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Stability Analysis of Fractional Chaotic and Fractional-Order Hyperchain Systems Using Lyapunov Functions

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Abstract. This study investigates the stability of nonlinear systems, particularly those characterized by eigenvalues. We introduce dynamic Lyapunov functions as a mechanism for stability analysis, especially when explicit solutions are not available. The authors provide stability criteria at the equilibrium point, demonstrating exponential stability and ensuring a return to equilibrium following disturbances. The results have a big effect on the design and analysis of control systems because they provide a new way to achieve stability without using complicated calculations or assumptions. The abstract delineates the Riemann–Liouville fractional integral, Caputo fractional integrals and derivatives, and the Mittag–Leffler function. The research employs the root–Horwitz criteria and introduces a novel formulation of the superchaotic Chen system. Fractional superchain systems (FHCS) represent a sophisticated framework for investigation.

2020 Mathematics Subject Classifications: 34C28, 37C25, 34A08

Key Words and Phrases: Caputo Derivative, Mittag-Leffler Function, Hurwitz Matrices, Stability Analysis, Hyperchain Systems

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1. Introduction

The study begins by emphasizing and reinforcing the importance of the basic ideas and properties of fractional calculus, including derivatives and integrals of incorrect order. It has become one of the most advanced fields in applied mathematics, physics, and many other scientific and technical disciplines. This rise is evident from the many important works that have contributed to the advancement and use of fractional calculus across different settings and disciplines. Lai et al.'s study[43][3][17], which encompasses control theory and dynamical systems, is one such example. The introduction highlights dynamical Liapunov functions as an innovative method for assessing the stability of nonlinear systems.

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This research looks at the differences between dynamical Liabanov functions and regular Liabanov functions. It demonstrates that we can create dynamical Liabanov functions without explicitly solving the differential equations they rely on. Fractional superchain systems (FHCS) also serve as a framework for analyzing complex systems characterized by fractal dynamics and strongly structured interactions. We use FHCS, the Adomian decomposition method (ADM), and the variational iteration method (VIM) to solve fractional differential equations. The Adomian decomposition method (ADM) provides a straightforward method for treating nonlinear differential equations, while the variational iteration method (VIM) offers an adaptable strategy for managing nonlinearity. Alternative techniques include the fractional Fourier transform, the Laplace transform, and numerical methods. [19][38][20][18]. By combining these methodologies, researchers can improve the modeling, analysis, and control of complex systems across many disciplines. The first section classifies fractional derivatives into several types, such as the Riemann–Liouville (RL) fractional integral, the Caputo (C) fractional derivative, the Caputo–Fabrizio (CF) fractional derivative, and the Atangana–Pagliano–Cabuto (ABC) fractional derivative. This is context for investigating fractional processes and their use in evaluating the stability of nonlinear systems[24]. A comprehensive understanding of the mathematical framework used in the research requires integrating each of these concepts. In Section 2, we talk about how important it is to do stability analysis in fractional systems. We note that for stability, the roots of a characteristic polynomial must have negative real parts, which is based on the Routh-Hurwitz criterion [7] [33]. The introduction gives a strong base for the discussion by looking at fractional operators, dynamical Lyapunov functions, and how important they are for figuring out the stability of nonlinear systems[1]. This adequately prepares the reader for further study. The text defines the fractional Caputo and Riemann-Liouville derivatives, which are fundamental to the description and behavior of fractIt is emphasized how important the Mittag-Leffler function is in fractional calculus, especially when looking at fractional systems, which can give us new information about how the system works[35][32]. This allows us to examine the synchronization between the Lorenz and Liou systems in Section 3, implying that comprehending these connections could provide valuable insights into chaotic dynamics and their control. This research begins with the definitions and properties of the fractional processes used in the study. The text also mentions fractional hyperchain systems (FHCS). The integration of fractional calculus into fractal hyperchain systems leads to significant advances in the design of control systems. By modeling system dynamics more accurately, enhancing stability and performance, and offering greater flexibility, (FHCS) provides a powerful framework for developing advanced control strategies suitable for complex, real-world applications. This leads to more effective and efficient control systems across a variety of engineering disciplines. This benefits diverse fields such as network theory, control systems, signal processing, biology, finance, thermodynamics, and materials science. [23][40]. In their study of stability, the authors look into how important it is to set criteria for the presence of dynamic Lyapunov functions at stable exponential equilibrium points. The goal is to determine the asymptotic stability of the systems under investigation.[25].

2. Preliminaries

This section delineates the definitions and characteristics of fractional operators that will be employed in our research.

