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On the Closability of Class Totally Paranormal Operators

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Abstract. This article explores the analysis of various spectral properties pertaining to totally paranormal closed operators, extending beyond the confines of boundedness and encompassing operators defined in a Hilbert space. Within this class, closed symmetric operators are included. Initially, we establish that the spectrum of such an operator is non-empty and provide a characterization of closed-range operators in terms of the spectrum. Building on these findings, we proceed to prove Weyl's theorem, demonstrating that for a densely defined closed totally paranormal operator T, the difference between the spectrum $\sigma(T)$ and the Weyl spectrum $\sigma_w(T)$ equals the set of all isolated eigenvalues with finite multiplicities, denoted by $\pi_{00}(T)$. In the final section, we establish the self-adjointness of the Riesz projection E_μ corresponding to any non-zero isolated spectral value μ of T. Furthermore, we show that this Riesz projection satisfies the relationships $\operatorname{ran}(E_\mu) = \ker(T - \mu I) = \ker(T - \mu I)^*$. Additionally, we demonstrate that if T is a closed totally paranormal operator with a Weyl spectrum $\sigma_w(T) = 0$, then T qualifies as a compact normal operator.

2020 Mathematics Subject Classifications: 47A10,47A53, 47B20

Key Words and Phrases: Densely defined operator, closed operator, totally paranormal, reduced minimum modulus, Riesz projection, Weyl's theorem

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1. Introduction

The class of normal operators is fundamental in operator theory, having been the subject of significant research. The spectral theorem for these operators confirms the existence of non-trivial invariant subspaces and provides insight into the operator's full structure. The category of bounded paranormal operators was initially investigated by Istrătescu, who referred to it as class N [13]. Later, Furuta coined the term "paranormal operator" [8]. Numerous researchers have since studied bounded paranormal operators, including

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works by Ando and others [3, 8, 13]. Specifically, Ando [3] provided a characterization of bounded paranormal operators, while Istrătescu demonstrated that normaloid operators generalize paranormal operators [13].

A continuous linear operator on a complex Banach space is said to be paranormal if $\|Tx\|^2 \leq \|T^2x\| \|x\|$ for all $x \in X$, where X is a Banach space. T is called totally paranormal [22] if $T - \mu I$ is paranormal for every $\mu \in \mathbb{C}$. That is, $\|(T - \mu I)x\|^2 \leq \|(T - \mu I)^2 x\| \|x\|$ for all $x \in X$ and $\mu \in \mathbb{C}$. Hence, We have the following inclusion relation between some subclasses and a generalized class of bounded totally paranormal operators.

$$\text{Normal} \subseteq \text{Hyponormal} \subseteq \text{Totally Paranormal} \subseteq \text{Paranormal} \subseteq \text{normaloid}.$$

The inclusion relationships mentioned above are strict. For additional information, see [8, 22]. Daniluk extended the concept of bounded paranormal operators to encompass unbounded operators, exploring the conditions for their closability [6].

In this paper, we focus on densely defined, closed totally paranormal operators in a Hilbert space $\mathcal{H}$ and establish the following results.

Let T be a densely defined closed totally paranormal operator in $\mathcal{H}$. Then

(i) spectrum of T is non-empty.

(ii) Every isolated spectral value of T is an eigenvalue.

(iii) In addition, if $\ker(T) = \ker(T^*)$, then

  (a) range of T is closed if and only if 0 is an isolated spectral value of T.

  (b) The minimum modulus, $m(T)$ is equal to the distance of 0 from spectrum of T.

(iv) T satisfies the Weyl's Theorem i.e. $\sigma(T) \setminus \sigma_w(T) = \pi_{00}(T)$. Here $\sigma_w(T)$ is the Weyl's spectrum and $\pi_{00}(T)$ consists of all isolated eigenvalues of T with finite multiplicity.

(v) If μ is a non-zero isolated spectral value of T, then the Riesz projection E_μ with respect to μ is self-adjoint and satisfies $\operatorname{ran}(E_\mu) = \ker(T - \mu I) = \ker(T - \mu I)^*$.

The study of Weyl's theorem and the self-adjointness of the Riesz projection for isolated spectral values has been explored for various operator classes. Coburn [5] established these properties for certain non-normal operators, including hyponormal and Toeplitz operators. Schmoeger [22] expanded this work to bounded totally paranormal operators, drawing on Ando's characterization [3] of paranormal operators. Since Ando's characterization does not extend to unbounded paranormal operators, and methods for bounded operators are unsuitable here, we aim to establish properties (iv) and (v) using an alternative approach.

Gupta and Mamtani [11] showed that closed hyponormal operators satisfy Weyl's theorem. In a follow-up study [10], they outlined key conditions required for the orthogonal direct sum of densely defined closed operators to fulfill Weyl's theorem.

The paper is organized into four sections for clarity. In Section 2, we introduce key notations and summarize relevant established results that will be used throughout the study.

Section 3 focuses on examining spectral properties associated with densely defined closed totally paranormal operators. Finally, Section 4 presents the proof of Weyl's theorem for these operators.

2. Notations and preliminaries

In this article, we explore intricate Hilbert spaces, represented as $\mathcal{H}, \mathcal{H}_1, \mathcal{H}_2$, and so forth. The inner product and the corresponding norm are symbolized by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$, respectively.

The set of all linear operators on $\mathcal{H}$ is denoted as $\mathcal{L}(\mathcal{H})$, while the collection of all bounded linear operators is represented as $\mathcal{B}(\mathcal{H})$. For a linear operator $T \in \mathcal{L}(\mathcal{H})$, we use $\mathfrak{D}(T)$, $\ker(T)$, and $\operatorname{ran}(T)$ to signify its domain, null space, and range space, respectively. A linear operator T is termed a *densely defined operator* if $\overline{\mathfrak{D}(T)} = \mathcal{H}$.

If $T \in \mathcal{L}(\mathcal{H})$ and $\mathbf{M}$ is a closed subspace of $\mathcal{H}$, then $\mathbf{M}$ is said to be invariant under T , if for every $x \in \mathfrak{D}(T) \cap \mathbf{M}$, Tx is in $\mathbf{M}$. We denote the identity operator on $\mathbf{M}$ by $I_{\mathbf{M}}$, the orthogonal projection on $\mathbf{M}$ by $P_{\mathbf{M}}$. The unit sphere of $\mathbf{M}$ is $\mathbb{T}_{\mathbf{M}} := \{x \in \mathbf{M} : \|x\| = 1\}$. The restriction of T to $\mathbf{M}$ is an operator $T|_{\mathbf{M}} : \mathbf{M} \cap \mathfrak{D}(T) \to \mathcal{H}$ defined by $T|_{\mathbf{M}}x = Tx$, for all $x \in \mathbf{M} \cap D(T)$. If $\mathbf{M}$ is invariant under T , then $T|_{\mathbf{M}}$ is an operator from $\mathfrak{D}(T) \cap \mathbf{M}$ into $\mathbf{M}$.

An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *closed* if for any sequence $\{x_n\} \subseteq \mathfrak{D}(T)$ with $x_n \to x$ and $Tx_n \to y$ then $x \in \mathfrak{D}(T)$ and $Tx = y$. In this document, the notation $C(\mathcal{H}_1, \mathcal{H}_2)$ will be employed to denote the collection of closed linear operators such that $\mathfrak{D}(T) \subseteq \mathcal{H}_1$ and $\operatorname{ran}(T) \subseteq \mathcal{H}_2$. In the case where $\mathcal{H}_1$ equals $\mathcal{H}_2$, we will use the shorthand $C(\mathcal{H})$.
It is known that every densely defined operator $T \in C(\mathcal{H}_1, \mathcal{H}_2)$ has a unique adjoint in $C(\mathcal{H}_2, \mathcal{H}_1)$, that is, there exists a unique $T^* \in C(\mathcal{H}_2, \mathcal{H}_1)$ such that $\langle Tx, y \rangle = \langle x, T^*y \rangle$ for all $x \in \mathfrak{D}(T)$ and $y \in \mathfrak{D}(T^*)$.

Remark 1. *By the closed graph theorem (cf. [18, Theorem 7.1, page 231], it follows that a closed operator $T \in C(\mathcal{H}_1, \mathcal{H}_2)$ with $\mathfrak{D}(T) = \mathcal{H}_1$ is bounded.*

Lemma 1. *[9] Let $T \in \mathcal{L}(\mathcal{H})$ be a densely defined closed operator. Then*

$$\overline{\mathfrak{D}(T) \cap \ker(T)^{\perp}} = \ker(T)^{\perp}.$$

If S and T are two closed operators, then S is called an extension of T (or T is a restriction of S), if $\mathfrak{D}(T) \subseteq \mathfrak{D}(S)$ and $Sx = Tx$ for all $x \in \mathfrak{D}(T)$. This is often denoted as $T \subseteq S$. Consequently, $S = T$ if and only if $\mathfrak{D}(S) = \mathfrak{D}(T)$ and $Sx = Tx$ for all $x \in \mathfrak{D}(S) = \mathfrak{D}(T)$.

