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Some Families of Differential Equations for Multivariate Hybrid Special Polynomials Associated with Frobenius-Genocchi Polynomials

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Abstract. This article introduces a new class of multivariate Hermite-Frobenius-Genocchi polynomials and explores various characterizations of these polynomials. We examine their properties, including recurrence relations and shift operators. Using the factorization method, we derive differential, partial differential, and integrodifferential equations satisfied by these polynomials. Furthermore, we present the Volterra integral equation associated with these multivariate Hermite-Frobenius-Genocchi polynomials, which improves our understanding and application of the factorization method in fields such as physics and engineering.

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1. Introduction and preliminaries

Special polynomial families of hybrid types are of profound importance due to their diverse and valuable attributes. These attributes include recurring and explicit relationships, functional and differential equations, summation formulas, symmetric and convolution properties, and determinant representations. Hybrid special polynomials serve as foundational elements in a wide range of mathematical and scientific disciplines, demonstrating their versatility and impact. Their unique properties facilitate the development of

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various theoretical and practical applications, making them essential tools in both research and applied contexts.

The applications of multi-variable hybrid special polynomials extend across several domains such as number theory, combinatorics, classical and numerical analysis, theoretical physics, and approximation theory. This broad applicability highlights their potential for practical implementation and further investigation. Many studies have systematically introduced and analyzed Apostol-type polynomials, including both traditional and generalized forms. These studies have employed a range of analytic techniques, as evidenced by the works of researchers such as [1, 2, 4, 8, 10, 14, 16]. Notably, recent research by Araci et al. [3] has provided a detailed examination of Hermite-Apostol-type polynomials, including Frobenius-Euler and Genocchi polynomials, using generating techniques as a systematic approach to their study.

A significant recent development in polynomial theory is the innovative approach to constructing Hermite polynomials, denoted as $\sigma_n^{[m]}(\eta_1, \eta_2, \eta_3, \cdots, \eta_m)$. This new class of polynomials was developed using generating relations, which are powerful tools for systematically exploring and analyzing mathematical functions. The use of generating relations represents a methodological advancement in polynomial theory, offering new insights and capabilities for investigating complex mathematical functions and their applications.

Expanding the scope of usefulness and increasing the already sufficient knowledge in this field of study is accomplished by using generalized Hermite-Apostol type Frobenius-Genocchi polynomials. These polynomials join together different areas of mathematics that, while seemingly unrelated on the surface, share a great deal in common underneath their structures, allowing mathematicians to transport concepts and methodologies between fields. This interdisciplinary method helps stimulate collaboration among different scholars and ensures idea exchanges that could bring forward new concepts, breakthroughs, or even practical applications within many sectors. Recent advancements in the study of multivariate Hermite polynomials, facilitated by the use of generating techniques, have significant implications for the field of polynomial mathematics. These polynomials have emerged as powerful tools for managing and analyzing complex multivariate systems. Their robust nature and unique properties make them indispensable for tackling intricate problems across various scientific and mathematical disciplines. The application of generating techniques has not only deepened our understanding of these polynomials but also opened new avenues for research and exploration.

The systematic study of these multivariate Hermite polynomials has introduced novel research goals and objectives. By leveraging generating relations, researchers can derive and investigate these polynomials in a structured manner, leading to a more comprehensive grasp of their characteristics and behaviors. This approach has revealed new insights and potential applications, highlighting the versatility and significance of these polynomials in a wide range of fields, from theoretical physics to numerical analysis.

The derivation of multivariate Hermite polynomials through generating relations exemplifies a methodical approach to polynomial theory. This process allows for the development of a rich framework for analyzing multi-dimensional problems, enhancing both theoretical understanding and practical application. As a result, the ongoing research into these polynomials promises to advance the field significantly, offering innovative solutions and contributing to the broader scientific and mathematical community. Thus, the generating relations for these polynomials are characterized by

$$\exp(\eta_1\xi + \eta_2\xi^2 + \cdots + \eta_m\xi^m) = \sum_{n=0}^{\infty} \sigma_n^{[m]}(\eta_1, \eta_2, \cdots, \eta_m)\frac{\xi^n}{n!},$$

with series representation as:

$$\sigma_n^{[m]}(\eta_1, \eta_2, \cdots \eta_m) = n! \sum_{r=0}^{[n/m]} \frac{p_m^r \sigma_{n-mr}^{[m]}(\eta_1, \eta_2, \cdots, p_{m-1})}{r!\ (n-mr)!}.$$

A special class of polynomials is introduced by the convolution between $\sigma_n^{[m]}(\eta_1, \eta_2, \cdots \eta_m)$, a multivariate Hermite polynomial and $\mathcal{F}_n(\eta_1; \lambda)$, a Frobenius-Genocchi polynomial. A completely new set of polynomials with unique characteristics results from the convolution of the two different types of polynomials. multivariate Hermite polynomials are based on the Hermite polynomials. These polynomials have been widely studied and used in different areas of mathematics and natural sciences. Therefore, it is true that these polynomials can be produced by combining the two types of polynomials with different characteristics and traits thus creating a new set which will have some novel mathematical properties and connections.

Our main concern here is to construct differential equations and integral equations within the scope of these polynomials. We are considering multivariate Hermite-Frobenius-Genocchi polynomials (MVHFGP) ${}_{\mathcal{G}}\mathcal{E}_n^{\mathcal{F}}(\eta_1, \eta_2, \cdots, \eta_m; \lambda)$ with $\lambda \in \mathbb{C}$, $\lambda \neq 1$, that follows:

$$\left(\frac{(1-\lambda)t}{e^t - \lambda}\right) e^{\eta_1 t + \eta_2 t^2 + \eta_3 t^3 + \cdots + \eta_m t^m} = \sum_{n=0}^{\infty} {}_{\mathcal{G}}\mathcal{E}_n^{\mathcal{F}}(\eta_1, \eta_2, \cdots, \eta_m; \lambda)\frac{t^n}{n!}. \tag{1}$$

Various analytical techniques can be used by researchers to study the properties, activities and uses of these complex functions (polynomials). In studying them, it may include analyzing the convergence properties, orthogonality, recurrence relations and generating functions among other important ones. This allows the Frobenius-Genocchi polynomials to be connected to multivariate Hermite polynomials through intricate polynomially. Convoluted polynomials provide a connection between the multivariate Hermite polynomials and Frobenius-Genocchi polynomials which enables bridge building between these two fields in terms of knowledge and methods used.

The convoluted special polynomials discussed above are very crucial because of their vital characteristics. For example, they have algebraic features as well as some summation formulas, symmetrical identities about convolution and reciprocity equations among others that comprise recurrence and explicit relations. These qualities make these kinds of polynomials useful in many mathematical applications hence making them easily adaptable. One interesting aspect of these polynomials is that they have connections and patterns. These connections allow for calculations and the ability to derive terms in a series, from earlier ones. This feature simplifies the analysis and manipulation of these polynomials facilitating research and computations. Additionally these polynomials can be represented as either infinite series due to the existence of summation formulas. These formulas enable the evaluation and estimation of polynomials leading to applications in fields such, as analysis and approximation theory.

Now let's explore some instances of the variable Hermite-Frobenius-Genocchi polynomials: ${}_{\mathcal{G}}\mathcal{E}_n^{\mathcal{F}}(\eta_1, \eta_2, \cdots, \eta_m; \lambda)$. These are given below:

Table 1. Special cases of ${}_{\mathcal{G}}\mathcal{E}_n^{\mathcal{F}}(\eta_1, \eta_2, \cdots, \eta_m; \lambda)$

S.No.	Cases	Name of polynomial	Generating function
I.	$\lambda = -1$	multivariate Hermite-Genocchi polynomials [12, 13]	$\left(\frac{2t}{e^t+1}\right) e^{\eta_1 t+\eta_2 t^2+\eta_3 t^3+\cdots+\eta_m t^m} = \sum_{n=0}^{\infty} {}_{\mathcal{G}}\mathcal{E}_n(\eta_1, \eta_2, \cdots, \eta_m) \frac{t^n}{n!}$
II.	$\lambda = -1,\ m = 3$	3-variable Hermite-Genocchi polynomials	$\left(\frac{2t}{e^t+1}\right) e^{\eta_1 t+\eta_2 t^2+\eta_3 t^3} = \sum_{n=0}^{\infty} {}_{\mathcal{G}}\mathcal{E}_n(\eta_1, \eta_2, \eta_3) \frac{t^n}{n!}$
III.	$\lambda = -1,\ m = 2,$	2-variable Hermite-Genocchi polynomials	$\left(\frac{2t}{e^t+1}\right) e^{\eta_1 t+\eta_2 t^2} = \sum_{n=0}^{\infty} {}_{\mathcal{G}}\mathcal{E}_n(\eta_1, \eta_2) \frac{t^n}{n!}$
	$\lambda = -1,\ \eta_1 = 2\eta_1,$ $\eta_2 = -1;\ m = 2$	Hermite-Genocchi polynomials	$\left(\frac{2t}{e^t+1}\right) e^{2\eta_1 t-t^2} = \sum_{n=0}^{\infty} {}_{\mathcal{G}}\mathcal{E}_n(\eta_1, \eta_2) \frac{t^n}{n!}$

The multivariate Hermite-Frobenius-Genocchi polynomials ${}_{\mathcal{G}}\mathcal{E}_n^{\mathcal{F}}(\eta_1, \eta_2, \cdots, \eta_m; \lambda)$ are represented by series:

$$ {}_{\mathcal{G}}\mathcal{E}_n^{\mathcal{F}}(\eta_1, \eta_2, \cdots, \eta_m; \lambda) = \sum_{k=0}^{n} \binom{n}{k} \mathcal{E}_{n-k}^{\mathcal{F}}(\lambda)\ \mathcal{G}_k(\eta_1, \eta_2, \cdots, \eta_m), $$

with $m = 2$, we find

$$ {}_{\mathcal{G}}\mathcal{E}_n^{\mathcal{F}}(\eta_1, \eta_2; \lambda) = n! \sum_{k=0}^{n} \sum_{r=0}^{[\frac{k}{2}]} \frac{\mathcal{E}_{n-k}^{\mathcal{F}}(\lambda)\ {\eta_1}^r\ {\eta_1}^{k-2r}}{(n-k)!\ r!\ (k-2r)}. $$

The various versions of Hermite Euler polynomials mentioned above hold importance in both applied mathematics as well, as physics particularly in the realms of quantum mechanics and probability theory. A wide range of issues and practical applications within these fields are closely intertwined with these forms.

