EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 1, Article Number 5583
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Advancing Solutions for Fractional Differential Equations: Integrating the Sawi Transform with Iterative Methods

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Abstract. This paper presents a powerful approach to solving fractional differential equations by combining the Sawi transform with iterative methods, particularly the Sawi iterative method. We begin by reviewing the fundamental properties and theoretical aspects of the Sawi transform, demonstrating its effectiveness in simplifying and solving fractional differential equations. The integration of the Sawi transform with the iterative method is applied to solve fractional delay differential equations, showcasing both analytical and approximate solutions through detailed examples and case studies. Our findings highlight that this combined approach not only streamlines the solution process but also significantly enhances the accuracy and applicability of solutions across a diverse range of differential equations. This study lays a robust foundation for further research and practical applications, offering valuable insights and tools for advancing scientific and engineering fields.

2020 Mathematics Subject Classifications: 26A33, 44A20, 65F08

Key Words and Phrases: Sawi transform, Fractional calculus, Caputo fractional derivative, Iterative method

1. Introduction

In the realm of applied mathematics, the quest for efficient and accurate methods to solve differential equations remains a pivotal challenge [6, 10]. Differential equations, both ordinary and fractional, are instrumental in modelling a wide array of physical phenomena across disciplines such as physics, engineering, biology, and finance [7, 22]. Traditional methods like the Runge-Kutta method, Taylor series expansions, finite difference methods, and various integral transforms have long been utilized for solving these equations [8], but they often fall short when dealing with more complex or nonlinear problems [15, 31]. For example, stiff problems and differential algebraic equations require specialized techniques

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DOI: https://doi.org/10.29020/nybg.ejpam.v18i1.5583

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[9], while parallel computing methods like domain decomposition and multigrid algorithms offer improved performance for complex problems [11, 12].

Several innovative methods have been proposed and refined to tackle the challenges posed by fractional differential equations. The Homotopy Analysis Method (HAM), for instance, has been utilized to find solutions of partial differential equations within fuzzy environments, enhancing analytical capabilities in uncertain systems [4, 19]. Similarly, the ARA-Residual Power Series Method has been effectively applied to solve systems of fractional differential equations, demonstrating its potential in handling complex fractional systems [14].

Researchers have also worked on establishing general formulas of integrals through master theorems, which are instrumental in the mathematical analysis of fractional equations [13, 29]. Analytical solutions to coupled nonlinear equations, such as the Hirota–Satsuma and Korteweg–de Vries (KdV) equations, have been derived, contributing to the understanding of nonlinear wave phenomena [30]. New schemes for solving fractional partial differential equations have been proposed, offering alternative approaches to existing methods [2].

The application of fractional calculus extends to modeling and analyzing chaotic systems. For example, the simplest chaotic circuit model has been studied using the Atangana–Baleanu Caputo fractional derivative, providing insights into the system's numerical behavior [5]. The dynamics of fractional discrete predator–prey models have been explored with a focus on chaos, control, and synchronization, highlighting the complex interactions within biological systems [28]. Furthermore, integrating machine learning techniques, such as physics-informed neural networks, has shown promise in predicting thermal distributions in convective wavy fins, bridging the gap between computational methods and practical applications [26].

Recently, the Sawi transform (SWT) has emerged as a promising tool, offering novel capabilities and enhanced flexibility in handling a broader class of differential equations. The SWT, which Mahgoub and Mohand introduced in 2019, has shown to have significant potential for simplifying and solving various types of differential equations [1, 3]. Its unique properties, including linearity, scaling, shifting, and convolution, make it particularly useful for transforming complex differential problems into more manageable algebraic forms. Moreover, the SWT has been extended to solve boundary value problems [20, 25], evaluate improper integrals, and integrate with iterative methods to solve nonlinear integro-differential equations, showcasing its versatility and effectiveness [16, 17, 24]. It is better to use iterative methods along with the SWT to solve problems, especially fractional differential equations and ordinary differential equations. Iterative methods, known for their efficiency in refining solutions and ensuring convergence, have been widely used in numerical analysis and computational mathematics. The combination of these methods with the SWT particularly, the Sawi iterative method (SIM), has proven effective in tackling nonlinear and complex differential equations, ensuring robust and accurate solutions [23, 27]. Recent research has highlighted the practical utility of this approach in solving delay differential equations and other complex mathematical problems, thereby expanding the toolkit available to researchers and practitioners [18, 21].

This paper aims to explore the extensive capabilities of the SWT by integrating it with iterative methods to enhance the solutions of both ordinary and fractional differential equations. By leveraging these combined techniques, we aim to provide more robust and precise solutions, thereby expanding the toolkit available to researchers and practitioners across various scientific and engineering disciplines.

2. Basic Definitions and Properties

This section presents the basic facts and properties related to SWT, that are essential in our work.