Definition 1. *[25] A densely defined operator $T \in \mathcal{L}(\mathcal{H})$ is said to be self-adjoint operator if $\mathfrak{D}(T) = \mathfrak{D}(T^*)$ and $T = T^*$. And a self-adjoint operator T is said to be positive if $\langle Tx, x \rangle \geq 0$ for all $x \in \mathfrak{D}(T)$.*

Definition 2. *[20, Page 365] If $T \in \mathcal{L}(\mathcal{H})$ is a closed operator, then the resolvent set of T is defined by*

$$\rho(T) = \left\{\mu \in \mathbb{C} : T - \mu I \text{ is invertible and } (T - \mu I)^{-1} \in \mathcal{B}(\mathcal{H})\right\}$$

and the spectrum of T, denoted by $\sigma(T)$, is defined by

$$\sigma(T) := \mathbb{C} \setminus \rho(T)$$

Note that $\sigma(T)$ is a closed subset of $\mathbb{C}$. Moreover $\sigma(T)$ can be empty set or the whole complex plane $\mathbb{C}$.
The spectrum of T decomposes as the disjoint union of *the point spectrum* $\sigma_p(T)$, *the continuous spectrum* $\sigma_c(T)$ and *the residual spectrum* $\sigma_r(T)$, where

$$\begin{aligned}
\sigma_p(T) &= \{\mu \in \mathbb{C} : T - \mu I \text{ is not injective}\},\\
\sigma_r(T) &= \{\mu \in \mathbb{C} : T - \mu I \text{ is not injective but } \operatorname{ran}(T - \mu I) \text{ is not dense in } \mathcal{H}\},\\
\sigma_c(T) &= \sigma(T) \setminus (\sigma_p(T) \cup \sigma_r(T)).
\end{aligned}$$

The spectral radius of $T \in \mathcal{B}(\mathcal{H})$ is defined by

$$r(T) := \sup\{|\mu| : \mu \in \sigma(T)\}.$$

An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be normaloid, if $r(T) = \|T\|$.
Recall that a linear operator $T \in \mathcal{L}(\mathcal{H})$ is compact, if T maps every bounded set in $\mathcal{H}$ to a pre-compact set in $\mathcal{H}$. For more details about compact operators, we refer to [21].

Definition 3. *[21, Page 156] A closed operator T in a densely defined space $\mathcal{H}$ is termed Fredholm if $\operatorname{ran}(T)$ is closed, and both the dimensions of $\ker(T)$ and its orthogonal complement $\operatorname{ran}(T)^{\perp}$ are finite. In such instances, the index of T, denoted by $\operatorname{ind}(T)$, is defined as $\operatorname{ind}(T) = \dim(\ker(T)) - \dim(\operatorname{ran}(T)^{\perp})$.*

Remark 2. *If $T \in \mathcal{L}(\mathcal{H})$ is a densely defined closed Fredholm operator and K is a compact operator, then $T + K$ is also Fredholm and $\operatorname{ind}(T + K) = \operatorname{ind}(T)$.*

Definition 4. *[21, Page 172] If $T \in \mathcal{L}(\mathcal{H})$ is a densely defined closed operator, then the Weyl's spectrum of T is defined by*

$$\sigma_w(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Fredholm of index 0}\}$$

and $\pi_{00}(T) = \{\lambda \in \sigma_p(T) : \lambda$ is isolated with $\dim(\ker(T - \lambda I)) < +\infty\}$.

Suppose $T \in \mathcal{L}(\mathcal{H})$ is a densely defined closed operator with $\sigma(T) = \sigma \cup \tau$, where σ is contained in some bounded domain Δ and τ is a subset of the complement of $\overline{\Delta}$. Let Λ be the boundary of Δ, then

$$E_\sigma = \frac{1}{2\pi i}\int_\Lambda (zI - T)^{-1} dz \tag{1}$$

is called the Riesz projection with respect to σ.

Theorem 1. *[12, Theorem 2.1, Page 326] Suppose $T \in \mathcal{L}(\mathcal{H})$ is a densely defined closed operator with $\sigma(T) = \sigma \cup \tau$, where σ is contained in some bounded domain and E_σ is the operator defined in Equation (1). Then*

(i) E_σ *is a projection.*

(ii) The subspaces $\operatorname{ran}(E_\sigma)$ *and* $\ker(E_\sigma)$ *are invariant under* T.

(iii) The subspace $\operatorname{ran}(E_\sigma)$ *is contained in* $\mathfrak{D}(T)$ *and* $T|_{\operatorname{ran}(E_\sigma)}$ *is bounded.*

(iv) $\sigma\left(T|_{\operatorname{ran}(E_\sigma)}\right) = \sigma$ *and* $\sigma(T|_{\ker(E_\sigma)}) = \tau$.

In particular, if μ is an isolated point of $\sigma(T)$, then there exist a positive real number r such that $\{z \in \mathbb{C} : |z - \mu| \leq r\} \cap \sigma(T) = \{\mu\}$. If we take $\Lambda = \{z \in \mathbb{C} : |z - \mu| = r\}$, then the Riesz projection with respect to μ is defined by

$$E_\mu = \frac{1}{2\pi i} \int_\Lambda (zI - T)^{-1} dz. \tag{2}$$

Definition 5. *[16] Let $T \in \mathcal{L}(\mathcal{H})$ be a closed operator. Then*

(i) the minimum modulus of T is defined by $m(T) := inf\left\{\|Tx\| : x \in \mathbb{T}_{\mathfrak{D}(T)}\right\}$. Then

(ii) the reduced minimum modulus of T isdenoted by $\gamma(T) := inf\left\{\|Tx\| : x \in \mathbb{T}_{\mathfrak{D}(T) \cap \ker(T)^\perp}\right\}$.

By the definition, it is clear that $m(T) \leq \gamma(T)$

The following characterization of closed range operators is frequently used in the article.

Theorem 2. *[1, Page 334] For a densely defined closed operator $T \in \mathcal{L}(\mathcal{H})$, the following are equivalent.*

(i) $\operatorname{ran}(T)$ *is closed.*

(ii) $\operatorname{ran}(T^*)$ *is closed.*

(iii) $\gamma(T) > 0$.

(iv) $S_0 = T|_{\mathfrak{D}(T) \cap \ker(T)^\perp}$ *has a bounded inverse.*

If $T \in \mathcal{L}(\mathcal{H})$ is a densely defined closed operator and $\ker(T) = \{0\}$, then the inverse operator, T^{-1} is the linear operator from $\mathcal{H}$ to $\mathcal{H}$, with $\mathfrak{D}(T^{-1}) = \operatorname{ran}(T)$ and $T^{-1}Tx = x$ for all $x \in \mathfrak{D}(T)$. In particular if T is a bijection, then by the closed graph theorem it follows that $T^{-1} \in \mathcal{B}(\mathcal{H})$. In addition, if T is normal then T has a bounded inverse if and only if $m(T) > 0$.

Theorem 3. *If $T \in \mathcal{B}(\mathcal{H})$ is a totally paranormal, then*

(i) T is normaloid.

(ii) T^{-1} is totally paranormal, if T is invertible.

(iii) T is unitary, if $\sigma(T)$ lies on the unit circle.

3. Spectral Properties

In this section, we study some spectral properties of densely defined closed totally paranormal operators.

Definition 6. *A densely defined operator $T \in \mathcal{L}(\mathcal{H})$ is said to be totally paranormal if*

$$\|(T-\lambda I)x\|^2 \leq \left\|(T-\lambda I)^2 x\right\| \|x\| \quad \text{for all } \lambda \in \mathbb{C} \text{ and } \mathfrak{D}((T-\lambda)^2) \subseteq \mathfrak{D}(T-\lambda).$$

Equivalently, T is totally paranormal if and only if $T-\lambda I$ is paranormal for all $\lambda \in \mathbb{C}$. And T is totally $*$-paranormal if $T-\lambda I$ is $*$-paranormal for all $\lambda \in \mathbb{C}$.

A densely defined operator T in $\mathcal{H}$ is said to be hyponormal if $\mathfrak{D}(T) \subseteq \mathfrak{D}(T^*)$ and $\|T^*x\| \leq \|Tx\|$ for $x \in \mathfrak{D}(T)$. And T is cohyponormal if T^* is hyponormal.

Proposition 1. *Let $T \in C(\mathcal{H})$. Then*

(i) If T is closed cohyponormal and closed totally $*$-paranormal, then T is closed totally paranormal.

(ii) If T is closed hyponormal and closed totally paranormal, then T is closed totally $*$-paranormal.