The study of equations encompasses applied mathematics, physics and engineering. In times there has been progress in the development of generalized and multi variable versions of special polynomials within mathematical physics. These polynomials offer avenues for analyzing categories of differential equations commonly encountered in physical problems. While practical mathematics focuses on validating methods, for approximating solutions pure mathematics delves into exploring the existence and uniqueness of solutions. Differential equations may be used to simulate many technical, biological, and physical processes, including the movements of celestial bodies, the building of bridges, and the connections between neurons. They play a crucial role in the development of the fundamental laws of chemistry and physics. In the domains of economics and biology, complex system behaviour is simulated using differential equations. The domains that give rise to these equations and the practical applications of their solutions have influenced the development of differential equation mathematics.

Recurrence relationships have their roots in population dynamics modelling and may be traced back to early uses, such as the use of Fibonacci numbers to depict the rise of the rabbit population. Their fundamental relevance in comprehending dynamic systems within ecological contexts is highlighted by this historical context. Recurrence relations are used for more than just numerical patterns; they are an effective tool for modelling intricate population dynamics and provide predictions and analysis that are vital for ecological research and conservation initiatives. The fact that these linkages were identified in early population modelling emphasises how important they are as a cornerstone of mathematical ecology. Recurrence relations are used in digital signal processing to simulate feedback processes present in systems where outputs at one time step are inputs at later time steps. Recurrence relations play a crucial role in the design and optimization of infinite impulse response digital filters. They simplify the modeling and analysis of systems with feedback loops, which is essential for developing effective digital filtering techniques. This is particularly valuable in signal processing applications, including audio, image processing, and telecommunications. This illustrates how recurrence relations are useful in contemporary engineering and technology.

Furthermore, linear recurrence relations are widely used in theoretical and empirical economics to represent a variety of economic events. These relationships give economists a mathematical framework to explain how economic variables interact dynamically across time, enabling them to foresee and assess economic trends and behaviours. In fields including macroeconomics, finance, and policy analysis, recurrence relations help economists create models that improve comprehension and decision-making by reflecting the temporal dependencies and feedback mechanisms present in economic systems. Recurrence relations are therefore essential instruments in the study of economics that help to progress both economic theory and practice by connecting theoretical ideas with empirical findings.

One of the most important techniques that many mathematicians and physicists use to solve eigenvalue problems is factorization, as explained in [11]. This method involves solving two main differential equations that, when combined, produce a secondary differential equation of equal significance. Moreover, it involves calculating transition probabilities that account for the manufacturing process. A broad foundation for proficiently addressing perturbation issues is provided by the factorization approach. This method essentially infers another differential equation of comparable relevance from the answers of two different classes of differential equations. It goes beyond basic computing by include transition probabilities, which describe how a system evolves over time.

Consider the polynomial sequence $\{\mathcal{P}_n(\eta_1)\}_{n=0}^{\infty}$, where n denotes the polynomial degree. Two sets of differential operators, Ψ_n^- and Ψ_n^+, influence the behavior of this polynomial sequence. These operators are defined by the following relations:

$$\mathcal{P}_{n-1}(\eta_1) = \psi_n^-(\mathcal{P}_n(\eta_1))$$

and

$$\mathcal{P}_{n+1}(\eta_1) = \Psi_n^+(\mathcal{P}_n(\eta_1)).$$

A key differential equation for this polynomial sequence is given by:

$$\mathcal{P}_n(\eta_1) = (\Psi_{n+1}^- \Psi_n^+)\{\mathcal{P}_n(\eta_1)\}. \tag{2}$$

Using the operators Ψ_n^- and Ψ_n^+ is key to deriving the differential equation outlined in expression (2). These operators play a crucial role in the factorization method, serving as fundamental tools in constructing differential equations. The main goal is to identify two distinct operators: Ψ_n^+ as the multiplicative operator and Ψ_n^- as the derivative operator. Accurate selection of these operators is essential to ensure that the equation (2) is satisfied.

The factorization process enables the transformation of the original equation (2) into a sequence of differential equations involving Ψ_n^- and Ψ_n^+. This method provides a structured approach to solving and analyzing the equation. By reframing the problem with these operators, new insights can be gained, leading to a clearer understanding and more effective solutions. Systematic construction of these differential equations simplifies the identification of appropriate operators, allowing for a more focused and methodical approach.

Integral equations are used in many scientific and engineering problems. They show up in several models of mathematical physics, including diffraction problems, quantum mechanical scattering, conformal mapping, and water wave phenomena. These models have proven useful in the research and creation of integral equations. A thorough investigation of the mathematical characteristics and behaviours of the multivariate Hermite-Frobenius-Genocchi polynomials is required for the analytical analysis of differential and integral equations for them. In this work, the multivariate Hermite-Frobenius-Genocchi polynomials are used to develop and analyse differential equations, integrodifferential equations, and integral equations that they satisfy. These polynomials and their differential equations shed light on their structures, connections, and solutions. Because the integrodifferential equations contain integral elements, they become much more complicated, necessitating a sophisticated comprehension of how differentiation and integration interact with these polynomials. The study focuses on deriving recurrence relations, shift operators, and explicit forms for the multivariate Hermite-Frobenius-Genocchi polynomials. Researchers also examine specific boundary conditions and constraints that influence the behavior of these polynomials. This detailed exploration enhances the understanding of their mathematical properties and can impact various fields such as mathematical physics, statistics, and other areas where these polynomials are applied. Additionally, the study may lead to the development of new mathematical techniques and methodologies that extend to other classes of polynomials and functions.

The research also involves presenting and analyzing differential and integral equations related to these polynomials. References such as [5–7, 9, 15, 17, 18] offer an overview of the differential and integral equations connected to these unique polynomial families.

These equations are not only instrumental in addressing emerging challenges across various scientific domains but also highlight key characteristics of the polynomials, contributing to their broader application and understanding.

The features and attributes of multivariate Hermite-Frobenius-Genocchi polynomials are extensively examined in this paper. The main goal is to use the factorization method to create sets of differential equations related to these polynomials. In order to comprehend these polynomials, Section 2 of the study explores essential concepts such the generating relation, recurrence relation, and shift operators. Section 3 provides a thorough explanation of the complex procedure for creating several families of differential equations customised for these polynomials. Moving on to Section 4, the Volterra integral equation is derived and shown to be fulfilled by multivariate Hermite-Frobenius-Genocchi polynomials. In order to shed light on the integral equivalents of these polynomials, this section clarifies the integral equation that captures their behaviour and characteristics. Finally, the concluding section offers a comprehensive summary of the key findings and contributions outlined in the paper. Through this analysis, the manuscript aims to enhance understanding and facilitate further exploration of multivariate Hermite-Frobenius-Genocchi polynomials and their associated differential and integral equations.

2. Iterative connection and displacement operators

For the multivariate Hermite-Frobenius-Genocchi polynomial ${}_{\mathcal{G}}\mathcal{E}_n^{\mathcal{F}}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda)$, we define the shift operators and iterative connections in this section. The expression of the polynomials in respect to one another provided by these recurrence relations allows for faster calculations and the detection of repeating patterns. Our comprehension of the properties and behaviours of the multivariate Hermite-Frobenius-Genocchi polynomial ${}_{\mathcal{G}}\mathcal{E}_n^{\mathcal{F}}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda)$ is improved by the formulation of these recurrence relations and shift operators. These findings could prove valuable for various computations, analyses, or applications of these polynomials within their relevant field of study. The subsequent result is employed to derive the recurrence relation for the function ${}_{\mathcal{G}}\mathcal{E}_n^{\mathcal{F}}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda)$:

Theorem 1. *The multivariate Hermite-Frobenius-Genocchi polynomials* ${}_{\mathcal{G}}\mathcal{E}_n^{\mathcal{F}}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda)$ *satisfy the following recurrence relation:*

$$\begin{aligned}&{}_{\mathcal{G}}\mathcal{E}_{n+1}^{\mathcal{F}}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda)=\left(\eta_1-\tfrac{n+1}{2(1-\lambda)}\right){}_{\mathcal{G}}\mathcal{E}_n^{\mathcal{F}}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda)+2n\eta_2\ {}_{\mathcal{G}}\mathcal{E}_{n-1}^{\mathcal{F}}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda)\\&+3n(n-1)\eta_3\ {}_{\mathcal{G}}\mathcal{E}_{n-2}^{\mathcal{F}}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda)+\cdots+n(n-1)(n-2)\cdots(n-m+1)\ \eta_m\ {}_{\mathcal{G}}\mathcal{E}_{n-m}^{\mathcal{F}}(\eta_1,\eta_2,\eta_3,\\&\cdots,\eta_m;\lambda)-\tfrac{1}{1-\lambda}\sum_{k=2}^{n+1}\tbinom{n+1}{k}{}_{\mathcal{G}}\mathcal{E}_{n-k+1}^{\mathcal{F}}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda)\quad \mathfrak{G}_k^{\mathcal{F}}(\lambda),\end{aligned} \tag{3}$$

where the expansion:

$$\mathfrak{G}_k^{\mathcal{F}}(\lambda):=-\sum_{i=0}^{k}\frac{1}{2^i}\binom{k}{i}\mathfrak{G}_{k-i}^{\mathcal{F}}\left(\frac{1}{2};\lambda\right),\quad \mathfrak{G}_0^{\mathcal{F}}=-1,\quad \mathfrak{G}_1^{\mathcal{F}}=\frac{1}{2}$$

expressed using numerical coefficients $\mathfrak{G}_n^{\mathcal{F}}(\lambda)$*, which are associated with the Frobenius-Genocchi polynomials* $\mathcal{G}_k^{\mathcal{F}}(\eta_1;\lambda)$*.*

