coining the term "harmonized fuzzy groups". These groups unify various subgroup types as special cases and provide a versatile foundation for further exploration. While their study addressed several properties of harmonized fuzzy groups, the extension to complex harmonized fuzzy groups remains an open area of research.

The motivation for constructing complex Fermatean fuzzy subgroups (CFFSG) lies in advancing the mathematical framework of fuzzy group theory. Specifically, it seeks to incorporate periodic information inherent in complex Fermatean fuzzy sets (CFFS) and leverage their capacity to represent larger values compared to complex Pythagorean fuzzy sets (CPFS) and complex intuitionistic fuzzy sets (CIFS). This development not only enriches the theoretical landscape but also opens pathways for practical applications, such as cryptographic primitives and generalized periodic algorithms.

Future work includes the construction of CFFSG and the development of cyclic CFFSG as a specialized extension. Additionally, integrating results from fixed-point theory (e.g., [1] and [34]) with complex Fermatean fuzzy algebra presents a promising avenue for novel applications. Such integration could address real-world problems by employing metric space frameworks to create innovative solutions.

This paper explores the concept of the complex Fermatean fuzzy subgroup (CFFSG) as an enhancement of both the complex Pythagorean fuzzy subgroup (CPFSG) and the Fermatean fuzzy subgroup (FFSG). Section 2 provides an overview of key definitions from relevant literature, establishing the foundational concepts. Section 3 introduces the formal definition of CFFSG and examines its fundamental properties. Section 4 extends the discussion to complex Fermatean fuzzy normal subgroups, detailing their characteristics. Section 5 analyzes homomorphisms within the context of CFFSG, highlighting their properties and implications. Finally, Section 6 summarizes the findings of this study and proposes potential directions for future research.

## 2. Preliminaries

Zadeh defined fuzzy set in 1965 [40].

**Definition 1.** *[40] A fuzzy set (FS)* $\mathbb{K}$ *of the universe of discourse* $\mathbb{X}$ *is defined by membership function;* $\mathbb{K}: \mathbb{X} \to [0,1]$*, whereas* $\mathbb{K}(x)$ *is a degree of membership for any* $x$ *in* $\mathbb{X}$*.*

Ramot et al. defined complex fuzzy set (CFS) on a crisp set in 2002 [30].

**Definition 2.** *[30] A complex fuzzy set (CFS)* $\mathbb{K}$ *of the universe of discourse* $\mathbb{X}$ *is defined by membership function;* $\mathbb{K}(x): \mathbb{X} \to \{z: z \in \mathbb{C}, |z| \leq 1\}$*, that assigns a degree of membership* $\mathbb{K}(x) = p(x)e^{2\pi i\omega(x)}$ *for any* $x$ *in* $\mathbb{X}$*, where the value of* $\mathbb{K}(x)$ *is defined by two variables* $p(x)$ *and* $\omega(x)$ *and both are located within zero and one.*

On the other hand, Atanassov (1986) [12] introduced the concept of an intuitionistic fuzzy set (IFS) by incorporating a non-membership degree, where the sum of the membership degree and the non-membership degree lies between zero and one. This framework was later expanded in various ways, one of which is the Pythagorean fuzzy set (PFS) [38], introduced by Yager in 2013, where the sum of the squares of the membership and non-membership degrees is constrained between zero and one. Subsequently, Senapti and Yager (2020) [33] defined the Fermatean fuzzy set (FFS), as described below.

**Definition 3.** *[33] Let* $\mathbb{X}$ *be a universe of discourse, then a fermatean fuzzy set* $\mathbb{F}$ *on* $\mathbb{X}$ *defined by* $\mathbb{F} = \{(x, \mathbb{K}(x), \mathbb{L}(x)): x \in \mathbb{X}\}$*. Such that* $\mathbb{K}(x) \in [0,1]$ *and* $\mathbb{L}(x) \in [0,1]$ *are the degree of membership and the degree of non-membership for any* $x \in \mathbb{X}$*, respectively, and* $0 \leq \mathbb{K}^3(x) + \mathbb{L}^3(x) \leq 1$*, for all* $x \in \mathbb{X}$*.*

Additionally, Yager (2017) [39] introduced a broader class of fuzzy sets known as q-rung orthopair fuzzy sets, in which the sum of the $q^{th}$ powers of the membership and non-membership degrees is constrained between zero and one. He demonstrated that as the value of $q$ increases, the space of acceptable orthopair sets expands, thereby providing users with greater flexibility in expressing their beliefs regarding the degree of membership.

On the other hand, the concept of the complex intuitionistic fuzzy set (CIFS) was introduced in 2012 [8], followed by the complex Pythagorean fuzzy set (CPFS) in 2019 [36], and more recently, the complex Fermatean fuzzy set (CFFS) was presented in 2021 [17]. These advancements involve extending traditional fuzzy sets to complex fuzzy sets for both membership and non-membership degrees. The definition of the complex Fermatean fuzzy set (CFFS) is provided below.

**Definition 4.** *[17] Let* $\mathbb{X}$ *be a universe of discourse and defined a complex fermatean fuzzy set* $\phi$ *on* $\mathbb{X}$*, where* $\phi = \{(x, \mathbb{K}(x), \mathbb{L}(x)): x \in \mathbb{X}\}$*. Such that* $\mathbb{K}(x): \mathbb{X} \to \{z: z \in \mathbb{C}, |z| \leq 1\}$ *and* $\mathbb{L}(x): \mathbb{X} \to \{z: z \in \mathbb{C}, |z| \leq 1\}$*, are the degree of membership and non-membership of* $x \in \mathbb{X}$*, respectively. Moreover,* $\mathbb{K}(x) = p(x)e^{2\pi i\omega(x)}$ *and* $\mathbb{L}(x) = q(x)e^{2\pi i\nu(x)}$ *are satisfying the conditions;* $0 \leq p^3(x) + q^3(x) \leq 1$ *and* $0 \leq \omega^3(x) + \nu^3(x) \leq 1$*.*

In the previous definition, if conditions become $0 \leq p^k(x) + q^k(x) \leq 1$ and $0 \leq \omega^k(x) + \nu^k(x) \leq 1$, for all $x \in \mathbb{X}$ with $k \geq 1$. Then $\phi$ define a complex q-rung orthopair fuzzy sets (2020) [25].

Another approach to fuzzy sets is the concept of fuzzy subgroups, first introduced by Rosenfeld in 1971 [32]. This concept was subsequently enhanced with the introduction of the intuitionistic fuzzy subgroup (IFSG) in 1989 [15]. In 2021, E.A. Abuhijleh et al. [3] defined the complex fuzzy subgroup (CFSG), while the complex intuitionistic fuzzy subgroup (CIFSG) was introduced in 2020 [20]. The Pythagorean fuzzy subgroup (PFSG) was presented in 2020 [14], followed by the complex Pythagorean fuzzy subgroup (CPFSG) in 2023 [7]. Additionally, the Fermatean fuzzy subgroup (FFSG) was defined in 2021 [35], with Onasanya et al. (2022) [28] also contributing to the development of the Fermatean fuzzy subgroup.

Here, we produce definitions of FSG, CFSG, and FFSG, respectively.

**Definition 5.** *[32] Let* $\mathbb{K} : \mathbb{X} \to [0,1]$ *defined a fuzzy subset of a group* $(\mathbb{X}, *)$. *Then* $\mathbb{K}$ *presented a fuzzy subgroup (FSG) of* $(\mathbb{X}, *)$, *if the following conditions hold:*
*i)* $\mathbb{K}(x * y) \geq \mathbb{K}(x) \wedge \mathbb{K}(y)$.
*ii)* $\mathbb{K}(x^{-1}) \geq \mathbb{K}(x)$, *for all* $x, y \in \mathbb{X}$

**Definition 6.** *[3] Let* $\mathbb{K}(x) : \mathbb{X} \to \{z : z \in \mathbb{C}, |z| \leq 1\}$ *be a complex fuzzy subset of a group* $(\mathbb{X}, *)$. *Then* $\mathbb{K}$ *presented a complex fuzzy subgroup, of* $(\mathbb{X}, *)$, *if the following conditions hold:*
*i)* $\mathbb{K}(x * y) \geq \mathbb{K}(x) \wedge \mathbb{K}(y)$.
*ii)* $\mathbb{K}(x^{-1}) \geq \mathbb{K}(x)$, *for all* $x, y \in \mathbb{X}$

*Equivalently, for any* $x, y \in \mathbb{X}$ *and* $\mathbb{K}(x) = p(x)e^{2\pi i\omega(x)}$, *we have:*
*i)* $p(x * y) \geq p(x) \wedge p(y)$ *and* $\omega(x * y) \geq \omega(x) \wedge \omega(y)$.
*ii)* $p(x^{-1}) \geq p(x)$ *and* $\omega(x^{-1}) \geq \omega(x)$.

**Definition 7.** *[35] Let* $(\mathbb{X}, *)$ *be a group and* $\mathbb{F} = (\mathbb{K}, \mathbb{L})$ *be a fermatean fuzzy set of* $\mathbb{X}$. *Then* $\mathbb{F}$ *is a fermatean fuzzy subgroup of* $\mathbb{X}$ *if the following conditions hold:*

*(i)* $\mathbb{K}^3(x * y) \geq \mathbb{K}^3(x) \wedge \mathbb{K}^3(y)$ *and* $\mathbb{L}^3(x * y) \leq \mathbb{L}^3(x) \vee \mathbb{L}^3(y)$.