Proof. (i) Assume that T is closed cohyponormal and closed totally $*$-paranormal. If T is closed cohyponormal, then so is $T-\lambda I$ for all $\lambda \in \mathbb{C}$. Hence,

$$\mathfrak{D}((T-\lambda I)^2) \subseteq \mathfrak{D}((T-\lambda I)^*) \subseteq \mathfrak{D}(T-\lambda I).$$

Let $x \in \mathfrak{D}((T-\lambda I)^2)$. Then we have

$$\|(T-\lambda I)x\|^2 \leq \|(T-\lambda I)^*x\|^2 \leq \left\|(T-\lambda I)^2 x\right\| \|x\|.$$

(ii) By the hypotheses on T, $\mathfrak{D}((T-\lambda I)^2) \subseteq \mathfrak{D}(T-\lambda I) \subseteq \mathfrak{D}((T-\lambda I)^*)$, and for each $x \in \mathfrak{D}((T-\lambda I)^2)$,

$$\|(T-\lambda I)^*x\|^2 \leq \|(T-\lambda I)x\|^2 \leq \left\|(T-\lambda I)^2 x\right\| \|x\|.$$

So, the proof is complete.

Proposition 2. *Let $T \in C(\mathcal{H})$ be totally $*$-paranormal, and let λ be any complex scalar. Then* $\ker(T-\lambda I) \subset \ker(T-\lambda I)^*$.

Proof. Let $x \in \mathfrak{D}(T)$ be a unit eigenvector of T associated to λ. Then $Tx = \lambda x$. By the hypotheses on T, $\|T^*x\| \leq |\lambda|$. Consequently,

$$\begin{aligned} 0 &\leq \left\|T^*x - \bar{\lambda}x\right\|^2 = \|T^*x\|^2 - \lambda \langle x, Tx\rangle - \bar{\lambda}\langle Tx, x\rangle + |\lambda|^2 \\ &= \|T^*x\|^2 - |\lambda|^2 \leq 0. \end{aligned}$$

Thus, $T^*x = \bar{\lambda}x$ and so $x \in \ker(T-\lambda)^*$.

Lemma 2. *Let $T \in C(\mathcal{H})$ be a totally $*$-paranormal, then there exists a contraction $Q_\lambda \in \mathcal{B}(\mathcal{H})$ such that $(T-\lambda I)^2 \subset (T-\lambda I)^* Q_\lambda$.*

Proof. Since $\|(T-\lambda I)^*x\|^2 \leq \left\|(T-\lambda I)^2 x\right\| \|x\|$ for all $x \in \mathfrak{D}((T-\lambda I)^2) \subseteq \mathfrak{D}((T-\lambda I)^*)$, there exists a contraction $K' \in \mathcal{B}\left(\overline{\operatorname{ran}((T-\lambda I)^2)}, \overline{\operatorname{ran}(T^*-\bar{\lambda} I)}\right)$ such that $K'(T-\lambda I)^2 \subset T^* - \bar{\lambda} I$. Let $K \in \mathcal{B}(\mathcal{H})$ be any contraction which extends K' (e.g. set $Kx = 0$ for $x \in \mathcal{H} \ominus \overline{\operatorname{ran}((T-\lambda I)^2)}$). Then $K(T-\lambda I)^2 \subset T^* - \bar{\lambda} I$. Taking adjoints in the last inclusion and exploiting the closability of T , we get $(T-\lambda I)^2 \subseteq \left((T-\lambda I)^2\right)^{**} \subseteq \left(K(T-\lambda I)^2\right)^* = (T-\lambda I)^{*2} K^* \subseteq (T-\lambda I)^* K^*$. This gives us the conclusion with $Q = K^*$.

A closed subspace $\mathbf{M}$ of $\mathcal{H}$ reduces $T \in C(\mathcal{H})$ if $\mathbf{M}$ and $\mathbf{M}^\perp$ are invariant under T. Stochel [23] proved if $T \in C(\mathcal{H})$ is hyponormal and $\mathbf{M}$ is a closed subspace of $\mathcal{H}$ which is invariant under T with $T|_{\mathbf{M}}$ is normal, then $\mathbf{M}$ reduces T.

Theorem 4. *Suppose $T \in C(\mathcal{H})$ is a densely defined totally $*$-paranormal. If $\mathbf{M}$ is a closed subspace of $\mathcal{H}$ which is invariant under T with $T|_{\mathbf{M}}$ is normal, then $\mathbf{M}$ reduces T.*

Proof. Let $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, where $\mathcal{H}_1 = \mathbf{M}$ and $\mathcal{H}_2 = \mathbf{M}^\perp$. Then T has the block matrix representation

$$\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix},$$

where $T_{ij} : \mathfrak{D}(T) \cap \mathcal{H}_j \to \mathcal{H}_i$ is defined by $T_{ij} = P|_{\mathcal{H}_i} T P|_{\mathcal{H}_j}|_{\mathfrak{D}(T)\cap H_k}$ for $k = 1, 2$. Here, $P|_{\mathcal{H}_i}$ denotes the orthogonal projection onto $\mathcal{H}_i$. Since $\mathbf{M}$ is invariant under T, we have

$$\begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix}.$$

Let $y \in \mathfrak{D}(T) \cap \mathbf{M}^\perp$. By Lemma 2, we have

$$(T-\lambda I)^2 \subset (T-\lambda I)^* Q_\lambda$$

for every $\lambda \in \mathbb{C}$. Thus, $\operatorname{ran}\left((T-\lambda I)^2\right) \subseteq \operatorname{ran}((T-\lambda I)^*)$for every $\lambda \in \mathbb{C}$. Then there exist a densely defined operator B such that $(T-\lambda I)^2 = (T-\lambda I)^* B$ (see [7]). Hence, $T_{12}(y) = (T_{11}-\lambda I)^* u$ for some $u \in \mathbf{M}$. We can choose v such that $(T_{11}-\lambda I)^* u = (T_{11}-\lambda I)v$. Therefore, $T_{12}(y) = (T_{11}-\lambda I)v$ for every $\lambda \in \mathbb{C}$. Consequently,

$$T_{12}(y) \in \bigcap_{\lambda \in \mathbb{C}} \operatorname{ran}(T_{11}-\lambda I).$$

Hence, $T_{12}(y) = 0$ for all $y \in \mathfrak{D}(T) \cap \mathbf{M}^\perp$ (see [19]) and so $T_{12} = 0$. This ends the proof.

The ascent $p(T)$ and descent $q(T)$ of an operator $C(\mathcal{H})$ are given by

$$\begin{aligned} p(T) &= \inf\left\{n : \ker(T^n) = \ker(T^{n+1})\right\} \text{ and} \\ q(T) &= \inf\left\{n : \operatorname{ran}(T^n) = \operatorname{ran}(T^{n+1})\right\} \end{aligned}$$

It follows from the definition of totally paranormal the following result holds.

Proposition 3. *Let $T \in \mathcal{L}(\mathcal{H})$ be a densely defined closed totally paranormal operator. Then the ascent $p(T)$ and descent $q(T)$ of T are finite for all $\lambda \in \mathbb{C}$.*

Definition 7. *[14] Let T be a non necessarily bounded operator with domain $\mathfrak{D}(T) \subset \mathcal{H}$. We say that λ is not in $\sigma(T)$ if $T - \lambda$ is injective and $(T-\lambda)^{-1} \in \mathcal{B}(\mathcal{H})$.*

The following results immediately follows from the definition.

Proposition 4. *Let $T \in \mathcal{L}(\mathcal{H})$ be a densely defined closed totally paranormal operator. Then $T - \alpha I$ and αT are totally paranormal operators for all $\alpha \in \mathbb{C}$.*

Proposition 5. *Let Let $T \in \mathcal{L}(\mathcal{H})$ be a densely defined closed totally paranormal operator. If $\sigma(T) = \{\mu\}$, then $T = \mu I$.*

We now give the counterexample of a closed densely defined operator T such that both T and T^* are one-to-one and totally paranormal, yet T is not normal.

Example 1. *The Hilbert space in question is $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$. From [24], we have an explicit example of a densely defined unbounded closed operator T for which $\mathfrak{D}(T^2) = \mathfrak{D}(T^{*2}) = \{0\}$ More precisely, T is defined by*

$$T = \begin{bmatrix} 0 & A^{-1} \\ B & 0 \end{bmatrix}$$

on $\mathfrak{D}(T) := \mathfrak{D}(B) \oplus \mathfrak{D}(A^{-1}) \subset L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$, and where A and B are two unbounded self-adjoint operators such that $\mathfrak{D}(A) \cap \mathfrak{D}(B) = \mathfrak{D}(A^{-1}) \cap \mathfrak{D}(B^{-1}) = \{0\}$, where A^{-1} and B^{-1} are not bounded (as in [15]). Hence

$$T^* = \begin{bmatrix} 0 & B \\ A^{-1} & 0 \end{bmatrix}$$

for A^{-1} and B are both self-adjoint. Observe now that both T and T^* are one-to-one since both A^{-1} and B are so. Both T and T^* are trivially totally paranormal thanks to the assumption $\mathfrak{D}(T^2) = \mathfrak{D}(T^{*2}) = \{0\}$. So tatally paranormality of both operators need only be checked at the zero vector and this is plain as $\|(T-\mu I)x\|^2 = \left\|(T-\mu I)^2 x\right\| \|x\| = 0$ and $\|(T-\mu I)^* x\|^2 = \left\|(T-\mu I)^2 x\right\| \|x\| = 0$ for $x = 0$. However, T cannot be normal for it were, T^2 would be normal too, in particular it would be densely defined which is impossible here

Here we discuss some basic results related to unbounded totally paranormal operators, which are often used in the article.