Definition 1. *[34] Consider a set $L^1(r_0, +\infty)$ given that 0^α is less than or equal to 1. The formal definition of the Riemann-Liouville (RL) fractional integral of function f is specified as:*

$$^{RL}I_{r_0}^{\alpha} g(r) = \frac{1}{\Gamma(\alpha)} \int_{r_0}^{r} (r-x)^{\alpha-1} g(x) dx \tag{1}$$

Let $\Gamma(\cdot)$ be the Gamma function.

Definition 2. *[45] Consider a function g in the closed set $H^1(r_0, +\infty)$ with $0^\alpha \leq 1$. The Caputo (C) fractional derivative of the function f is defined as follows[29]:*

$$^{C}D_{r}^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_{r_0}^{r} \frac{f'(x)}{(r-x)^{\alpha}} dx \tag{2}$$

Definition 3. *[39] Let $g \in H^1(r_0, +\infty)$ and $0 < \alpha \leq 1$. An expression for the Caputo-Fabrizio (CF) fractional derivative of function g is given as[12]:*

$$CFD_{r}^{\alpha} g(r) = \frac{1}{2} \frac{B(\alpha)(2-\alpha)}{1-\alpha} \int_{r_0}^{t} f'(x) \exp\left[-\frac{\alpha}{1-\alpha}(r-x)\right] dx \tag{3}$$

Consider a normalizing function $B(\alpha)$ defined by the equation $B(0) = B(1) = 1$. The definition of the fractional integral associated with the fractional derivative of CF is provided under

$$^{CF}_{r}I_{r_0}^{\alpha} g(r) = \frac{2(1-\alpha)}{B(\alpha)(2-\alpha)} g(r) + \frac{2\alpha}{B(\alpha)(2-\alpha)} {}^{RL}_{r}I_{r_0}^{1} g(r) \tag{4}$$

Definition 4. *[37]Consider a function g in the closed set $H^1(t_0, +\infty)$ with $0^\alpha \leq 1$.The fractional derivative of function f according to the Atangana-Baleanu-Caputo (ABC) method is expressed as:*

$$^{ABC}_{r_0}D_{r}^{\alpha} g(r) = \frac{B(\alpha)}{1-\alpha} \int_{r_0}^{r} g'(x) E_{\alpha}\left[-\frac{\alpha}{1-\alpha}(r-x)^{\alpha}\right] dx \tag{5}$$

A precise specification of the fractional integral that regulates the ABC fractional derivative is presented.

$$^{AB}_{r}I_{r_0}^{\alpha} g(r) = \frac{1-\alpha}{B(\alpha)} f(r) + \frac{\alpha}{B(\alpha)} {}^{RL}_{t}I_{r_0}^{\alpha} g(r) \tag{6}$$

Definition 5. *[28] Let α be more than or equal to zero. A Mittag-Leffler function is defined using two parameters, denoted as α and β[13].*

$$E_{\alpha,\beta}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j + \beta)}, z \in \mathbb{C}$$

Remark 1. [14] Assuming that $\beta = 1$, we may assert Assuming that $\beta = 1$, we may assert

$$E_{\alpha,1}(z) = E_{\alpha}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j + 1)}$$

In this analysis, we examine the Mittag-Leffler function with a single parameter α. Provided that $\alpha = \beta = 1$, one may derive

$$E_{1,1}(z) = \sum_{j=0}^{\infty} \frac{z^j}{j!} = \exp(z)$$

Theorem 1. [41] A characterization of The derivative of the Mittag-Leffler function is as follows:

$$\frac{dE_{\alpha,\beta}}{dz}(z) = E^2_{\alpha,\alpha+\beta}(z)$$

3. Stability and the Routh-Hurwitz criteria of fractional-order

This section presents a nonlinear fractional-order system. This study employs an enhanced direct Lyapunov approach using contemporary fractional derivative components to analyzes the stability of fractional-order systems and the Rath-Hurtwitz criterion for chaotic fractional-order systems. The stability of the hyperbolic equilibrium point in any dynamical system with an integer derivative is ascertained by the signs of the real parts of the eigenvalues of its Jacobian matrix[30]. If all eigenvalues of the Jacobian matrix possess negative real portions, the hyperbolic equilibrium point is asymptotically stable[11]. This is analogous to the Routh-Hurwitz criteria for algebraic methods. The Routh-Hurwitz criteria is a recognized method for assessing the stability of linear systems represented as $\dot{x} = \Lambda x(r)$, where $x(r) \in \mathbb{R}^n$ and Λ is a $n \times n$ real matrix, without resolving roots. This criterion also assesses stability by examining the system's characteristic equation, which may be expressed as $p(\theta) = \theta^n + a_1\theta^{n-1} + a_2\theta^{n-2} + ... + a^n = 0$ where all coefficients a_i are real constants. The n Hurwitz matrices are defined as

