EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 17, No. 4, 2024, 3687-3707
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Gronwall-Type Inequalities and Qualitative Studies on Higher-Variable Orders of Atangana-Baleanu Fractional Operators via Increasing Functions

Hasanen A. Hammad[1,2,*], Manuel De la Sen[3]

[1] *Department of Mathematics, College of Science, Qassim University, Buraydah 51452, Saudi Arabia*
[2] *Department of Mathematics, Faculty of Science, Sohag University, Sohag 82524, Egypt*
[3] *Institute of Research and Development of Processes, Department of Electricity and Electronics, Faculty of Science and Technology, Univesity of the Basque Country, 48940-Leioa (Bizkaia), Spain*

Abstract. This paper introduces a novel extension of Caputo-Atangana-Baleanu and Riemann-Atangana-Baleanu fractional derivatives from constant to increasing variable order. We generalize the fractional order from a fixed value in $(0, 1]$ to a time-dependent function in $(k, k + 1]$, where $k \geq 0$. The corresponding Atangana-Baleanu fractional integral is also extended. Key properties of these new definitions are explored, including a generalized Gronwall inequality. We then delve into the analysis of higher-variable initial fractional differential equations using the Caputo-Atangana-Baleanu operator with an increasing function, establishing existence and uniqueness results via Picard's iterative method. The findings presented in this work are expected to stimulate further research on inequalities and fractional differential equations related to Atangana-Baleanu fractional calculus with respect to increasing functions. Concrete examples are provided to illustrate the practical applications of our results.

2020 Mathematics Subject Classifications: 74H10, 54H25, 34A08, 34A12

Key Words and Phrases: Fractional derivatives, variable order derivatives, fixed point techniques, existence results, differential equations

1. Introduction

For a better explanation of chaotic complex systems, fractional calculus has drawn the attention of numerous authors in a variety of fields over the past three decades. These fields have many applications in qualitative theories, electrical networks, etc. For more information, see [16, 18, 22]. The reason why this trend has so many readers is that the fractional differentiation of the function produces its complete spectrum which includes

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v17i4.5592

Email addresses: `h.abdelwareth@qu.edu.sa` (H.A. Hammad), `manuel.delasen@ehu.eus` (M. De la Sen)

the corresponding integer-order counterpart as a special case. In addition, the use of these equations and formulas in its mathematical models contributes fundamentally to real-world applications because it generates full dynamics of the topic under study and benefits from higher degrees of freedom. Regarding applications of viscoelasticity, physics, and dynamics, reputable results can be found in [1, 7, 13, 21].

Some scientists and engineers have adapted fractional calculus to singular and nonsingular kernels in order to recognize and explain the genuine phenomena in their respective domains. A novel definition of a fractional operator with an exponential kernel was studied by Caputo and Fabrizio [8]. The Atangana-Baleanu (AB) fractional operator was introduced by Atangana and Baleanu [5] and has a fresh and intriguing definition of a Mittag-Leffler (ML) kernel. The AB fractional operator was extended to higher arbitrary orders by Abdeljawad [2]. Following that, a number of researchers examined the qualitative characteristics and approximate solutions of fractional differential equations (FDEs) utilizing Atangana-Baleanu-Caputo (ABC) fractional operators, Caputo-Fabrizio derivatives, and others applied the technique of FP theory to find the existence solutions to theses operators; for more information, see [6, 9, 10, 12, 14, 15, 23–26, 28].

Recently, a fractional derivative of a function with respect to (w.r.t.) another function with a ML kernel was proposed by Fernandez and Baleanu [11], and it is actually thought of as a generalized AB fractional operator. By establishing the appropriate AB-fractional integral of a function w.r.t. another function, authors [20] established a link between the AB fractional operator and the Riemann-Liouville (RL) fractional integral w.r.t. another function. Following that, Kashuri [17] introduced a fractional integral operator known as the Atangana-Baleanu-Kashuri (ABK) fractional integral.

Inspired of the above works, in this article, we increase the fractional derivatives of ABC and RAB with respect to an increasing function from a fractional order $\varpi \in (0,1]$ to an arbitrary variable order $\varpi(\tau) \in (k, k+1]$, $k \geq 0$. Several characteristics and uses of these concepts are also studied. Further, in the framework of the AB fractional integrals, a brand-new generalized Gronwall inequality is also demonstrated. Moreover, Picard's iterative approach is used to establish the existence and uniqueness results of a higher-variable order ABC fractional issue under initial boundary constraints our paper extends and generalizes the results of [3]. Finally, illustrative examples are provided to support our results.

