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Fourth-Order Differential Equations: Asymptotic and Oscillatory Behaviors of Solutions

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Abstract. Our aim in this work is to derive conditions and criteria for the oscillation of some differential equations of p-Laplace type with a delayed term. Therefore, we develop these criteria that confirm to us that the equations studied are oscillatory by applying comparison with lower-order equations and Riccati techniques . Finally, we can elucidate the meaning of the new inequalities by applying our findings to a few particular cases of the studied equation. Our findings build on earlier findings that looked at equations with a delay term and operators of the p-Laplace type. To demonstrate the importance of the acquired results, we provide an example.

2020 Mathematics Subject Classifications: 34C10, 34K11

Key Words and Phrases: Asymptotic behavior, p-Laplacian, Fourth-order, Delay differential equations

1. Introduction

The study of systems influenced by their historical behavior requires the use of functional equations (FDEs), which are equations where the variables' current values are dependent on their past or future states. Delay differential equations are important categories of these equations. These formulas are essential for simulating intricate systems in disciplines like biology, engineering, and physics. For instance, in control theory, FDEs regulate feedback systems to maintain stability, and in ecology, they aid in the analysis of population dynamics based on historical states. Continuous research in DDEs is essential for creating theoretical underpinnings and some techniques to address real-world issues as contemporary systems become more complicated [12]-[19].
In our work, we focus on the oscillation of

$$\left(\varkappa(\eta)\left|z'''(\eta)\right|^{p-2}z'''(\eta)\right)' + \sum_{i=1}^{j} a_i(\eta) f(z(b_i(\eta))) = 0,\ \eta \ge \eta_0, \tag{1}$$

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where $\varkappa \in C^1([\eta_0,\infty),\mathbb{R})$, $\varkappa(\eta) > 0$, $\varkappa'(\eta) \geq 0, a_i \in C[\eta_0,\infty), a(\eta) > 0, b_i \in C[\eta_0,\infty), b_i(\eta) \leq \eta, \lim_{\eta\to\infty} b_i(\eta) = \infty;\ i = 1,2,..,j, f \in C(\mathbb{R},\mathbb{R})$ such that

$$f(\eta)/\eta^{p-1} \geq \ell > 0,\ \text{for } \eta \neq 0,\ p > 1 \text{is a constant}, \tag{2}$$

and under the condition

$$\int_{\eta_0}^{\infty} \frac{1}{\varkappa^{1/(p-1)}(\eta)} \mathrm{d}\eta = \infty. \tag{3}$$

Definition 1. *[6] If $\varkappa(\eta)(z'''(\eta))^{p-1} \in C^1[\eta_z,\infty)$, and $z(\eta)$ satisfies (1) on $[\eta_z,\infty)$, then a function $z \in C^3[\eta_z,\infty)$, $\eta_z \geq \eta_0$, is a solution of (1). If a solution of (1) contains arbitrarily large zeros on $[\eta_z,\infty)$, it is said to be oscillatory; if not, it is said to be nonoscillatory. If all of its solutions are oscillatory, the equations (1) are considered oscillatory.*

The oscillation criteria for equtions with p-Laplace type have drawn a lot of interest from scientists, engineers, and researchers in the study of the oscillation to DDEs, which has grown in importance and prominence in recent years. Some advanced models based on delays differential equations with fractional characteristics have been developed as a result of this interest and have shown value in a variety of sectors [7]-[2]. This technique is effective for researching the transmission of ultrasound, mimicking the behavior of proteins and polymers, and examining the mechanical behavior of human tissues under stress. We can better comprehend numerous biological and physical processes thanks to these models, which also enable scientists come up with creative solutions for challenging practical problems (see [1]-[11]).

In order to apply mathematical methods to practical or real-world issues, the issue must be stated in mathematical terms. This entails developing a model a mathematical description of the issue. It is known mathematically that derivatives describe rates of change, so equations that link functions and their derivatives are often included in mathematical models. These equations, also referred to as differential equations, are used in many scientific domains, including economics, chemistry, physics, and biology [20]-[21].