Proof. Taking the derivatives of (1) w.r.t. t, we find

$$\sum_{n=0}^{\infty} {}_{\mathcal{G}}\mathcal{E}_{n+1}^{\mathcal{F}}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda)\frac{t^n}{n!} = \left(\eta_1 + 2\ \eta_2 t + 3\ \eta_3 t^2 + \cdots + m\ \eta_m t^{m-1}\right)$$
$$\sum_{n=0}^{\infty} {}_{\mathcal{G}}\mathcal{E}_{n}^{\mathcal{F}}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda)\frac{t^n}{n!}$$
$$-\frac{1}{1-\lambda}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} {}_{\mathcal{G}}\mathcal{E}_{k}^{\mathcal{F}}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda)\mathfrak{G}_k^{\mathcal{F}}(\lambda)\frac{t^{n+k}}{n!\ k!}.$$

The Cauchy product rule is then applied to the simplified right-hand side, leading to the following conclusion:

$$\sum_{n=0}^{\infty} {}_{\mathcal{G}}\mathcal{E}_{n+1}^{\mathcal{F}}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\eta_1\ {}_{\mathcal{G}}\mathcal{E}_{n}^{\mathcal{F}}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda)\frac{t^n}{n!}+$$
$$\sum_{n=0}^{\infty}2n\ \eta_2 {}_{\mathcal{G}}\mathcal{E}_{n-1}^{\mathcal{F}}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda)\frac{t^n}{n!}$$
$$+\sum_{n=0}^{\infty}3n(n-1)\eta_3\ {}_{\mathcal{G}}\mathcal{E}_{n-2}^{\mathcal{F}}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda)\frac{t^n}{n!}+\cdots+\sum_{n=0}^{\infty}n(n-1)\cdots(n-m+1)\ m\eta_m {}_{\mathcal{G}}\mathcal{E}_{n-m}^{\mathcal{F}}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda)$$
$$-\frac{1}{1-\lambda}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k} {}_{\mathcal{G}}\mathcal{E}_{n-k}^{\mathcal{F}}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda)\mathfrak{G}_k^{\mathcal{F}}(\lambda)\frac{t^n}{n!}.$$

The resulting expression is derived by equating the coefficients of matching powers of t on both sides of the previously discussed equation:

$${}_{\mathcal{G}}\mathcal{E}_{n+1}^{\mathcal{F}}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda) = \eta_1\ {}_{\mathcal{G}}\mathcal{E}_{n}^{\mathcal{F}}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda)+2n\ \eta_2\ {}_{\mathcal{G}}\mathcal{E}_{n-1}^{\mathcal{F}}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda)$$
$$+3n(n-1)\ \eta_3 {}_{\mathcal{G}}\mathcal{E}_{n-2}^{\mathcal{F}}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda)+\cdots+n(n-1)\cdots(n-m+1)\ m\eta_m\ {}_{\mathcal{G}}\mathcal{E}_{n-m}^{\mathcal{F}}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda)$$
$$-\frac{1}{1-\lambda}\sum_{k=0}^{n}\binom{n}{k} {}_{\mathcal{G}}\mathcal{E}_{n-k}^{\mathcal{F}}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda)\mathfrak{G}_k^{\mathcal{F}}(\lambda).$$

Assertion (3) is obtained after replacing $n \to n+1$ and taking $k = 0, 1$ in the aforementioned equation and putting $\mathfrak{G}_0^{\mathcal{F}} = -1$, $\mathfrak{G}_1^{\mathcal{F}} = \frac{1}{2}$ into the resultant equation.

In the subsequent examination, we illustrate the development of shift operators for the multivariate Hermite-Frobenius-Genocchi polynomial ${}_{\mathcal{G}}\mathcal{E}_{n}^{\mathcal{F}}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda)$ through the derivation of the following outcome:

Theorem 2. *The MVHFGP* ${}_{\mathcal{G}}\mathcal{E}_n^{\mathcal{F}}(\eta_1, \eta_2, \eta_3, \cdots, \eta_m; \lambda)$ *satisfy the listed shift operators:*

$$ {}_{\eta_1}\pounds_n^- := \frac{1}{n} D_{\eta_1}, \tag{4}$$

$$ {}_{\eta_2}\pounds_n^- := \frac{1}{n} D_{\eta_1}^{-1} D_{\eta_2}, \tag{5}$$

$$ {}_{\eta_3}\pounds_n^- := \frac{1}{n} D_{\eta_1}^{-2} D_{\eta_3}, \tag{6}$$

$$\vdots \qquad\qquad \vdots$$

$$ {}_{\eta_m}\pounds_n^- := \frac{1}{n} D_{\eta_1}^{-(m-1)} D_{\eta_m}, \tag{7}$$

$$ {}_{\eta_1}\pounds_n^+ := \left(\eta_1 - \frac{n+1}{2(1-\lambda)}\right) + 2\eta_2 D_{\eta_1} + 3\eta_3 D_{\eta_1}^2 + \cdots + m\ \eta_m D_{\eta_1}^{m-1} - \frac{n+1}{1-\lambda} \sum_{k=2}^{n+1} D_{\eta_1}^{(k-1)} \frac{\mathfrak{G}_k^{\mathcal{F}}(\lambda)}{k!} \tag{8}$$

$$ {}_{\eta_2}\pounds_n^+ := \left(\eta_1 - \frac{n+1}{2(1-\lambda)}\right) + 2\eta_2\ D_{\eta_1}^{-1} D_{\eta_2} + 3\eta_3\ D_{\eta_1}^{-2} D_{\eta_2}^2 + \cdots + m\eta_m\ D_{\eta_1}^{-(m-1)} D_{\eta_2}^{m-1} - \frac{n+1}{1-\lambda} \sum_{k=2}^{n+1} D_{\eta_1}^{-(k-1)} D_{\eta_2}^{k-1} \frac{\mathfrak{G}_k^{\mathcal{F}}(\lambda)}{k!} \tag{9}$$

$$ {}_{\eta_3}\pounds_n^+ := \left(\eta_1 - \frac{n+1}{2(1-\lambda)}\right) + 2\eta_2 D_{\eta_1}^{-2} D_{\eta_3} + 3\eta_3\ D_{\eta_1}^{-4} D_{\eta_3}^2 + \cdots + m\eta_m\ D_{\eta_1}^{-2(m-1)} D_{\eta_3}^{m-1} - \frac{n+1}{1-\lambda} \sum_{k=2}^{n+1} D_{\eta_1}^{-2(k-1)} D_{\eta_3}^{k-1} \frac{\mathfrak{G}_k^{\mathcal{F}}(\lambda)}{k!} \tag{10}$$

$$\vdots \qquad\qquad \vdots$$

$$ {}_{\eta_m}\pounds_n^+ := \left(\eta_1 - \frac{n+1}{2(1-\lambda)}\right) + 2\eta_2 D_{\eta_1}^{-(m-1)} D_{\eta_m} + 3\eta_3\ D_{\eta_1}^{-2(m-1)} D_{\eta_m}^2 + \cdots + m\eta_m\ D_{\eta_1}^{-(m-1)^2} D_{\eta_m}^{m-1} - \frac{n+1}{1-\lambda} \sum_{k=2}^{n+1} D_{\eta_1}^{-(m-1)(k-1)} D_{\eta_m}^{k-1} \frac{\mathfrak{G}_k^{\mathcal{F}}(\lambda)}{k!} \tag{11}$$

where

$$D_{\eta_1} := \frac{\partial}{\partial \eta_1}, \quad D_{\eta_2} := \frac{\partial}{\partial \eta_2}, \quad D_{\eta_3} := \frac{\partial}{\partial \eta_3} \quad \textit{and} \quad D_{\eta_1}^{-1} := \int_0^{\eta_1} f(\eta) d\eta.$$

Proof. By differentiating equation (1) concerning η_1 and subsequently juxtaposing the coefficients corresponding to similar powers of t on both sides of the ensuing equation, we arrive at the following expression:

$$\frac{\partial}{\partial \eta_1}\{{}_{\mathcal{G}}\mathcal{E}_n^{\mathcal{F}}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda)\} = n\ {}_{\mathcal{G}}\mathcal{E}_{n-1}^{\mathcal{F}}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda).$$

As a result of the steps outlined above, we reach the subsequent expression:

$${}_{\eta_1}\pounds_n^-\{{}_{\mathcal{G}}\mathcal{E}_n^{\mathcal{F}}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda)\} = \frac{1}{n}D_{\eta_1}\{{}_{\mathcal{G}}\mathcal{E}_n^{\mathcal{F}}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda)\} = {}_{\mathcal{G}}\mathcal{E}_{n-1}^{\mathcal{F}}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda), \tag{12}$$

and thereby confirming the assertion made in (4).

By differentiating equation (1) with respect to η_2 and then equating the coefficients of corresponding powers of t on both sides, the resulting expression is:

$$\frac{\partial}{\partial \eta_2}\{{}_{\mathcal{G}}\mathcal{E}_n^{\mathcal{F}}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda)\} = n(n-1)\ {}_{\mathcal{G}}\mathcal{E}_{n-2}^{\mathcal{F}}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda).$$

The earlier expression can be alternatively stated as:

$$\frac{\partial}{\partial \eta_2}\{{}_{\mathcal{G}}\mathcal{E}_n^{\mathcal{F}}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda)\} = n\ \frac{\partial}{\partial \eta_1}\{{}_{\mathcal{G}}\mathcal{E}_{n-1}^{\mathcal{F}}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda)\},$$

and eventually provides

$${}_{\eta_2}\pounds_n^-\{{}_{\mathcal{G}}\mathcal{E}_n^{\mathcal{F}}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda)\} = \frac{1}{n}D_{\eta_1}^{-1}D_{\eta_2}\{{}_{\mathcal{G}}\mathcal{E}_n^{\mathcal{F}}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda)\} = {}_{\mathcal{G}}\mathcal{E}_{n-1}^{\mathcal{F}}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda). \tag{13}$$

thus, the affirmation in (5) is substantiated.