2. Preliminaries

This part is devoted to present some crucial foundational material for fractional calculus. Let us denote by $C^k(\Im, \mathbb{R})$ the BS of all the kth continuously differentiable functions κ equipped with usual norm $\|\kappa\| = \sup\{|\kappa(r)| : r \in \Im = [\varkappa, \varrho]\}$.

Definition 1. *[4] Let $\psi : \Im \to \mathbb{R}$ be an increasing and differentiable function. For the integrable function $\xi : \Im \to \mathbb{R}$, the ϖth left-sided $\psi-$RL fractional integral w.r.t. another* *function* $\psi(z)$, *is described as*

$$^{RL}\Re_{\varkappa}^{\varpi,\psi}\xi(z)=\frac{1}{\Gamma(\varpi)}\int_{\varkappa}^{z}(\psi(z)-\psi(r))\,\psi'(r)\xi(r)\,dr,$$

for all $z\in\Im=[\varkappa,\varrho]$, *where* $\Gamma(\varpi)=\int_{0}^{\infty}e^{-r}r^{\varpi-1}dr$, $\varpi>0$.

Definition 2. *[5] For the function* $\xi\in H^{1}(\varkappa,\varrho)$ *and* $\varpi\in(0,1]$, *the* ϖ*th left-sided RL-AB fractional derivative is defined by*

$$\left(^{RLAB}D_{\varkappa}^{\varpi}\xi\right)z=\frac{\Lambda(\varpi)}{1-\varpi}\frac{d}{dz}\int_{\varkappa}^{z}L_{\varpi}\left(\frac{-\varpi}{1-\varpi}(z-r)^{\varpi}\right)\xi(r)\,dr,\ \ z\in\Im,$$

where $\Lambda(\varpi)$ *is the normalization function with* $\Lambda(0)=\Lambda(1)=1$, *and* L_{ϖ} *is the ML function given by*

$$L_{\varpi}(s)=\sum_{j=0}^{\infty}\frac{s^{j}}{\Gamma(1+\varpi j)},\ \ Re(\varpi)>0,\ \ s\in\mathbb{C}.$$

Definition 3. *[5] For the function* $\xi\in H^{1}(\varkappa,\varrho)$ *and* $\varpi\in(0,1]$, *the* ϖ*th left-sided ABC fractional derivative is proposed by*

$$\left(^{CAB}D_{\varkappa}^{\varpi}\xi\right)z=\frac{\Lambda(\varpi)}{1-\varpi}\int_{\varkappa}^{z}L_{\varpi}\left(\frac{-\varpi}{1-\varpi}(z-r)^{\varpi}\right)\xi'(r)\,dr,\ \ z\in\Im.$$

Definition 4. *[5] For the function* $\xi\in H^{1}(\varkappa,\varrho)$ *and* $\varpi\in(0,1]$, *the* ϖ*th left-sided RL-AB fractional integral is formed as*

$$\left(^{RLAB}\Re_{\varkappa}^{\varpi}\xi\right)z=\frac{1-\varpi}{\Lambda(\varpi)}\xi(z)+\frac{\varpi}{\Lambda(\varpi)}{}^{RL}\Re_{\varkappa}^{\varpi}\xi(z),\ \ z\in\Im.$$

Definition 5. *[17] For the function* $\xi\in H_{t}^{q}(\varkappa,\varrho)$, (*where* $1\le q<\infty,\ t\in\mathbb{R}$), *and* $\varpi\in(0,1]$, *the* ϖ*th left-sided KAB fractional integral is written as*

$$\left(_{\varkappa}^{KAB}\Re^{\varpi,\eta}\xi\right)z=\frac{1-\varpi}{\Lambda(\varpi)}\xi(z)+\frac{\varpi}{\Lambda(\varpi)}\frac{1}{\Gamma(\varpi)}\int_{\varkappa}^{z}r^{\eta-1}\left(\frac{z^{\eta}-r^{\eta}}{\eta}\right)^{\varpi-1}\xi(r)\,dr,\ \ z\in\Im,\ \ \eta>0.$$