$$\boldsymbol{M_2} = \begin{pmatrix} a_1 & 1 \\ a_3 & a_2 \end{pmatrix}, \boldsymbol{M_3} = \begin{pmatrix} a_1 & 1 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{pmatrix},$$

$$\begin{pmatrix} a_1 & 1 & 0 & & \\ a_3 & a_2 & a_1 & \cdots & \\ a_5 & a_4 & a_3 & & \\ & \vdots & & \ddots & \vdots \\ \vdots & \vdots & & 0 & 0 \\ 0 & \cdots & & & \end{pmatrix}$$

when $a_j = 0$ exceeds n. Many terms $P(\theta)$ possess negative real portions if and only if the determinant of all Horowitz matrices is positive. Consequently, employing integer arithmetic, a theoretical analysis is a crucial endeavor in fractional differential computation.

Matignon understands the subsequent fractional differential equations and formulae when the order of the derivative lies between 0 and 1[31].
Theorem 2. [25] The fractional-order nonlinear system is characterised as

$$ {}_0D_r^{\alpha} x(r) = g(r, x(r)) \tag{7}$$

here $g = (g_1, g_2, ..., g_n)^T$ Let $x(r) = (x_1(r), x_2(r), \ldots, x_n(r))^T \in \mathbb{R}^n$ with initial value $x(0) = (b_1, b_2, \ldots, b_n)^T$.The function $g_i(i = 1, 2, \ldots, n)$ denotes continuous differential nonlinear functions, whereas α indicates the fractional order of the derivative. The operator ${}_0D_r$ denotes the Caputo (or Riemann-Liouville) fractional-order derivative operator. Denotes the Caputo (or Riemann-Liouville) derivative. Assuming no loss of generality, let $x^* = 0$ be the equilibrium point, and let $x(r) \in \mathbb{R}^n$. There exists a positive definite matrix P that fulfils

$$ x_n(r)^T p g(x(r)) \leq 0 \tag{8}$$

then the origin of the system 8is asymptotically stable.
Lemma 1. [27]Let us consider a real continuous and differentiable function denoted as $u(r)$. Taking into account any $t \geq t_0$ and $0^{\alpha} \leq 1$, It is possible to infer the following:

$$ {}_{r_0}^{ABC}D_r^{\alpha}\left(u^2(r)\right) \leq 2u(r){}_{r_0}^{ABC}D_r^{\alpha}u(r), \tag{9}$$

$$ {}_{r_0}^{CF}D_r^{\alpha}\left(u^2(r)\right) \leq 2u(r){}_{r_0}^{CF}D_t^{\alpha}u(r), \tag{10}$$

$$ {}_{r_0}^{C}D_r^{\alpha}\left(u^2(r)\right) \leq 2u(r){}_{r_0}^{C}D_t^{\alpha}u(r). \tag{11}$$

Proposition 1.[8]

- If m_1, m_2, m_3 are Routh–Hurwitz determinants which are defined as follows:

  $m_1 = n_1$, $m_2 = n_1 n_2 n_3$, $m_3 = n_{1n2}n - 3n_1^2 n_4 n_3^2$

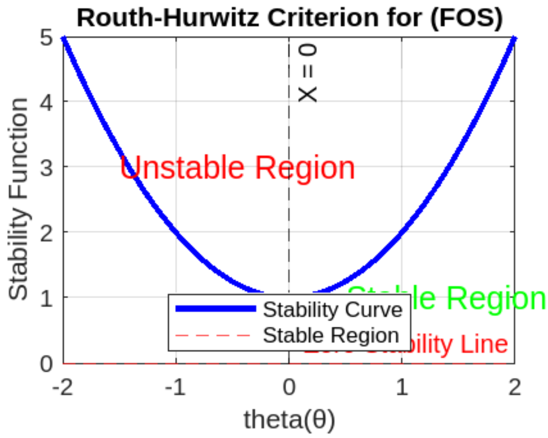
- If $D(P) > 0, n_1 > 0, n_2 < 0$ and $\alpha > 2/3$ then the equilibrium point $E$is unstable.