By differentiating equation (1) with respect to η_3 and then comparing the coefficients of like powers of t on both sides of the resulting equation, we obtain the following expression:

$$\frac{\partial}{\partial \eta_3}\{{}_{\mathcal{G}}\mathcal{E}_n^{\mathcal{F}}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda)\} = n(n-1)(n-2)\ {}_{\mathcal{G}}\mathcal{E}_{n-3}^{\mathcal{F}}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda). \tag{14}$$

The earlier expression (14) can be presented in the form

$$\frac{\partial}{\partial \eta_3}\{{}_{\mathcal{G}}\mathcal{E}_n^{\mathcal{F}}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda)\} = n\ \frac{\partial^2}{\partial \eta_1^2}\{{}_{\mathcal{G}}\mathcal{E}_{n-1}^{\mathcal{F}}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda)\},$$

and thus eventually provides

$${}_{\eta_3}\pounds_n^-\{{}_{\mathcal{G}}\mathcal{E}_n^{\mathcal{F}}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda)\} = \frac{1}{n}D_{\eta_1}^{-2}D_{\eta_3}\{{}_{\mathcal{G}}\mathcal{E}_n^{\mathcal{F}}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda)\} = {}_{\mathcal{G}}\mathcal{E}_{n-1}^{\mathcal{F}}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda). \tag{15}$$

Thus, the assertion in (6) is validated.

Finally, by differentiating equation (1) with respect to η_m and equating the coefficients of corresponding powers of t on both sides of the resulting equation, we derive the following expression:

$$\frac{\partial}{\partial \eta_m}\{{}_{\mathcal{G}}\mathcal{E}_n^{\mathcal{F}}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda)\} = n(n-1)(n-2)(n-m+1){}_{\mathcal{G}}\mathcal{E}_{n-m}^{\mathcal{F}}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda),$$

and further presented as

$$\frac{\partial}{\partial \eta_m}\{{}_{\mathcal{G}}\mathcal{E}_n^{\mathcal{F}}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda)\} = n\ \frac{\partial^{m-1}}{\partial \eta_1^{m-1}}\{{}_{\mathcal{G}}\mathcal{E}_{n-1}^{\mathcal{F}}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda)\},$$

thus eventually gives

$$_{\eta_m}\pounds_n^{-}\{{}_{\mathcal{G}}\mathcal{E}_n^{\mathcal{F}}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda)\} = \frac{1}{n}D_{\eta_1}^{-(m-1)}D_{\eta_m}\{{}_{\mathcal{G}}\mathcal{E}_n^{\mathcal{F}}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda)\} = {}_{\mathcal{G}}\mathcal{E}_{n-1}^{\mathcal{F}}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda). \tag{16}$$

Thus, the statement in (7) is validated.

To derive the equation for the raising operator in (8), we use the following expression:

$${}_{\mathcal{G}}\mathcal{E}_{n-m}^{\mathcal{F}}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda) = ({}_{\eta_1}\pounds_{n-m+1}^{-}\ {}_{\eta_1}\pounds_{n-m+2}^{-}\cdots{}_{\eta_1}\pounds_{n-1}^{-}\ {}_{\eta_1}\pounds_{n}^{-})\{{}_{\mathcal{G}}\mathcal{E}_n^{\mathcal{F}}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda)\}. \tag{17}$$

Therefore, considering expression (12), we can represent expression (17) in a simplified form as:

$${}_{\mathcal{G}}\mathcal{E}_{n-m}^{\mathcal{F}}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda) = \frac{(n-m)!}{m!}D_{\eta_1}^{m}\{{}_{\mathcal{G}}\mathcal{E}_n^{\mathcal{F}}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda)\}. \tag{18}$$

By substituting equation (18) into the recurrence relation (3), we deduce that:

$${}_{\mathcal{G}}\mathcal{E}_{n+1}^{\mathcal{F}}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda) = \Bigg(\left(\eta_1 - \frac{n+1}{2(1-\lambda)}\right) + 2\eta_2 D_{\eta_1} + 3\eta_3 D_{\eta_1}^2 + \cdots + m\ \eta_m D_{\eta_1}^{m-1} - \frac{n+1}{1-\lambda}\sum_{k=2}^{n+1} D_{\eta_1}^{(k-1)}\frac{\mathfrak{G}_k^{\mathcal{F}}(\lambda)}{k!}\Bigg).$$

Thus, the correctness of the raising operator ${}_{\eta_1}\pounds_n^{+}$ in (8) is confirmed.

To demonstrate the raising operator in (9), we examine the following relationship:

$${}_{\mathcal{G}}\mathcal{E}_{n-m}^{\mathcal{F}}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda) = ({}_{\eta_2}\pounds_{n-m+1}^{-}\ {}_{\eta_2}\pounds_{n-m+2}^{-}\cdots{}_{\eta_2}\pounds_{n-1}^{-}\ {}_{\eta_2}\pounds_{n}^{-})\{{}_{\mathcal{G}}\mathcal{E}_n^{\mathcal{F}}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda)\}.$$

Given equation (13), the above expression can be expanded as follows:

$${}_{\mathcal{G}}\mathcal{E}_{n-m}^{\mathcal{F}}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda) = \frac{(n-m)!}{m!}D_{\eta_1}^{-(m-1)}D_{\eta_2}^{(m-1)}\{{}_{\mathcal{G}}\mathcal{E}_n^{\mathcal{F}}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda)\}. \tag{19}$$

By substituting equation (19) into the recurrence relation (3), we can conclude that:

$$
{}_{\mathcal{G}}\mathcal{E}^{\mathcal{F}}_{n+1}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda) = \Bigg(\Big(\eta_1 - \frac{n+1}{2(1-\lambda)}\Big) + 2\eta_2\ D_{\eta_1}^{-1}D_{\eta_2} + 3\eta_3\ D_{\eta_1}^{-2}D_{\eta_2}^{2} + \cdots
+ m\eta_m\ D_{\eta_1}^{-(m-1)}D_{\eta_2}^{m-1} - \frac{n+1}{1-\lambda}\sum_{k=2}^{n+1} D_{\eta_1}^{-(k-1)}D_{\eta_2}^{k-1}\frac{\mathfrak{G}_k^{\mathcal{F}}(\lambda)}{k!}\Bigg).
$$

Thus, we have successfully confirmed the validity of Assertion (9) for the raising operator ${}_{\eta_2}\pounds_n^+$.

To illustrate the raising operator ${}_{\eta_3}\pounds_n^+$, we analyze the following expression:

$$
{}_{\mathcal{G}}\mathcal{E}^{\mathcal{F}}_{n-m}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda) = ({}_{\eta_3}\pounds^-_{n-m+1}\ {}_{\eta_3}\pounds^-_{n-m+2}\cdots {}_{\eta_3}\pounds^-_{n-1}\ {}_{\eta_3}\pounds^-_{n})\{{}_{\mathcal{G}}\mathcal{E}^{\mathcal{F}}_{n}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda)\},
$$

Given equation (15), the above expression can be expanded in the following manner:

$$
{}_{\mathcal{G}}\mathcal{E}^{\mathcal{F}}_{n-m}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda) = \frac{(n-m)!}{m!}D_{\eta_1}^{-2(m-1)}D_{\eta_3}^{(m-1)}\{{}_{\mathcal{G}}\mathcal{E}^{\mathcal{F}}_{n}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda)\}. \tag{20}
$$

By substituting equation (20) into the recurrence relation (3), we find that:

$$
{}_{\mathcal{G}}\mathcal{E}^{\mathcal{F}}_{n+1}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda) = \Bigg(\Big(\eta_1 - \frac{n+1}{2(1-\lambda)}\Big) + 2\eta_2 D_{\eta_1}^{-2}D_{\eta_3} + 3\eta_3\ D_{\eta_1}^{-4}D_{\eta_3}^{2} + \cdots
+ m\eta_m\ D_{\eta_1}^{-2(m-1)}D_{\eta_3}^{m-1} - \frac{n+1}{1-\lambda}\sum_{k=2}^{n+1} D_{\eta_1}^{-2(k-1)}D_{\eta_3}^{k-1}\frac{\mathfrak{G}_k^{\mathcal{F}}(\lambda)}{k!}\Bigg).
$$

Thus, we have effectively verified the validity of Assertion (10) for the raising operator ${}_{\eta_3}\pounds_n^+$.

In summary, to validate the raising operator ${}_{\eta_m}\pounds_n^+$, we examine the following expression:

$$
{}_{\mathcal{G}}\mathcal{E}^{\mathcal{F}}_{n-m}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda) = ({}_{\eta_m}\pounds^-_{n-m+1}\ {}_{\eta_m}\pounds^-_{n-m+2}\cdots {}_{\eta_m}\pounds^-_{n-1}\ {}_{\eta_m}\pounds^-_{n})\{{}_{\mathcal{G}}\mathcal{E}^{\mathcal{F}}_{n}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda)\},
$$

Given equation (16), the above expression can be expanded as follows:

$$
{}_{\mathcal{G}}\mathcal{E}^{\mathcal{F}}_{n-m}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda) = \frac{(n-m)!}{m!}D_{\eta_1}^{-(m-1)^2}D_{\eta_m}^{(m-1)}\{{}_{\mathcal{G}}\mathcal{E}^{\mathcal{F}}_{n}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda)\}. \tag{21}
$$

By substituting equation (21) into the recurrence relation (3), we deduce that:

$$
{}_{\mathcal{G}}\mathcal{E}^{\mathcal{F}}_{n+1}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda) = \Bigg(\Big(\eta_1 - \frac{n+1}{2(1-\lambda)}\Big) + 2\eta_2 D_{\eta_1}^{-(m-1)}D_{\eta_m} + 3\eta_3\ D_{\eta_1}^{-2(m-1)}D_{\eta_m}^{2} + \cdots
+ m\eta_m\ D_{\eta_1}^{-(m-1)^2}D_{\eta_m}^{m-1} - \frac{n+1}{1-\lambda}\sum_{k=2}^{n+1} D_{\eta_1}^{-(m-1)(k-1)}D_{\eta_m}^{k-1}\frac{\mathfrak{G}_k^{\mathcal{F}}(\lambda)}{k!}\Bigg).
$$

Thus, the validity of expression (11) for the raising operator ${}_{\eta_m}\pounds_n^+$ is established.