Theorem 5. *Let $T \in \mathcal{L}(\mathcal{H})$ be a densely defined closed totally paranormal operator and not a multiple of the identity. Then the following holds.*

(i) If $\mathbf{M}$ is a closed invariant subspace of T , then $T|_{\mathbf{M}}$ is totally paranormal.

(ii) If $0 \notin \sigma(T)$, then T^{-1} totally paranormal

(iii) $\sigma(T)$ *is nonempty.*

Proof. (i) For every $\lambda \in \mathbb{C}$, as $\mathbf{M}$ is invariant under T, we have

$$\begin{aligned}
\mathfrak{D}((T-\lambda I)^2|_{\mathbf{M}}) &= \mathfrak{D}\left((T-\lambda I)^2\right) \cap \mathbf{M} \\
&= \{x \in \mathfrak{D}(T-\lambda I) : (T-\lambda I)x \in \mathfrak{D}(T-\lambda I)\} \cap \mathbf{M} \\
&= \{x \in \mathfrak{D}(T-\lambda I) \cap \mathbf{M} : (T-\lambda I)x \in \mathfrak{D}(T-\lambda I) \cap \mathbf{M}\} \\
&\quad \text{(since } T(\mathfrak{D}(T-\lambda I) \cap \mathbf{M}) \subseteq \mathbf{M}) \\
&= \{x \in \mathfrak{D}((T-\lambda I)|_{\mathbf{M}}) : Tx \in \mathfrak{D}((T-\lambda I)|_{\mathbf{M}})\} \\
&= \mathfrak{D}\left(((T-\lambda I)|_{\mathbf{M}})^2\right).
\end{aligned}$$

Thus, $(T-\lambda I)^2|_{\mathbf{M}} = ((T-\lambda)|_{\mathbf{M}})^2$. Now the result follows from the below inequality;

$$\begin{aligned}
\|(T-\lambda I)|_{\mathbf{M}}x\|^2 &= \|(T-\lambda I)x\|^2 \leq \left\|(T-\lambda I)^2 x\right\| \\
&= \left\|(T-\lambda I)^2|_{\mathbf{M}}x\right\| = \left\|((T-\lambda I)|_{\mathbf{M}})^2 x\right\|, \forall x \in \mathbb{T}_{\mathfrak{D}\left(((T-\lambda I)|_{\mathbf{M}})^2\right)}.
\end{aligned}$$

(ii) Existence of T^{-1} implies $\operatorname{ran}(T) = \mathcal{H}$ and consequently $\operatorname{ran}\left((T)^2\right) = \mathcal{H}$. As T is totally paranormal, we get $\ker(T) = \ker((T)^2)$, so T^2 is bijective and $\left((T)^2\right)^{-1}$ exists. Also $\mathfrak{D}(\left((T)^{-1}\right)^2) = \mathcal{H} = \operatorname{ran}\left((T)^2\right)$. If $y \in \mathcal{H}$, then there exist $x \in \mathfrak{D}\left((T)^2\right)$, such that $y = T^2 x$. Now,

$$\begin{aligned}
\left\|T^{-1}y\right\|^2 &= \|Tx\|^2 \leq \left\|T^2 x\right\| \|x\| \\
&= \|y\| \left\|T^{-2}y\right\|.
\end{aligned}$$

Hence T^{-1} is totally paranormal since totally paranormal has invariant translation property.
(iii) Suppose on the contrary that $\sigma(T) = \emptyset$. Then T is invertible and $T \in \mathcal{B}(\mathcal{H})$.
First, we show that $\sigma(T^{-1}) = \{0\}$. For any complex number $\mu \neq 0$, consider the operator $S = \mu^{-1}(T - \mu^{-1}I)^{-1}$. Here S can also be written as the sum of two bounded operators, $S = \mu^{-1}(I + \mu^{-1}(T - \mu^{-1}I)^{-1})$, so S is bounded. By a simple computation we can show that S is the bounded inverse of $\mu I - T^{-1}$. Thus $\sigma(T^{-1}) \subseteq \{0\}$. As $T^{-1} \in \mathcal{B}(\mathcal{H})$, this implies $\sigma(T^{-1})$ is non-empty, so we conclude that $\sigma(T^{-1}) = \{0\}$. By (ii), T^{-1} is bounded totally paranormal operator and consequently normaloid by Theorem 3. Hence $\left\|T^{-1}\right\| = 0$, which implies $T^{-1} = 0$, a contradiction. Hence $\sigma(T)$ is non-empty.

Now we discuss about isolated spectral values of totally paranormal operators.

Theorem 6. *Let* T *be a densely defined closed totally paranormal operator. If* μ *is an isolated point of* $\sigma(T)$*, then* $\ker(T - \mu I) = \operatorname{ran}(E_\mu)$.

Proof. It follows from [4, Lemma 3.4] that $\ker(T - \mu I) \subseteq \operatorname{ran}(E_\mu)$. To complete the proof we have to show that $\ker(T - \mu I) \supseteq \operatorname{ran}(E_\mu)$.

As a consequence of Theorem 1 and Theorem 5, we know that $T|_{\operatorname{ran}(E_\mu)}$ is bounded

and totally paranormal. By Theorem 3 it follows that $T|_{\text{ran}(E_\mu)}$ is normaloid.
If $\mu = 0$, then $\sigma\left(T|_{\text{ran}(E_0)}\right) = \{0\}$. This implies $\left\|T|_{\text{ran}(E_0)}\right\| = 0$ and consequently $T|_{\text{ran}(E_0)} = 0$. Hence $\text{ran}(E_0) \subseteq \ker(T)$.
If $\mu \neq 0$, then $\sigma\left(\mu^{-1}T|_{\text{ran}(E_\mu)}\right) = \{1\}$. By Theorem 3, it follows that $\mu^{-1}T|_{\text{ran}(E_\mu)}$ is unitary. Thus $T|_{\text{ran}(E_\mu)} - \mu I_{\text{ran}(E_\mu)}$ is normal and $\sigma\left(T|_{\text{ran}(E_\mu)} - \mu I_{\text{ran}(E_\mu)}\right) = \{0\}$. Since every normal operator is normaloid, we conclude that $T|_{\text{ran}(E_\mu)} - \mu I_{\text{ran}(E_\mu)} = 0$. Hence $\text{ran}(E_\mu) \subseteq \ker(T - \mu I)$.

Theorem 7. *Let $T \in \mathcal{L}(\mathcal{H})$ be a densely defined closed totally paranormal operator and μ be an isolated point of $\sigma(T)$. Then $\ker(E_\mu) = \text{ran}(T - \mu I)$.*

Proof. By Theorem 1, $\mu \notin \sigma\left(T|_{\ker(E_\mu)}\right)$. This implies that $\text{ran}(T - \mu I)|_{\ker(E_\mu)} = n(E_\mu)$ and consequently $n(E_\mu) \subseteq \text{ran}(T - \mu I)$.
Let $y \in \text{ran}(T - \mu I)$. There exist $x \in \mathfrak{D}(T)$ such that $y = (T - \mu I)x$. Since $\mathcal{H} = \text{ran}(E_\mu) + \ker(E_\mu)$ and $\text{ran}(E_\mu) \cap \ker(E_\mu) = \{0\}$, we have $x = p + q$, where $p \in \text{ran}(E_\mu)$ and $q \in \ker(E_\mu)$.
It follows from Theorem 6, that $p \in \ker(T - \mu I) \subseteq \mathfrak{D}(T)$ and consequently $q = x - p \in \mathfrak{D}(T)$. As we know from Theorem 1 that $\ker(E_\mu)$ is invariant under T, we have

$$y = (T - \mu I)x = (T - \mu I)q \in (T - \mu I)(\ker(E_\mu)) \subseteq \ker(E_\mu).$$

Hence $\text{ran}(T - \mu I) \subseteq \ker(E_\mu)$. This proves the result.

The following results are consequences of Theorem 7 which gives a characterization for closed range totally paranormal operators.

Corollary 1. *Suppose $T \in \mathcal{L}(\mathcal{H})$ is a densely defined closed totally paranormal operator. If 0 is an isolated point of $\sigma(T)$, then $\text{ran}(T)$ is closed.*

In general, the converse of Corollary 1 is not true. We have the following example to illustrate this.

Example 2. *Let $T : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ be defined by*

$$T(x_1, x_2, \cdots) = (0, x_1, x_2, \cdots), \text{ for all } (x_n) \in \ell^2(\mathbb{N})$$

Then $\sigma(T) = \{z \in \mathbb{C} : |z| \leq 1\}$, $\text{ran}(T) = \ell^2(\mathbb{N}) \setminus span\{e_1\}$. Here $\text{ran}(T)$ is closed but 0 is not an isolated point of $\sigma(T)$. Clearly, T is a totally paranormal operator.

Next result gives a sufficient condition under which the converse of Corollary 1 is also true.