The qualitative theory of differential equations has an important place in the study of applied as well as theoretical mathematics. It introduces dynamical systems, a popular area of mathematics in recent years, and it is works as an expansion and generalization of some types of ordinary equations. It also comes in quite handy when dealing with complicated differential equations that are impossible to solve with traditional techniques. Making assumptions on the solutions behavior without actually solving them is the basic idea underlying qualitative analysis of differential equations [16, 23].

In mathematics, a delay differential equation is a kind of FDEs that expresses the derivative of some function at a given time in types of the function's values at previous times. These equations are called hereditary systems, dead time systems, aftereffects systems, time delay systems. Additionally, there are sophisticated differential equations that can be used in a variety of real-world scenarios where the rate at which a system's state where these changes are based on the current and future situation. The equation can be changed

to highlight the effects of possible future actions. Population dynamics, mechanical control engineering and economic issues are a few domains where these equations are frequently applied. NDEs are used in many areas of natural and technological inquiry[3, 14].
One subfield of qualitative theory, oscillation theory, examines the qualitative characteristics of differential equation solutions, including stability, oscillation, and others, without actually solving the problems [8]-[9]. According to [**?**]-[15], the solutions of the examined equation are divided into three distinct classes: oscillatory solutions, positive and negative eventually solutions. Researchers started studying the equations of fourth-order after the oscillation for the second-order equations developed, see [17, 22].

Bazighifan and colleagues [6], we employed some techniques to ontain the adequate and required criteria for the oscillation of

$$\left(\varkappa(\eta)\left|z'''(\eta)\right|^{p-2}z'''(\eta)\right)'+a(\eta)f(z(b(\eta)))=0, \tag{4}$$

under the condition

$$\int_{\eta_0}^{\infty}\frac{1}{\varkappa^{1/p-1}(\eta)}\mathrm{d}\eta=\infty. \tag{5}$$

New standards were presented by Bazighifan and Thabet [5] to evaluate the oscillatory of fourth-order DEs.
Li et al. [13] concentrated on the oscillation of

$$\left(\left(z^{(u-1)}(\eta)\right)^{p-1}\right)'+a(\eta)f(z(b(\eta)))=0, \tag{6}$$

by utilizing the integral averaging method with Riccati technique and found new criteria for oscillation.

Theorem 1. *([1]) If*

$$\limsup_{\eta\to\infty}\int_{\eta_0}^{\eta}\left(\mu(s)a(s)-\lambda\varphi\frac{(\mu'(s))^{(p-1)+1}}{(\mu(s)b^{u-2}(s)b'(s))^{p-1}}\right)\mathrm{d}s=\infty, \tag{7}$$

where $\lambda:=(1/((p-1)+1))^{(p-1)+1}(2(u-1)!)^{p-1}, \mu\in C^1([\eta_0,\infty),(0,\infty))$ *and* $\varphi>1$, *then every solution of (6) is oscillatory.*

Theorem 2. *([4]) Let* $f\left(\eta^{1/p-1}\right)/\eta\geq 1$ *for* $0<\eta\leq 1,\ h\in(0,1)$ *such that*

$$\lim\inf_{\eta\to\infty}\int_{b_i(\eta)}^{\eta}a(s)f\left(\frac{h}{(u-1)!}\frac{b^{u-1}(s)}{\varkappa^{1/p-1}(b(s))}\right)\mathrm{d}s>\frac{1}{\mathrm{e}}, \tag{8}$$

then (6) is oscillatory.

The goal of researching this work is to enhance and supplement the findings of [18].

The structure of the paper is as follows. We provide a few lemmas in Section 2 that will be helpful in demonstrating our findings. We provide a new standards of oscillation for (1) by the use of generalized Riccati transformations in Section 3. Lastly, a few examples are taken into consideration to highlight the main findings.

2. Preliminary Results

The lemmas, and presumptions presented in this part are crucial for streamlining the mathematical computations utilized in this work.