We begin a thorough investigation of the families of differential equations that the Multivariate Hermite-Frobenius-Genocchi polynomials satisfy in the next section. This involves a thorough analysis covering many types of differential equations, such as partial, integrodifferential, and differential. By carefully using the factorization process, these equations are derived, providing an explanation of the complex characteristics and connections included in the polynomial solutions. By dissecting the various mathematical processes that these polynomials capture, this analytical project hopes to increase knowledge of these polynomials' importance and function in mathematical analysis and application.

3. Differential Equations

We cover an extensive spectrum of differential equations in this part, elucidating their intricate structures and emphasising their relationships to the MVHFGP ${}_{\mathcal{G}}\mathcal{E}_n^{\mathcal{F}}(\eta_1, \eta_2, \eta_3, \cdots, \eta_m; \lambda)$. By means of painstaking examination, we want to provide a refined understanding of the basic characteristics of these equations and their complex interactions, thereby clarifying their significance in the context of MVHFGP. We want to reveal the underlying mathematical linkages and patterns that lead to a greater comprehension of the equations and the polynomials by investigating their associations. The purpose of this analytical project is to improve understanding and appreciation of the role that MVHFGP plays in mathematical analysis and problem resolution.

For the Multivariate Hermite-Frobenius-Genocchi polynomials (MVHFGP) ${}_{\mathcal{G}}\mathcal{E}_n^{\mathcal{F}}(\eta_1, \eta_2, \eta_3, \cdots, \eta_m; \lambda)$, we establish differential, integrodifferential, and partial differential equations. Moreover, we derive the differential equation for the MVHFGP ${}_{\mathcal{G}}\mathcal{E}_n^{\mathcal{F}}(\eta_1, \eta_2, \eta_3, \cdots, \eta_m; \lambda)$ through the following conclusion:

Theorem 3. *The MVHFGP* ${}_{\mathcal{G}}\mathcal{E}_n^{\mathcal{F}}(\eta_1, \eta_2, \eta_3, \cdots, \eta_m; \lambda)$ *satisfy the following differential equation:*

$$\left(\Big(\eta_1 - \frac{n+1}{2(1-\lambda)}\Big)D_{\eta_1} + 2\eta_2 D_{\eta_1}^2 + 3\eta_3 D_{\eta_1}^3 + \cdots + m\ \eta_m D_{\eta_1}^m - \frac{n+1}{1-\lambda}\sum_{k=2}^{n+1} D_{\eta_1}^k \frac{\mathfrak{G}_k^{\mathcal{F}}(\lambda)}{k!} - n\right) \times {}_{\mathcal{G}}\mathcal{E}_n^{\mathcal{F}}(\eta_1, \eta_2, \eta_3, \cdots, \eta_m; \lambda) = 0. \quad (22)$$

Proof. The expressions (4) and (8) for the shift operators are utilized in the factorization formula, given by:

$${}_{\eta_1}\pounds_{n+1}^{-}\ {}_{\eta_1}\pounds_n^{+}\{{}_{\mathcal{G}}\mathcal{E}_n^{\mathcal{F}}(\eta_1, \eta_2, \eta_3, \cdots, \eta_m; \lambda)\} = {}_{\mathcal{G}}\mathcal{E}_n^{\mathcal{F}}(\eta_1, \eta_2, \eta_3, \cdots, \eta_m; \lambda),$$

After simplifying the mathematical expression, the statement in (22) is confirmed.

Theorem 4. *The MVHFGP* ${}_{\mathcal{G}}\mathcal{E}_n^{\mathcal{F}}(\eta_1, \eta_2, \eta_3, \cdots, \eta_m; \lambda)$ *satisfy the following integrodifferential equations:*

$$\Bigg\{\Big(\eta_1 - \frac{n+1}{2(1-\lambda)}\Big)D_{\eta_2} + 2\eta_2\ D_{\eta_1}^{-1}D_{\eta_2}^2 + 3\eta_3\ D_{\eta_1}^{-2}D_{\eta_2}^3 + \cdots + m\eta_m\ D_{\eta_1}^{-(m-1)}D_{\eta_2}^m$$

$$- \frac{n+1}{1-\lambda}\sum_{k=2}^{n+1} D_{\eta_1}^{-(k-1)} D_{\eta_2}^{k} \frac{\mathfrak{G}_k^{\mathcal{F}}(\lambda)}{k!} - (n+1)D_{\eta_1}\Bigg\}{}_{\mathcal{G}}\mathcal{E}_n^{\mathcal{F}}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda)=0, \quad (23)$$

$$\Bigg\{\Big(\eta_1 - \frac{n+1}{2(1-\lambda)}\Big)D_{\eta_3} + 2\eta_2\ D_{\eta_1}^{-1} D_{\eta_2} D_{\eta_3} + 3\eta_3\ D_{\eta_1}^{-2} D_{\eta_2} D_{\eta_3} + \cdots + m\eta_m\ D_{\eta_1}^{-(m-1)} D_{\eta_2}^{m-1} D_{\eta_3}$$
$$- \frac{n+1}{1-\lambda}\sum_{k=2}^{n+1} D_{\eta_1}^{-(k-1)} D_{\eta_2}^{k-1} D_{\eta_3} \frac{\mathfrak{G}_k^{\mathcal{F}}(\lambda)}{k!} - (n+1)D_{\eta_1}^2\Bigg\}{}_{\mathcal{G}}\mathcal{E}_n^{\mathcal{F}}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda)=0, \quad (24)$$

$$\Bigg\{\Big(\eta_1 - \frac{n+1}{2(1-\lambda)}\Big)D_{\eta_m} + 2\eta_2\ D_{\eta_1}^{-1} D_{\eta_2} D_{\eta_m} + 3\eta_3\ D_{\eta_1}^{-2} D_{\eta_2} D_{\eta_m} + \cdots + m\eta_m\ D_{\eta_1}^{-(m-1)} D_{\eta_2}^{m-1} D_{\eta_m}$$
$$- \frac{n+1}{1-\lambda}\sum_{k=2}^{n+1} D_{\eta_1}^{-(k-1)} D_{\eta_2}^{k-1} D_{\eta_m} \frac{\mathfrak{G}_k^{\mathcal{F}}(\lambda)}{k!} - (n+1)D_{\eta_m}^{m-1}\Bigg\}{}_{\mathcal{G}}\mathcal{E}_n^{\mathcal{F}}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda)=0, \quad (25)$$

$$\Bigg\{\Big(\eta_1 - \frac{n+1}{2(1-\lambda)}\Big)D_{\eta_2} + 2\eta_2 D_{\eta_1}^{-2} D_{\eta_2} D_{\eta_3} + 3\eta_3\ D_{\eta_1}^{-4} D_{\eta_2} D_{\eta_3}^2 + \cdots + m\eta_m\ D_{\eta_1}^{-2(m-1)} D_{\eta_2} D_{\eta_3}^{m-1}$$
$$- \frac{n+1}{1-\lambda}\sum_{k=2}^{n+1} D_{\eta_1}^{-2(k-1)} D_{\eta_2} D_{\eta_3}^{k-1} \frac{\mathfrak{G}_k^{\mathcal{F}}(\lambda)}{k!} - (n+1)D_{\eta_1}\Bigg\}{}_{\mathcal{G}}\mathcal{E}_n^{\mathcal{F}}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda)=0, \quad (26)$$

$$\Bigg\{\Big(\eta_1 - \frac{n+1}{2(1-\lambda)}\Big)D_{\eta_3} + 2\eta_2 D_{\eta_1}^{-2} D_{\eta_3}^2 + 3\eta_3\ D_{\eta_1}^{-4} D_{\eta_3}^3 + \cdots + m\eta_m\ D_{\eta_1}^{-2(m-1)} D_{\eta_3}^{m}$$
$$- \frac{n+1}{1-\lambda}\sum_{k=2}^{n+1} D_{\eta_1}^{-2(k-1)} D_{\eta_3}^{k} \frac{\mathfrak{G}_k^{\mathcal{F}}(\lambda)}{k!} - (n+1)D_{\eta_1}^2\Bigg\}{}_{\mathcal{G}}\mathcal{E}_n^{\mathcal{F}}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda)=0, \quad (27)$$

$$\Bigg\{\Big(\eta_1 - \frac{n+1}{2(1-\lambda)}\Big)D_{\eta_m} + 2\eta_2 D_{\eta_1}^{-2} D_{\eta_3} D_{\eta_m} + 3\eta_3\ D_{\eta_1}^{-4} D_{\eta_3}^2 D_{\eta_m} + \cdots + m\eta_m\ D_{\eta_1}^{-2(m-1)} D_{\eta_3}^{m-1} D_{\eta_m}$$
$$- \frac{n+1}{1-\lambda}\sum_{k=2}^{n+1} D_{\eta_1}^{-2(k-1)} D_{\eta_3}^{k-1} D_{\eta_m} \frac{\mathfrak{G}_k^{\mathcal{F}}(\lambda)}{k!} - (n+1)D_{\eta_1}^{m-1}\Bigg\}{}_{\mathcal{G}}\mathcal{E}_n^{\mathcal{F}}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda)=0, \quad (28)$$

$$\vdots$$

$$\Big\{\Big(\eta_1-\frac{n+1}{2(1-\lambda)}\Big)D_{\eta_2}+2\eta_2D_{\eta_1}^{-(m-1)}D_{\eta_2}D_{\eta_m}+3\eta_3\ D_{\eta_1}^{-2(m-1)}D_{\eta_2}D_{\eta_m}^2+\cdots+m\eta_m\ D_{\eta_1}^{-(m-1)^2}D_{\eta_2}D_{\eta_m}^{m-1}$$
$$-\frac{n+1}{1-\lambda}\sum_{k=2}^{n+1}D_{\eta_1}^{-(m-1)(k-1)}D_{\eta_2}D_{\eta_m}^{k-1}\frac{\mathfrak{G}_k^{\mathcal{F}}(\lambda)}{k!}-(n+1)D_{\eta_1}\Big\}{}_{\mathcal{G}}\mathcal{E}_n^{\mathcal{F}}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda)=0, \tag{29}$$