Theorem 8. *Let $T \in \mathcal{L}(\mathcal{H})$ be a densely defined closed totally paranormal operator with $\ker(T) = \ker(T^*)$ and $0 \in \sigma(T)$. Then 0 is an isolated point of $\sigma(T)$ if and only if $\text{ran}(T)$ is closed.*

Proof. The necessary condition follows from Corollary 1. To prove the sufficient condition. Assume that $\operatorname{ran}(T)$ is closed. Consider $S_0 = T|_{\ker(T)^\perp} : \ker(T)^\perp \cap \mathfrak{D}(T) \to \ker(T)^\perp$. Clearly S_0 is injective and $\operatorname{ran}(S_0) = \operatorname{ran}(T)$ is closed. Also $\operatorname{ran}(S_0) = \ker(T^*)^\perp = \ker(T)^\perp$, consequently S_0 is bijective and $S_0^{-1} \in \mathcal{B}(\ker(T)^\perp)$. Thus $0 \notin \sigma(S_0)$. Applying [2, Theorem 5.4, Page 289], $\sigma(T) \subseteq \{0\} \cup \sigma(S_0)$. Since $0 \in \sigma(T)$, we have $\sigma(T) = \{0\} \cup \sigma(S_0)$ and hence 0 is an isolated point of $\sigma(T)$.

Note that Theorem 8 does not hold if we drop the condition $\ker(T) = \ker(T^*)$. Consider the operator T defined in Example 2. Clearly $\ker(T) = \{0\} \neq span\{e_1\} = \ker(T^*)$, and $\operatorname{ran}(T)$ is closed but 0 is not an isolated point of $\sigma(T)$.

Theorem 9. *Let $T \in \mathcal{L}(\mathcal{H})$ be a densely defined closed totally paranormal operator. If $\ker(T) = \ker(T^*)$, then $m(T) = d(0, \sigma(T))$, the distance between 0 and $\sigma(T)$.*

Proof. We will prove this result by considering the following two cases, which exhaust all the possibilities.
Case (1): T is not injective. Clearly $m(T) = 0$ and $0 \in \sigma_p(T)$. Hence $m(T) = 0 = d(0, \sigma(T))$.
Case (2): T is injective. It suffices to show that $\gamma(T) = d(0, \sigma(T))$ because $m(T) = \gamma(T)$.

First assume that $\gamma(T) = 0$. It follows from Theorem 2 that $\operatorname{ran}(T)$ is not closed and consequently $0 \in \sigma_c(T)$. Thus $d(0, \sigma(T)) = 0 = \gamma(T)$.
Now assume that $\gamma(T) > 0$. As a consequence of Theorem 2, $\operatorname{ran}(T)$ is closed. Note that $0 \notin \sigma(T)$, otherwise Theorem 9 and Theorem 6 implies that $0 \in \sigma_p(T)$. But this is not true, as T is injective. Thus $0 \notin \sigma(T)$ and T^{-1} is bounded totally paranormal operator, by Theorem 5. Hence T^{-1} is normaloid and [17, Proposition 2.12] implies that

$$\begin{aligned}\gamma(T) &= \frac{1}{\|T^{-1}\|} = \frac{1}{r(T^{-1})} = \frac{1}{\sup\{|\mu| : \mu \in \sigma(T^{-1})\}} \\ &= \inf\{|\nu| : \nu \in \sigma(T)\} = d(0, \sigma(T)).\end{aligned}$$

This completes the proof.

As a consequence of Theorem 9 we have the following result.

Corollary 2. *If $T \in \mathcal{L}(\mathcal{H})$ is a densely defined closed totally paranormal operator and $\ker(T) = \ker(T^*)$, then $\gamma(T) = d(T) := \inf\{|\mu| : \mu \in \sigma(T) \setminus \{0\}\}$.*

Proof. Consider the operator $S_0 = T|_{\ker(T)^\perp} : \mathfrak{D}(T) \cap \ker(T)^\perp \to \ker(T)^\perp$. By Theorem 5 and Theorem 9, S_0 is paranormal and $\gamma(T) = m(S_0) = d(0, \sigma(S_0)) = d(T)$. This proves the result

The following example shows the following facts:

- Theorem 9 does not hold if $\ker(T) \neq \ker(T^*)$.
- It is well known that the residual spectrum of a closed densely defined normal operator is empty. But this is not true in the case of totally paranormal operators.

Example 3. *Let $T : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ be defined by*

$$T(x_1, x_2, \cdots) = (0, x_1, 2x_2, 3x_3, \cdots),$$

where $\mathfrak{D}(T) = \left\{(x_1, x_2, \cdots) \in \ell^2(\mathbb{N}) : \sum_{j=1}^{+\infty} \|jx_j\|^2 < +\infty\right\}$.
As c_0, the space of all complex sequences consisting of at most finitely many non zero terms is a subset of $\mathfrak{D}(T)$ and is dense in $\ell^2(\mathbb{N})$, we can conclude that T is densely defined. Hence T^* is well defined. Note that T is a closed operator. We can show that

$$T^*(x_1, x_2, \cdots) = (x_2, 2x_3, 3x_4, \cdots)$$

with $D(T^*) = \left\{(x_n) \in \ell^2(\mathbb{N}) : \sum_{j=2}^{+\infty} \|(j-1)x_j\|^2 < +\infty\right\}$.
For any $x = (x_n) \in \mathfrak{D}((T - \lambda I)^2)$ and $\lambda \in \mathbb{C}$, we have

$$\begin{aligned} \|(T - \lambda I)x\|^2 &= \sum_{j=1}^{+\infty} \|(j - \lambda)x_j\|^2 \leq \sum_{j=1}^{+\infty} (j + 1 - \lambda)(j - \lambda) \|x_j\|^2 \\ &\leq \left(\sum_{j=1}^{+\infty} ((j + 1 - \lambda)(j - \lambda) \|x_j\|)^2\right)^{\frac{1}{2}} \left(\sum_{j=1}^{+\infty} \|x_j\|^2\right)^{\frac{1}{2}} \\ &\leq \left\|(T - \lambda I)^2 x\right\| \|x\| . \end{aligned}$$

Hence T is totally paranormal.
Since $\|Tx\| \geq \|x\|$ for all $x \in \mathfrak{D}(T)$ and $\|Te_1\| = \|e_1\|$, we get $m(T) = 1$. Also it can be easily verified that T is injective, $\operatorname{ran}(T) = \ell^2(\mathbb{N}) \setminus span\{e_1\}$ is closed but $\operatorname{ran}(T) \neq \mathcal{H}$, so $0 \in \sigma(T)$. Hence $d(0, \sigma(T)) = 0 \neq 1 = m(T)$.

Now we will show that $\sigma(T) = \mathbb{C}$. To prove this, we show that $T - \mu I$ is injective and $\ker(T - \mu i)^* \neq \{0\}$, for all $\mu \in \mathbb{C}$.

Let $\mu \in \mathbb{C} \setminus \{0\}$ and $(T - \mu I))x = 0$ for some $x = (x_n) \in \mathfrak{D}(T)$. Then

$$(-\mu x_1, x_1 - \mu x_2, 2x_2 - \mu x_3, \cdots) = 0.$$

Equating component-wise we get $x = 0$. This implies that $T - \mu I$ injective. Let $y = (y_n) \in \mathfrak{D}(T^*)$ be such that $(T - \mu I)^* y = 0$. That is,

$$(y_2 - \bar{\mu} y_1, 2y_3 - \bar{\mu} y_2, y_4 - \bar{\mu} y_3, \cdots) = 0.$$

From this we get

$$y = \left(1, \bar{\mu}, \frac{(\bar{\mu})^2}{2!}, \frac{(\bar{\mu})^3}{3!}, \cdots\right) y_1. \tag{3}$$

If $\mu = 0$, then $\ker(T^*) = span\{e1\}$. If $\mu \neq 0$, then we will show that y obtained in Equation (3) belongs to $\ker(T - \mu I)^*$. Consider $z_n = \frac{\bar{\mu}^{2n}}{(n!)^2}$. Then

$$\left|\frac{z_{n+1}}{z_n}\right| = \frac{|\mu|^2}{n+1} \to 0 \text{ as } n \to +\infty.$$

By the ratio test we conclude that $\sum_{j=1}^{+\infty} z_n$ *is absolutely convergent, that is* $\sum_{n=1}^{+\infty}\left(\frac{|\mu|^n}{n!}\right)^2 < +\infty$. *Thus* $y \in \ell^2(\mathbb{N})$. *On the similar lines we can show that* $\sum_{n=1}^{+\infty}\left(\frac{|\mu|^n}{(n-1)!}\right)^2 < +\infty$. *Hence* $\ker(T-\mu I))^* \neq \{0\}$.
For every $\mu \in \mathbb{C}$, $\ker(T-\mu I) = \{0\}$ *and* $\operatorname{ran}(T-\mu I) = (\ker(T-\mu I)^*)^{\perp} \neq \ell^2(\mathbb{N})$. *Hence we conclude that* $\mu \in \sigma_r(T)$, *and* $\sigma(T) = \mathbb{C}$. *We also have* $\gamma(T) = 1 \neq 0 = d(T)$. *From this we conclude that Corollary 2 is also not true if the condition,* $\ker(T) = \ker(T^*)$ *is dropped.*

Theorem 10. *Suppose* $T \in \mathcal{L}(\mathcal{H})$ *is a densely defined closed totally paranormal operator,* $\ker(T) = \ker(T^*)$ *and 0 is an isolated spectral value of* T . *Then* $0 \in \sigma_p(T)$.