Definition 6. *[8] Assume that* $\varpi\in(0,1]$. *For the function* $\xi\in H^{1}(\varkappa,\varrho)$, *the* ϖ*th left-sided* $\psi-$*RL-AB fractional derivative under an increasing differentiable function* $\psi:\Im\to\mathbb{R}$ *with* $\psi'(z)\neq0$, *for all* $z\in\Im$ *is given by*

$$\left(^{RLAB}D_{\varkappa}^{\varpi,\psi}\xi\right)(z)=\frac{\Lambda(\varpi)}{(1-\varpi)\,\psi'(z)}\frac{d}{dz}\int_{\varkappa}^{z}\psi'(r)\left(\frac{-\varpi}{1-\varpi}(\psi(z)-\psi(r))^{\varpi}\right)\xi(r)\,dr,\ \ z\in\Im.$$

Definition 7. *[8] Assume that $\varpi \in (0,1]$. For the function $\xi \in H^1(\varkappa, \varrho)$, the ϖth left-sided $\psi-$ABC fractional derivative under an increasing differentiable function $\psi : \Im \to \mathbb{R}$ with $\psi'(z) \neq 0$, for all $z \in \Im$ is defined by*

$$\left({}^{CAB}D_{\varkappa}^{\varpi,\psi}\xi\right)(z) = \frac{\Lambda(\varpi)}{(1-\varpi)}\int\limits_{\varkappa}^{z}\psi'(r)L_{\varpi}\left(\frac{-\varpi}{1-\varpi}(\psi(z)-\psi(r))^{\varpi}\right)\xi'_{\psi}(r)\,dr,\ z \in \Im,$$

where $\xi'_{\psi}(r) = \frac{\xi'(z)}{\psi'(z)}$.

Definition 8. *[20] Assume that $\varpi \in (0,1]$. For the function $\xi \in H^1(\varkappa, \varrho)$, the ϖth left-sided $\psi-$RL-AB fractional integral under an increasing differentiable function $\psi : \Im \to \mathbb{R}$ with $\psi'(z) \neq 0$, for all $z \in \Im$ is described as*

$$\left({}^{RLAB}\Re_{\varkappa}^{\varpi,\psi}\xi\right)(z) = \frac{1-\varpi}{\Lambda(\varpi)}\xi(z) + \frac{\varpi}{\Lambda(\varpi)}\left({}^{RL}\Re_{\varkappa}^{\varpi,\psi}\xi\right)(z),\ z \in \Im.$$

Remark 1. *It should be noted that*

(*i*) *Definitions 6, 7 and 8 reduce to Definitions 2, 3 and 4, respectively, by taking $\psi(z) = z$.*

(*ii*) *Definition 8 follows immediately from Definition 5, by considering $\psi(z) = \frac{z^{\eta}}{\eta}$.*

Lemma 1. *[4] Assume that $\varpi, \nu > 0$ and $\xi : \Im \to \mathbb{R}$. Then*

(1) ${}^{RL}\Re_{\varkappa}^{\varpi,\psi}\left(\xi(z) - \xi(\varkappa)\right)^{\nu-1} = \frac{\Gamma(\nu)}{\Gamma(\varpi+\nu)}\left(\xi(z) - \xi(\varkappa)\right)^{\varpi+\nu-1};$

(2) ${}^{RL}\Re_{\varkappa}^{\varpi,\psi}\ {}^{RL}\Re_{\varkappa}^{\nu,\psi}\xi(z) = {}^{RL}\ \Re_{\varkappa}^{\varpi+\nu,\psi}\xi(z);$

(3) $\left(\left(\frac{1}{\psi(z)}\frac{d}{dz}\right)^{k}\ {}^{RL}\Re_{\varkappa}^{k,\psi}\xi\right)(z) = \xi(z),\ k \in \mathbb{N}.$

Lemma 2. *[20] For $\varpi \in (0,1]$ and $\xi : \Im \to \mathbb{R}$, the relations below are satisfied:*

(i) $\left({}^{RLAB}\Re_{\varkappa}^{\varpi}\ {}^{RLAB}D_{\varkappa}^{\varpi,\psi}\xi\right)(z) = \xi(z);$

(ii) $\left({}^{RLAB}D_{\varkappa}^{\varpi,\psi}\ {}^{RLAB}\Re_{\varkappa}^{\varpi}\xi\right)(z) = \xi(z).$