$$\Big\{\Big(\eta_1-\frac{n+1}{2(1-\lambda)}\Big)D_{\eta_3}+2\eta_2D_{\eta_1}^{-(m-1)}D_{\eta_3}D_{\eta_m}+3\eta_3\ D_{\eta_1}^{-2(m-1)}D_{\eta_3}D_{\eta_m}^2+\cdots+m\eta_m\ D_{\eta_1}^{-(m-1)^2}D_{\eta_3}D_{\eta_m}^{m-1}$$
$$-\frac{n+1}{1-\lambda}\sum_{k=2}^{n+1}D_{\eta_1}^{-(m-1)(k-1)}D_{\eta_3}D_{\eta_m}^{k-1}\frac{\mathfrak{G}_k^{\mathcal{F}}(\lambda)}{k!}-(n+1)D_{\eta_1}^2\Big\}{}_{\mathcal{G}}\mathcal{E}_n^{\mathcal{F}}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda)=0, \tag{30}$$

$$\Big\{\Big(\eta_1-\frac{n+1}{2(1-\lambda)}\Big)D_{\eta_m}+2\eta_2D_{\eta_1}^{-(m-1)}D_{\eta_m}^2+3\eta_3\ D_{\eta_1}^{-2(m-1)}D_{\eta_m}^3+\cdots+m\eta_m\ D_{\eta_1}^{-(m-1)^2}D_{\eta_m}^{m}$$
$$-\frac{n+1}{1-\lambda}\sum_{k=2}^{n+1}D_{\eta_1}^{-(m-1)(k-1)}D_{\eta_m}^{k}\frac{\mathfrak{G}_k^{\mathcal{F}}(\lambda)}{k!}-(n+1)D_{\eta_1}^{m-1}\Big\}{}_{\mathcal{G}}\mathcal{E}_n^{\mathcal{F}}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda)=0. \tag{31}$$

Proof. By utilizing the expression:

$$\pounds_{n+1}^{-}\ \pounds_n^{+}\{{}_{\mathcal{G}}\mathcal{E}_n^{\mathcal{F}}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda)\}={}_{\mathcal{G}}\mathcal{E}_n^{\mathcal{F}}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda). \tag{32}$$

Inserting expressions (5) and (9) into the factorization formula (32) confirms the validity of Assertion (23).

Utilizing expressions (6) and (9) within the factorization formula (31) supports the truth of Assertion (24).

Applying expressions (7) and (9) to the factorization formula (31) affirms the correctness of Assertion (25).

By using expressions (5), (6), and (7) along with expression (10), we can independently substantiate Assertions (26), (27), and (28).

Employing expressions (5), (6), and (7) in combination with expression (11) allows for the independent verification of Assertions (29), (30), and (31).

Theorem 5. *The MVHFGP* ${}_{\mathcal{G}}\mathcal{E}_n^{\mathcal{F}}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda)$ *satisfy the following partial differential equations:*

$$\Big\{\Big(\eta_1-\frac{n+1}{2(1-\lambda)}\Big)D_{\eta_1}^{n}D_{\eta_2}+2\eta_2\ D_{\eta_1}^{n-1}D_{\eta_2}^{2}+3\eta_3\ D_{\eta_1}^{n-2}D_{\eta_2}^{3}+\cdots+m\eta_m\ D_{\eta_1}^{n-(m-1)}D_{\eta_2}^{m}$$
$$-\frac{n+1}{1-\lambda}\sum_{k=2}^{n+1}D_{\eta_1}^{n-(k-1)}D_{\eta_2}^{k}\frac{\mathfrak{G}_k^{\mathcal{F}}(\lambda)}{k!}-(n+1)D_{\eta_1}^{n+1}\Big\}{}_{\mathcal{G}}\mathcal{E}_n^{\mathcal{F}}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda)=0, \quad (33)$$

$$\Big\{\Big(\eta_1-\frac{n+1}{2(1-\lambda)}\Big)D_{\eta_1}^{n}D_{\eta_3}+2\eta_2\ D_{\eta_1}^{n-1}D_{\eta_2}D_{\eta_3}+3\eta_3\ D_{\eta_1}^{n-2}D_{\eta_2}D_{\eta_3}+\cdots+m\eta_m\ D_{\eta_1}^{n-(m-1)}D_{\eta_2}^{m-1}D_{\eta_3}$$
$$-\frac{n+1}{1-\lambda}\sum_{k=2}^{n+1}D_{\eta_1}^{n-(k-1)}D_{\eta_2}^{k-1}D_{\eta_3}\frac{\mathfrak{G}_k^{\mathcal{F}}(\lambda)}{k!}-(n+1)D_{\eta_1}^{n+2}\Big\}{}_{\mathcal{G}}\mathcal{E}_n^{\mathcal{F}}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda)=0, \quad (34)$$

$$\Big\{\Big(\eta_1-\frac{n+1}{2(1-\lambda)}\Big)D_{\eta_1}^{2n}D_{\eta_m}+2\eta_2\ D_{\eta_1}^{2n-1}D_{\eta_2}D_{\eta_m}+3\eta_3\ D_{\eta_1}^{2n-2}D_{\eta_2}D_{\eta_m}+\cdots+m\eta_m\ D_{\eta_1}^{2n-(m-1)}D_{\eta_2}^{m-1}D_{\eta_m}$$
$$-\frac{n+1}{1-\lambda}\sum_{k=2}^{n+1}D_{\eta_1}^{2n-(k-1)}D_{\eta_2}^{k-1}D_{\eta_m}\frac{\mathfrak{G}_k^{\mathcal{F}}(\lambda)}{k!}-(n+1)D_{\eta_m}^{2n+m-1}\Big\}{}_{\mathcal{G}}\mathcal{E}_n^{\mathcal{F}}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda)=0, \quad (35)$$

$$\Big\{\Big(\eta_1-\frac{n+1}{2(1-\lambda)}\Big)D_{\eta_1}^{2n+2}D_{\eta_2}+2\eta_2D_{\eta_1}^{2n}D_{\eta_2}D_{\eta_3}+3\eta_3\ D_{\eta_1}^{2n-2}D_{\eta_2}D_{\eta_3}^{2}+\cdots+m\eta_m\ D_{\eta_1}^{2n+4-2m}D_{\eta_2}D_{\eta_3}^{m-1}$$
$$-\frac{n+1}{1-\lambda}\sum_{k=2}^{n+1}D_{\eta_1}^{2n+4-k}D_{\eta_2}D_{\eta_3}^{k-1}\frac{\mathfrak{G}_k^{\mathcal{F}}(\lambda)}{k!}-(n+1)D_{\eta_1}^{2n+2}\Big\}{}_{\mathcal{G}}\mathcal{E}_n^{\mathcal{F}}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda)=0, \quad (36)$$

$$\Big\{\Big(\eta_1-\frac{n+1}{2(1-\lambda)}\Big)D_{\eta_1}^{2n+2}D_{\eta_3}+2\eta_2D_{\eta_1}^{2n}D_{\eta_3}^{2}+3\eta_3\ D_{\eta_1}^{2n-2}D_{\eta_3}^{3}+\cdots+m\eta_m\ D_{\eta_1}^{2n+4-m}D_{\eta_3}^{m}$$
$$-\frac{n+1}{1-\lambda}\sum_{k=2}^{n+1}D_{\eta_1}^{2n+4-2k}D_{\eta_3}^{k}\frac{\mathfrak{G}_k^{\mathcal{F}}(\lambda)}{k!}-(n+1)D_{\eta_1}^{2n+4}\Big\}{}_{\mathcal{G}}\mathcal{E}_n^{\mathcal{F}}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda)=0,$$

$$\Big\{\Big(\eta_1-\frac{n+1}{2(1-\lambda)}\Big)D_{\eta_1}^{2n+2}D_{\eta_m}+2\eta_2D_{\eta_1}^{2n}D_{\eta_3}D_{\eta_m}+3\eta_3\ D_{\eta_1}^{2n-2}D_{\eta_3}^{2}D_{\eta_m}+\cdots+m\eta_m\ D_{\eta_1}^{2n+4-2m}D_{\eta_3}^{m-1}D_{\eta_m}$$
$$-\frac{n+1}{1-\lambda}\sum_{k=2}^{n+1}D_{\eta_1}^{2n+4-2k}D_{\eta_3}^{k-1}D_{\eta_m}\frac{\mathfrak{G}_k^{\mathcal{F}}(\lambda)}{k!}-(n+1)D_{\eta_1}^{2n+m+1}\Big\}{}_{\mathcal{G}}\mathcal{E}_n^{\mathcal{F}}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda)=0, \quad (37)$$

$$\vdots$$

$$\Bigg\{\Big(\eta_1-\frac{n+1}{2(1-\lambda)}\Big)D_{\eta_1}^{n^2+1}D_{\eta_2}+2\eta_2D_{\eta_1}^{n^2+1-(m-1)}D_{\eta_2}D_{\eta_m}+3\eta_3\ D_{\eta_1}^{n^2+1-2(m-1)}D_{\eta_2}D_{\eta_m}^2+\cdots+m\eta_m\ D_{\eta_1}^{n^2+1-(m-1)^2}$$
$$\times D_{\eta_2}D_{\eta_m}^{m-1}-\frac{n+1}{1-\lambda}\sum_{k=2}^{n+1}D_{\eta_1}^{n^2+1-(m-1)(k-1)}D_{\eta_2}D_{\eta_m}^{k-1}\frac{\mathfrak{G}_k^{\mathcal{F}}(\lambda)}{k!}-(n+1)D_{\eta_1}^{n^2+2}\Bigg\}{}_{\mathcal{G}}\mathcal{E}_n^{\mathcal{F}}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda)=0, \tag{38}$$

$$\Bigg\{\Big(\eta_1-\frac{n+1}{2(1-\lambda)}\Big)D^{n^2+1}D_{\eta_3}+2\eta_2D_{\eta_1}^{n^2+1-(m-1)}D_{\eta_3}D_{\eta_m}+3\eta_3\ D_{\eta_1}^{n^2+1-2(m-1)}D_{\eta_3}D_{\eta_m}^2+\cdots+m\eta_m\ D_{\eta_1}^{n^2+1-(m-1)^2}$$
$$\times D_{\eta_3}D_{\eta_m}^{m-1}-\frac{n+1}{1-\lambda}\sum_{k=2}^{n+1}D_{\eta_1}^{n^2+1-(m-1)(k-1)}D_{\eta_3}D_{\eta_m}^{k-1}\frac{\mathfrak{G}_k^{\mathcal{F}}(\lambda)}{k!}-(n+1)D_{\eta_1}^{n^2+3}\Bigg\}{}_{\mathcal{G}}\mathcal{E}_n^{\mathcal{F}}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda)=0, \tag{39}$$

$$\Bigg\{\Big(\eta_1-\frac{n+1}{2(1-\lambda)}\Big)D^{n^2+2}D_{\eta_m}+2\eta_2D_{\eta_1}^{n^2+2-(m-1)}D_{\eta_m}^2+3\eta_3\ D_{\eta_1}^{n^2+2-2(m-1)}D_{\eta_m}^3+\cdots+m\eta_m\ D_{\eta_1}^{n^2+2-(m-1)^2}D_{\eta_m}^m$$
$$-\frac{n+1}{1-\lambda}\sum_{k=2}^{n+1}D_{\eta_1}^{n^2+2-(m-1)(k-1)}D_{\eta_m}^{k}\frac{\mathfrak{G}_k^{\mathcal{F}}(\lambda)}{k!}-(n+1)D_{\eta_1}^{n^2+1+m}\Bigg\}{}_{\mathcal{G}}\mathcal{E}_n^{\mathcal{F}}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda)=0. \tag{40}$$