There are just two instances when examining the asymptotic behavior of the positive solutions of (1).

$$\begin{array}{llll} \text{Case (1)}: & z^{(m)}(\eta) > 0 & \text{for} & m = 0,1,2,3; \\ \text{Case (2)}: & z^{(m)}(\eta) > 0 & \text{for} & m = 0,1,3 \text{ and } z''(\eta) < 0. \end{array}$$

For convenience, we denote

$$R(\eta) := \int_{\eta}^{\infty} \frac{1}{\varkappa^{1/p-1}(s)} \mathrm{d}s, \ \ F_{+}(\eta) := \max\{0, F(\eta)\},$$

$$\varrho(\eta) := \mu(\eta) \left(\ell \sum_{i=1}^{j} a_i(\eta) \left(\frac{b_i^3(\eta)}{\eta^3} \right)^{p-1} + \frac{\varepsilon \nu_1^{(1+(p-1))/p-1} \eta^2 - 2\nu_1 p - 1}{2\varkappa^{\frac{1}{p-1}}(\eta) R^{(p-1)+1}(\eta)} \right),$$

$$\sigma(\eta) := \frac{\mu'_{+}(\eta)}{\mu(\eta)} + \frac{((p-1)+1)\nu_1^{1/p-1} \varepsilon \eta^2}{2\varkappa^{\frac{1}{(p-1)}}(\eta) R(\eta)}, \ \ \sigma^*(\eta) := \frac{\varsigma'_{+}(\eta)}{\varsigma(\eta)} + \frac{2\nu_2}{R(\eta)},$$

and

$$\varrho^*(\eta) := \varsigma(\eta) \left(\int_{\eta}^{\infty} \left(\frac{\ell}{\varkappa(v)} \int_{v}^{\infty} \sum_{i=1}^{j} a_i(s) \frac{b_i^{p-1}(s)}{s^{p-1}} \mathrm{d}s \right)^{1/p-1} \mathrm{d}v + \frac{\nu_2^2 - \nu_2 \varkappa^{\frac{-1}{p-1}}(\eta)}{R^2(\eta)} \right),$$

where $\mu, \varsigma \in C^1([\eta_0, \infty), (0, \infty))$ and ν_1, ν_2 are constants.

Remark 1. *The generalized Riccati substitutions are defined by us.*

$$\zeta(\eta) := \mu(\eta) \left(\frac{\varkappa(\eta) (z''')^{p-1}(\eta)}{z^{p-1}(\eta)} + \frac{\nu_1}{R^{p-1}(\eta)} \right), \tag{9}$$

and

$$w(\eta) := \varsigma(\eta) \left(\frac{z'(\eta)}{z(\eta)} + \frac{\nu_2}{R(\eta)} \right). \tag{10}$$

Lemma 1. *[2] Assume that $V > 0$, U be constant, and v is the ratio of two odd values. Then*

$$P^{(\nu+1)/\nu} - (P-a)^{(\nu+1)/\nu} \leq \frac{1}{\nu} a^{1/\nu} [(1+\nu)P - a], \quad Pa \geq 0, \ \nu \geq 1$$

and

$$Uz - Vz^{(\nu+1)/\nu} \leq \frac{\nu^{\nu}}{(\nu+1)^{\nu+1}} \frac{U^{\nu+1}}{V^{\nu}}.$$

Lemma 2. *[23] Suppose that* $g \in C^u\left([\eta_0, \infty), (0, \infty)\right)$, $g^{(u)}$ *is of a fixed sign on* $[\eta_0, \infty)$, $g^{(u)}$ *not identically zero and there exists a* $\eta_1 \geq \eta_0$ *such that*

$$g^{(u-1)}(\eta)\, g^{(u)}(\eta) \leq 0,$$

for all $\eta \geq \eta_1$. *If we have* $\lim_{\eta\to\infty} g(\eta) \neq 0$, *then there exists* $\eta_\nu \geq \eta_1$ *such that*

$$g(\eta) \geq \frac{\nu}{(u-1)!}\eta^{u-1}\left|g^{(u-1)}(\eta)\right|,$$

for all $\nu \in (0,1)$ *and* $\eta \geq \eta_\nu$.