*(ii)* $\mathbb{K}^3(x^{-1}) \geq \mathbb{K}^3(x)$ *and* $\mathbb{L}^3(x^{-1}) \leq \mathbb{L}^3(x)$

, $\forall\ x,\ y \in \mathbb{X}$

In the previous definition, if the power $k = 1,\ 2$ we get intuitionistic fuzzy subgroup (IFSG) [15] and Pythagorean fuzzy subgroup (PFSG) [14], respectively.

The union, intersection, and complement of CFFS defined in 2021 [17], as follows.

**Definition 8.** *Let* $\phi_1 = (\mathbb{K}_1, \mathbb{L}_1)$ *and* $\phi_2 = (\mathbb{K}_2, \mathbb{L}_2)$ *be two CFFSs on* $\mathbb{X}$, *where:*
$\mathbb{K}_j(x) : \mathbb{X} \to \{p_j(x)e^{2\pi i\omega_j(x)} : 0 \leq p_j(x),\ \omega_j(x) \leq 1\}$, *and* $\mathbb{L}_j(x) : \mathbb{X} \to \{q_j(x)e^{2\pi i\nu_j(x)} : 0 \leq q_j(x),\ \nu_j(x) \leq 1\}$, *for* $j = 1,\ 2$, *then:*

*1.* $\phi_1 \cap \phi_2 = (\mathbb{K}_1 \cap \mathbb{K}_2, \mathbb{L}_1 \cap \mathbb{L}_2)$, *where* $(\mathbb{K}_1 \cap \mathbb{K}_2)(x) = (p_1(x) \wedge p_2(x))e^{2\pi i(\omega_1(x) \wedge \omega_2(x))}$ *and* $(\mathbb{L}_1 \cap \mathbb{L}_2)(x) = (q_1(x) \vee q_2(x))e^{2\pi i(\nu_1(x) \vee \nu_2(x))}$.

*2.* $\phi_1 \cup \phi_2 = (\mathbb{K}_1 \cup \mathbb{K}_2, \mathbb{L}_1 \cup \mathbb{L}_2)$, *where* $(\mathbb{K}_1 \cup \mathbb{K}_2)(x) = (p_1(x) \vee p_2(x))e^{2\pi i(\omega_1(x) \vee \omega_2(x))}$ *and* $(\mathbb{L}_1 \cup \mathbb{L}_2)(x) = (q_1(x) \wedge q_2(x))e^{2\pi i(\nu_1(x) \wedge \nu_2(x))}$.

*3.* $\phi^c = \bar{\phi} = (\mathbb{L}, \mathbb{K}) = (q(x)e^{2\pi i\nu(x)},\ p(x)e^{2\pi i\omega(x)})$.

## 3. Complex Fermatean Fuzzy Subgroups

A generalization of Fermatean fuzzy subgroups [35] and complex Fermatean fuzzy sets [17] is presented in the following definition, which also serves as a generalization of complex Pythagorean fuzzy subgroups [7].

**Definition 9.** *Let* $(\mathbb{X}, *)$ *be a group and* $\phi = (p\ e^{2\pi i\omega},\ q\ e^{2\pi i\nu})$ *be a CFFS of* $\mathbb{X}$*. Then* $\phi$ *is complex fermatean fuzzy subgroup (CFFSG) of* $\mathbb{X}$*, where* $p^3 + q^3 \leq 1$ *and* $\omega^3 + \nu^3 \leq 1$*, if the following holds:*

*1a.* $p^3(x\ *\ y)e^{2\pi i\omega^3(x\ *\ y)} \geq p^3(x)e^{2\pi i\omega^3(x)} \wedge p^3(y)e^{2\pi i\omega^3(y)}$.
*where,* $p^3(x\ *\ y) \geq p^3(x) \wedge p^3(y)$ *and* $\omega^3(x\ *\ y) \geq \omega^3(x) \wedge \omega^3(y)$.
*1b.* $q^3(x\ *\ y)e^{2\pi i\nu^3(x\ *\ y)} \leq q^3(x)e^{2\pi i\nu^3(x)}\ \vee\ q^3(y)e^{2\pi i\nu^3(y)}$
*where,* $q^3(x\ *\ y) \leq q^3(x) \vee q^3(y)$ *and* $\nu^3(x\ *\ y) \leq \nu^3(x) \vee \nu^3(y)$

*2a.* $p^3(x^{-1})e^{2\pi i\omega^3(x^{-1})} \geq p^3(x)e^{2\pi i\omega^3(x)}$
*where,* $p^3(x^{-1}) \geq p^3(x)$ *and* $\omega^3(x^{-1}) \geq \omega^3(x)$.
*2b.* $q^3(x^{-1})e^{2\pi i\nu^3(x^{-1})} \leq q^3(x)e^{2\pi i\nu^3(x)}$
*where,* $q^3(x^{-1}) \leq q^3(x)$ *and* $\nu^3(x^{-1}) \leq \nu^3(x)$

In the previous definition, if the power $k = 1,\ 2$ we get a complex intuitionistic fuzzy subgroups (CIFSGs) [20], and a complex Pythagorean fuzzy subgroups (CPFSGs) [7], respectively.

**Proposition 1.** *Let* $\phi = (p\ e^{2\pi i\omega},\ q\ e^{2\pi i\nu})$ *be a CFFSG of a group* $(\mathbb{X}, *)$*, then the following holds:*

*(i)* $p^3(id)e^{2\pi i\omega^3(id)} \geq p^3(x)e^{2\pi i\omega^3(x)}$, *where* $p^3(id) \geq p^3(x)$ *and* $\omega^3(id) \geq \omega^3(x)$.

*(ii)* $q^3(id)e^{2\pi i\nu^3(id)} \leq q^3(x)e^{2\pi i\nu^3(x)}$, *where* $q^3(id) \leq q^3(x)$ *and* $\nu^3(id) \leq \nu^3(x)$.

*(iii)* $p^3(x^{-1})e^{2\pi i\omega^3(x^{-1})} = p^3(x)e^{2\pi i\omega^3(x)}$, *where* $p^3(x^{-1}) = p^3(x)$ *and* $\omega^3(x^{-1}) = \omega^3(x)$.

*(iv)* $q^3(x^{-1})e^{2\pi i\nu^3(x^{-1})} = q^3(x)e^{2\pi i\nu^3(x)}$, *where* $q^3(x^{-1}) = q^3(x)$ *and* $\nu^3(x^{-1}) = \nu^3(x)$.

*for all* $x \in \mathbb{X}$*, where id is the identity of all elements.*

*Proof.* Since $\phi$ is CFFSG then by Definition 9:
"1" and "2" can be proved as follow, $p^3(id)e^{2\pi i\omega^3(id)} = p^3(x\ o\ x^{-1})e^{2\pi i\omega^3(x\ o\ x^{-1})} \geq \min\{p^3(x)e^{2\pi i\omega^3(x)}\ ,\ p^3(x^{-1})e^{2\pi i\omega^3(x^{-1})}\} = \min\{p^3(x), p^3(x^{-1})\}e^{2\pi i \min\{\omega^3(x), \omega^3(x^{-1})\}} = p^3(x)e^{2\pi i\omega^3(x)}$.

In addition, $q^3(id)e^{2\pi i\nu^3(id)} = q^3(x\,o\,x^{-1})e^{2\pi i\nu^3(x\ o\ x^{-1})} \leq \max\{q^3(x)e^{2\pi i\omega_q^3(x)},\ q^3(x^{-1})e^{2\pi i\omega_q^3(x^{-1})}\}$ $= \max\{q^3(x), q^3(x^{-1})\}e^{2\pi i \max\{\nu^3(x),\nu^3(x^{-1})\}} = q^3(x)e^{2\pi i\nu^3(x)}$.
"3" and "4" can be prove in the same manner of "1" and "2".

The following is an example of complex fermatean fuzzy subgroup (CFFSG).

**Example 1.** *For the set* $\mathbf{D_2}$ *which define a dihedral group of order four and isomorphic to the direct sum of two cyclic group* $\mathbb{Z}_2$*. Hence,* $\mathbf{D_2} = \{(0,0),(1,0),(0,1),(1,1)\}$*. Also, let* $\phi = (\mathbb{K},\mathbb{L})$ *be a CFFS on* $\mathbf{D_2}$*, such that:*
$\phi((0,0)) = (0.8e^{2\pi i(0.85)},\ 0.5e^{2\pi i(0.6)})$,
$\phi((1,0)) = (0.78e^{2\pi i(0.83)},\ 0.6e^{2\pi i(0.65)})$,
$\phi((0,1)) = (0.75e^{2\pi i(0.8)},\ 0.65e^{2\pi i(0.68)}) \ = \ \phi(xy)$.
*At first, by Definition 4, it is easy to check that* $\phi$ *is CFFS, but it is not CPFS. For example in* $\phi((0,0))$*, we have* $\omega^3+\nu^3 = (0.614+0.216) = 0.83 \leq 1$ *and* $\omega^2+\nu^2 = 0.723+0.36 \nleq 1$.