- If $D(P) < 0, n_1 > 0, n_2 > 0, n_3 > 0, n_4 > 0$, and $\alpha < 1/3$, then the equilibrium point E is locally asymptotically stable. Also, if $D(P) < 0, n_1 < 0, n_2 > 0, n_3 < 0, n_4 > 0$, then the equilibrium point $\bar{E}$is unstable.

- $If D(P) < 0, n_1 > 0, n_2 > 0, n_3 > 0, n_4 > 0 and n_2 = n_1 n_4 n_3 + n_3 n_1$, then the equilibrium point $\bar{E}$is locally asymptotically stable, for all$\alpha \in (0, 1)$.

- $n_4 > 0$is the necessary condition for the equilibrium point $E$to be locally asymptotically stable.

then for$\alpha = 1$, the equilibrium point$\bar{E}$ of system 11 is locally asymptotically stable if and only if $m_1 > 0, m_2 > 0, m_3 = 0, n_4 > 0$. (7) Moreover, the conditions of (7) are sufficient conditions for the equilibrium point$\bar{E}$ to be locally asymptotically stable for all $\alpha \in [0, 1)$.



4. Simulation and Analysis

This section analyzes the synchronization between the Lorenz and Liu systems, identifying the Lorenz system as the main system and the Liu system as the responsive system. It incorporates the proposed unified system by Lu et al. into the Lorenz, Chen, and Liu systems, offering a thorough analytical framework. By examining the extensive equations of the unified fractional degree system,[5].

4.1. Stabilization of Unified Chaotic System

In 2002, Lü et al. introduced the unified system 10, which integrates the Chen, Lü, and Lorenz systems. The fractional representation is specified as follows. Chen and Ueta discovered a similar yet separate chaotic attractor, identified as the counterpart of the Lorenz system. Recently, the L¨u system was formulated, establishing a connection between the Lorenz system and the Chen system. Lü et al. have recently created a unified chaotic system that includes the Lorenz and Chen systems as two dual systems at the boundaries of its parameter range. The written workThe system illustrates the continuous shift from the Lorenz system to the Chen system and demonstrates chaotic behavior throughout the spectrum of the crucial system parameter[26]. The fractional-order unified chaotic system is a member of the fractional-order Lorenz chaotic system family.The administration of various chaotic attractors within Lorenz system families is enhanced by the use of a fractional order unified chaotic system. The fractional order unified chaotic system is described as follows:

$$\begin{aligned} {}_0D_r^{\alpha} x_1 &= (25\vartheta + 10)\left(x_2 - x_1\right), \\ {}_0D_r^{\alpha} x_2 &= (28 - 35\vartheta)x_1 + (29\vartheta - 1)x_2 - x_1 x_3, \\ {}_0D_r^{\alpha} x_3 &= x_1 x_2 - (\vartheta + 8)/3x_3, \end{aligned} \tag{12}$$

In this context, ϑ belongs to the set of real numbers, and $0 < \alpha < 1$. The system is referred to as the Lorenz system when $\vartheta = 0$ and as the Chen system when $\vartheta =$

1. It denotes a continuous chaotic system as ϑ ranges from 0 to 1. We discern three equilibrium positions inside the system: $(0,0,0)$, $(\sqrt{72-7\vartheta^2}, \sqrt{72-7\vartheta-2\vartheta^2}, 6\vartheta)$, and $(-\sqrt{72-7\vartheta^2}, -\sqrt{72-7\vartheta-2\vartheta^2}, -6\vartheta)$. Employing a fractional order of $\alpha = 0.9$ and a starting condition of (2,2,1), the unified system 13 exhibits chaotic behavior, characterized by various equilibria as ϑ varies from 0 to 1. Figure 1 depicts an unstable equilibrium[6].

The controlled stabilization of a Unified Chaotic System 12 can be reformulated into the form 13.

$$ {}_0\mathrm{D}_{\mathrm{r}}^{\alpha}\boldsymbol{x}(\mathrm{r}) = \boldsymbol{\Lambda x} + \mathbf{f}(\boldsymbol{x}(\mathrm{r}))\boldsymbol{x}(\mathrm{r}) - \mathbf{B}x(\mathrm{r}) \tag{13} $$

where

$$ \boldsymbol{\Lambda} = \begin{pmatrix} -(25+\vartheta) & 25+\vartheta & 0 \\ 28-35\vartheta & 29\vartheta-1 & 0 \\ 0 & 0 & -\frac{8+\vartheta}{3} \end{pmatrix}, $$