Proof. Since 0 is an isolated spectral value of T , $d(T) > 0$. Hence by Corollary 2, $\gamma(T) >)$ so that by Theorem 8, $\operatorname{ran}(T)$ is closed. If $0 \notin \sigma_p(T)$, then $\ker(T) = \{0\}$ so that we also have $\operatorname{ran}(T) = \overline{\operatorname{ran}(T)} = \ker(T)^{\perp} = \mathcal{H}$, making T bijective and hence $0 \notin \sigma(T)$, a contradiction. Hence $0 \in \sigma_p(T)$.

Example 4. *The converse of Theorem 10 need not be true. To see this, consider* $T : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ *defined by*

$$T(x_1, x_2, x_3, \cdots) = \left(0, 2x_2, \frac{1}{3}x_3, 4x_4, \frac{1}{5}x_5, \cdots\right),$$

where $\mathfrak{D}(T) = \left\{x \in \ell^2(\mathbb{N}) : \left(0, 2x_2, \frac{1}{3}x_3, 4x_4, \frac{1}{5}x_5, \cdots\right) \in \ell^2(\mathbb{N})\right\}$. *Here* T *is a densely defined closed totally paranormal operator. Since* T *is not one to one,* $0 \in \sigma_p(T)$ *but it is not an isolated point of the spectrum* $\sigma(T) = \left\{0, 2, \frac{1}{3}, 4, \frac{1}{5}, \cdots\right\}$.

Theorem 11. *Suppose* $T \in \mathcal{L}(\mathcal{H})$ *is a densely defined closed totally paranormal operator,* $\ker(T) = \ker(T^*)$. *Then* $\operatorname{ran}(T)$ *is closed if and only if 0 is not an accumulation point of* $\sigma(T)$.

Proof. By Theorem 2, $\operatorname{ran}(T)$ is closed if and only if $\gamma(T) > 0$ and by Corollary 2, $\gamma(T) = d(T)$. Hence, $\operatorname{ran}(T)$ is closed if and only if $d(T) > 0$ if and only if 0 is not an accumulation point of $\sigma(T)$.

4. Weyl's theorem for totally paranormal operators

In this section, we demonstrate the fulfillment of Weyl's theorem by a densely defined closed operator T that is totally paranormal. Additionally, we establish the self-adjointness of the Riesz projection E_μ corresponding to any non-zero isolated spectral value μ of T. For a Hilbert space $\mathcal{H}$ decomposed as $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, where $T \in \mathcal{L}(\mathcal{H})$ is a closed operator, we ascertain the block matrix representation of T.

$$T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \tag{4}$$

where $T_{ij} : \mathfrak{D}(T) \cap \mathcal{H}_j \to \mathcal{H}_i$ is defined by $T_{ij} = P\mathcal{H}_i T P\mathcal{H}_j|_{\mathfrak{D}(T)\cap\mathcal{H}_j}$ for $i, j = 1, 2$. Here $P_{\mathcal{H}_i}$ is an orthogonal projection onto $\mathcal{H}_i$.
For $(x_1, x_2) \in (\mathcal{H}_1 \cap \mathfrak{D}(T) \oplus (\mathcal{H}_2 \cap \mathfrak{D}(T)),$

$$T(x_1, x_2) = (T_{11}x_1 + T_{12}x_2, T_{21}x_1 + T_{22}x_2).$$

Remark 3. *Let T be as defined in Equation (4). If $\mathcal{H}_1 = \ker(T) \neq \{0\}$ and $\mathcal{H}_2 = \ker(T)^{\perp}$, then*

$$T = \begin{bmatrix} 0 & T_{12} \\ 0 & T_{22} \end{bmatrix}. \tag{5}$$

- *If T is densely defined closed operator then by Lemma 1, $T_{22} \in \mathcal{L}(\ker(T)^{\perp})$ is also densely defined closed operator.*

- *It can be easily checked that $\operatorname{ran}(T_{22}) = \operatorname{ran}(T) \cap \ker(T)^{\perp}$. If $\operatorname{ran}(T)$ is closed, then $\operatorname{ran}(T_{22})$ is closed in $\ker(T)^{\perp}$.*

We say a closed operator $T \in \mathcal{L}(\mathcal{H})$ satisfy the Weyl's theorem if the Weyl's spectrum, $\sigma_w(T)$ consists of all spectral values of T except the isolated eigenvalues of finite multiplicity. That is, $\sigma(T) \setminus \sigma_w(T) = \pi_{00}(T)$.

Coburn [5] established that Weyl's theorem applies to all bounded hyponormal and Toeplitz operators. Schmoeger [22] subsequently extended this finding to include bounded totally paranormal operators. In this study, we seek to verify the applicability of Weyl's theorem to unbounded totally paranormal operators.

Theorem 12. *Let $T \in \mathcal{L}(\mathcal{H})$ be a densely defined closed totally paranormal operator. Then Weyl's theorem holds for T, that is, $\sigma(T) \setminus \sigma_w(T) = \pi_{00}(T)$.*

Proof. Let $\mu \in \sigma(T)\setminus\sigma_w(T)$. So, we have $\dim(\ker(T-\mu I)) = \dim(\ker(T-\mu I)^*) < +\infty$ and $\operatorname{ran}(T - \mu I)$ is closed.
On $\mathcal{H} = \ker(T - \mu I) \oplus \ker(T - \mu I)^{\perp}$, $T - \mu I$ can be decomposed as

$$T - \mu I = \begin{bmatrix} 0 & T_{12} \\ 0 & T_{22} - \mu I|_{\ker(T-\mu I)^{\perp}} \end{bmatrix},$$

where $T_{22} = P|_{\ker(T-\mu I)^{\perp}} T|_{\ker(T-\mu I)^{\perp}}$. By Remark 3, $T_{22} - \mu I_{\ker(T-\mu I)^{\perp}}$ is a densely defined closed operator with domain $\mathfrak{D}(T-\mu I) \cap \ker(T-\mu I)^{\perp}$ and $\operatorname{ran}(T_{22} - \mu I_{\ker(T-\mu I)^{\perp}})$ is closed. As $\ker(T-\mu I)$ is finite dimensional, this implies T_{12} is finite rank operator and by Remark 2, $\operatorname{ind}(T - \mu I) = \operatorname{ind}\left(T_{22} - \mu I|_{\ker(T-\mu I)^{\perp}}\right) = 0$.
Since $\ker(T_{22} - \mu I|_{\ker(T-\mu I)^{\perp}}) = \{0\}$ and $\operatorname{ind}\left(T_{22} - \mu I|_{\ker(T-\mu I)^{\perp}}\right) = 0$, we get $\ker(T_{22} - \mu I|_{\ker(T-\mu I)^{\perp}})^* = \{0\}$ and consequently $\overline{\operatorname{ran}(T_{22} - \mu I|^{\perp}_{\ker(T-\mu I)})} = \ker(T - \mu I)^{\perp}$. Thus $T_{22} - \mu I|^{\perp}_{\ker(T-\mu I)}$ has bounded inverse and hence $\mu \notin \sigma(T_{22})$. As $\sigma(T) \subseteq \{\mu\} \cup \sigma(T_{22})$, this implies that μ is an isolated point of $\sigma(T)$. Hence $\mu \in \pi_{00}(T)$. Conversely, let $\mu \in \pi_{00}(T)$. Now consider the Riesz projection E_μ with respect to μ. By Theorem 1 and

Theorem 7, $\mu \notin \sigma(T|_{\ker(E_\mu)})$ and $\operatorname{ran}(T-\mu I) = \operatorname{ran}\left((T-\mu I)|_{\ker(E_\mu)}\right) = \ker(E_\mu)$. Since $\mu \notin \sigma(T|_{\ker(E_\mu)})$, we have that $\operatorname{ran}((T-\mu I)|_{\ker(E_\mu)}) = \ker(E_\mu)$. Hence $\operatorname{ran}(T-\mu I)$ is closed. Also $((T-\mu I)|_{\ker(E_\mu)})^{-1} \in \mathcal{B}(\ker(E_\mu))$. Thus we get

$$\begin{aligned} \dim\ker(T-\mu I)^* &= \dim\left(\operatorname{ran}(T-\mu I)^\perp\right) = \dim\left(\ker(E_\mu)^\perp\right) \\ &= \dim\left(\operatorname{ran}(E_\mu)\right) = \dim\left(\ker(T-\mu I)\right). \end{aligned}$$

Note that $\dim\left(\ker(E_\mu)^\perp\right) = \dim\left(\operatorname{ran}(E_\mu)\right)$ but the spaces, $\ker(E_\mu)^\perp$ and $\operatorname{ran}(E_\mu)$ need not be the same. Hence $T-\mu I$ is Fredholm operator of index zero. This proves our result.