*In the second part, it suffices to prove that the set* $\phi(\ell) = (p(\ell)e^{2\pi i\omega(\ell)},\ q(\ell)e^{2\pi i\nu(\ell)})$ *is CFFSG on* $\mathbf{D_2}$*, for any* $\ell\ \in \mathbf{D_2}$*. So that check conditions (1a) and (1b) in the Definition 9, as follows:*
*Consider* $\ell_1 = (1,0)$*,* $\ell_2 = (0,1)$*, then* $(1,0) \oplus (0,1) = (1,1)$ *1a.* $p^3((1,0) \oplus (0,1)) = p^3((1,1)) = 0.422 \geq \min\{p^3((1,0)), p^3((0,1))\} = 0.422$ *and* $\omega^3((1,0)\oplus(0,1)) = \omega^3((1,1)) = 0.512 \geq \min\{\omega^3((1,0)), \omega^3((0,1))\} = 0.512$.
*1b.* $q^3((1,0) \oplus (0,1)) = q^3((1,1)) = 0.275 \leq \max\{q^3((1,0)), q^3((0,1))\} = 0.275$ *and* $\nu^3((1,0) \oplus (0,1)) = \nu^3((1,1)) = 0.314 \leq \max\{\nu^3((1,0)), \nu^3((0,1))\} = 0.314$.
*Then the property satisfied at* $\ell_1 = (1,0)$*,* $\ell_2 = (0,1)$*, and one can go through all* $\ell_i$ *and check conditions.*

*In addition, for conditions (2a) and (2b) in the Definition 9, they satisfied too, since* $\ell = \ell^{-1}$ *for any* $\ell\ \in \mathbf{D_2}$*. Note that, it is easy to see that* $\phi$ *is satisfied the previous proposition.*

The following theorem prove that any CPFSG is a CFFSG.

**Theorem 1.** *If* $\phi$ *is a CPFSG of the group* $(\mathbb{X},*)$*, then* $\phi$ *is a CFFSG of the group* $(\mathbb{X},*)$.

*Proof.* At first, to show that $p^3(x\ *\ y)e^{2\pi i\omega^3(x\ *\ y)} \geq p^3(x)e^{2\pi i\omega^3(x)} \wedge p^3(y)e^{2\pi i\omega^3(y)}$ and $q^3(x\ *\ y)e^{2\pi i\nu^3(x\ *\ y)} \leq q^3(x)e^{2\pi i\nu^3(x)} \vee q^3(y)e^{2\pi i\nu^3(y)}$. We know that $\phi$ is a CPFSG, then $p^2(x\ *\ y)e^{2\pi i\omega^2(x\ *\ y)} \geq p^2(x)e^{2\pi i\omega^2(x)} \wedge p^2(y)e^{2\pi i\omega^2(y)}$ and $q^2(x\ *\ y)e^{2\pi i\nu^2(x\ *\ y)} \leq q^2(x)e^{2\pi i\nu^2(x)} \vee q^2(y)e^{2\pi i\nu^2(y)}$, where $p^2+q^2 \leq 1$ and $\omega^2+\nu^2 \leq 1$. Then, we have four cases to consider:

a) Let $p^2(x)e^{2\pi i\omega^2(x)} \geq p^2(y)e^{2\pi i\omega^2(y)}$ and $q^2(x)e^{2\pi i\nu^2(x)} \geq q^2(y)e^{2\pi i\nu^2(y)}$, then $p^2(x\ *\ y)e^{2\pi i\omega^2(x\ *\ y)} \geq p^2(y)e^{2\pi i\omega^2(y)}$. Now consider $p^3(x\ *\ y)e^{2\pi i\omega^3(x\ *\ y)} \geq p^3(y)e^{2\pi i\omega^3(y)}$ $= p^3(x)e^{2\pi i\omega^3(x)} \wedge p^3(y)e^{2\pi i\omega^3(y)}$. Moreover, $q^2(x\ *\ y)e^{2\pi i\nu^2(x\ *\ y)} \leq q^2(x)e^{2\pi i\nu^2(x)}$. Now consider $q^3(x\ *\ y)e^{2\pi i\nu^3(x\ *\ y)} \leq q^3(x)e^{2\pi i\nu^3(x)} = q^3(x)e^{2\pi i\nu^3(x)}\ \vee q^3(y)e^{2\pi i\omega_q{}^3(y)}$.

b) Let $p^2(x)e^{2\pi i\omega^2(x)} \leq p^2(y)e^{2\pi i\omega^2(y)}$ and $q^2(x)e^{2\pi i\nu^2(x)} \leq q^2(y)e^{2\pi i\nu^2(y)}$, then with same argument of case a, we get the result.

c) Let $p^2(x)e^{2\pi i\omega^2(x)} \leq p^2(y)e^{2\pi i\omega^2(y)}$ and $q^2(x)e^{2\pi i\nu^2(x)} \geq q^2(y)e^{2\pi i\nu^2(y)}$, then $p^2(x * y)e^{2\pi i\omega^2(x * y)} \geq p^2(x)e^{2\pi i\omega^2(x)}$. Now consider $p^3(x * y)e^{2\pi i\omega^3(x * y)} \geq p^3(x)e^{2\pi i\omega^3(x)} = p^3(x)e^{2\pi i\omega^3(x)} \wedge p^3(y)e^{2\pi i\omega^3(y)}$. Moreover, $q^2(x * y)e^{2\pi i\nu^2(x * y)} \leq q^2(x)e^{2\pi i\nu^2(x)}$. Now consider $q^3(x * y)e^{2\pi i\nu^3(x * y)} \leq q^3(x)e^{2\pi i\nu^3(x)} = q^3(x)e^{2\pi i\nu^3(x)} \vee q^3(y)e^{2\pi i\nu^3(y)}$.

d) Let $p^2(x)e^{2\pi i\omega^2(x)} \geq p^2(y)e^{2\pi i\omega^2(y)}$ and $q^2(x)e^{2\pi i\nu^2(x)} \leq q^2(y)e^{2\pi i\nu^2(y)}$, then with same argument of case c, we get the result.

Secondly, since $p^2(x^{-1})e^{2\pi i\omega^2(x^{-1})} \geq p^2(x)e^{2\pi i\omega^2(x)}$ and $q^2(x^{-1})e^{2\pi i\nu^2(x^{-1})} \leq q^2(x)e^{2\pi i\nu^2(x)}$, then $p^3(x^{-1})e^{2\pi i\omega^3(x^{-1})} \geq p^3(x)e^{2\pi i\omega^3(x)}$ and $q^3(x^{-1})e^{2\pi i\nu^3(x^{-1})} \leq q^3(x)e^{2\pi i\nu^3(x)}$ too.

The converse of Theorem 1 is not always true, please see the following example.

**Example 2.** *For the set $\mathbb{X} = \{1, -1, i, -i\}$, define a group $(\mathbb{X}, .)$, where $'.'$ is the known multiplication. Also define $\phi = (\mathbb{K}, \mathbb{L})$ be a CFFS on $\mathbb{X}$, where:*
$\phi(1) = (0.8e^{2\pi i(0.75)},\ 0.3e^{2\pi i(0.3)})$,
$\phi(-1) = (0.8e^{2\pi i(0.5)},\ 0.65e^{2\pi i(0.45)})$,
$\phi(i) = (0.7e^{2\pi i(0.45)},\ 0.9e^{2\pi i(0.6)} = \phi(-i)$.
*Now, it is easy to check that for all $x \in \mathbb{X}$, that $\phi$ is CFFS, c.f. Definition 4. But, $\phi$ is not CPFS, for example at $x = -1$ we have $p^2 + q^2 = 0.64 + 0.4225 \nleq 1$.*

*Now, it suffices to prove that the set $\phi(x) = (p(x)\ e^{2\pi i\omega(x)},\ q(x)\ e^{2\pi i\nu(x)})$ is CFFSG:*
*i) First, for any $x,\ y\ \in \mathbb{X}$, we check that:*
*a.* $p^3(x * y) \geq \min\{p^3(x),\ p^3(y)\}$ *and* $\omega^3(x * y) \geq \min\{\omega^3(x),\ \omega^3(y)\}$.
*b.* $q^3(x * y) \leq \max\{q^3(x),\ q^3(y)\}$ *and* $\nu^3(x * y) \leq \max\{\nu^3(x),\ \nu^3(y)\}$.
*Hence, consider $x = i,\ y = -i$, then $i. - i =\ 1$:*
*a.* $p^3(i * -i) = p^3(1) = 0.512 \geq \min\{p^3(i), p^3(-i)\} = 0.343$ *and* $\omega^3(i * -i) = \omega^3(1) = 0.42188 \geq \min\{\omega^3(i), \omega^3(-i)\} = 0.09112$.
*b.* $q^3(i * -i) = q^3(1) = 0.027 \leq \max\{q^3(i), q^3(-i)\} = 0.729$ *and* $\nu^3(i * -i) = \nu^3(1) = 0.027 \leq \max\{\nu^3(i), \nu^3(-i)\} = 0.216$.
*, then the property satisfied at $x = i,\ y = -i$; one can go through all $x$ and see that property satisfied.*
*ii) Second, since $1 = 1^{-1},\ -1 = -1^{-1}$ and $i = -i^{-1}$, where $\phi(i) = \phi(-i)$, then property 2 in of CFFSG is satisfied too.*

Note that, since CIFSG is subclass of CPFSG [7], then CIFSG is subclass of CFFSG.