is a linear matrix

$$ \mathbf{f}(\boldsymbol{x}(\mathrm{t})) = \begin{pmatrix} 0 \\ x_1x_3 \\ x_1x_2 \end{pmatrix} $$

is a nonlinear term,

$$ \mathbf{B} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} $$

is the control matrix

4.2. Regulating The Erratic Dynamics of the Liu system

The Liu system is an obscure chaotic system that has intricate and chaotic dynamics akin to the more renowned Lorenz and Chen systems. examined the fractional-order chaotic Liu system, characterized as[4][36]:

$$ \begin{aligned} {}_0^cD_r^{\alpha}x_1(r) &= -a(x_2x_1), \\ {}_0^cD_r^{\alpha}x_2(r) &= bx_1 - cx_1x_3, \\ {}_0^cD_r^{\alpha}x_3(r) &= -dx_3 + hx_1^2 \end{aligned} \tag{14} $$

When the parameters are set to $a = 10, b = 40, c = 10, d = 2.8, h = 4, and \alpha = 1.05$, system 13 demonstrates chaotic behavior, as illustrated in Figure 2. The controlled fractional-order chaotic system14 may be reformulated into the structure of 14.

$$ {}_0^cD_r^{\alpha}x(r) = Ax(r) + f(x(r))x(r) - Kx(r) \tag{15} $$

where

$$\mathbf{\Lambda} = \begin{pmatrix} -\mathrm{a} & 0 & 0 \\ 0 & \mathrm{b} & 0 \\ 0 & 0 & -\mathrm{d} \end{pmatrix}, \quad \mathbf{f}(x(\mathrm{r})) = \begin{pmatrix} 0 \\ -cx_1x_3 \\ hx_1^2 \end{pmatrix} \quad x(\mathrm{r}) = \begin{pmatrix} x_1(r) \\ x_2(r) \\ x_3(r) \end{pmatrix}$$

Considering the system's chaotic characteristics and the constrained state variables, we can easily determine that $|x_1(r)| < 2$ using numerical simulation. Consequently, the expression $x^{\top}(\mathrm{r})\mathbf{g}(\boldsymbol{x}(\mathrm{r}))\boldsymbol{x}(\mathrm{t}) = -ex_1x_2^2 < 2x_2^2$ necessitates that c = 2. asserts that if the matrix $\mathbf{S} = \frac{\mathbf{\Lambda}^{\top}+\mathbf{\Lambda}}{2} - h\mathbf{I}$ is negative definite, then the system 14 exhibits asymptotic stability. The maximum eigenvalue of the matrix $\frac{\mathbf{\Lambda}^{\top}+\mathbf{\Lambda}}{2}$ is $\lambda_{\max} = 2.5$. If the feedback gain matrix $\mathbf{H} = h\mathbf{I}$ satisfies the condition $h > 4.5$, the system attains asymptotic stability at the origin[26].

4.3. Challenges in Stabilising the Fractional-Order Hyperchaotic Chen System (FHCS)

Fractional Hyper Chain Systems (FHCS) provide an innovative methodology for stability analysis in fractional differential systems, providing numerous benefits compared to conventional techniques such as the Adomian Decomposition Method (ADM) and the Variational Iteration Method (VIM). FHCS offers advanced modelling capabilities that accurately represent the intricate behaviours of real-world systems characterised by non-integer order dynamics. Its highly organised structure facilitates a more thorough examination of systems with interconnected components, especially beneficial in networked systems or multi-agent situations. FHCS additionally provides comprehensive stability analysis, using novel stability criteria that surpass conventional Lyapunov methods in efficacy. This facilitates improved design and control tactics, augmenting versatility in controller design. FHCS can streamline computing demands and offer analytical insights, rendering it appropriate for various applications in engineering and applied sciences. The study illustrates the practical applications of FHCS in addressing intricate issues in engineering and applied sciences, connecting theory with practice.[22][21][16][10] [6]:

$$\begin{aligned} \dot{x}_1 &= a_1\left(x_2 - x_1\right) + x_4, \\ \dot{x}_2 &= d_1x_1 - x_1x_3 + c_1x_2, \\ \dot{x}_3 &= x_1x_2 - b_1x_3, \\ \dot{x}_4 &= x_2x_3 + rx_4 \end{aligned} \tag{16}$$

where $a_1, b_1, c_1, d_1, k \in R$. When $a_1 = 35, b_1 = 3, c_1 = 12, d_1 = 7$, and $0.085 < r \leqslant 0.798$, the system 16 is hyperchaotic[15]. Subsequently, we examine the fractional variant of this system, which is represented by