Lemma 3. *[10] If* $\kappa^{(j)} > 0$ *and* $\kappa^{(u+1)} < 0$, *then*

$$\frac{u!}{\eta^u}\kappa(\eta) - \frac{(u-1)!}{\eta^{u-1}}\frac{d}{d\eta}\kappa(\eta) \geq 0,$$

for all $j = 0, 1, ..., u$.

3. Oscillation criteria

We will define various oscillation criterion for equation (1) in this part.

Lemma 4. *Let* z *be a positive solution of (1) in the end, and for all* $r = 1, 2, 3$, $z^{(r)}(\eta) > 0$. *In the case where* $\mu \in C^1\left([\eta_0, \infty), (0, \infty)\right)$, *and* $\zeta \in C^1[\eta, \infty)$ *defined as (9), then*

$$\zeta'(\eta) \leq -\varrho(\eta) + \sigma(\eta)\zeta(\eta) - \frac{\varepsilon\eta^2(p-1)}{2\left(\varkappa(\eta)\mu(\eta)\right)^{1/p-1}}\left(\zeta(\eta)\right)^{\frac{p}{p-1}}, \tag{11}$$

for all $\eta > \eta_1$.

Proof. Let $z > 0$. From Lemma 2, wefind

$$z'(\eta) \geq \frac{\varepsilon}{2}\eta^2 z'''(\eta), \varepsilon \in (0,1). \tag{12}$$

By (9), we find $\zeta(\eta) > 0$ for $\eta \geq \eta_1$, and

$$\begin{aligned}\zeta'(\eta) &= \mu'(\eta)\left(\frac{\varkappa(\eta)\left(z'''\right)^{p-1}(\eta)}{z^{p-1}(\eta)} + \frac{\nu_1}{R^{p-1}(\eta)}\right) + \mu(\eta)\frac{\left(\varkappa\left(z'''\right)^{p-1}\right)'(\eta)}{z^{p-1}(\eta)} \\ &\quad -(p-1)\mu(\eta)\frac{z^{(p-1)-1}(\eta)\, z'(\eta)\,\varkappa(\eta)\left(z'''\right)^{p-1}(\eta)}{z^{2p-1}(\eta)} + \frac{(p-1)\nu_1\mu(\eta)}{\varkappa^{\frac{1}{p-1}}(\eta)\, R^p(\eta)}.\end{aligned}$$

Using (12) and (9), we obtain

$$\zeta'(\eta) \leq \frac{\mu'_+(\eta)}{\mu(\eta)}\zeta(\eta) + \mu(\eta)\frac{\left(\varkappa(\eta)\left(z'''(\eta)\right)^{p-1}\right)'}{z^{p-1}(\eta)}$$

$$
\begin{aligned}
& -(p-1)\mu(\eta)\frac{\varepsilon}{2}\eta^2\frac{\varkappa(\eta)(z'''(\eta))^p}{z^p(\eta)}+\frac{(p-1)\nu_1\mu(\eta)}{\varkappa^{\frac{1}{p-1}}(\eta)R^p(\eta)} \\
\leq\ & \frac{\mu'(\eta)}{\mu(\eta)}\zeta(\eta)+\mu(\eta)\frac{\left(\varkappa(\eta)(z'''(\eta))^{p-1}\right)'}{z^{p-1}(\eta)} \\
& -(p-1)\mu(\eta)\frac{\varepsilon}{2}\eta^2\varkappa(\eta)\left(\frac{\zeta(\eta)}{\mu(\eta)\varkappa(\eta)}-\frac{\nu_1}{\varkappa(\eta)R^{p-1}(\eta)}\right)^{\frac{p}{p-1}}+\frac{(p-1)\nu_1\mu(\eta)}{\varkappa^{\frac{1}{p-1}}(\eta)R^p(\eta)} \quad (13)
\end{aligned}
$$