**Proposition 2.** *For a CFFS $\phi = (p\ e^{2\pi i\omega}, q\ e^{2\pi i\nu})$ of a group $(\mathbb{X}, *)$, it is a CFFSG if and only if:*
*1.* $p^3(x * y^{-1})e^{2\pi i\omega^3(x * y^{-1})} \geq\ p^3(x)e^{2\pi i\omega^3(x)} \wedge p^3(y)e^{2\pi i\omega^3(y)}$
*, where* $p^3(x * y^{-1}) \geq p^3(x) \wedge p^3(y)$ *and* $\omega^3(x * y^{-1}) \geq \omega^3(x) \wedge \omega^3(y)$

*2.* $q^3(x \ * \ y^{-1})e^{2\pi i\nu^3(x \ * \ y^{-1})} \leq q^3(x)e^{2\pi i\nu^3(x)} \vee q^3(y)e^{2\pi i\nu^3(y)}$
*, where* $q^3(x \ * \ y^{-1}) \leq q^3(x) \vee q^3(y)$ *and* $\nu^3(x \ * \ y^{-1}) \leq \nu^3(x) \vee \nu^3(y)$

*Proof.* ($\Longrightarrow$) According to Proposition 1, we have $p^3(x^{-1})e^{2\pi i\omega^3(x^{-1})} = p^3(x)e^{2\pi i\omega^3(x)}$ and $q^3(x^{-1})e^{2\pi i\nu^3(x^{-1})} = q^3(x)e^{2\pi i\nu^3(x)}$ for all $x \in \mathbb{X}$, then results follow by Definition 9.

($\Longleftarrow$) First, $\phi$ is CFFS and is defined on group $(\mathbb{X}, *)$, then:
(i) $p^3(id)e^{2\pi i\omega^3(id)} = p^3(x * x^{-1})e^{2\pi i\omega^3(x \ * \ x^{-1})} \geq p^3(x)e^{2\pi i\omega^3(x)}$, where $p^3(x * x^{-1}) \geq p^3(x)$ and $\omega^3(x \ * \ x^{-1}) \geq \omega^3(x)$.
(ii) $p^3(x^{-1})e^{2\pi i\omega^3(x^{-1})} = p^3(id \ * \ x^{-1})e^{2\pi i\omega^3(id \ * \ x^{-1})} \geq \min\{p^3(id)e^{2\pi i\omega^3(di)}, p^3(x)e^{2\pi i\omega^3(x)}\}$ $= \min\{p^3(id), p^3(x)\}e^{2\pi i \min\{\omega^3(di),\omega^3(x)\}} = p^3(x)e^{2\pi i\omega^3(x)}$, by (i).
(ii) $p^3(x \ * \ y)e^{2\pi i\omega^3(x \ * \ y)} = p^3(x \ * \ (y^{-1})^{-1})e^{2\pi i\omega^3(x \ * \ (y^{-1})^{-1})}$
$\geq p^3(x)e^{2\pi i\omega^3(x)} \wedge p^3(y^{-1})e^{2\pi i\omega^3(y^{-1})} \geq p^3(x)e^{2\pi i\omega^3(x)} \wedge p^3(y)e^{2\pi i\omega^3(y)}$, by (ii).

Similarly, we have:
(iv) $q^3(id)e^{2\pi i\nu^3(id)} = q^3(x * x^{-1})e^{2\pi i\nu^3(x \ * \ x^{-1})} \leq q^3(x)e^{2\pi i\nu^3(x)}$, where $q^3(x * x^{-1}) \leq q^3(x)$ and $\nu^3(x \ * \ x^{-1}) \leq \nu^3(x)$.
(v) $q^3(x^{-1})e^{2\pi i\nu^3(x^{-1})} = q^3(id \ * \ x^{-1})e^{2\pi i\nu^3(id \ * \ x^{-1})} \leq \max\{q^3(id)e^{2\pi i\nu^3(di)}, q^3(x)e^{2\pi i\nu^3(x)}\}$ $= \max\{q^3(id), q^3(x)\}e^{2\pi i \max\{\nu^3(di),\nu^3(x)\}} = q^3(x)e^{2\pi i\nu^3(x)}$, by (iv).
(vi) $q^3(x \ * \ y)e^{2\pi i\nu^3(x \ * \ y)} = q^3(x \ * \ (y^{-1})^{-1})e^{2\pi i\nu^3(x \ * \ (y^{-1})^{-1})}$
$\leq q^3(x)e^{2\pi i\nu^3(x)} \ \vee \ q^3(y^{-1})e^{2\pi i\nu^3(y^{-1})} \leq q^3(x)e^{2\pi i\nu^3(x)} \ \vee \ q^3(y)e^{2\pi i\nu^3(y)}$, by (v).

Finally, by (iii) and (vi) the first condition was satisfied, and by (ii) and (v) the second condition was satisfied in the Definition 9, hence $\phi$ is CFFSG of a group $(\mathbb{X}, *)$.

**Proposition 3.** *The intersection of two CFFSGs of a group* $(\mathbb{X}, *)$ *is a CFFSG.*

*Proof.* Let $S_1, \ S_2$ be two CFFSGs of $\mathbb{X}$ and using previous proposition, then:
i) $p^3_{S_1 \cap S_2}(x \ * \ y^{-1})e^{2\pi i\omega^3_{S_1 \cap S_2}(x \ * \ y^{-1})}$
$= (p^3_{S_1}(x \ * \ y^{-1}) \wedge \ p^3_{S_2}(x \ * \ y^{-1}))e^{2\pi i(\omega^3_{S_1}(x \ * \ y^{-1}) \wedge \ \omega^3_{S_2}(x \ * \ y^{-1}))}$
$\geq (\min\{p^3_{S_1}(x), p^3_{S_1}(y)\} \wedge \ \min\{p^3_{S_2}(x), p^3_{S_2}(y)\})e^{2\pi i(\min\{\omega^3_{S_1}(x),\omega^3_{S_1}(y)\} \wedge \ \min\{\omega^3_{S_2}(x),\omega^3_{S_2}(y)\})}$
$= (\min\{p^3_{S_1}(x), p^3_{S_2}(x)\} \wedge \ \min\{p^3_{S_1}(y), p^3_{S_2}(y)\})e^{2\pi i(\min\{\omega^3_{S_1}(x),\omega^3_{S_2}(x)\} \wedge \ \min\{\omega^3_{S_1}(y),\omega^3_{S_2}(y)\})}$
$= (p^3_{S_1 \cap S_2}(x) \wedge \ p^3_{S_1 \cap S_2}(y))e^{2\pi i(\omega^3_{S_1 \cap S_2}(x) \wedge \ \omega^3_{S_1 \cap S_2}(y))}$
$= p^3_{S_1 \cap S_2}(x)e^{2\pi i\omega^3_{S_1 \cap S_2}(x)} \wedge \ p^3_{S_1 \cap S_2}(y)e^{2\pi i\omega^3_{S_1 \cap S_2}(y)}$ .

ii) $q^3_{S_1 \cap S_2}(x \ * \ y^{-1})e^{2\pi i\nu^3_{S_1 \cap S_2}(x \ * \ y^{-1})}$
$= (q^3_{S_1}(x \ * \ y^{-1}) \vee \ q^3_{S_2}(x \ * \ y^{-1}))e^{2\pi i(\nu^3_{S_1}(x \ * \ y^{-1}) \vee \ \nu^3_{S_2}(x \ * \ y^{-1}))}$
$\leq (\max\{q^3_{S_1}(x), q^3_{S_1}(y)\} \vee \ \max\{q^3_{S_2}(x), q^3_{S_2}(y)\})e^{2\pi i(\max\{\nu^3_{S_1}(x),\nu^3_{S_1}(y)\} \vee \ \max\{\nu^3_{S_2}(x),\nu^3_{S_2}(y)\})}$
$= (\max\{q^3_{S_1}(x), q^3_{S_2}(x)\} \vee \ \max\{q^3_{S_1}(y), q^3_{S_2}(y)\})e^{2\pi i(\max\{\nu^3_{S_1}(x),\nu^3_{S_2}(x)\} \vee \ \max\{\nu^3_{S_1}(y),\nu^3_{S_2}(y)\})}$
$= (q^3_{S_1 \cap S_2}(x) \vee q^3_{S_1 \cap S_2}(y))e^{2\pi i(\nu^3_{S_1 \cap S_2}(x) \vee \nu^3_{S_1 \cap S_2}(y))}$

$= q^2_{A\cap B}(x)e^{2\pi i\nu^2_{A\cap B}(x)} \vee \ q^2_{A\cap B}(y)e^{2\pi i\nu^2_{A\cap B}(y)}$.

The union of two CFFSG is not necessary a CFFSG, see the following example.

**Example 3.** *Let* $(\mathbb{X},*) = (\mathbb{Z},+)$ *be a group, also* $\phi_1 = 5\mathbb{Z}$ *and* $\phi_2 = 2\mathbb{Z}$ *be two CFFSG of* $\mathbb{Z}$*. Where,* $\phi_j = (\mathbb{K}_{\phi_j} = p_{\phi_j}(x)e^{2\pi i\omega_{\phi_j}(x)},\ \mathbb{L}_{\phi_j} = q_{\phi_j}(x)e^{2\pi i\nu_{\phi_j}(x)})$*;* $j = 1,\ 2$.