Definition 1. *The SWT of the function* $w(t)$*, defiend on* $[0,\infty)$ *, is denoted by* $S[w(t)]$ *and given by*

$$S[w(t)] = R(v) = \frac{1}{v^2}\int_0^\infty w(t)\, e^{-\frac{t}{v}} dt. \quad (1)$$

If $S[w(t)] = R(v)$*, then* $w(t)$*, is referred to as the inverse SWT of* $R(v)$*, and is denoted by* $S^{-1}[R(v)] = w(t)$

$$S^{-1}[R(v)] = \frac{-1}{2\pi i}\int_{c-i\infty}^{c+i\infty} R(v)\ e^{\frac{t}{v}}\ dv, \quad c \in \mathbb{R}. \quad (2)$$

Note that, if $S[w_1(t)] = R_1(v)$ and, $S[w_2(t)] = R_2(v)$, then

$$S[\ aw_1(t) + bw_2(t)] = a\ S[w_1(t)] + bS[w_2(t)] = aR_1(v) + bR_2(v),$$

where a & b are arbitrary constants. Moreover, the inverse of SWT is linear. If

$$S^{-1}[R_1(v)] = w_1(t)$$

and,

$$S^{-1}[R_2(v)] = w_2(t),$$

then

$$S^{-1}[aR_1(v) + bR_2(v)] = aS^{-1}[R_1(v)] + bS^{-1}[R_2(v)] = aw_1(t) + bw_2(t). \quad (3)$$

Theorem 1. *Let* $w(t)$ *be a continuous function defined for* $t > 0$ *and has exponential order* α *property;* $|w(t))| \le \mu e^{\alpha t}$ *where* $\mu > 0$*. Then, the SWT* $S[w(t)]$ *exists for* $Re\left(\frac{1}{v}\right) > \alpha$*.*

The SWT is a well-known transform that satisfies the following properties

- If $S[w(t)] = R(v)$, then $S[\ w(at)] = a\ R(av)$.
- $S\left[e^{at}w(t)\right] = \frac{1}{(1-av)^2}R\left(\frac{v}{1-av}\right)$.
- $S[w_1(t) * w_2(t)] = v^2\ R_1(v)\ R_2(v)$, where

$$w_1(t) * w_2(t) = \int_0^t w_1(\tau)\ w_2(t-\tau)d\tau.$$

Theorem 2. *Let* $R(v)$ *be SWT of* $w(t)$*. Then*

(i) $S[w'(t)] = \frac{R(v)}{v} - \frac{w(0)}{v^2}$.

(ii) $S[w''(t)] = \frac{R(v)}{v^2} - \frac{w(0)}{v^3} - \frac{w'(0)}{v^2}$.

(iii) $S\left[w^{(n)}(t)\right] = \frac{R(v)}{v^n} - \sum_{k=0}^{n-1} \frac{w^{(k)}(0)}{v^{n-k+1}}$.

The following Table 1, introduces SWT for some basic functions

Table 1: SWT of Some Elementary Functions.

Sr. No.	$\boldsymbol{W}(t)$	$\boldsymbol{S}[w(t)]$
1	1	$\frac{1}{v}$
2	t	1
3	$t^n,\ n \in \mathbb{N}$	$n!v^{n-1}$
4	$t^\alpha,\ \alpha \in \mathbb{R}^+$	$\Gamma(\alpha+1)\, v^{\alpha-1}$
5	e^{at}	$\frac{1}{v(1-av)}$
6	$\sin(at)$	$\frac{a}{1+a^2v^2}$
7	$\cos(at)$	$\frac{1}{v(1+a^2v^2)}$
8	$\sinh(at)$	$\frac{a}{1-a^2v^2}$
9	$\cosh(at)$	$\frac{1}{v(1-a^2v^2)}$

3. The SWT of Some Fractional Operators

The Riemann-Liouville integral is motivated from Cauchy formula for repeated integration and, the Mittage-Leffler function is one of the important special functions, which is considered a generalization of the exponential function and frequently used in the solutions of fractional differential equations and systems of fractional differential equations.

Definition 2. *The Riemann–Liouville fractional integral of a function* $w(t)$ *of order* $\alpha > 0$ *is defined by:*

$$I^\alpha w(t) = \frac{1}{\Gamma(\alpha)} \int_\alpha^t (t-\tau)^{\alpha-1}\; w(\tau)\, d\tau. \tag{4}$$

Definition 3. *The Caputo fractional derivative of a function* $w(t)$ *of order* $\alpha > 0$ *is defined by:*

$$D^\alpha w(t) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{w^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}} d\tau, & m-1 < \alpha < m, \\ w^{(m)}(t), & \alpha = m,\ m \in \mathbb{N}. \end{cases} \tag{5}$$

Theorem 3. *If* $R(v)$ *is the SWT of* $w(t)$*, then SWT of Riemann-Liouville fractional integral is given by*

$$S[I^\alpha w(t)] = v^\alpha R(v). \tag{6}$$

Proof. From the definition of Riemann-Liouville integral, we have

$$I^a w(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} w(\tau)\, d\tau = \frac{1}{\Gamma(\alpha)} \left(t^{\alpha-1} * w(t)\right). \tag{7}$$

Taking SWT to both sides of (7), we obtain

$$S\left[I^\alpha w(t)\right] = \frac{1}{\Gamma(\alpha)}\, S\left[t^{\alpha-1} * w(t)\right].$$