$$\begin{aligned} {}_0D_t^{\alpha}x_1 &= a\left(x_2 - x_1\right) + x_4, \\ {}_0D_t^{\alpha}x_2 &= dx_1 - x_1x_3 + cx_2, \\ {}_0D_t^{\alpha}x_3 &= x_1x_2 - bx_3, \quad {}_0 \\ D_t^{\alpha}x_4 &= x_2x_3 + rx_4 \end{aligned} \tag{17}$$

where α denotes the fractional order. For the values $a = 35, b = 3, c = 12, d = 7,$ and $r = 0.5$, this system possesses a singular equilibrium point at $(0,0,0,0)$. In the commensurate system, computer simulations indicate that the minimum order required to produce hyperchaos is 0.946. The regulated fractional-order chaotic system (4.6) can be reformulated as (3.4).

$$ {}_0D_t^{\alpha}x(t) = \boldsymbol{A}x(t) + g(x(t))x(t) - Kx(t) $$

where

$$ \boldsymbol{A} = \begin{pmatrix} -\mathrm{a} & \mathrm{a} & 0 & 1 \\ \mathrm{d} & \mathrm{c} & 0 & 0 \\ 0 & 0 & -\mathrm{b} & \\ 0 & 0 & 0 & \mathrm{r} \end{pmatrix}, \quad \mathbf{g}(\boldsymbol{x}(\mathrm{t})) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\mathrm{x}_1 & 0 \\ 0 & \mathrm{x}_1 & 0 & 0 \\ 0 & 0 & \mathrm{x}_2 & 0 \end{pmatrix} $$

Given the chaotic nature of the system and the bounded state variables, we can readily derive

$$ x^{\top}(t)g(x(t))x(t) = x_2x_3x_4 = \frac{x_2x_3}{x_4}x_4^2 \leqslant 43x_4^2 $$

through numerical simulation, namely, c = 43. Therefore, if the matrix $S = \frac{\boldsymbol{\Lambda}^{\top}+\boldsymbol{\Lambda}}{2} - \mathrm{hI}$ is negative definite, the system (4.7) is asymptotically stable. The maximal eigenvalue of the matrix $\frac{\Lambda^{\top}+\Lambda}{2}$ is approximately $\lambda_{\max} \approx 20$. If the feedback gain matrix $H = hI$ meets the condition > 63, the system is asymptotically stable at the origin.

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Figure 1: Chaotic attractors in the Lorenz system with $\alpha = 1$ and chaotic attractors in the Lorenz system with $\alpha = 0.94$, respectively.

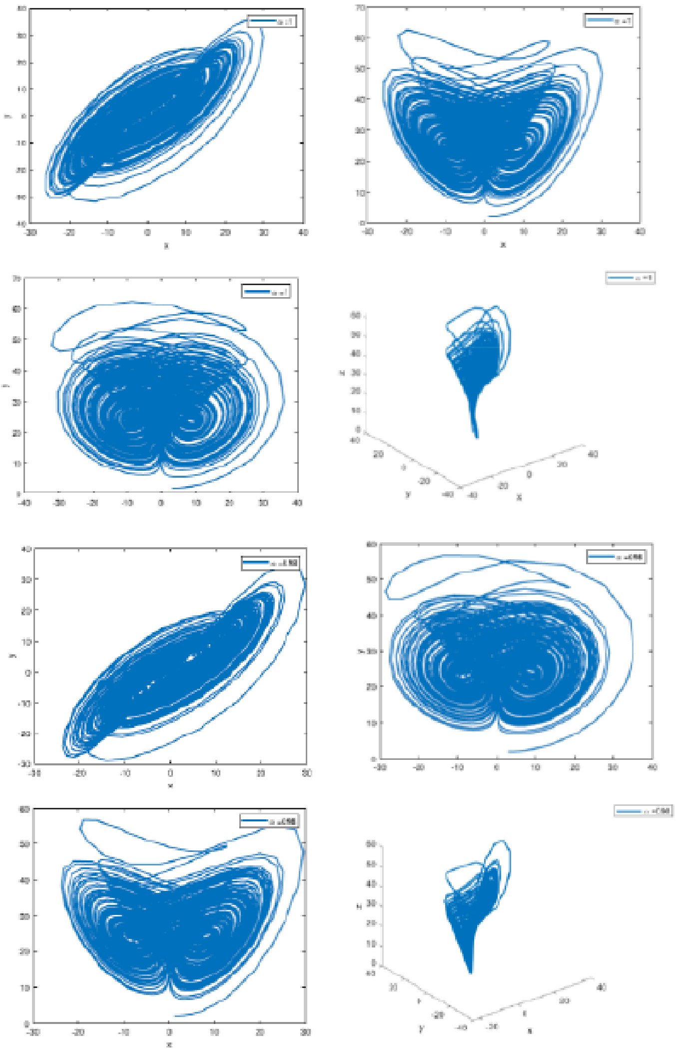

Figure 2:Chaotic attractors in Lü system with $\alpha = 1$, Chaotic attractors in Lü system with $\alpha = 0.94$.