Using Lemma 1 with $P=\zeta(\eta)/(\mu(\eta)\varkappa(\eta))$, $a=\nu_1/\left(\varkappa(\eta)R^{(p-1)}(\eta)\right)$ and $\nu=(p-1)$, we get

$$
\begin{aligned}
\left(\frac{\zeta(\eta)}{\varkappa(\eta)\mu(\eta)}-\frac{\nu_1}{\varkappa(\eta)R^{(p-1)}(\eta)}\right)^{\frac{p}{p-1}} \geq\ & \left(\frac{\zeta(\eta)}{\mu(\eta)\varkappa(\eta)}\right)^{\frac{p}{p-1}} \\
& -\frac{\nu_1^{1/(p-1)}}{(p-1)\varkappa^{\frac{1}{(p-1)}}(\eta)R(\eta)}\left(((p-1)+1)\frac{\zeta(\eta)}{\mu(\eta)\varkappa(\eta)}-\frac{\nu_1}{\varkappa(\eta)R^{(p-1)}(\eta)}\right) \quad (14)
\end{aligned}
$$

From Lemma 3, we have that $z(\eta)\geq\frac{\eta}{3}z'(\eta)$ and hence,

$$
\frac{z(b_i(\eta))}{z(\eta)}\geq\frac{b_i^3(\eta)}{\eta^3}. \quad (15)
$$

From (1), (13) and (14), we obtain

$$
\begin{aligned}
\zeta'(\eta) \leq\ & \frac{\mu'_+(\eta)}{\mu(\eta)}\zeta(\eta)-\ell\mu(\eta)\sum_{i=1}^{j}a_i(\eta)\left[\frac{b_i^3(\eta)}{\eta^3}\right]^{p-1}-(p-1)\mu(\eta)\frac{\varepsilon}{2}\eta^2\varkappa(\eta)\left(\frac{\zeta(\eta)}{\mu(\eta)\varkappa(\eta)}\right)^{\frac{p}{(p-1)}} \\
& -(p-1)\mu(\eta)\frac{\varepsilon}{2}\eta^2\varkappa(\eta)\left(\frac{-\nu_1^{1/(p-1)}}{(p-1)\varkappa^{\frac{1}{(p-1)}}(\eta)R(\eta)}\left(\frac{p\zeta(\eta)}{\mu(\eta)\varkappa(\eta)}-\frac{\nu_1}{\varkappa(\eta)R^{(p-1)}(\eta)}\right)\right)+\frac{(p-1)\nu_1\mu(\eta)}{\varkappa^{\frac{1}{(p-1)}}(\eta)R^p(\eta)}
\end{aligned}
$$

This implies that

$$
\begin{aligned}
\zeta'(\eta) \leq\ & \left(\frac{\mu'_+(\eta)}{\mu(\eta)}+\frac{p\nu_1^{1/(p-1)}\varepsilon\eta^2}{2\varkappa^{\frac{1}{(p-1)}}(\eta)R(\eta)}\right)\zeta(\eta)-\frac{\varepsilon\eta^2(p-1)}{2\varkappa^{1/(p-1)}(\eta)\mu^{1/(p-1)}(\eta)}\zeta^{\frac{p}{p-1}}(\eta) \\
& -\mu(\eta)\left(\ell\sum_{i=1}^{j}a_i(\eta)\left(\frac{b_i^3(\eta)}{\eta^3}\right)^{(p-1)}+\frac{\varepsilon\nu_1^{p/(p-1)}\eta^2-2\nu_1(p-1)}{2\varkappa^{\frac{1}{(p-1)}}(\eta)R^p(\eta)}\right).
\end{aligned}
$$

Thus,

$$
\zeta'(\eta)\leq-\varrho(\eta)+\sigma(\eta)\zeta(\eta)-\frac{(p-1)\varepsilon\eta^2}{2(\varkappa(\eta)\mu(\eta))^{1/(p-1)}}\zeta^{\frac{p}{(p-1)}}(\eta).
$$

The proof is complete.

Lemma 5. *Let z be a positive solution of (1) in the end and Case* (2) *hold, then*

$$w'(\eta) \leq -\varrho^{*}(\eta) + \sigma^{*}(\eta) w(\eta) - \frac{1}{\varsigma(\eta)} w^{2}(\eta), \tag{16}$$

where $\varsigma \in C^{1}([\eta_0, \infty), (0, \infty))$.