By using convolution property of SWT, we obtain

$$S\left[I^\alpha w(t)\right] = \frac{v^2}{\Gamma(\alpha)} S[t^{\alpha-1}] S[w(t)] = \frac{v^2}{\Gamma(\alpha)} \Gamma(\alpha)\, v^{\alpha-2}\, R(v) = v^\alpha R(v). \square \tag{8}$$

Theorem 4. *If $R(v)$ is SWT of the $w(t)$, then SWT of Caputo functional derivative of a function $w(t)$, is given by*

$$S\left[D^\alpha w(t)\right] = \frac{1}{v^\alpha}\, R(v) - \sum_{k=0}^{m-1} p \left(\frac{1}{v}\right)^{m-(k-1)} w^{(k)}(0), \tag{9}$$

where, $m-1 < \alpha \le m, \quad m \in \mathbb{N}$.

Proof. The definition of Caputo derivative of a function w(t) is

$$D^\alpha w(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{w^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}} d\tau = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} w^{(m)}(\tau)\, d\tau,$$
$$m-1 < \alpha < m$$

which can be written as,

$$D^\alpha w(t) = \frac{1}{\Gamma(m-\alpha)} (t^{m-\alpha-1} * w^{(m)}(t)). \tag{10}$$

Taking SWT to both sides of Eq (10), we obtain

$$S\left[D^\alpha w(t)\right] = \frac{1}{\Gamma(m-\alpha)} S[t^{m-\alpha-1} * w^{(m)}(t)].$$

By using convolution property of SWT,

$$\begin{aligned} S\left[D^\alpha w(t)\right] &= \frac{v^2}{\Gamma(m-\alpha)}\, S\left[t^{m-\alpha-1}\right]\, S\left[w^{(m)}(t)\right] \\ &= \frac{v^2}{\Gamma(m-\alpha)} \Gamma(m-\alpha)\, v^{m-\alpha-2} \left(\frac{1}{v^m} R(v) - \sum_{k=0}^{m-1} \left(\frac{1}{v}\right)^{m-(k-1)} w^{(k)}(0)\right) \\ &= v^{m-\alpha} \left(\frac{1}{v^m} R(v) - \sum_{k=0}^{m-1} \left(\frac{1}{v}\right)^{m-(k-1)} w^{(k)}(0)\right) \\ &= \frac{1}{v^\alpha} R(v) - \sum_{k=0}^{m-1} \left(\frac{1}{v}\right)^{m-(k-1)} w^{(k)}(0). \square \end{aligned}$$

4. Iterative Method

In this section, we discuss the iterative method, and in the second one we present SIM for solving delay differential equation (DDE). Let us consider the following general functional equation

$$w\left(t\right)=N\left(w\right)+g\left(t\right), \tag{11}$$

where N is a nonlinear operator from a Banach space $S\longrightarrow S$, and $g(t)$ is a known function. We are looking for a solution $w(t)$ of Eq (11), having the series form:

$$w\left(t\right)=\sum_{i=0}^{\infty}w_i\left(t\right). \tag{12}$$

The nonlinear operator N can be decomposed as:

$$N\left[\sum_{i=0}^{\infty}w_i(t)\right]=N\left(w_0\right)+\sum_{i=1}^{\infty}\left(N\left[\sum_{k=0}^{i}pw_k(t)\right]-N\left[\sum_{k=0}^{i-1}w_k(t)\right]\right). \tag{13}$$

From Eq (12) and (13), the Eq (11) is equivalent to

$$\sum_{i=0}^{\infty}w_i(t)=g(t)+N\left(w_0(t)\right)+\sum_{i=0}^{\infty}\left(N\left[\sum_{k=0}^{i}w_i(t)\right]-N\left[\sum_{k=0}^{i-1}w_k(t)\right]\right). \tag{14}$$

Now, we define the recurrence relation:

$$\begin{aligned}
w_0&=g\left(t\right),\\
w_1&=N\left[w_0\right],\\
w_2&=N\left[w_0+w_1\right]-N\left[w_0\right],\\
w_3&=N\left[w_0+w_1+w_2\right]-N\left[w_0+w_1\right],\\
w_4&=N\left[w_0+w_1+w_2+w_3\right]-N\left[w_0+w_1+w_2\right],\\
&\vdots\\
w_{m+1}&=N\left[w_0+\cdots+w_m\right]-N\left[w_0+\cdots+w_{m-1}\right],
\end{aligned}$$

where, $m=1,2,3,\ldots$.Thus,

$$w_1+w_2+\cdots+w_{m+1}=N\left[w_0+\cdots+w_m\right],$$

where, $m=1,2,\ldots$, and

$$w\left(t\right)=\sum_{i=0}^{\infty}w_i\left(t\right).$$

For the convergence of the iterative method, we introduce the following two theorems.