*They defined by:*

$$\phi_1(x) = \ (\mathbb{K}_{\phi_1}, \mathbb{L}_{\phi_1}) = \begin{cases} (0.8e^{2\pi i\ 0.6} & ,0.7e^{2\pi i\ 0.3}): \quad x \ \in \ 5\mathbb{Z} \\ (0.0e^{2\pi i\ 0.0} & ,0.5e^{2\pi i\ 0.4}): \quad elsewhere \end{cases}$$

$$\phi_2(x) = \ (\mathbb{K}_{\phi_2}, \mathbb{L}_{\phi_2}) = \begin{cases} (0.8e^{2\pi i\ 0.7} & ,0.4e^{2\pi i\ 0.2}): \quad x \ \in \ 7\mathbb{Z} \\ (0.1e^{2\pi i\ 0.5} & ,0.6e^{2\pi i\ 0.9}): \quad elsewhere \end{cases}$$

*Then, we get:*

$$\phi = \phi_1 \cup \phi_2 = \begin{cases} (0.8e^{2\pi i\ 0.7} & ,0.4e^{2\pi i\ 0.2}): \quad x \ \in \ 7\mathbb{Z} \\ (0.8e^{2\pi i\ 0.6} & ,0.7e^{2\pi i\ 0.3}): \quad x \ \in \ 5\mathbb{Z} - 7\mathbb{Z} \\ (0.0e^{2\pi i\ 0.0} & ,0.5e^{2\pi i\ 0.4}): \quad elsewhere \end{cases}$$

*Now, it is easy to check that* $\phi$ *is CFFS. Then, consider definition 9 to check if* $\phi$ *is CFFSG:*
*For* $x_1 = 15$ *and* $x_2 = -7$*, then:*
$< \mathbb{K}^3_\phi(15\ +\ -7),\ \mathbb{L}^3_\phi(15\ +\ -7) > = < \mathbb{K}^3_\phi(8),\ \mathbb{L}^3_\phi(8) >$
$= < 0.0e^{2\pi i\ 0.0},\ 0.125e^{2\pi i\ 0.064} >$
*and,*
$< \mathbb{K}^3_\phi(15) \wedge \mathbb{K}^3_\phi(-7),\ \mathbb{L}^3_\phi(15) \vee \mathbb{L}^3_\phi(-7) >$
$= < 0.512e^{2\pi i\ 0.216} \wedge 0.512e^{2\pi i\ 0.343},\ 0.343e^{2\pi i\ 0.027} \vee 0.064e^{2\pi i\ 0.008} >$
$= < 0.512e^{2\pi i\ 0.216},\ 0.343e^{2\pi i\ 0.027} >$
*But,* $\mathbb{K}^3_\phi(15\ +\ -7) \ngeq \mathbb{K}^3_\phi(15) \wedge \mathbb{K}^3_\phi(-7)$*;* $0.0e^{2\pi i\ 0.0} \ngeq 0.512e^{2\pi i\ 0.216}$
*and*
$\mathbb{L}^3_\phi(15\ +\ -7) \nleq \mathbb{L}^3_\phi(15) \vee \mathbb{L}^3_\phi(-7)$*;* $0.125e^{2\pi i\ 0.064} \nleq 0.343e^{2\pi i\ 0.027}$.
*Therefore,* $\phi = \varphi_1 \cup \varphi_2$ *is not a CFFSG of* $(\mathbb{Z},+)$.

**Proposition 4.** *For a CFFS* $\phi = (pe^{2\pi i\omega}, qe^{2\pi i\nu})$ *of a group* $(\mathbb{X},*)$*. Then* $p^3(x*x*\cdots*x)e^{2\pi i\omega^3(x*x*\cdots*x)} \geq p^3(x)e^{2\pi i\omega^3(x)}$*, where* $p^3(x*x*\cdots*x) \geq p^3(x)$ *and* $\omega^3(x*x*\cdots*x) \geq \omega^3(x)$*. Also,* $q^3(x*x*\cdots*x)e^{2\pi i\nu^3(x*x*\cdots*x)} \leq q^3(x)e^{2\pi i\nu^3(x)}$*, where* $q^3(x*x*\cdots*x) \leq q^3(x)$ *and* $\nu^3(x*x*\cdots*x) \leq \nu^3(x)$.

*Proof.* By induction the results will follow, such that $p^3(x*x)e^{2\pi i\omega^3(x*x)} \geq p^3(x)e^{2\pi i\omega^3(x)}$, where $p^3(x*x) \geq p^3(x)$ and $\omega^3(x*x) \geq \omega^3(x)$. Also, $q^3(x*x)e^{2\pi i\nu^3(x*x)} \leq q^3(x)e^{2\pi i\nu^3(x)}$, where $q^3(x*x) \leq q^3(x)$ and $\nu^3(x*x) \leq \nu^3(x)$.

**Theorem 2.** *For a CFFS* $\phi = (pe^{2\pi i\omega}, qe^{2\pi i\nu})$ *of a group* $(\mathbb{X},*)$*. The set* $\mathbb{M} = \{x \in \mathbb{X} : p^3(id)e^{2\pi i\omega^3(id)} = p^3(x)e^{2\pi i\omega^3(x)}$ *and* $q^3(id)e^{2\pi i\nu^3(id)} = q^3(x)e^{2\pi i\nu^3(x)}\}$*, is a subgroup of* $\mathbb{X}$*, where id is the identity of it.*

*Proof.* At first, we have $id \in \mathbb{M}$, hence $\mathbb{M}$ is not empty. Moreover, we need to show that $x * y^{-1} \in \mathbb{M}$ for all $x,\ y \in \mathbb{X}$.
Assume that $x,\ y \in \mathbb{M}$, where $\phi$ is CFFSG of $\mathbb{X}$, then, by Proposition 2, $p^3(x*y^{-1})e^{2\pi i\omega^3(x*y^{-1})} \geq p^3(x)e^{2\pi i\omega^3(x)} \wedge p^3(y)e^{2\pi i\omega^3(y)} = p^3(id)e^{2\pi i\omega^3(id)}$, according to definition of $\mathbb{M}$. But, $x * y^{-1} \in \phi$, hence $p^3(id)e^{2\pi i\omega^3(id)} \geq p^3(x*y^{-1})e^{2\pi i\omega^3(x*y^{-1})})$, by Proposition 1. So that equality holds and $p^3(id)e^{2\pi i\omega^3(id)} = p^3(x*y^{-1})e^{2\pi i\omega^3(x*y^{-1})})$. Similarly, we can prove that $q^3(id)e^{2\pi i\nu^3(id)} = q^3(x*y^{-1})e^{2\pi i\nu^3(x*y^{-1})}$, by Proposition 1 and Proposition 2. So that $x * y^{-1} \in \mathbb{M}$ and $\mathbb{M}$ is subgroup of $\mathbb{X}$.

## 4. Complex Fermatean fuzzy normal subgroup

In this section, we define complex fermatean fuzzy normal subgroup (CFFNSG) and give equivalent conditions and some properties for it.

**Definition 10.** *Let $\phi = (pe^{2\pi i\omega}, qe^{2\pi i\nu})$ be a CFFSG of a group $(\mathbb{X}, *)$. Then for $z \in \mathbb{X}$, the complex fermatean fuzzy left coset of $\phi$ is the CFFS $z\phi = ((zp)e^{2\pi i(z\omega)}, (zq)e^{2\pi i(z\nu)})$, which defined for membership by, $(zp^3(x))e^{2\pi i(z\omega^3(x))} = p^3(z^{-1} * x)e^{2\pi i\omega^3(z^{-1} *x)}$. Also, for nonmembership it is defined by, $(zq^3(x))e^{2\pi i(z\nu^3(x))} = q^3(z^{-1} * x)e^{2\pi i\nu^3(z^{-1} *x)}$.*
*In the same manner, the complex fermatean fuzzy right coset of $\phi$ is the CFFS $\phi z = ((pz)e^{2\pi i(\omega z)}, (qz)e^{2\pi i(\nu z)})$ and is defined by $(p^3(x)z)e^{2\pi i(\omega^3(x)z)} = p^3(x * z^{-1})e^{2\pi i\omega^3(x * z^{-1})}$ and $(q^3(x)z)e^{2\pi i(\nu^3(x)z)} = q^3(x * z^{-1})e^{2\pi i\nu^3(x * z^{-1})}$, for membership and nonmembership, respectively.*

**Definition 11.** *Let $\phi = (p(x)e^{2\pi i\omega(x)}, q(x)e^{2\pi i\nu(x)})$ be a CFFSG of a group $(\mathbb{X}, *)$. Then $\phi$ is a complex fermatean fuzzy normal subgroup, of the group $(\mathbb{X}, *)$ if every complex fermatean fuzzy left coset is complex fermatean fuzzy right coset of $\phi$ in $\mathbb{X}$, equivalently, $z\phi = \phi z$.*

**Example 4.** *Let $(\mathbb{X}, *) = (\mathbb{Z}_3, +_3)$ be a group with addition integer modulo 3. Define a CFFS $\phi$, as follows:*

$$\phi = \ (\mathbb{K}(x),\ \mathbb{L}(x)) = \begin{cases} (0.9e^{2\pi i\ 0.7}, & 0.8e^{2\pi i\ 0.8}): \quad x = 0 \\ (0.8e^{2\pi i\ 0.8}, & 0.7e^{2\pi i\ 0.6}): \quad x = 1 \\ (0.3e^{2\pi i\ 0.6}, & 0.5e^{2\pi i\ 0.6}): \quad x = 2 \end{cases}$$