Theorem 13. *Let $T \in \mathcal{L}(\mathcal{H})$ be a densely defined closed totally paranormal operator and μ be a non-zero isolated point of $\sigma(T)$. Then the Riesz projection E_μ with respect to μ satisfy*

$$\operatorname{ran}(E_\mu) = \ker(T-\mu I) = \ker(T-\mu I)^*.$$

Moreover, E_μ is self-adjoint.

Proof. Let μ be a non-zero isolated point of $\sigma(T)$. By Theorem 1 and Theorem 7, $\mu \notin \sigma\left(T|_{\ker(E_\mu)}\right)$ and $\operatorname{ran}(T-\mu I) = \ker(E_\mu)$. That means $(T-\mu I)|_{\ker(E_\mu)} : \ker(E_\mu) \cap \mathfrak{D}(T) \to \ker(E_\mu) = \operatorname{ran}(T-\mu I)$ is a bijection. Also $(T-\mu I)|_{\ker(T-\mu I)^\perp} \cap \mathfrak{D}(T) : \ker(T-\mu I) \cap \mathfrak{D}(T) \to \operatorname{ran}(T-\mu I)$ is a bijection, we have $\ker(E_\mu) \cap \mathfrak{D}(T) \subseteq \ker(T-\mu I)^\perp \cap \mathfrak{D}(T)$.
Now we claim that $\ker(E_\mu) \cap \mathfrak{D}(T) = \ker(T-\mu I)^\perp \cap \mathfrak{D}(T)$. Let $x \in \ker(T-\mu I)^\perp \cap \mathfrak{D}(T)$ and $E_\mu x = p + q$, where $p \in \ker(T-\mu I)$, $q \in \ker(T-\mu I)^\perp$. Operating E_μ on both sides, we get

$$p + q = E_\mu x = p + E_\mu q.$$

This implies $E_\mu q = q \in \operatorname{ran}(E_\mu) \cap \ker(T-\mu I)^\perp = \{0\}$, by Theorem 6. From this we conclude that $E_\mu x = p = E_\mu p$, that is, $x - p \in \ker(E_\mu) \cap \mathfrak{D}(T) \subseteq \ker(T-\mu I)^\perp \cap \mathfrak{D}(T)$. As $x \in \ker(T-\mu I)^\perp$, we get $p \in \ker(T-\mu I) \cap \ker(T-\mu I)^\perp = \{0\}$. Consequently $E_\mu x = 0$. So $\ker(T-\mu I)^\perp \cap \mathfrak{D}(T) \subseteq \ker(E_\mu) \cap \mathfrak{D}(T)$. Hence $\ker(T-\mu I)^\perp \cap \mathfrak{D}(T) = \ker(E_\mu) \cap \mathfrak{D}(T)$. By Lemma 1 and Theorem 7, we get

$$\begin{aligned} \ker(T-\mu I)^\perp &= \overline{\ker(T-\mu I)^\perp \cap \mathfrak{D}(T)} = \overline{\ker(E_\mu) \cap \mathfrak{D}(T)} \\ &= \overline{\operatorname{ran}(T-\mu I) \cap \mathfrak{D}(T)} = \overline{(\ker(T-\mu I)^*)^\perp \cap \mathfrak{D}(T)} \subseteq (\ker(T-\mu I)^*)^\perp. \end{aligned}$$

Hence $\ker(T-\mu I)^* \subseteq \ker(T-\mu I)$. By Theorem 7, $\ker(E_\mu)^\perp = \operatorname{ran}(T-\mu I)^\perp = \ker((T-\mu I)^*) \subseteq \ker(T-\mu I) = \operatorname{ran}(E_\mu)$. Hence $\ker(E_\mu)^\perp = \operatorname{ran}(T-\mu I)^\perp = \ker((T-\mu I)^*) \subseteq \ker(T-\mu I) = \operatorname{ran}(E_\mu)$. Hence $\ker(E_\mu)^\perp \subseteq \operatorname{ran}(E_\mu)$. If $x \in \operatorname{ran}(E_\mu)$, then $x = u + v$ where $u \in \ker(E_\mu)$ and $v \in \ker(E_\mu)^\perp$. As $\ker(E_\mu)^\perp \subseteq \operatorname{ran}(E_\mu)$, we get $u = x - v \in \ker(E_\mu) \cap \operatorname{ran}(E_\mu) = \{0\}$. Thus we get $\ker(E_\mu)^\perp = \operatorname{ran}(E_\mu)$, which is equivalent to say that $\ker(T-\mu I) = \ker(T-\mu I)^*$. As $\ker(E_\mu)^\perp = \operatorname{ran}(E_\mu)$, we have that E_μ is an orthogonal projection. Hence E_μ is self-adjoint.

From the proof of Theorem 13, we have

Corollary 3. *Let $T \in \mathcal{L}(\mathcal{H})$ be a densely defined closed paranormal operator. Then*

$$\ker(T - \mu I)^* \subset \ker(T - \mu I)$$

for all $\mu \in \mathbb{C}$.

Applying the concept of Birkhoff–James orthogonality, we show that for a paranormal operator, the eigenspaces corresponding to distinct isolated eigenvalues are entirely independent of one another. Specifically, let M be a subspace within a Banach space X. We say that M is Birkhoff–James orthogonal to another subspace N in X if $\|m\| \leq \|m + n\|$ for all $m \in M$ and $n \in N$. This definition aligns with the concept of Birkhoff–James orthogonality, which, in the case of a Hilbert space X, matches the traditional notion of orthogonality.

Proposition 6. *Let $T \in \mathcal{L}(\mathcal{H})$ be a densely defined closed totally paranormal operator. If μ_1 and μ_2 are two non zero distinct isolated points of $\sigma(T)$, then $\ker(T - \mu_1 I)$ is orthogonal to $\ker(T - \mu_2 I)$.*

Proof. Without loss of generality, assume that $|\mu_1| < |\mu_2|$. For any $x \in \ker(T - \mu_1 I)$ and $y \in \ker(T - \mu_2 I)$, consider the set $M = span\{x, y\}$. As M is invariant subspace for T , it follows that $T|_M$ is totally paranormal operator and $\|T|_M\| = |\mu_2|$. We have the following.

$$\begin{aligned} \left\| \frac{\mu_1^n}{\mu_2^n} x + y \right\| &= \frac{1}{|\mu_2^n|} \|\mu_1^n x + \mu_2^n y\| \\ &\leq \frac{\|T|_M\|^n}{|\mu_2^n|} \|x + y\| = \|x + y\| \end{aligned}$$

Taking the limit $n \to +\infty$, we get $\|y\| \leq \|x + y\|$, for every $x \in \ker(T - \mu_1 I)$ and $y \in \ker(T - \mu_2 I)$. Hence $\ker(T - \mu_2 I)$ is orthogonal to $\ker(T - \mu_1 I)$.
Next, if $|\mu_1| = |\mu_2|$, then for every $n \in \mathbb{N}$

$$\begin{aligned} \left\| \left(\frac{\mu_1 + \mu_2}{2\mu_2} \right)^n x + y \right\| &= \left\| \frac{(\mu_1 + \mu_2)^n x + (\mu_1 + \mu_2)^n y}{(2\mu_2)^n} \right\| \\ &\leq \frac{1}{(2|\mu_2|)^n} \sum_{j=0}^{n} \binom{n}{j} |\mu_2|^j \left\| \mu_1^{n-j} x + \mu_2^{n-j} y \right\| \\ &= \frac{1}{(2|\mu_2|)^n} \sum_{j=0}^{n} \binom{n}{j} |\mu_2|^j \left\| (T|_M)^{n-j} (x + y) \right\| \\ &\leq \frac{1}{(2|\mu_2|)^n} \sum_{j=0}^{n} \binom{n}{j} |\mu_2|^n \|x + y\| \\ &= \|x + y\| . \end{aligned}$$

As $\mu_1 \neq \mu_2$, we have $\left| \frac{\mu_1 + \mu_2}{2\mu_2} \right| < 1$. Now as $n \to +\infty$ in the above inequality we get that $\|y\| \leq \|x + y\|$. This proves the result

Proposition 7. *Let $T \in \mathcal{L}(\mathcal{H})$ be a densely defined closed totally paranormal operator and T^2 be a compact operator. Then T is also compact and normal.*

Proof. Assume that T is a totally paranormal operator . Hence,

$$\|Tx\|^2 \leq \left\|T^2 x\right\| \|x\| \text{ for every } x \in \mathfrak{D}(T^2). \tag{6}$$

Let $\{x_m\} \in \mathcal{H}$ be weakly convergent sequence with limit 0 in $\mathfrak{D}(T)$. From the compactness of T^2 and the relation (6) we get the following relation:

$$\|Tx_m\|^2 \to 0, \ m \to +\infty.$$

From the last relation it follows that T is compact. Since T is compact $\sigma(T)$ is finite set or countable infinite with 0 as the unique limit point of it. Let $\sigma(T) \setminus \{0\} = \{\lambda_n\}$ with

$$|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n| \geq |\lambda_{n+1}| \geq \cdots \geq 0, \text{ and } \lambda_n \to 0 \ (n \to +\infty).$$

By the compactness of T or isoloidness of T, $\lambda_n \in \sigma_p(T)$ and $dim \ker(T - \lambda_n) < +\infty$ for all n. Since $\ker(T - \lambda_n) \subset \ker(T - \lambda_n)^*$, $\mathbf{M} := \bigoplus_{n=1}^{+\infty} \ker(T - \lambda_n)$ reduces T, and T is of the form

$$T = \left(\bigoplus_{n=1}^{+\infty} \lambda_n\right) \oplus T' \text{ on } \mathcal{H} = \mathbf{M} \oplus \mathbf{M}^{\perp}.$$

By the construction, T' is totally paranormal and $\sigma(T') = \{0\}$ hence $T' = 0$. This shows that

$$T = \left(\bigoplus_{n=1}^{+\infty} \lambda_n\right) \oplus 0$$

and it is normal.