Theorem 5. *If N is a continuously differentiable functional in a neighbourhood of w_0 and*

$$\| N^{(n)}(w_0) \| = \sup \left\{ N^{(n)}(w_0)(h_1, h_2, \cdots, h_n) : \| h_i \| \leq 1,\ 1 \leq i \leq n \right\} \leq L,$$

for each n and for some real $L > 0$ and, $\| w_i \| \leq M < \frac{1}{e}$, $i = 1, 2, 3, \cdot$, then the series $\sum_{i=0}^{\infty} w_{i+1}$ is absolutely convergent. Moreover;

$$\| w_{i+1} \| \leq L M^n e^{n-1}(e-1),\ n = 1, 2, \cdots .$$

Theorem 6. *If N is continuously differentiable functional in a neighbourhood of w_0 and $\| N^{(n)}(w_0) \| \leq M \leq \frac{1}{e}$, then the series $\sum_{i=0}^{\infty} w_{i+1}$ is absolutely convergent.*

5. SIM for Fractional Equations

Let us consider the form of fractional nonlinear differential equation

$$D^{\alpha} w(t) + L[w(t)] + N[w(\lambda t)] = q(t), \tag{15}$$

with the initial condition

$$w(0) = a, \tag{16}$$

where $0 < \alpha \leq 1$, and L refers to the linear operator, N refers to the nonlinear operator and, $D^{\alpha} w(t)$ is the Caputo fractional derivative of $w(t)$. Now, we introduce the steps of SIM to find the solution of the problem (15) and (16), by using SIM for fractional equations.

Step (1): Applying SWT on Eq (15)

$$S[D^{\alpha} w(t)] + S[L[w(t)]] + S[N[w(\lambda t)]] = S[q(t)]. \tag{17}$$

By running SWT on Eq (17), we obtain

$$\frac{R(v)}{v^{\alpha}} - \frac{w(0)}{v^{\alpha+1}} + S[L[w(t)]] + S[N[w(\lambda t)]] = S[q(t)]. \tag{18}$$

Substituting the initial condition Eq (16) into Eq (18), we get

$$\frac{R(v)}{v^{\alpha}} - \frac{a}{v^{\alpha+1}} + S[L[w(t)]] + S[N[w(\lambda t)]] = S[q(t)],$$

$$R(v) = v^{\alpha}\left(\frac{a}{v^{\alpha+1}} + S[q(t)] - S[L[w(t)]]\right) - v^{\alpha}(S[N[w(\lambda t)]]). \tag{19}$$

Step (2): By taking the inverse SWT on both sides of Eq (19), we obtain

$$S^{-1}[R(v)] = S^{-1}\left[v^{\alpha}\left(\frac{a}{v^{\alpha+1}} + S[q(t)] - S[L[w(t)]]\right) - v^{\alpha}(S[N[w(\lambda t)]])\right],$$

$$w(t) = S^{-1}\left[v^{\alpha}\left(\frac{a}{v^{\alpha+1}} + S[q(t)] - S[L[w(t)]]\right)\right] - S^{-1}[v^{\alpha}(S[N[w(\lambda t)]])].$$

Step (3): The nonlinear operator can be decomposed as

$$N\left[w\left(\lambda t\right)\right]=N\left[w_0\left(\lambda t\right)\right]+\sum_{n=1}^{\infty}\left(N\left[\sum_{i=0}^{k}w_i\left(\lambda t\right)\right]-N\left[\sum_{i=0}^{k-1}w_i\left(\lambda t\right)\right]\right).$$

Thus,

$$\begin{aligned}w\left(t\right)=&S^{-1}\left[v^{\alpha}\left(\frac{a}{v^{\alpha+1}}+S\left[q\left(t\right)\right]-S\left[L\left[w\left(t\right)\right]\right]\right)\right]\\&-S^{-1}\left[v^{\alpha}\left(S\left[N\left[w_0\left(\lambda t\right)\right]+\sum_{n=1}^{\infty}\left[N\left[\sum_{i=0}^{k}w_i\left(\lambda t\right)\right]-N\left[\sum_{i=0}^{k-1}w_i\left(\lambda t\right)\right]\right]\right]\right)\right].\end{aligned}$$

Step (4): Find the general form of the solution as follows,

$$\begin{aligned}\sum_{i=0}^{\infty}w_i\left(t\right)=&S^{-1}\left[v^{\alpha}\left(\frac{a}{v^{\alpha+1}}+S\left[q\left(t\right)\right]-S\left[L\left[w_0\left(t\right)\right]\right]\right)\right]\\&-S^{-1}\left[v^{\alpha}\left(S\left[N\left[w_0\left(\lambda t\right)\right]\right]\right)\right]\\&-S^{-1}\left[v^{\alpha}S\left[\sum_{k=1}^{\infty}\left[N\left[\sum_{i=0}^{k}w_i\left(\lambda t\right)\right]-N\left[\sum_{i=0}^{k-1}w_i\left(\lambda t\right)\right]\right]\right]\right].\end{aligned}$$

$$\begin{aligned}w_0\left(t\right)&=S^{-1}\left[v^{\alpha}\left(\frac{a}{v^{\alpha+1}}+S\left[q\left(t\right)\right]-S\left[L\left[w_0\left(t\right)\right]\right]\right)\right],\\w_1(t)&=-S^{-1}\left[v^{\alpha}\left(S\left[N\left[w_0\left(\lambda t\right)\right]\right]\right)\right],\\w\left(t\right)&=-S^{-1}\left[v^{\alpha}S\left[\sum_{k=1}^{\infty}\left[N\left[\sum_{i=0}^{k}w_i\left(\lambda t\right)\right]-N\left[\sum_{i=0}^{k-1}w_i\left(\lambda t\right)\right]\right]\right]\right],\end{aligned}$$

where $n=1,2,3,\cdots$.