Definition 9. *[22] For the function $\xi \in L([0,T])$,*

(a) the variable order of the RL fractional integral is remembered as

$$\Re_{+0}^{\varpi(z)}\xi(z) = \frac{1}{\Gamma(\varpi(z))}\int_0^z (z-r)^{\varpi(r)-1}\xi(r)dr;$$

(b) the variable order of the Caputo fractional derivative is given by

$${}^{C}D_0^{\varpi(z)}\xi(z) = \frac{1}{\Gamma(k-\varpi(z))}\int_0^z (z-r)^{k-\varpi(r)-1}\xi^{(k)}(r)dr.$$

where $\varpi : [0,T] \to (0,1]$, $T > 0$ is a continuous function.

3. Derivatives of higher-variable orders

In this section, we consider $\psi : \Im \to \mathbb{R}$ to be an increasing function with $\psi'(z) \neq 0$ to investigate the definitions of higher-variables order fractional derivatives and integrals within the AB framework with regard to a function ψ.

Consider a partition of $\Im = [\varkappa, \varrho]$ as

$$\{\Im_1 = [\varkappa, \tau_1],\ \Im_2 = (\tau_1, \tau_2],\ \Im_3 = (\tau_2, \tau_3], \cdots, \Im_k = (\tau_{k-1}, \varrho]\},$$

and assume that $\varpi : \Im \to (k, k+1]$ is a piecewise function such that

$$\varpi(\tau) = \sum_{u=1}^{k} \varpi_u(\tau) I_u(\tau) = \begin{cases} \varpi_1, \text{ if } \tau \in \Im_1 \\ \varpi_2, \text{ if } \tau \in \Im_2 \\ \quad \vdots \\ \varpi_k, \text{ if } \tau \in \Im_k, \end{cases}$$

where I_u is the indicator function of $\Im_u = (\tau_{u-1}, \tau_k]$ and $k < \varpi_u < k+1$ are constants with $u = 1, 2, \cdots, k$, such that $\tau_0 = \varkappa$ and $\tau_k = \varrho$ and

$$I_u(\tau) = \begin{cases} 1 \text{ for } \tau \in \Im_u, \\ 0 \text{ otherwise.} \end{cases}$$

Assume that $C^k(\Im_u, \mathbb{R})$ refers to the space of all kth continuously differentiable functions ξ. Clearly, it is a Banach space under the norm $\|\xi\| = \sup\{|\xi(z)| : z \in \Im = [\varkappa, \varrho]\}$. Here, we shall write for simplicity $\varpi_u(\tau) = \varpi_u$ and $\theta_u(\tau) = \theta_u$ for all $\tau \in \Im_u$.

Definition 10. *Let $\varpi_u \in (k, k+1]$ and $\theta_u = \varpi_u - k$, for $k \geq 0$, $u \geq 1$. For the function $\wp \in H^1(\varkappa, \varrho)$, the ϖ_uth left-sided $\psi-$RL-AB fractional derivative under the function ψ with $\psi'(z) \neq 0$, for all $z \in \Im$ is defined by*

$$\begin{aligned}
&\left({}^{RLAB}D_{\varkappa}^{\varpi_u,\psi}\wp\right)(z) \\
= &\left(\frac{1}{\psi'(z)}\frac{d}{dz}\right)^k \left({}^{RLAB}D_{\varkappa}^{\theta_u,\psi}\wp(z)\right) \\
= &\left(\frac{1}{\psi'(z)}\frac{d}{dz}\right)^k \frac{\Lambda(\theta_u)}{(1-\theta_u)\,\psi'(z)}\frac{d}{dz}\int_{\varkappa}^{z} \psi'(r) L_{\theta_u}\left(\frac{-\theta_u}{1-\theta_u}(\psi(z)-\psi(r))^{\theta_u}\right)\wp(r)\,dr \\
= &\left(\frac{1}{\psi'(z)}\frac{d}{dz}\right)^{k+1} \frac{\Lambda(\varpi_u - k)}{(k+1-\varpi_u)\,\psi'(z)} \\
&\times \int_{\varkappa}^{z} \psi'(r) L_{\varpi_u - k}\left(\frac{-(\varpi_u - k)}{k+1-\varpi_u}(\psi(z)-\psi(r))^{\varpi_u - k}\right)\wp(r)\,dr.
\end{aligned}$$