Proof. By taking partial derivatives n times with respect to η_1 of the integrodifferential expressions (23) and (24), Assertions (33) and (34) are confirmed.

Similarly, by differentiating the integrodifferential expression (25) $2n$ times with respect to η_1, Assertion (35) is validated.

Moreover, by taking partial derivatives $2n+2$ times with respect to η_1 of the integrodifferential expressions (26) through (28), Assertions (36) to (37) are validated.

In addition, by differentiating the integrodifferential expressions (29) and (30) n^2+1 times with respect to η_1, Assertions (38) and (39) are confirmed.

Furthermore, through partial differentiation n^2+2 times with respect to η_1 of the integrodifferential expression (31), Assertion (40) is substantiated.

4. Volterra integral equations

Volterra integral equations are highly significant in the study of special functions, particularly in the context of integral transforms and functional analysis. They offer a robust framework for capturing complex interactions between functions and are essential for solving differential equations involving special functions. Within the domain of special functions, Volterra integral equations frequently appear as integral representations of solutions to differential equations, aiding in the examination of their characteristics and behavior.

These equations are pivotal in the analysis of orthogonal polynomials and special functions, serving as a means to derive integral representations and related integral equations. Additionally, Volterra integral equations are utilized in the examination of integral transforms, including the Laplace, Fourier, and Mellin transforms, which are fundamental in the study of special functions.

Moreover, Volterra integral equations have applications across various fields such as physics, engineering, and applied mathematics, where special functions naturally arise in the description of physical phenomena. They are employed in modeling dynamic processes, such as heat conduction, wave propagation, and quantum mechanics, where special functions are crucial for expressing solutions to the differential equations that describe these phenomena. Consequently, Volterra integral equations serve as a versatile and powerful mathematical tool for exploring special functions, enhancing the analysis and understanding of their properties and applications in diverse scientific areas.

For the Multivariate Hermite-Frobenius-Genocchi polynomials (MVHFGP) $\mathcal{G}\mathcal{E}_n^{\mathcal{F}}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda)$, we derive the integral equation by establishing the following conclusion:

Theorem 6. *The MVHFGP* ${}_{\mathcal{G}}\mathcal{E}_n^{\mathcal{F}}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda)$ *satisfy the following homogeneous Volterra integral equation:*

$$\begin{aligned}
&\Psi(\eta_1)=-\tfrac{m!(1-\lambda)}{(n+1)\mathfrak{G}_m^{\mathbb{F}}(\lambda)}\Big(m\eta_m n(n-1)(n-2)\cdots(n-m+1)_{\mathcal{H}}\mathcal{E}_{n-m}^{\mathcal{F}}(\sigma,\Sigma,\lambda)+\cdots+3\eta_3 n(n-1)(n-2)\\
&{}_{\mathcal{H}}\mathcal{E}_{n-3}^{\mathcal{F}}(\sigma,\Sigma,\lambda)+2\eta_2 n(n-1)(n-2)\ {}_{\mathcal{H}}\mathcal{E}_{n-3}^{\mathcal{F}}(\sigma,\Sigma,\lambda)\eta_1\\
&+2\eta_1 n(n-1)\ {}_{\mathcal{H}}\mathcal{E}_{n-2}^{\mathcal{F}}(\sigma,\Sigma,\lambda)+\Big(\eta_1-\tfrac{1}{1-\lambda}\Big)\Big(n(n-1)\cdots(n-m+1)\ {}_{\mathcal{H}}\mathcal{E}_{n-m}^{\mathcal{F}}(\sigma,\Sigma,\lambda)\tfrac{\eta_1^m}{m!}+\cdots+n(n-1)\\
&{}_{\mathcal{H}}\mathcal{E}_{n-2}^{\mathcal{F}}(\sigma,\Sigma,\lambda)x+n\ {}_{\mathcal{H}}\mathcal{E}_{n-1}^{\mathcal{F}}(\sigma,\Sigma,\lambda)\Big)-n(n-1)\cdots(n-m+1)\ {}_{\mathcal{H}}\mathcal{E}_{n-m}^{\mathcal{F}}(\sigma,\Sigma,\lambda)\tfrac{\eta_1^m}{2!\ m!}-\cdots\\
&-n(n-1)\ {}_{\mathcal{H}}\mathcal{E}_{n-2}^{\mathcal{F}}(\sigma,\Sigma,\lambda)\tfrac{\eta_1^2}{2!}-n\ {}_{\mathcal{H}}\mathcal{E}_{n-1}^{\mathcal{F}}(\sigma,\Sigma,\lambda)\eta_1-{}_{\mathcal{H}}\mathcal{E}_{n}^{\mathcal{F}}(\sigma,\Sigma,\lambda)\Big)+\int_0^{\eta_1}\Big(\tfrac{m!(1-\lambda)}{(n+1)\mathfrak{G}_m^{\mathbb{F}}(\lambda)}\Big(3\eta_3+2\eta_2\\
&(\eta_1-\xi)+\Big(\eta_1-\tfrac{1}{1-\lambda}\Big)\tfrac{(\eta_1-\xi)^2}{2!}\Big)-n\tfrac{(\eta_1-\xi)^3}{3!}\Big)\Psi(\xi)d\xi.
\end{aligned} \tag{41}$$

Proof. We start by looking at the MVHFGP's ${}_{\mathcal{G}}\mathcal{E}_n^{\mathcal{F}}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda)$ fourth-order differential equation of the following form:

$$\Big(\Big(D_{\eta_1}^m+\cdots+\tfrac{m!(1-\lambda)}{(n+1)\mathfrak{G}_m^{\mathbb{F}}(\lambda)}\Big(3\eta_3\ D_{\eta_1}^3+2\eta_2 D_{\eta_1}^2+\Big(\eta_1-\tfrac{n+1}{2(1-\lambda)}\Big)D_{\eta_1}-n\Big)\Big)\ {}_{\mathcal{G}}\mathcal{E}_n^{\mathcal{F}}(\eta_1,\eta_2,\eta_3,\cdots,\eta_m;\lambda)=0. \tag{42}$$

For initial conditions, we find

$$
\begin{aligned}
&{}_{\mathcal{G}}\mathcal{E}_n^{\mathcal{F}}(\eta_1,\eta_2,0,\cdots,0;\lambda) = {}_{\mathcal{G}}\mathcal{E}_n^{\mathcal{F}}(\eta_1,\eta_2;\lambda) = n!\sum_{k=0}^{n}\sum_{r=0}^{[\frac{k}{2}]}\tfrac{\mathbb{E}^{\mathbb{F}}_{n-k}(\lambda)\ {\eta_1}^r\ {\eta_2}^{k-2r}}{(n-k)!\ r!\ (k-2r)}\\
&:= {}_{\mathcal{H}}\mathcal{E}_n^{\mathcal{F}}(\sigma,\Sigma,\lambda),\\
&\tfrac{d}{d\eta_1}{}_{\mathcal{G}}\mathcal{E}_n^{\mathcal{F}}(\eta_1,\eta_2,0,\cdots,0;\lambda) = n\ {}_{\mathcal{G}}\mathcal{E}_{n-1}^{\mathcal{F}}(\eta_1,\eta_2,0,\cdots,0;\lambda) = n(n-1)!\sum_{k=0}^{n-1}\sum_{r=0}^{[\frac{k}{2}]}\tfrac{\mathbb{E}^{\mathbb{F}}_{n-1-k}(\lambda)\ {\eta_1}^r\ {\eta_2}^{k-2r}}{(n-1-k)!\ r!\ (k-2r)}\\
&:= n\ {}_{\mathcal{H}}\mathcal{E}_{n-1}^{\mathcal{F}}(\sigma,\Sigma,\lambda),\\
&\tfrac{d^2}{d^2_{\eta_1}}{}_{\mathcal{G}}\mathcal{E}_n^{\mathcal{F}}(\eta_1,\eta_2,0,\cdots,0;\lambda) = n(n-1)\ {}_{\mathcal{G}}\mathcal{E}_{n-1}^{\mathcal{F}}(\eta_1,\eta_2,0,\cdots,0;\lambda) = n(n-1)(n-2)!\\
&\times\sum_{k=0}^{n-2}\sum_{r=0}^{[\frac{k}{2}]}\tfrac{\mathbb{E}^{\mathbb{F}}_{n-2-k}(\lambda)\ {\eta_1}^r\ {\eta_2}^{k-2r}}{(n-2-k)!\ r!\ (k-2r)} := n(n-1)\ {}_{\mathcal{H}}\mathcal{E}_{n-2}^{\mathcal{F}}(\sigma,\Sigma,\lambda),\\
&\tfrac{d^3}{d^3_{\eta_1}}{}_{\mathcal{G}}\mathcal{E}_n^{\mathcal{F}}(\eta_1,\eta_2,0,\cdots,0;\lambda) = n(n-1)(n-2)\ {}_{\mathcal{G}}\mathcal{E}_{n-3}^{\mathcal{F}}(\eta_1,\eta_2,0,\cdots,0;\lambda) = n(n-1)(n-2)(n-3)!\\
&\sum_{k=0}^{n-3}\sum_{r=0}^{[\frac{k}{2}]}\tfrac{\mathbb{E}^{\mathbb{F}}_{n-3-k}(\lambda)\ {\eta_1}^r\ {\eta_2}^{k-2r}}{(n-3-k)!\ r!\ (k-2r)}\\
&:= n(n-1)(n-2)\ {}_{\mathcal{H}}\mathcal{E}_{n-3}^{\mathcal{F}}(\sigma,\Sigma,\lambda),\\
&\vdots\\
&\tfrac{d^m}{d^m_{\eta_1}}{}_{\mathcal{G}}\mathcal{E}_n^{\mathcal{F}}(\eta_1,\eta_2,0,\cdots,0;\lambda) = n(n-1)(n-2)\cdots(n-m+1)\ {}_{\mathcal{G}}\mathcal{E}_{n-m}^{\mathcal{F}}(\eta_1,\eta_2,0,\cdots,0;\lambda)\\
&= n(n-1)(n-2)\cdots(n-m+1)!\sum_{k=0}^{n-m+1}\sum_{r=0}^{[\frac{k}{2}]}\tfrac{\mathbb{E}^{\mathbb{F}}_{n-m-k}(\lambda)\ {\eta_1}^r\ {\eta_2}^{k-2r}}{(n-m-k)!\ r!\ (k-2r)}\\
&:= n(n-1)(n-2)\cdots(n-m+1)\ {}_{\mathcal{H}}\mathcal{E}_{n-m}^{\mathcal{F}}(\sigma,\Sigma,\lambda),
\end{aligned}
\tag{43}
$$