6. Illustrative Examples (Fractional Case)

Example 1. *Consider the following nonlinear equation DDE:*

$$D^{\alpha}w\left(t\right)=2w^2\left(\frac{t}{2}\right), \tag{20}$$

with the initial condition

$$w\left(0\right)=1,\ \ and\ \ 0<\alpha\leq 1. \tag{21}$$

Solution. Taking SWT to both sides of Eq (20), we get

$$S\left[D^{\alpha}w\left(t\right)\right]=S\left[2w^2\left(\frac{t}{2}\right)\right]. \tag{22}$$

By running SWT on both sides of Eq (22), and substituting the initial condition Eq (21), we get

$$\frac{R(v)}{v^{\alpha}} - \left(\frac{1}{v}\right)^{\alpha+1} w_0(t) = S\left[2w^2\left(\frac{t}{2}\right)\right],$$

which implies

$$R(v) = \frac{1}{v} - v^{\alpha} S\left[2w^2\left(\frac{t}{2}\right)\right]. \tag{23}$$

By taking inverse SWT on both sides of Eq (23), we obtain

$$w(t) = 1 - S^{-1}\left[v^{\alpha} S\left[2\, w^2\left(\frac{t}{2}\right)\right]\right]. \tag{24}$$

Thus, $w_0(t) = 1$ and $w_0\left(\frac{t}{2}\right) = 1$. To find $w_1(t)$, we compute

$$\begin{aligned} w_1(t) = N\left[w_0\left(\frac{t}{2}\right)\right] &= -S^{-1}\left[v^{\alpha} S\left[2\, w_0^2\left(\frac{t}{2}\right)\right]\right] = -S^{-1}\left[v^{\alpha} S\left[2(1)^2\right]\right] \\ &= -S^{-1}\left[v^{\alpha} S[2]\right] = -S^{-1}\left[v^{\alpha}\left(\frac{2}{v}\right)\right] = -S^{-1}\left[\frac{2v^{\alpha}}{v}\right] = -\frac{2t^{\alpha}}{\Gamma(\alpha+1)}. \end{aligned}$$

Hence,

$$w_1\left(\frac{t}{2}\right) = -\frac{2t^{\alpha}}{2^{\alpha}\Gamma(1+\alpha)}.$$

To find $w_2(t)$, we compute

$$\begin{aligned} w_2(t) = N\left[w_0\left(\frac{t}{2}\right) + w_1\left(\frac{t}{2}\right)\right] - N\left[w_0\left(\frac{t}{2}\right)\right] \\ &= -S^{-1}\left[2\, v^{\alpha} S\left[\frac{2^{2-2\alpha}t^{2\alpha}}{\Gamma(1+\alpha)^2} - \frac{2^{2-\alpha}t^{\alpha}}{\Gamma(\alpha+1)}\right]\right] \\ &= -\frac{2^{3-2\alpha}t^{3\alpha}}{\Gamma(1+\alpha)^2} + \frac{2^{3-\alpha}t^{2\alpha}}{\Gamma(1+\alpha)}. \end{aligned}$$

Hence,

$$w_2\left(\frac{t}{2}\right) = -\frac{2^{3-2\alpha}t^{3\alpha}}{2^{3\alpha}\Gamma(1+\alpha)^2} + \frac{2^{3-\alpha}t^{2\alpha}}{2^{2\alpha}\Gamma(1+\alpha)}.$$

To find $w_3(t)$, we compute

$$\begin{aligned} w_3(t) &= N\left[w_0\left(\frac{t}{2}\right)+w_1\left(\frac{t}{2}\right)+w_2\left(\frac{t}{2}\right)\right]-N\left[w_0\left(\frac{t}{2}\right)+w_1\left(\frac{t}{2}\right)\right] \\ &= -S^{-1}\left[v^{\alpha}S\left[2\left(1-\frac{2t^{\alpha}}{\Gamma(\alpha+1)}-\frac{2^{3-2\alpha}t^{3\alpha}}{2^{3\alpha}\Gamma(1+\alpha)^2}+\frac{2^{3-\alpha}t^{2\alpha}}{2^{2\alpha}\Gamma(1+\alpha)}\right)^2\right.\right. \\ &\quad \left.\left.-\left(1-\frac{2t^{\alpha}}{\Gamma(\alpha+1)}\right)^2\right]\right] \\ &= -S^{-1}\left[v^{\alpha}-\frac{4v^{-1+2\alpha}}{\Gamma(\alpha+1)^2}+\frac{4v^{-1+3\alpha}}{\Gamma(\alpha+1)^2\Gamma(2\alpha+1)}+\frac{2^{5-3\alpha}v^{-1+3\alpha}}{\Gamma(\alpha+1)\Gamma(2\alpha+1)}\right. \\ &\quad -\frac{2^{5-8\alpha}v^{-1+4\alpha}}{\Gamma(\alpha+1)^2\Gamma(3\alpha+1)}-\frac{2^{6-3\alpha}v^{-1+4\alpha}}{\Gamma(\alpha+1)^2\Gamma(3\alpha+1)}+\frac{2^{6-8\alpha}v^{-1+5\alpha}}{\Gamma(\alpha+1)^3\Gamma(4\alpha+1)} \\ &\quad \left.+\frac{2^{7-6\alpha}v^{-1+5\alpha}}{\Gamma(\alpha+1)^2\Gamma(4\alpha+1)}-\frac{2^{8-11\alpha}v^{-1+6\alpha}}{\Gamma(\alpha+1)^3\Gamma(5\alpha+1)}+\frac{2^{7-16\alpha}v^{-1+7\alpha}}{\Gamma(\alpha+1)^4\Gamma(6\alpha+1)}\right]. \end{aligned}$$