Definition 11. *Let $\varpi_u \in (k, k+1]$ and $\theta_u = \varpi_u - k$, for $k \geq 0$, $u \geq 1$. For the function $\wp^{(k)} \in H^1(\varkappa, \varrho)$, the ϖ_uth left-sided $\psi-ABC$ fractional derivative under the function ψ with $\psi'(z) \neq 0$, for all $z \in \Im$ is described as*

$$\begin{aligned}\left({}^{CAB}D_{\varkappa}^{\varpi_u,\psi}\wp\right)(z) &= \left({}^{CAB}D_{\varkappa}^{\theta_u,\psi}\wp_{\psi}^{(k)}\right)(z) \\ &= \frac{\Lambda(\theta_u)}{(1-\theta_u)}\int_{\varkappa}^{z}\psi'(r)L_{\theta_u}\left(\frac{-\theta_u}{1-\theta_u}(\psi(z)-\psi(r))^{\theta_u}\right)\wp_{\psi}^{(k+1)}(r)\,dr \\ &= \frac{\Lambda(\varpi_u-k)}{(k+1-\varpi_u)} \\ &\quad\times\int_{\varkappa}^{z}\psi'(r)L_{\varpi_u-k}\left(\frac{-(\varpi_u-k)}{k+1-\varpi_u}(\psi(z)-\psi(r))^{\varpi_u-k}\right)\wp_{\psi}^{(k+1)}(r)\,dr,\end{aligned}$$

where $\wp_{\psi}^{(k)}(z) = \left(\frac{1}{\psi'(z)}\frac{d}{dz}\right)^k \wp(z)$ and $\wp_{\psi}^{(0)}(z) = \wp(z)$. If $\varpi_u = n \in \mathbb{N}$, then $\left({}^{CAB}D_{\varkappa}^{\varpi_u,\psi}\wp\right)(z) = \wp_{\psi}^{(n)}(z)$.

Definition 12. *Let $\varpi_u \in (k, k+1]$ and $\theta_u = \varpi_u - k$, for $k \geq 0$, $u \geq 1$. For the function $\wp \in H^1(\varkappa, \varrho)$, the ϖ_uth left-sided $\psi-RL$-AB fractional integral under the function ψ with $\psi'(z) \neq 0$, for all $z \in \Im$ is defined by*

$$\begin{aligned}\left({}^{RLAB}\Re_{\varkappa}^{\varpi_u,\psi}\wp\right)(z) &= \left({}^{RL}\Re_{\varkappa}^{k,\psi}\ {}^{AB}\Re_{\varkappa}^{\theta_u,\psi}\wp\right)(z) = \left({}^{AB}\Re_{\varkappa}^{\theta_u,\psi}\ {}^{RL}\Re_{\varkappa}^{k,\psi}\wp\right)(z) \\ &= \frac{k+1-\varpi_u}{\Lambda(\varpi_u-k)}{}^{RL}\Re_{\varkappa}^{k,\psi}\wp(z) + \frac{\varpi_u-k}{\Lambda(\varpi_u-k)}{}^{RL}\Re_{\varkappa}^{\varpi_u,\psi}\wp(z),\end{aligned}$$

where $\Re_{\varkappa}^{k,\psi}$ takes the form

$$\Re_{\varkappa}^{k,\psi}\wp(z) = \frac{1}{\Gamma(k)}\int_{\varkappa}^{z}\psi'(r)(\psi(z)-\psi(r))^{k-1}\xi(r)\,dr.$$

Remark 2. *For $u \geq 1$, it is clear that*

(i) if we take $\varpi_u = \varpi \in (0, 1]$ in Definitions 10, 11 and 12, then we have Definitions 6, 7 and 8, respectively.

(ii) if we put $\varpi_u = \varpi = k+1$, then $\varpi_u = 1$ and hence the following is true for our generalization to the higher-variable order cases:

$$\begin{aligned}\left({}^{RLAB}D_{\varkappa}^{\varpi_u,\psi}\wp\right)(z) &= \left(\frac{1}{\psi'(z)}\frac{d}{dz}\right)^k\left({}^{RLAB}D_{\varkappa}^{1,\psi}\wp(z)\right) = \wp_{\psi}^{(k+1)}(z), \\ \left({}^{CAB}D_{\varkappa}^{\varpi_u,\psi}\wp\right)(z) &= \left({}^{CAB}D_{\varkappa}^{1,\psi}\wp_{\psi}^{(k)}\right)(z) = \wp_{\psi}^{(k+1)}(z), \\ \left({}^{RLAB}\Re_{\varkappa}^{\varpi_u,\psi}\wp\right)(z) &= \left({}^{RL}\Re_{\varkappa}^{k,\psi}\ {}^{AB}\Re_{\varkappa}^{1,\psi}\wp\right)(z) = \left({}^{RL}\Re_{\varkappa}^{k+1,\psi}\wp\right)(z).\end{aligned}$$