Proof. Let z ultimately be a positive solution of (1) and Case (2) hold. Lemma 3 gives us the result that $z(\eta) \geq \eta z'(\eta)$. This inequality can be integrated from $b_i(\eta)$ to η to obtain

$$z(b_i(\eta)) \geq \frac{b_i(\eta)}{\eta} z(\eta).$$

From (2), we so have

$$f(z(b_i(\eta))) \geq \ell \frac{b_i^{(p-1)}(\eta)}{\eta^{(p-1)}} z^{(p-1)}(\eta). \tag{17}$$

Integrating (1) from η to κ and using $z'(\eta) > 0$, we obtain

$$\begin{aligned} \varkappa(\kappa)\left(z'''(\kappa)\right)^{(p-1)} - \varkappa(\eta)\left(z'''(\eta)\right)^{(p-1)} &= -\int_{\eta}^{\kappa} \sum_{i=1}^{j} a_i(s) f(z(b_i(s))) ds \\ &\leq -\ell z^{(p-1)}(\eta) \int_{\eta}^{\kappa} \sum_{i=1}^{j} a_i(s) \frac{b_i^{(p-1)}(s)}{s^{(p-1)}} \mathrm{d}s. \end{aligned}$$

Letting $\kappa \to \infty$, we find

$$\varkappa(\eta)\left(z'''(\eta)\right)^{(p-1)} \geq \ell z^{(p-1)}(\eta) \int_{\eta}^{\infty} \sum_{i=1}^{j} a_i(s) \frac{b_i^{(p-1)}(s)}{s^{(p-1)}} \mathrm{d}s$$

and so

$$z'''(\eta) \geq z(\eta) \left(\frac{\ell}{\varkappa(\eta)} \int_{\eta}^{\infty} \sum_{i=1}^{j} a_i(s) \frac{b_i^{(p-1)}(s)}{s^{(p-1)}} \mathrm{d}s \right)^{1/(p-1)}.$$

Once more integrating from η to ∞, we obtain

$$z''(\eta) \leq -z(\eta) \int_{\eta}^{\infty} \left(\frac{\ell}{\varkappa(v)} \int_{v}^{\infty} \sum_{i=1}^{j} a_i(s) \frac{b_i^{(p-1)}(s)}{s^{(p-1)}} \mathrm{d}s \right)^{1/(p-1)} \mathrm{d}v. \tag{18}$$

By differentiating $w(\eta)$, we find

$$w'(\eta) = \frac{\varsigma'(\eta)}{\varsigma(\eta)} w(\eta) + \varsigma(\eta) \frac{z''(\eta)}{z(\eta)} - \varsigma(\eta) \left(\frac{w(\eta)}{\varsigma(\eta)} - \frac{\nu_2}{R(\eta)} \right)^{2} + \frac{\varsigma(\eta)\nu_2}{\varkappa^{1/(p-1)}(\eta) R^{2}(\eta)}. \tag{19}$$

Using Lemma 1 with $P = w(\eta)/\varsigma(\eta)$, $a = \nu_2/R(\eta)$ and $\nu = 1$, we get

$$\left(\frac{w(\eta)}{\varsigma(\eta)} - \frac{\nu_2}{R(\eta)} \right)^{2} \geq \left(\frac{w(\eta)}{\varsigma(\eta)} \right)^{2} - \frac{\nu_2}{R(\eta)} \left(\frac{2w(\eta)}{\varsigma(\eta)} - \frac{\nu_2}{R(\eta)} \right). \tag{20}$$

By (1), (19) and (20), we get

$$\begin{aligned} w'(\eta) &\le \frac{\varsigma'(\eta)}{\varsigma(\eta)} w(\eta) - \varsigma(\eta) \int_{\eta}^{\infty} \left(\frac{\ell}{\varkappa(v)} \int_{v}^{\infty} \sum_{i=1}^{j} a_i(s) \frac{b_i^{(p-1)}(s)}{s^{(p-1)}} \mathrm{d}s \right)^{1/(p-1)} \mathrm{d}v \\ & -\varsigma(\eta) \left(\left(\frac{w(\eta)}{\varsigma(\eta)} \right)^2 - \frac{\nu_2}{R(\eta)} \left(\frac{2w(\eta)}{\varsigma(\eta)} - \frac{\nu_2}{R(\eta)} \right) \right) + \frac{\nu_2 \varsigma(\eta)}{\varkappa^{\frac{1}{(p-1)}}(\eta) R^2(\eta)}. \end{aligned}$$