Theorem 14. *Let $T \in \mathcal{L}(\mathcal{H})$ be a densely defined closed totally paranormal operator with $\sigma_w(T) = \{0\}$. Then T is a compact normal operator.*

Proof. By Theorem 12, T satisfy Weyl's theorem and this implies that each element in $\sigma(T) \setminus \sigma_w(T) = \sigma(T) \setminus \{0\}$ is an eigenvalue of T with finite multiplicity, and is isolated in $\sigma(T)$. Hence $\sigma(T) \setminus \{0\}$ is a finite set or a countable set with 0 as its only accumulation point. Put $\sigma(T) \setminus \{\lambda_n\}$, where $\lambda_n \neq \lambda_m$ whenever $n \neq m$ and $\{|\lambda_n|\}$ is a non-increasing sequence. Since T is normaloid, we have $|\lambda_1| = \|T\|$. By Corollary 3, we have $(T - \lambda_1 I)x = 0$ implies $(T - \lambda_1 I)^* x = 0$. Hence $\ker(T - \lambda_1 I)$ is a reducing subspace of T. Let E_1 be the orthogonal projection onto $\ker(T - \lambda_1 I)$. Then $T = \lambda_1 I \oplus T_1$ on $\mathcal{H} = \operatorname{ran}(E_1) \oplus \operatorname{ran}(I - E_1)$. Since T_1 is totally paranormal by Theorem 5 (i) and $\sigma_p(T) = \sigma_p(T_1) \cup \{\lambda_1\}$, we have $\lambda_2 \in \sigma_p(T_1)$. By the same argument as above, $\ker(T - \lambda_2 I) = \ker(T_1 - \lambda_2 I)$ is a finite dimensional reducing subspace of T which is included in $\operatorname{ran}(I - E_1)$. Put E_2 be the othogonal projection onto $\ker(T - \lambda_2 I)$. Then $T = \lambda_1 E_1 \oplus \lambda_2 E_2 \oplus T_2$ on $\mathcal{H} = \operatorname{ran}(E_1) \oplus \operatorname{ran}(E_2) \oplus \operatorname{ran}(I - E_1 - E_2)$. By repeating above argument, each $\ker(T - \lambda_n I)$

is a reducing subspace of T and $\left\| T - \bigoplus_{k=1}^{n} \lambda_k E_k \right\| = \|T_n\| = |\lambda_{n+1}| \to 0$ as $n \to +\infty$. Here E_k is the orthogonal projection onto $\ker(T - \lambda_k)$ and $T = (\bigoplus_{k=1}^{n} \lambda_k E_k) \oplus T_n$ on $\mathcal{H} = \bigoplus_{k=1}^{n} \operatorname{ran}(E_k) \oplus (1 - \sum_{k=1}^{n} \operatorname{ran}(E_k)$. Hence $T = \bigoplus_{k=1}^{+\infty} \lambda_k E_k$ is compact and normal because each E_k is a finite rank orthogonal projection which satisfies $E_k E_t = 0$ whenever $k \neq t$ by Proposition 6 and $\lambda_n \to 0$ as $n \to +\infty$.

5. Conclusion

To conclude, this article provides a thorough analysis of the spectral properties of totally paranormal closed operators within Hilbert spaces. The study goes beyond standard constraints on boundedness, also considering closed symmetric operators.

The initial focus was on establishing the non-emptiness of the spectrum for such operators, accompanied by a characterization of closed-range operators based on the spectrum. Building on these foundational results, Weyl's theorem was proven for densely defined closed totally paranormal operators. Specifically, it was demonstrated that the difference between the spectrum $\sigma(T)$ and the Weyl spectrum $\sigma_w(T)$ is precisely the set of isolated eigenvalues with finite multiplicities, denoted as $\pi_{00}(T)$.

The final section of the article explored the self-adjointness of the Riesz projection E_μ corresponding to any non-zero isolated spectral value μ of the operator T. The relationships $\operatorname{ran}(E_\mu) = \ker(T - \mu I) = \ker(T - \mu I)^*$ were established for this Riesz projection. Furthermore, it was shown that if a closed totally paranormal operator T has a Weyl spectrum $\sigma_w(T) = 0$, then T qualifies as a compact normal operator.

In terms of future work, potential avenues include exploring applications of these spectral properties in specific mathematical or physical contexts. Additionally, investigating the implications of these results on related areas of operator theory or functional analysis could provide valuable insights. Further developments in the understanding of totally paranormal operators and their spectral characteristics may contribute to advancements in various mathematical disciplines.

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References

[1] Thomas N.E. Greville Adi Ben-Israel. *Generalized Inverses: Theory and Applications.* Springer, 2 edition, 2003.

[2] D.C. Lay A.E. Taylor. *Introduction to Functional Analysis.* Wiley, 2 edition, 1986.

[3] T. Ando. Operators with a norm condition. *Acta Sci. Math. (Szeged)*, 33:169–178, 1972.

[4] Neeru Bala and G. Ramesh. Weyl's theorem for paranormal closed operators. *Annals of Functional Analysis*, 11:567 – 582, 2018.

[5] L. Coburn. Weyl's theorem for nonnormal operators. *Michigan Math. J.*, 20, 01 1966.

[6] Andrzej Daniluk. On the closability of paranormal operators. *Journal of Mathematical Analysis and Applications*, 376(1):342–348, 2011.

[7] R. G. Douglas. On majorization, factorization, and range inclusion of operators on hilbert space. *Proceedings of the American Mathematical Society*, 17(2):413–415, 1966.

[8] T. Furuta. On the class of paranormal operators. *Proceedings of the Japan Academy*, 43:594–598, 1967.

[9] Ramesh G and S. Kulkarni. The carrier graph topology. *Banach Journal of Mathematical Analysis*, 5:56–69, 01 2011.

[10] A. Gupta and K. Mamtani. Variants of weyl's theorem for direct sums of closed linear operators. *Advances in Operator Theory*, 2:409–418, 09 2017.

[11] Anuradha Gupta and Karuna Mamtani. Weyl type theorems for unbounded hyponormal operators. *Kyungpook mathematical journal*, 55:531–540, 09 2015.

[12] Marinus A. Kaashoek Israel Gohberg, Seymour Goldberg. *Classes of Linear Operators, vol. I, Operator Theory: Advances and Applications*, volume 49. Birkhäuser Basel, 199.

[13] V. Istrătescu. On some hyponormal operators. *Pacific Journal of Mathematics*, 22:413–417, 1967.

[14] T. Kato. *Perturbation Theory for Linear Operators*, volume 132. Springer, Berlin, 2 edition, 198.

[15] Hideki Kosaki. On intersections of domains of unbounded positive operators. *Kyushu Journal of Mathematics*, 60, 03 2006.

[16] S. Kulkarni and Ramesh G. On the denseness of minimum attaining operators. *Operators and Matrices*, 12, 09 2016.

[17] S. Kulkarni, M. Thamban Nair, and Ramesh G. Some properties of unbounded operators with closed range. *Proceedings Mathematical Sciences*, 118:613–625, 11 2008.

[18] M.T. Nair. *Funcitonal Analysis: A First Course.* PHI Learning Pvt. Ltd., 2021.

[19] C. R. Putnam. Ranges of normal and subnormal operators. *Michigan Mathematical Journal*, 18(1):33 – 36, 1971.

[20] W. Rudin. *Functional Analysis.* McGraw-Hill, 1973.

[21] M. Schechter. *Principles of Functional Analysis.* American Mathematical Society, 2 edition, 1971.

[22] C. Schmoeger. On totally paranormal operators. *Bull. Austral. Math. Soc.*, 66(3):425–441, 2002.

[23] Jan Stochel. An asymmetric putnam–fuglede theorem for unbounded operators. *Proceedings of the American Mathematical Society*, 129:2261–2271, 03 2001.

[24] Schŏichi Ōta and Konrad Schmüdgen. Some classes of unbounded operators. *Integral Equations and Operator Theory*, 12:211–226, 03 1989.

[25] R. Walter. *Functional analysis: International Series in Pure and Applied Mathematics.* New York: McGraw-Hill Inc., 2 edition, 1991.