*First, it is easy to see at $x = 0$, $\phi$ is CFFS and not CPFS (so, not CIFS too). Second, to prove that $\phi$ is CFFNSG:*
*Assume that $z = 2$ and $x = 0$ and take $2\phi$, then*

$$\begin{aligned} < (2\mathbb{K})^3(0), (2\mathbb{L})^3(0) > &= < \mathbb{K}^3(2^{-1} +_3 0),\ \mathbb{L}^3(2^{-1} +_3 0) > \\ &= < p^3(2^{-1} +_3 0)e^{2\pi i\omega^3(2^{-1} +_3 0)}, q^3(2^{-1} +_3 0)e^{2\pi i\nu^3(2^{-1} +_3 0)} > \\ &= < p^3(1 +_3 0)e^{2\pi i\omega^3(1 +_3 0)}, q^3(1 +_3 0)e^{2\pi i\nu^3(1 +_3 0)} > \\ &= < p^3(1)e^{2\pi i\omega^3(1)}, q^3(1)e^{2\pi i\nu^3(1)} > \end{aligned}$$

$$=< p^3(0{+_3}1)e^{2\pi i\omega^3(0+_31)}, q^3(0{+_3}1)e^{2\pi i\nu^3(0+_31)} >$$
$$=< p^3(0{+_3}2^{-1})e^{2\pi i\omega^3(0+_32^{-1})}, q^3(0{+_3}2^{-1})e^{2\pi i\nu^3(0+_32^{-1})} >$$
$$=< \mathbb{K}^3(0{+_3}2^{-1}), \mathbb{L}^3(0{+_3}2^{-1}) >$$
$$=< (\mathbb{K}2)^3(0), (\mathbb{L}2)^3(0) >$$

*So that, $2\phi = \phi 2$. Hence, for all $x = 0, 1, 2$ we can verify that $z\phi = \phi z$, where $z \in \mathbb{X}$, i.e. $\phi$ is CFFNSG of the group $(\mathbb{Z}_3, +_3)$.*

**Proposition 5.** *Let $\phi = (pe^{2\pi i\omega}, qe^{2\pi i\nu})$ be a CFFSG of a group $(\mathbb{X}, *)$. Then $\phi$ is a CFFNSG of $\mathbb{X}$ if and only if $p^3(z_1 * z_2)e^{2\pi i\omega^3(z_1 * z_2)} = p^3(z_2 * z_1)e^{2\pi i\omega^3(z_2 * z_1)}$ and $q^3(z_1 * z_2)e^{2\pi i\nu^3(z_1 * z_2)} = q^3(z_2 * z_1)e^{2\pi i\nu^3(z_2 * z_1)}$.*

*Proof.*
$\Rightarrow$ Assume that $\phi$ is a CFFNSG of $\mathbb{X}$, then $(z_2p^3(z_1))e^{2\pi i(z_2\omega^3(z_1))} = (p^3(z_1)z_2)e^{2\pi i(\omega^3(z_1)z_2)}$, for all $z_1, z_2 \in \mathbb{X}$. Equivalently, $p^3(z_2^{-1} * z_1)e^{2\pi i\omega^3(z_2^{-1} * z_1)} = p^3(z_1 * z_2^{-1})e^{2\pi i\omega^3(z_1 * z_2^{-1})}$. Hence, $p^3(z_2 * z_1)e^{2\pi i\omega^3(z_2 * z_1)} = p^3((z_2^{-1})^{-1} * z_1)e^{2\pi i\omega^3((z_2^{-1})^{-1} * z_1)} = p^3(z_1 * (z_2^{-1})^{-1})e^{2\pi i\omega^3(z_1 * (z_2^{-1})^{-1})} = p^3(z_1 * z_2)e^{2\pi i\omega^3(z_1 * z_2)}$. Similarly, we can verify that $q^3(z_1 * z_2)e^{2\pi i\nu^3(z_1 * z_2)} = q^3(z_2 * z_1)e^{2\pi i\nu^3(z_2 * z_1)}$.
$\Leftarrow$ Assume that $z_3 = z_1^{-1}$, then for arbitrary $z_1, z_2 \in \mathbb{X}$. We have $p^3(z_1 * z_2)e^{2\pi i\omega^3(z_1 * z_2)} = p^3(z_2 * z_1)e^{2\pi i\omega^3(z_2 * z_1)}$, hence $p^3(z_3^{-1} * z_2)e^{2\pi i\omega^3(z_3^{-1} * z_2)} = p^3(z_2 * z_3^{-1})e^{2\pi i\omega^3(z_2 * z_3^{-1})}$ for any $z_3, z_2 \in \mathbb{X}$.

So that, $(z_3p^3(z_2))e^{2\pi i(z_3\omega^3(z_2))} = (p^3(z_2)z_3)e^{2\pi i(\omega^3(z_2)z_3)}$. Similarly, we can prove that $(z_3q^3(z_2))e^{2\pi i(z_3\nu^3(z_2))} = (q^3(z_2)z_3)e^{2\pi i(\nu^3(z_2)z_3)}$, then $z_3\phi = \phi z_3$ for any $z_3 \in \mathbb{X}$, which implies that $\phi$ is CFFNSG of a group $(\mathbb{X}, *)$.

**Proposition 6.** *For a group $(\mathbb{X}, *)$ that was defined on CFFSG, $\phi = (pe^{2\pi i\omega}, qe^{2\pi i\nu})$. Then $\phi$ is a CFFNSG of $\mathbb{X}$ if and only if $p^3(x)e^{2\pi i\omega^3(x)} = p^3(z * x * z^{-1})e^{2\pi i\omega^3(z*x*z^{-1})}$, and $q^3(x)e^{2\pi i\nu^3(x)} = q^3(z * x * z^{-1})e^{2\pi i\nu^3(z*x*z^{-1})}$, for all $z, x \in \mathbb{X}$*

*Proof.* First consider, $p^3(x)e^{2\pi i\omega^3(x)} = p^3(x*id)e^{2\pi i\omega^3(x*id)} = p^3(x*z*z^{-1})e^{2\pi i\omega^3(x*z*z^{-1})} = p^3(x * (z * z^{-1}))e^{2\pi i\omega^3(x*(z*z^{-1}))} = p^3((x * z) * z^{-1})e^{2\pi i\omega^3((x*z)*z^{-1})} = p^3(z^{-1} * (x * z))e^{2\pi i\omega^3(z^{-1}*(x*z))}$, whereas $\phi$ is CFFNSG of $\mathbb{X}$. But $z = (z^{-1})^{-1}$ and by similarity $p^3(x)e^{2\pi i\omega^3(x)} = p^3(z * x * z^{-1})e^{2\pi i\omega^3(z*x*z^{-1})}$. Also, it is easy to show that $q^3(x)e^{2\pi i\nu^3(x)} = q^3(z * x * z^{-1})e^{2\pi i\nu^3(z*x*z^{-1})}$ too.

Conversely, $p^3(z*x)e^{2\pi i\omega^3(z*x)} = p^3(z*x*id)e^{2\pi i\omega^3(z*x*id)} = p^3(z*(x*z)*z^{-1})\ e^{2\pi i\omega^3(z*(x*z)*z^{-1})} = p^3(x * z)e^{2\pi i\omega^3(x*z)}$. Also, it is easy to show that $q^3(z*x)e^{2\pi i\nu^3(z*x)} = q^3(x*z)e^{2\pi i\nu^3(x*z)}$. Then by previous proposition, $\phi$ is CFFNSG of $\mathbb{X}$.

**Theorem 3.** *Let $\phi$ be a CFFNSG of a group $(\mathbb{X}, *)$. Then the set $\mathbb{M} = \{y \in \mathbb{X} : p^3(id)e^{2\pi i\omega^3(id)} = p^3(y)e^{2\pi i\omega^3(y)}$ and $q^3(id)e^{2\pi i\nu^3(id)} = q^3(y)e^{2\pi i\nu^3(y)}\}$, is a normal subgroup of $\mathbb{X}$, where id is the identity of it.*

*Proof.* At first $id \in \mathbb{M}$, i.e. $\mathbb{M}$ is not empty. Moreover, it is subgroup of $\mathbb{X}$, by Theorem 2. So that, $p^3(id)e^{2\pi i\omega^3(id)} = p^3(y * z^{-1})e^{2\pi i\omega^3(y * z^{-1})}$ and $q^3(id)e^{2\pi i\nu^3(id)} = q^3(y * z^{-1})e^{2\pi i\nu^3(y * z^{-1})}$. But, $\phi$ is a CFFNSG of $(\mathbb{X}, *)$. Then $p^3(y * z^{-1})e^{2\pi i\omega^3(y * z^{-1})} = p^3(z^{-1} * y)e^{2\pi i\omega^3(z^{-1} * y)}$ and $q^3(y * z^{-1})e^{2\pi i\nu^3(y * z^{-1})} = q^3(z^{-1} * y)e^{2\pi i\nu^3(z^{-1} * y)}$. Hence, $(z^{-1} * y) \in \mathbb{M}$ and $\mathbb{M}$ is a normal subgroup of $\mathbb{M}$.