This implies that

$$\begin{aligned} w'(\eta) &\le \left(\frac{\varsigma_+'(\eta)}{\varsigma(\eta)} + \frac{2\nu_2}{R(\eta)} \right) w(\eta) - \frac{1}{\varsigma(\eta)} w^2(\eta) \\ & -\varsigma(\eta) \left(\int_{\eta}^{\infty} \left(\frac{\ell}{\varkappa(v)} \int_{v}^{\infty} \sum_{i=1}^{j} a_i(s) \frac{b_i^{(p-1)}(s)}{s^{(p-1)}} \mathrm{d}s \right)^{1/(p-1)} \mathrm{d}v + \frac{\nu_2^2 - \nu_2 \varkappa^{\frac{-1}{(p-1)}}(\eta)}{R^2(\eta)} \right). \end{aligned}$$

Thus,

$$w'(\eta) \le -\varrho^*(\eta) + \sigma^*(\eta) w(\eta) - \frac{1}{\varsigma(\eta)} w^2(\eta).$$

The proof is finished.

Lemma 6. *Let z be a positive solution of (1) in the end. If*

$$\int_{\eta_0}^{\infty} \left(\varrho(s) - \left(\frac{2}{\varepsilon s^2} \right)^{(p-1)} \frac{\varkappa(s) \mu(s) (\sigma(s))^p}{p^p} \right) \mathrm{d}s = \infty, \tag{21}$$

where $\mu \in C([\eta_0, \infty))$ and $\varepsilon \in (0,1)$, then z doesn't fulfill Case (1).

Proof. Let z be a positive solution of (1) in the end. By Lemma 4, we see (11) holds and from Lemma 1 with

$$U = \sigma(\eta), \ V = (p-1)\varepsilon\eta^2 / \left(2 (\varkappa(\eta) \mu(\eta))^{1/(p-1)} \right) \text{ and } \ \eta = \zeta,$$

we get

$$\zeta'(\eta) \le -\varrho(\eta) + \left(\frac{2}{\varepsilon \eta^2} \right)^{(p-1)} \frac{\varkappa(\eta) \mu(\eta) (\sigma(\eta))^{(p-1)+1}}{p^p}. \tag{22}$$

Once more integrating from η_1 to η, we find

$$\int_{\eta_1}^{\eta} \left(\varrho(s) - \left(\frac{2}{\varepsilon s^2} \right)^{(p-1)} \frac{\varkappa(s) \mu(s) (\sigma(s))^p}{p^p} \right) \mathrm{d}s \le \zeta(\eta_1),$$

which contradicts (21). So, The proof is finished.

Lemma 7. *Let z be an eventually positive solution of (1) and Case (2) hold. If*

$$\int_{\eta_0}^{\infty}\left(\varrho^*(s)-\frac{1}{4}\varsigma(s)(\sigma^*(s))^2\right)\mathrm{d}s=\infty,\ \textit{where } \varsigma\in C([\eta_0,\infty)) \tag{23}$$

then z doesn't fulfill Case (2).

Proof. Let z be a positive solution of (1) in the end. (16) holds according to Lemma 5. Lemma 1 is used with

$$U=\sigma^*(\eta),\ V=1/\varsigma(\eta),p=2\text{ and }\eta=w,$$

We obtainThe formula is

$$\zeta'(\eta)\leq-\varrho^*(\eta)+\frac{1}{4}\varsigma(\eta)(\sigma^*(\eta))^2. \tag{24}$$

Once more integrating from η_1 to η, we find

$$\int_{\eta_1}^{\eta}\left(\varrho^*(s)-\frac{1}{4}\varsigma(s)(\sigma^*(s))^2\right)\mathrm{d}s\leq\zeta(\eta_1).$$

This runs counter to (23). The proof is finished.