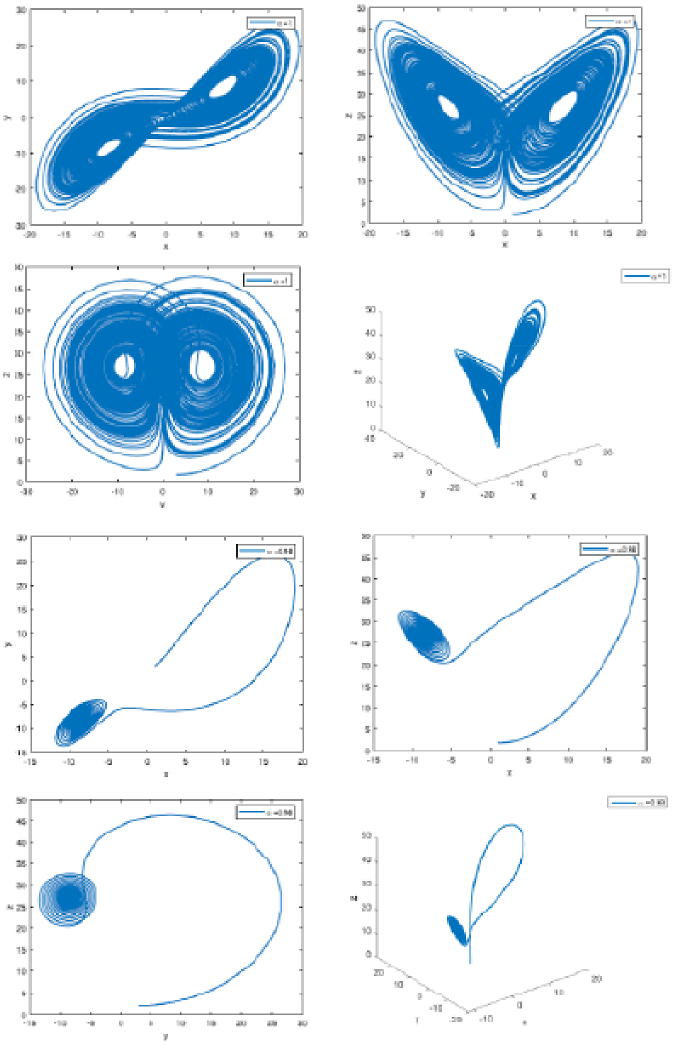

Figure 3: Chaotic attractors in hyperchaotic Chensystem with $\alpha = 1$, Chaotic attractors in hyperchaotic Chen system with $\alpha = 0.94$. respectively.

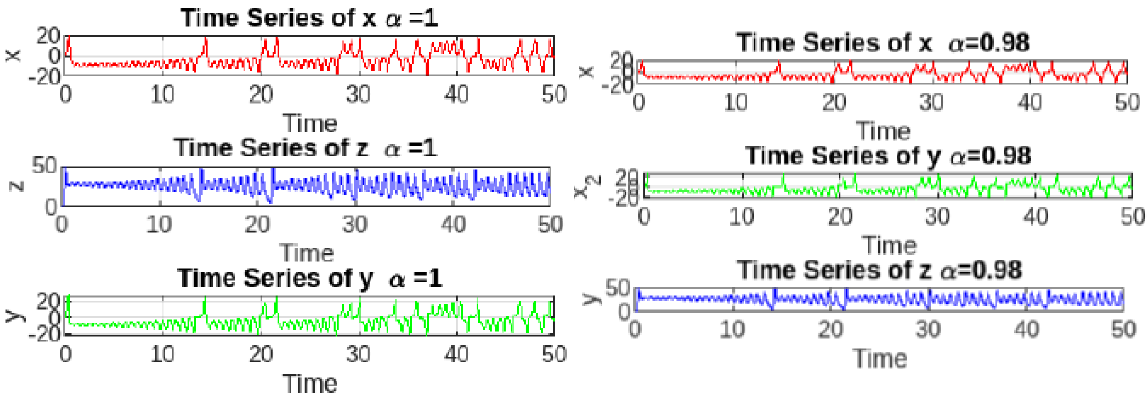

Figure 4:Time series considering a Caputo Lorenz system with$\alpha = 1$ and Time series considering a Lü system system with $\alpha = 0.94$ respectively