## 5. Homomorphism on complex Fermatean fuzzy subgroup

In this section, we discuss the effect of homomorphism on CFFSG.

**Definition 12.** *A homomorphism function $h: \mathbb{X} \to \mathbb{U}$ from group $\mathbb{X}$ to group $\mathbb{U}$. Let $A$ be CFFSG of $\mathbb{X}$ and $B$ be CFFSG of $\mathbb{U}$. Let $x \in \mathbb{X}$ and $y \in \mathbb{U}$, then we have:*
*$h(A)(y) = \{(y, h(\mathbb{K}_A)(y), h(\mathbb{L}_A)(y))\}$, is the image of $A$, where:*

$$h(\mathbb{K}^3_A) = \begin{cases} \sup\limits_{x \in h^{-1}(y)} \mathbb{K}^3_A(x) & , \quad h(x) = y \\ 0 & , \quad otherwise. \end{cases}$$

$$= \begin{cases} (\sup\limits_{x \in h^{-1}(y)} p^3_A(x))e^{2\pi i(\sup\limits_{x \in h^{-1}(y)} \omega^3_A(x))} & , \quad h(x) = y \\ 0\ e^{2\pi i\ 0} & , \quad otherwise. \end{cases}$$

$$h(\mathbb{L}^3_A) = \begin{cases} \inf\limits_{x \in h^{-1}(y)} \mathbb{L}^3_A(x) & , \quad h(x) = y \\ 1 & , \quad otherwise. \end{cases}$$

$$= \begin{cases} (\inf\limits_{x \in h^{-1}(x)} q^3_A(x))e^{2\pi i(\inf\limits_{x \in h^{-1}(y)} \nu^3_A(x))} & , \quad h(x) = y \\ 1 & , \quad otherwise. \end{cases}$$

*And the set of pre-image of $B$ is $h^{-1}(B)(x) = \{(x, h^{-1}(\mathbb{K}_B)(x), h^{-1}(\mathbb{L}_B)(x))\}$, where:*

$$h^{-1}(\mathbb{K}^3_B)(x) = (\mathbb{K}_B)^3(h(x)) = p^3_B(h(x))e^{2\pi i\omega^3_B(h(x))}$$
$$h^{-1}(\mathbb{L}^3_B)(x) = (\mathbb{L}_B)^3(h(x)) = q^3_B(h(x))e^{2\pi i\nu^3_B(h(x))}, \ \forall\ x \in \mathbb{X}.$$

**Lemma 1.** *Let $h: \mathbb{X} \to \mathbb{U}$ be a homomorphism from group $\mathbb{X}$ to group $\mathbb{U}$, and let $A$ be CFFSG of $\mathbb{X}$, $B$ be CFFSG of $\mathbb{U}$. Then:*
*1) $h(\mathbb{K}^3_A)(y) = h(p^3_A)(y)e^{2\pi ih(\omega^3_A)(y)}\ \forall\ y \in \mathbb{U}$.*
*2) $h(\mathbb{L}^3_A)(y) = h(q^3_A)(y)e^{2\pi if(\nu^3_A)(y)}\ \forall\ y \in \mathbb{U}$.*
*3) $h^{-1}(\mathbb{K}^3_B)(x) = h^{-1}(p^3_B)(x)e^{2\pi ih^{-1}(\omega^3_B)(x)}\ \forall\ x \in \mathbb{X}$.*
*4) $h^{-1}(\mathbb{L}^3_B)(x) = h^{-1}(q^3_B)(x)e^{2\pi ih^{-1}(\nu^3_B)(x)}\ \forall\ x \in \mathbb{X}$.*

*Proof.*

$$1)\quad h(\mathbb{K}^3_A)(y) = \sup_{x \in h^{-1}(y)} \{\mathbb{K}^3_A(x);\ h(x) = y\}$$
$$= \sup_{x \in h^{-1}(y)} \{p^3_A(x)e^{2\pi i\omega^3_A(x)};\ h(x) = y\}$$

$$= \sup_{x\in h^{-1}(y)}\{p_A^3(x)\}e^{2\pi i \sup\limits_{x\in h^{-1}(y)}\{\omega_A^3(x)\}}$$
$$= h(p_A^3)(y)e^{2\pi i h(\omega_A^3)(y)}.$$

2) $$h(\mathbb{L}_A^3)(y) = \inf_{x\in h^{-1}(y)}\{\mathbb{L}_A^3(x);\ h(x)=y\}$$
$$= \inf_{x\in h^{-1}(y)}\{q_A^3(x)e^{2\pi i\nu_A^3(x)};\ h(x)=y\}$$
$$= \inf_{x\in h^{-1}(y)}\{q_A^3(x)\}e^{2\pi i \inf\limits_{x\in h^{-1}(y)}\{\nu_A^3(x)\}}$$
$$= h(q_A^3)(y)e^{2\pi i h(\nu_A^3)(y)}.$$

3) $$h^{-1}(\mathbb{K}_B^3)(x) = (\mathbb{K}_B)^3(h(x))$$
$$= p_B^3(h(x))e^{2\pi\omega_B^3(h(x))}$$
$$= h^{-1}(p_B^3)(x)e^{2\pi i h^{-1}(\omega_B^3)(x)}$$

4) $$h^{-1}(\mathbb{L}_B^3)(x) = (\mathbb{L}_B)^3(h(x))$$
$$= q_B^3(h(x))e^{2\pi\nu_B^3(h(x))}$$
$$= h^{-1}(q_B^3)(x)e^{2\pi i h^{-1}(\nu_B^3)(x)}$$

**Example 5.** *Let $(\mathbb{Z}_3, +_3)$ and $(\mathbb{Z}, +)$ be complex fermatean fuzzy group (CFFG), where we define $(\mathbb{Z}_3, +_3)$ as in example 4.*
*The map $h : (\mathbb{Z}, +) \to (\mathbb{Z}_3, +_3)$ is complex fermatean fuzzy homomorphism. Consider $A = \{1, 4, 5, 8, 11, 12\} \subseteq \mathbb{Z}$, then $h(A) = (x, h(\mathbb{K}_A)(x), h(\mathbb{L}_A)(x))$. Then:*

1) $$h(\mathbb{K}_A^3)(z) = \sup_{x\in h^{-1}(z)}\{\mathbb{K}_A^3(x);\ h(x)=z\ (mod\ 3)\ \}$$
$$= \sup\{\mathbb{K}_A^3(1), \mathbb{K}_A^3(4), \mathbb{K}_A^3(5), \mathbb{K}_A^3(8), \mathbb{K}_A^3(11), \mathbb{K}_A^3(12)\}$$
$$= \sup\{p_A^3(1)e^{2\pi i\omega_A^3(1)}, p_A^3(4)e^{2\pi i\omega_A^3(4)}, p_A^3(5)e^{2\pi i\omega_A^3(5)},$$
$$, p_A^3(8)e^{2\pi i\omega_A^3(8)}, p_A^3(11)e^{2\pi i\omega_A^3(11)}, p_A^3(12)e^{2\pi i\omega_A^3(12)}\}$$
$$= \sup\{p_A^3(1), \dots, p_A^3(12)\}e^{2\pi i \sup\{\omega_A^3(1),\dots,\omega_A^3(12)\}}$$
$$= \sup\{0.512, 0.027, 0.729\}e^{2\pi i\ \sup\{0.512,0.216,0.343\}} = 0.729e^{2\pi i\ 0.512}$$

2) $$h(\mathbb{L}_A^3)(z) = \inf_{x\in h^{-1}(z)}\{\mathbb{L}_A^3(x);\ h(x)=z\ (mod\ 3)\ \}$$

$$= \inf\{\mathbb{L}_A^3(1), \mathbb{L}_A^3(4), \mathbb{L}_A^3(5), \mathbb{L}_A^3(8), \mathbb{L}_A^3(11), \mathbb{L}_A^3(12)\}$$
$$= \inf\{q_A^3(1)e^{2\pi i\nu_A^3(1)}, q_A^3(4)e^{2\pi i\nu_A^3(4)}, q_A^3(5)e^{2\pi i\nu_A^3(5)},$$
$$, q_A^3(8)e^{2\pi i\nu_A^3(8)}, q_A^3(11)e^{2\pi i\nu_A^3(11)}, q_A^3(12)e^{2\pi i\nu_A^3(12)}\}$$
$$= \inf\{q_A^3(1), \dots, q_A^3(12)\}e^{2\pi i \inf\{\nu_A^3(1),\dots,\nu_A^3(12)\}}$$
$$= \inf\{0.343, 0.125, 0.512\}e^{2\pi i \ \inf\{0.216, 0.512\}} = 0.125e^{2\pi i \ 0.216}$$

**Theorem 4.** *Let* $h: \mathbb{X} \xrightarrow{epimorphism} \mathbb{U}$, *from* $(\mathbb{X}, *_1)$ *to* $(\mathbb{U}, *_2)$, *and let* $A$ *be CFFSG of* $\mathbb{X}$. *Then* $h(A)$ *is CFFSG of* $\mathbb{U}$.