respectively, where

$$
{}_{\mathcal{H}}\mathcal{E}_s^{\mathcal{F}}(\sigma,\Sigma,\lambda) := s!\sum_{k=0}^{s}\sum_{r=0}^{[\frac{k}{2}]}\frac{\mathbb{E}^{\mathbb{F}}_{s-k}(\lambda)\ {\eta_1}^r\ {\eta_2}^{k-2r}}{(s-k)!\ r!\ (k-2r)},\quad s=n,\ n-1,\ n-2,\ n-3\ \cdots\ n-m+1.
$$

Consider

$$
D^m_{\eta_1}{}_{\mathcal{G}}\mathcal{E}_n^{\mathcal{F}}(\eta_1,\eta_2,\cdots,\eta_m;\lambda) = \Psi(\eta_1).
$$

By integrating the given equation and applying the initial conditions specified in equation (43), we derive the following expression:

$$\frac{d^m}{d\eta_1{}^m}{}_{\mathcal{G}}\mathcal{E}_n^{\mathcal{F}}(\eta_1,\eta_2,\cdots,\eta_m;\lambda)=\int\limits_0^{\eta_1}\Psi(\xi)d\xi+n(n-1)\cdots(n-m+1)\ {}_{\mathcal{H}}\mathcal{E}_{n-m}^{\mathcal{F}}(\sigma,\Sigma,\lambda),$$

$$\vdots$$

$$\frac{d^3}{d\eta_1{}^3}{}_{\mathcal{G}}\mathcal{E}_n^{\mathcal{F}}(\eta_1,\eta_2,\cdots,\eta_m;\lambda)=\int\limits_0^{\eta_1}\Psi(\xi)d\xi+n(n-1)(n-2)\ {}_{\mathcal{H}}\mathcal{E}_{n-3}^{\mathcal{F}}(\sigma,\Sigma,\lambda),$$

$$\frac{d^2}{dq{-}1^2}{}_{\mathcal{G}}\mathcal{E}_n^{\mathcal{F}}(\eta_1,\eta_2,\cdots,\eta_m;\lambda)=\int\limits_0^{\eta_1}\Psi(\xi)d\xi^2+n(n-1)(n-2)\ {}_{\mathcal{H}}\mathcal{E}_{n-3}^{\mathcal{F}}(\sigma,\Sigma,\lambda)\eta_1+n(n-1)\ {}_{\mathcal{H}}\mathcal{E}_{n-2}^{\mathcal{F}}(\sigma,\Sigma,\lambda),$$

$$\begin{aligned}\frac{d}{d_{\eta_1}}{}_{\mathcal{G}}\mathcal{E}_n^{\mathcal{F}}(\eta_1,\eta_2,\cdots,\eta_m;\lambda)=&\int\limits_0^{\eta_1}\Psi(\xi)d\xi^3+n(n-1)(n-2)\ {}_{\mathcal{H}}\mathcal{E}_{n-3}^{\mathcal{F}}(\sigma,\Sigma,\lambda)\frac{\eta_1{}^2}{2!}+n(n-1)\ {}_{\mathcal{H}}\mathcal{E}_{n-2}^{\mathcal{F}}(\sigma,\Sigma,\lambda)\eta_1\\&+n\ {}_{\mathcal{H}}\mathcal{E}_{n-1}^{\mathcal{F}}(\sigma,\Sigma,\lambda),\end{aligned}$$

$$\begin{aligned}{}_{\mathcal{G}}\mathcal{E}_n^{\mathcal{F}}(\eta_1,\eta_2,\cdots,\eta_m;\lambda)=&\int\limits_0^{\eta_1}\Psi(\xi)d\xi^4+n(n-1)(n-2)\ {}_{\mathcal{H}}\mathcal{E}_{n-3}^{\mathcal{F}}(\sigma,\Sigma,\lambda)\frac{\eta_1{}^3}{2!\ 3!}+n(n-1)\ {}_{\mathcal{H}}\mathcal{E}_{n-2}^{\mathcal{F}}(\sigma,\Sigma,\lambda)\frac{q{-}1^2}{2!}\\&+n\ {}_{\mathcal{H}}\mathcal{E}_{n-1}^{\mathcal{F}}(\sigma,\Sigma,\lambda)\eta_1+{}_{\mathcal{H}}\mathcal{E}_n^{\mathcal{F}}(\sigma,\Sigma,\lambda).\end{aligned}$$

In light of the previous expression in (42), we have

$$\begin{aligned}&\Psi(\eta_1)=-\frac{m!(1-\lambda)}{(n+1)\mathfrak{G}_m^{\mathbb{F}}(\lambda)}\Big(m\eta_m\Big(\int\limits_0^{\eta_1}\Psi(\xi)d\xi+n(n-1)\cdots(n-m+1)\ {}_{\mathcal{H}}\mathcal{E}_{n-m}^{\mathcal{F}}(\sigma,\Sigma,\lambda)\Big)+\cdots\\&+2\eta_2\Big(\int\limits_0^{\eta_1}\Psi(\xi)d\xi^3+n(n-1)(n-2)\ {}_{\mathcal{H}}\mathcal{E}_{n-3}^{\mathcal{F}}(\sigma,\Sigma,\lambda)\eta_1+n(n-1)\ {}_{\mathcal{H}}\mathcal{E}_{n-2}^{\mathcal{F}}(\sigma,\Sigma,\lambda)\Big)+\Big(\eta_1-\frac{1}{1-\lambda}\Big)\\&\Big(\int\limits_0^{\eta_1}\Psi(\xi)d\xi^3+n(n-1)(n-2)\ {}_{\mathcal{H}}\mathcal{E}_{n-3}^{\mathcal{F}}(\sigma,\Sigma,\lambda)\frac{\eta_1{}^2}{2!}+n(n-1)\ {}_{\mathcal{H}}\mathcal{E}_{n-2}^{\mathcal{F}}(\sigma,\Sigma,\lambda)\eta_1+n\ {}_{\mathcal{H}}\mathcal{E}_{n-1}^{\mathcal{F}}(\sigma,\Sigma,\lambda)\Big)\Big)\\&+\frac{6n(1-\lambda)}{\mathfrak{G}_3^{\mathbb{F}}(\lambda)}\Big(\int\limits_0^{\eta_1}\Psi(\xi)d\xi^4+n(n-1)(n-2)\ {}_{\mathcal{H}}\mathcal{E}_{n-3}^{\mathcal{F}}(\sigma,\Sigma,\lambda)\frac{\eta_1{}^3}{2!\ 3!}+n(n-1)\ {}_{\mathcal{H}}\mathcal{E}_{n-2}^{\mathcal{F}}(\sigma,\Sigma,\lambda)\frac{\eta_1{}^2}{2!}\\&+n\ {}_{\mathcal{H}}\mathcal{E}_{n-1}^{\mathcal{F}}(\sigma,\Sigma,\lambda)q_1+{}_{\mathcal{H}}\mathcal{E}_n^{\mathcal{F}}(\sigma,\Sigma,\lambda)\Big).\end{aligned}$$

Therefore, by using the following method, after simplifying and integrating the resulting equation

$$\int\limits_b^{q_1}f(\eta)\ d\eta^n=\int\limits_b^{q_1}\frac{(q_1-\eta)^{n-1}}{(n-1)!}f(\eta)d\eta,$$

result (41) is demonstrated.

5. Conclusion

This paper introduces a new family of hybrid multidimensional polynomials generated by convolving Frobenius-Genocchi and Hermite polynomials. The study thoroughly investigates the properties of these polynomials, leading to the development of a recurrence relation and a set of shift operators that these multivariate Hermite-Frobenius-Genocchi polynomials satisfy. We also demonstrate that these polynomials adhere to a differential equation and a series of partial and integrodifferential equations. Additionally, we identify the specific Volterra integral equation that this polynomial family satisfies. This research makes a substantial contribution to polynomial theory by proposing and analyzing the characteristics of this novel polynomial family.

Further exploration and research could uncover additional features of these polynomials. Investigating symmetric identities, extended and generalized forms, and other properties may lead to new insights and applications. Future studies might need to address potential challenges related to computational issues, especially when dealing with new datasets and tackling determinant forms and summation equations.

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Competing interests

The authors declare no competing interests.

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