After simple calculations, we get the value of $w_3(t)$ as

$$\begin{aligned} w_3(t) \approx & -\frac{t^{1+\alpha}}{\Gamma(\alpha+2)}+\frac{4t^{2\alpha}}{\Gamma(2\alpha+1)\Gamma(\alpha+1)^2}-\frac{4t^{3\alpha}}{\Gamma(\alpha+1)^2\Gamma(2\alpha+1)\Gamma(3\alpha+1)} \\ & -\frac{2^{5-3\alpha}t^{3\alpha}}{\Gamma(\alpha+1)\Gamma(2\alpha+1)\Gamma(3\alpha+1)}+\frac{2^{5-8\alpha}t^{4\alpha}}{\Gamma(\alpha+1)^2\Gamma(3\alpha+1)\Gamma(4\alpha+1)} \\ & +\frac{2^{6-3\alpha}t^{4\alpha}}{\Gamma(\alpha+1)^2\Gamma(3\alpha+1)\Gamma(4\alpha+1)}-\frac{2^{6-8\alpha}t^{5\alpha}}{\Gamma(\alpha+1)^3\Gamma(4\alpha+1)\Gamma(5\alpha+1)} \\ & -\frac{2^{7-6\alpha}t^{5\alpha}}{\Gamma(\alpha+1)^2\Gamma(4\alpha+1)\Gamma(5\alpha+1)}+\frac{2^{8-11\alpha}t^{6\alpha}}{\Gamma(\alpha+1)^3\Gamma(5\alpha+1)\Gamma(6\alpha+1)} \\ & -\frac{2^{7-16\alpha}t^{7\alpha}}{\Gamma(\alpha+1)^4\Gamma(6\alpha+1)\Gamma(7\alpha+1)}. \end{aligned}$$

Hence,

$$\begin{aligned} w_3\left(\frac{t}{2}\right) \approx & -\frac{t^{1+\alpha}}{2^{\alpha}\Gamma(\alpha+2)}+\frac{4t^{2\alpha}}{2^{2\alpha}\Gamma(2\alpha+1)\Gamma(\alpha+1)^2}-\frac{4t^{3\alpha}}{2^{3\alpha}\Gamma(\alpha+1)^2\Gamma(2\alpha+1)\Gamma(3\alpha+1)} \\ & -\frac{2^{5-3\alpha}t^{3\alpha}}{2^{3\alpha}\Gamma(\alpha+1)\Gamma(2\alpha+1)\Gamma(3\alpha+1)}+\frac{2^{5-8\alpha}t^{4\alpha}}{2^{4\alpha}\Gamma(\alpha+1)^2\Gamma(3\alpha+1)\Gamma(4\alpha+1)} \\ & +\frac{2^{6-3\alpha}t^{4\alpha}}{2^{4\alpha}\Gamma(\alpha+1)^2\Gamma(3\alpha+1)\Gamma(4\alpha+1)}-\frac{2^{6-8\alpha}t^{5\alpha}}{2^{5\alpha}\Gamma(\alpha+1)^3\Gamma(4\alpha+1)\Gamma(5\alpha+1)} \\ & -\frac{2^{7-6\alpha}t^{5\alpha}}{2^{5\alpha}\Gamma(\alpha+1)^2\Gamma(4\alpha+1)\Gamma(5\alpha+1)}+\frac{2^{8-11\alpha}t^{6\alpha}}{2^{6\alpha}\Gamma(\alpha+1)^3\Gamma(5\alpha+1)\Gamma(6\alpha+1)} \\ & -\frac{2^{7-16\alpha}t^{7\alpha}}{2^{7\alpha}\Gamma(\alpha+1)^4\Gamma(6\alpha+1)\Gamma(7\alpha+1)}. \end{aligned}$$