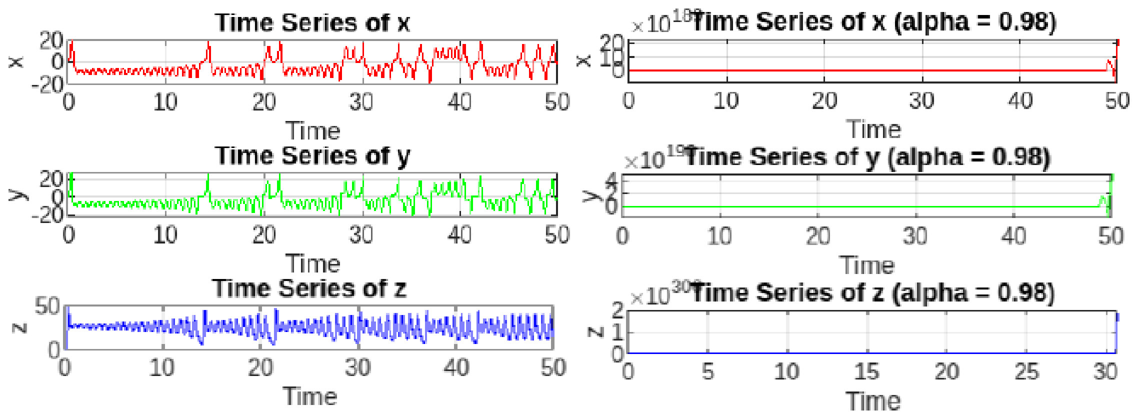

Figure 5: Time series considering a Lü system system with$\alpha = 1$ Time series considering a Lü system system with $\alpha = 0.94$.

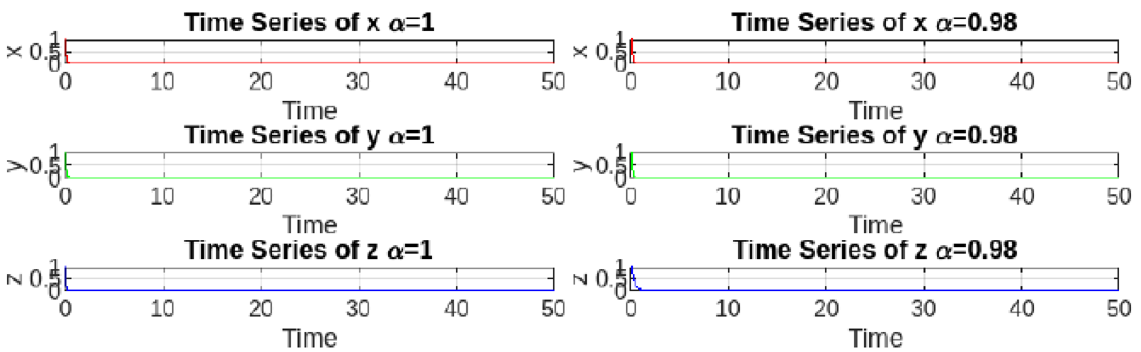

Figure 6: Time series considering a hyperchaotic Chen system system with$\alpha = 1$ Time series considering a hyperchaotic Chen system system with $\alpha = 0.94$.

5. Conclusion

The paper comprehensively investigates the synchronization of Lorenz and Liu systems, and establishes a framework for understanding their dynamics using a unified chaotic system. Fractal superchain systems (FHCS) are used as a framework for analyzing complex systems characterized by fractal dynamics and highly ordered interactions[9]. It is shown that the lowest fractal order is necessary to achieve superchaos, highlighting the importance of initial conditions in influencing the behavior of the system. The results improve the understanding and stability of chaotic systems. This indicates the chaotic character of the system and its use in chaos control[42][44]. The results demonstrate the potential for better analysis of system models and identification of attractor chaos and hybrid systems that integrate fractal chaotic dynamics and neural networks for control and prediction purposes. Thus, the proposed methods can be used for other systems to obtain more accurate results. These methods can provide numerical simulations and model solutions, which enhance the control and reliability for future research and engineering applications[28][2].

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