Theorem 3. *Assume that (21) and (23) hold. Then (1) is oscillatory.*

The oscillation requirements that result from applying $\mu(\eta)=\eta^3$ and $\varsigma(\eta)=\eta$ to Theorem 3 are as follows:

Corollary 1. *Let (3) hold. Assume that*

$$\limsup_{\eta\to\infty}\int_{\eta_1}^{\eta}\left(\beta(s)-\left(\frac{2}{\varepsilon s^2}\right)^{(p-1)}\frac{\varkappa(s)\mu(s)(\beta(s))^p}{p^p}\right)\mathrm{d}s=\infty, \tag{25}$$

for some $\varepsilon\in(0,1)$. If

$$\limsup_{\eta\to\infty}\int_{\eta_1}^{\eta}\left(\beta_1(s)-\frac{1}{4}\varsigma(s)(\beta_1(s))^2\right)\mathrm{d}s=\infty, \tag{26}$$

where

$$\beta(\eta)\ :\ =\eta^3\left(\ell\sum_{i=1}^{j}a_i(\eta)\left(\frac{b_i^3(\eta)}{\eta^3}\right)^{(p-1)}+\frac{\varepsilon\nu_1^{p/(p-1)}\eta^2-2\nu_1(p-1)}{2\varkappa^{\frac{1}{(p-1)}}(\eta)R^p(\eta)}\right)$$

$$\beta(\eta)\ :\ =\frac{3}{\eta}+\frac{((p-1)+1)\nu_1^{1/(p-1)}\varepsilon\eta^2}{2\varkappa^{\frac{1}{(p-1)}}(\eta)R(\eta)},\quad\beta_1(\eta):=\frac{1}{\eta}+\frac{2\nu_2}{R(\eta)}$$

and

$$\beta_1(\eta):=\eta\left(\int_{\eta}^{\infty}\left(\frac{\ell}{\varkappa(v)}\int_{v}^{\infty}\sum_{i=1}^{j}a_i(s)\frac{b_i^{(p-1)}(s)}{s^{(p-1)}}\mathrm{d}s\right)^{1/(p-1)}\mathrm{d}v+\frac{\nu_2^2-\nu_2\varkappa^{\frac{-1}{(p-1)}}(\eta)}{R^2(\eta)}\right),$$

then (1) is oscillatory.

Example 1. *Let equation*

$$z^{(4)}(\eta) + \frac{c_0}{\eta^4} z\left(\frac{1}{2}\eta\right) = 0, \ \eta \geq 1, \tag{27}$$

where $p = 2, \varkappa(\eta) = 1, \ c_0 > 0, \ a(\eta) = c_0/\eta^4$ *and* $b(\eta) = \eta/2$. *Hence, we have*

$$R(\eta_0) = \infty, \ \beta(s) = \frac{c_0}{8s}.$$

If we set $\ell = \nu_1 = 1$, *then condition (25) becomes*

$$\begin{aligned} \limsup_{\eta\to\infty} \int_{\eta_1}^{\eta} \left(\beta(s) - \left(\frac{2}{\varepsilon s^2}\right)^{(p-1)} \frac{\varkappa(s)\,\mu(s)\,(\beta(s))^p}{p^p}\right) \mathrm{d}s &= \limsup_{\eta\to\infty} \int_{\eta_1}^{\eta} \left(\frac{c_0}{8s} - \frac{9}{2s}\right) ds \\ &= \infty \quad if \quad c_0 > 36. \end{aligned}$$

Therefore, from Corollary 1, we see that (27) is oscillatory if $c_0 > 36$.

4. Conclusion

The asymptotic behavior oscillatory characteristics of a fourth-order DDE with a p-Laplacian were examined in our work. The goal of this work is to apply the findings of [18] to equations that have a sublinear delay term and canonical operators. Furthermore, our work streamlines and builds upon previous discoveries in the literature while also advancing current knowledge. Building on these discoveries, we created new standards that ensure all solutions to the examined equations oscillate. This contribution offers a strong basis for upcoming investigations and is essential for developing the theoretical framework of delay differential equations. To illustrate the strength of our findings, an example was provided.

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