Thus, we get the value of $w(t)$ as:

$$
\begin{aligned}
w(t) &= w_0(t) + w_1(t) + w_2(t) + w_3(t) + \cdots \\
&= 1 - \frac{2t^{\alpha}}{\Gamma(\alpha+1)} - \frac{2^{3-2\alpha}t^{3\alpha}}{2^{3\alpha}\Gamma(1+\alpha)^2} + \frac{2^{3-\alpha}t^{2\alpha}}{2^{2\alpha}\Gamma(1+\alpha)} - \frac{t^{1+\alpha}}{\Gamma(\alpha+2)} \\
&\quad + \frac{4t^{2\alpha}}{\Gamma(2\alpha+1)\Gamma(\alpha+1)^2} - \frac{4t^{3\alpha}}{\Gamma(\alpha+1)^2\Gamma(2\alpha+1)\Gamma(3\alpha+1)} \\
&\quad - \frac{2^{5-3\alpha}t^{3\alpha}}{\Gamma(\alpha+1)\Gamma(2\alpha+1)\Gamma(3\alpha+1)} + \frac{2^{5-8\alpha}t^{4\alpha}}{\Gamma(\alpha+1)^2\Gamma(3\alpha+1)\Gamma(4\alpha+1)} \\
&\quad + \frac{2^{6-3\alpha}t^{4\alpha}}{\Gamma(\alpha+1)^2\Gamma(3\alpha+1)\Gamma(4\alpha+1)} - \frac{2^{6-8\alpha}t^{5\alpha}}{\Gamma(\alpha+1)^3\Gamma(4\alpha+1)\Gamma(5\alpha+1)} \\
&\quad - \frac{2^{7-6\alpha}t^{5\alpha}}{\Gamma(\alpha+1)^2\Gamma(4\alpha+1)\Gamma(5\alpha+1)} \\
&\quad + \frac{2^{8-11\alpha}t^{6\alpha}}{\Gamma(\alpha+1)^3\Gamma(5\alpha+1)\Gamma(6\alpha+1)} - \frac{2^{7-16\alpha}t^{7\alpha}}{\Gamma(\alpha+1)^4\Gamma(6\alpha+1)\Gamma(7\alpha+1)} \\
&\quad + \cdots .
\end{aligned}
$$

We use Mathematica version 13.0 to simplify the expressions.

In the following Figure 1, we sketch the approximate solution of Example 1 for different values of $\alpha = 0.75, 0.8, 0.85, 0.9, 0.95, 1.0$.

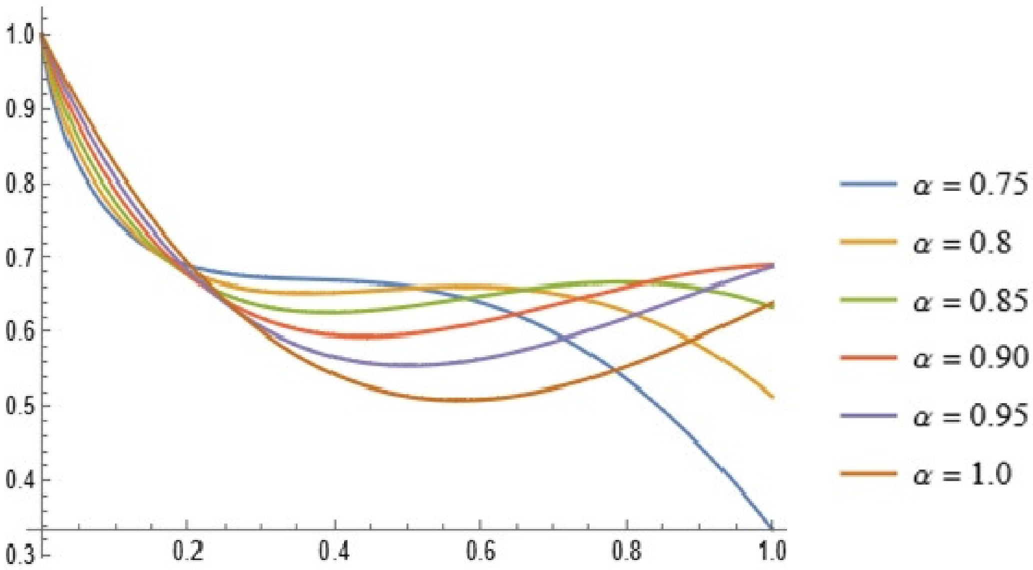

Figure 1: Approximate solution with different values of α for Example 1.

Example 2. *Consider the nonlinear DDE*

$$D^{\alpha}w(t) = 2tw\left(\frac{t}{2}\right), \tag{25}$$

with the initial condition,

$$w\left(0\right)=1,\ \ where\ 0<\alpha<1. \tag{26}$$

Solution. Applying SWT on both sides of Eq (25), we get

$$S\left[D^{a}w(t)\right]=S\left[2tw\left(\frac{t}{2}\right)\right]. \tag{27}$$

Running SWT on both sides of Eq (27), and using the initial condition Eq (26), we get

$$\frac{R(v)}{v^{\alpha}}-\left(\frac{1}{v^{\alpha+1}}\right)(1)=S\left[2tw\left(\frac{t}{2}\right)\right],$$

$$R\left(v\right)=\frac{1}{v}+v^{\alpha}\ S\left[2tw\left(\frac{t}{2}\right)\right]. \tag{28}$$

By taking inverse SWT on both sides of Eq (28), we get

$$w\left(t\right)=1+S^{-1}\left[v^{\alpha}S\left[2t\ w\left(\frac{t}{2}\right)\right]\right].$$