*Proof.* Consider a two groups $(\mathbb{X}, *_1)$ and $(\mathbb{U}, *_2)$, with $A = (\mathbb{K}_A, \mathbb{L}_A)$ is CFFSG, and want to show that $h(A) = (h(\mathbb{K}_A), h(\mathbb{L}_A))$
$= (h(p_A)(y)e^{2\pi i h(\omega_A)(y)}, h(q_A)(y)e^{2\pi i h(\nu_A)(y)})$ is CFFSG.
At first, the set $S_1 = \{(x, p_A(x), q_A(x)) : x \in \mathbb{X}, 0 \le p_A^3(x) + q_A^3(x) \le 1\}$ and $S_2 = \{(x, \omega_A(x), \nu_A(x)) : x \in \mathbb{X}, 0 \le \omega_A^3(x) + \nu_A^3(x) \le 1\}$ are the amplitude and phase terms of CFFSG, since **A** is CFFSG and using Lemma 1. Then by Theorem[6.1] [35] and $h$ is homomorphism, we have:
i) a) $h(p_A^3)(x_1 *_2 x_2) = (h(p_A))^3(x_1 *_2 x_2) \ge (h(p_A))^3(x_1) \wedge (h(p_A))^3(x_2)$,
b) $h(q_A^3)(x_1 *_2 x_2) = (h(q_A))^3(x_1 *_2 x_2) \le (h(q_A))^3(x_1) \vee (h(q_A))^3(x_2)$,
c) $h(\omega_A^3)(x_1 *_2 x_2) = (h(\omega_A))^3(x_1 *_2 x_2) \ge (h(\omega_A))^3(x_1) \wedge (h(\omega_A))^3(x_2)$,
d) $h(\nu_A^3)(x_1 *_2 x_2) = (h(\nu_A))^3(x_1 *_2 x_2) \le (h(\nu_A))^3(x_1) \vee (h(\nu_A))^3(x_2)$.
ii) a) $h(p_A^3)(x^{-1}) = (h(p_A))^3(x^{-1}) = (h(p_A))^3(x^{-1})$,
b) $h(q_A^3)(x^{-1}) = (h(q_A))^3(x^{-1}) = (h(q_A))^3(x^{-1})$,
c) $h(\omega_A^3)(x^{-1}) = (h(\omega_A))^3(x^{-1}) = (h(\omega_A))^3(x^{-1})$,
d) $h(\omega_A^3)(x^{-1}) = (h(\nu_A))^3(x^{-1}) = (h(\nu_A))^3(x^{-1})$.

Consequently and by Lemma 1, we have:
1) $(h(\mathbb{K}_A))^3(x_1 *_2 x_2) = h(\mathbb{K}_A^3)(x_1 *_2 x_2) =$
$h(p_A^3)(x_1 *_2 x_2)e^{2\pi i \ h(\omega_A^3)(x_1 *_2 x_2)} \ge$
$(h(p_A^3)(x_1) \wedge h(p_A^3)(x_2)) * e^{2\pi i(h(\omega_A^3)(x_1) \wedge h(\omega_A^3)(x_2))} =$
$\{h(p_A^3)(x_1)e^{2\pi i \ h(\omega_A^3)(x_1)} \wedge h(p_A^3)(x_2)e^{2\pi i \ f(\gamma_A^3)(x_2)}\} =$
$h(\mathbb{K}_A^3)(x_1) \wedge h(\mathbb{K}_A^3)(x_2) = (h(\mathbb{K}_A))^3(x_1) \wedge (h(\mathbb{K}_A))^3(x_2)$.
2) $(h(\mathbb{L}_A))^3(x_1 *_2 x_2) = h(\mathbb{L}_A^3)(x_1 *_2 x_2) =$
$h(q_A^3)(x_1 *_2 x_2)e^{2\pi i \ f(\nu_A^3)(x_1 *_2 x_2)} \le$
$(h(q_A^3)(x_1) \vee h(q_A^3)(x_2)) * e^{2\pi i(h(\nu_A^3)(x_1) \vee h(\nu_A^3)(x_2))} =$
$\{h(q_A^3)(x_1)e^{2\pi i \ h(\nu_A^3)(x_1)} \vee h(q_A^3)(x_2)e^{2\pi i \ h(\nu_A^3)(x_2)}\} =$
$h(\mathbb{L}_A^3)(x_1) \vee h(\mathbb{L}_A^3)(x_2) = (h(\mathbb{L}_A))^3(x_1) \vee (h(\mathbb{L}_A))^3(x_2)$.
3) $(h(\mathbb{K}_A))^3(x^{-1}) = h(\mathbb{K}_A^3)(x^{-1}) =$
$h(p_A^3)(x^{-1})e^{2\pi i \ h(\omega_A^3)(x^{-1})} = h(p_A^3)(x)e^{2\pi i \ h(\omega_A^3)(x)} =$
$h(\mathbb{K}_A^3)(x) = (h(\mathbb{K}_A))^3(x)$.
4) $(h(\mathbb{L}_A))^3(x^{-1}) = h(\mathbb{L}_A^3)(x^{-1}) =$

$h(q_A^3)(x^{-1})e^{2\pi i\ h(\nu_A^3)(x^{-1})} = h(q_A^3)(x)e^{2\pi i\ f(\nu_A^3)(x)} =$
$h(\mathbb{L}_A^3)(x) = (h(\mathbb{L}_A))^3(x)$. Hence result follows.

**Theorem 5.** *Let* $f: \mathbb{X} \xrightarrow{isomorphism} \mathbb{U}$*, from* $(\mathbb{X}, *_1)$ *to* $(\mathbb{U}, *_2)$*, and let* $B$ *be CFFSG of* $\mathbb{U}$*. Then* $h^{-1}(B)$ *is CFFSG of* $\mathbb{X}$*.*

*Proof.* The proof will be similar to previous theorem, and that by using Lemma 1 with Theorem[6.2] [35].

**Theorem 6.** *Let* $h: \mathbb{X} \xrightarrow{epimorphism} \mathbb{U}$*, from* $(\mathbb{X}, *_1)$ *to* $(\mathbb{U}, *_2)$*, and let* $A$ *be CFFNSG of* $\mathbb{X}$*. Then* $h(A)$ *is CFFNSG of* $\mathbb{U}$*.*

*Proof.* According to Proposition 5 with Theorem[6.3] [35], and inspiring the proof of Theorem 4, result will follows.

**Theorem 7.** *Let* $h: \mathbb{X} \xrightarrow{isomorphism} \mathbb{U}$*, from* $(\mathbb{X}, *_1)$ *to* $(\mathbb{U}, *_2)$*, and let* $B$ *be CFFNSG of* $\mathbb{U}$*. Then* $h^{-1}(B)$ *is CFFNSG of* $\mathbb{X}$*.*

*Proof.* According to Proposition 5 with Theorem[6.4] [35], and inspiring the proof of Theorem 4, result will follows.

**Example 6.** *Let* $(\mathbb{X}, *_1) = \mathbf{K_4} =< a,\ b|a^2 = b^2 = (ab)^2 = 1 >$*, i.e. the Klein four-group, and* $(\mathbb{U}, *_2) = \mathbf{D_2} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 = \{(0,0),(1,0),(0,1),(1,1)\}$*. In addition, for* $B =< \ell, p(\ell)e^{2\pi i\omega(\ell)}, q(\ell)e^{2\pi i\nu(\ell)} >$ *that defined a CFFSG on* $\mathbf{D_2}$*, see Example 1. Then* $B$ *defined CFFNSG of* $\mathbf{D_2}$*, where the dihedral group* $\mathbf{D_2}$ *is an abelian group. Hence, we can find homomorphism function that is bijection;* $h: \mathbf{K_4} \xrightarrow{isomorphism} \mathbf{D_2}$*, with* $h(1) = (0,0)$*,* $h(a) = (1,0)$*,* $h(b) = (0,1)$*,* $h(ab) = (1,1)$*. Now, according to definition 12* $h^{-1}(\mathbb{K}_B^3)(u) = p_B^3(h(u))e^{2\pi i\omega_B^3(h(u))}$ *and* $h^{-1}(\mathbb{L}_B^3)(u) = q_B^3(h(u))e^{2\pi i\nu_B^3(h(u))}$*,* $\forall\ u \in \mathbf{K_4}$*. Hence, by this definition we get that* $h^{-1}(B)$ *is CFFNSG of* $\mathbf{K_4}$*.*

## 6. Conclusion

This research provides a theoretical foundation for the complex Fermatean fuzzy subgroup (CFFSG) and examines its algebraic properties. The concepts of complex Fermatean fuzzy normal subgroups and complex Fermatean fuzzy cosets were introduced. Additionally, the conditions under which a complex Fermatean fuzzy subgroup can be a complex Fermatean fuzzy normal subgroup were explored. A homomorphism between two complex Fermatean fuzzy subgroups and its properties were also discussed. As a direction for future research, we plan to refine the definition of CFFSG by replacing the minimum and maximum operations with T-norm and S-norm functions, respectively. Furthermore, fixed-point theory could be integrated and extended within the context of CFFSG. The approach presented here can be progressively applied to other algebraic structures, such as integral domains, fields, rings, and factor groups. By incorporating periodic information into the CFFSG framework, the current structure could facilitate the development of cryptographic primitives and be applied to the generalization of new algorithms. Another avenue for future work is upgrading CFFSG to the complex q-rung orthopair fuzzy subgroup.

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