We can conclude that

$$w_0(t)=1,$$

and

$$w_0\left(\frac{t}{2}\right)=1.$$

To find $w_1(t)$, we compute

$$\begin{aligned}N\left[w_0\left(\frac{t}{2}\right)\right]&=-S^{-1}\left[v^{\alpha}\ S\left[2t\ w_0\left(\frac{t}{2}\right)\right]\right]=-S^{-1}\left[v^{\alpha}\ S\left[2t(1)\right]\right]\\&=-S^{-1}\left[v^{\alpha}\left(2\right)\right]=-S^{-1}\left[2v^{\alpha}\right]=\frac{2t^{\alpha+1}}{\Gamma\left(\alpha+2\right)}.\end{aligned}$$

Hence,

$$w_1\left(\frac{t}{2}\right)=-\frac{2t^{\alpha+1}}{2^{\alpha}\ \Gamma\left(\alpha+2\right)}\ .$$

To find $w_2(t)$, we compute

$$\begin{aligned}w_2(t)&=N\left[w_0\left(\frac{t}{2}\right)+w_1\left(\frac{t}{i}\right)\right]-N\left[w_0\left(\frac{t}{2}\right)\right]\\&=S^{-1}\left[v^{\alpha}S\left[2t\left(\left(1-\frac{2t^{\alpha+1}}{2^{\alpha}\Gamma(\alpha+2)}\right)-(1)\right)\right]\right]\\&=S^{-1}\left[\frac{4v^{2\alpha+1}\Gamma(\alpha+3)}{2^{\alpha}\ \Gamma(\alpha+2)}\right]\\&=\frac{4t^{2\alpha+2}\cdot\Gamma\left(\alpha+3\right)}{2^{\alpha}\ \Gamma\left(2\alpha+3\right)\Gamma\left(\alpha+2\right)}.\end{aligned}$$

Thus we get

$$\begin{aligned} w(t) &= w_0(t) + w_1(t) + w_2(t) + \cdots \\ &= 1 + \frac{2t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{4t^{2\alpha+2}\Gamma(\alpha+3)}{2^{\alpha}\,\Gamma(2\alpha+3)\Gamma(\alpha+2)} + \cdots . \end{aligned}$$

We use Mathematica version 13.0 to simplify the expressions.

In the following Figure 2, we sketch the approximate solution of Example 2 for different values of $\alpha = 0.6, 0.7, 0.8, 0.9, 0.95, 1.0$.

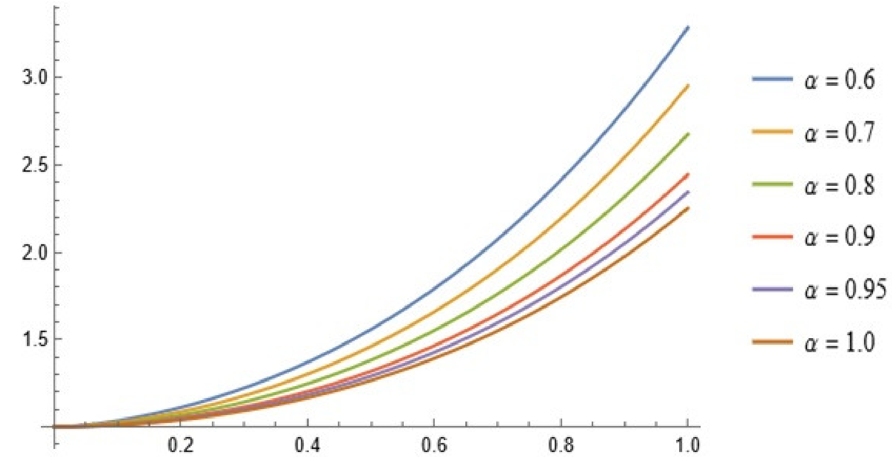

Figure 2: Approximate solution with different values of α for Example 2.

7. Conclusion

This study has demonstrated the substantial potential of integrating the SWT with iterative methods, particularly the SIM, to solve fractional differential equations and other complex differential equations. By leveraging the unique properties of the SWT such as linearity, scaling, shifting, and convolution, we have shown that it is possible to transform complex differential problems into simpler algebraic forms, facilitating more efficient and precise solutions. The application of iterative methods in conjunction with the SWT has proven to enhance both the accuracy and convergence of solutions, particularly in challenging scenarios involving non-linear and delay differential equations. Through detailed examples and case studies, we have illustrated the practical utility of this combined approach, emphasizing its effectiveness in tackling a broad range of mathematical problems.

The results of this research contribute significantly to the expanding body of knowledge in the field of integral transforms and iterative methods. The enhanced solutions derived from the integration of SWT and iterative methods hold great promise for various scientific and engineering disciplines, offering new tools and methodologies for addressing real-world problems with greater efficiency and accuracy. Future research can build upon these findings by exploring additional applications of the SWT in other areas of differential equations and further refining iterative methods to improve their efficiency and convergence. The continued development and application of these techniques are likely to advance the field of applied mathematics and expand its practical applications in numerous disciplines.

Acknowledgements

This research is funded partially by Zarqa University-Jordan.

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