Lemma 3. *For $\varpi_u = \varpi \in (0,1]$, $u \geq 1$, the equations below hold*

(i) $\left({}^{RLAB}\Re_{\varkappa}^{\varpi,\psi}\ {}^{CAB}D_{\varkappa}^{\varpi,\psi}\wp\right)(z) = \wp(z) - \wp(\varkappa)$.

(ii) $\left({}^{CAB}D_{\varkappa}^{\varpi,\psi}\ {}^{RLAB}\Re_{\varkappa}^{\varpi,\psi}\wp\right)(z) = \wp(z) - \wp(\varkappa)L_{\varpi}\left(\frac{-\varpi}{1-\varpi}(\psi(z)-\psi(\varkappa))^{\varpi}\right)$.

Proof. (i) Utilizing Definitions 7 and 8, we get

$$\begin{aligned}
&\left({}^{RLAB}\Re_{\varkappa}^{\varpi,\psi}\ {}^{CAB}D_{\varkappa}^{\varpi,\psi}\wp\right)(z)\\
=\;& \frac{\Lambda(\varpi)}{(1-\varpi)}\left({}^{CAB}D_{\varkappa}^{\varpi,\psi}\wp\right)(z)\\
&+\frac{\varpi}{\Lambda(\varpi)}\left({}^{RL}\Re_{\varkappa}^{\varpi,\psi}\ {}^{CAB}D_{\varkappa}^{\varpi,\psi}\wp\right)(z)\\
=\;& \sum_{j=0}^{\infty}\left(\frac{-\varpi}{1-\varpi}\right)^{j}\int_{\varkappa}^{z}\psi'(r)\left(\frac{\psi(z)-\psi(r)}{\Gamma(1+j\varpi)}\right)^{j\varpi}\wp'_{\psi}(r)\,dr\\
&+\frac{\varpi}{1-\varpi}{}^{RL}\Re_{\varkappa}^{\varpi,\psi}\sum_{j=0}^{\infty}\left(\frac{-\varpi}{1-\varpi}\right)^{j}\int_{\varkappa}^{z}\psi'(r)\frac{[\psi(z)-\psi(r)]^{j\varpi}}{\Gamma(1+j\varpi)}\wp'_{\psi}(r)\,dr\\
=\;& \sum_{j=0}^{\infty}\left(\frac{-\varpi}{1-\varpi}\right)^{j}\ {}^{RL}\Re_{\varkappa}^{j\varpi+1,\psi}\frac{\wp'(z)}{\psi'(z)}+\frac{\varpi}{1-\varpi}{}^{RL}\Re_{\varkappa}^{\varpi,\psi}\sum_{j=0}^{\infty}\left(\frac{-\varpi}{1-\varpi}\right)^{j}\ {}^{RL}\Re_{\varkappa}^{j\varpi+1,\psi}\frac{\wp'(z)}{\psi'(z)}\\
=\;& \sum_{j=0}^{\infty}\left(\frac{-\varpi}{1-\varpi}\right)^{j}\ {}^{RL}\Re_{\varkappa}^{j\varpi+1,\psi}\frac{\wp'(z)}{\psi'(z)}-\sum_{j=0}^{\infty}\left(\frac{-\varpi}{1-\varpi}\right)^{j+1}\ {}^{RL}\Re_{\varkappa}^{j\varpi+\varpi+1,\psi}\frac{\wp'(z)}{\psi'(z)}\\
=\;& {}^{RL}\Re_{\varkappa}^{1,\psi}\frac{\wp'(z)}{\psi'(z)}=\int_{\varkappa}^{z}\wp'(r)dr=\wp(z)-\wp(\varkappa).
\end{aligned}$$

(ii) Again, utilizing Definitions 7, 8 and the identity

$${}^{RL}\Re_{\varkappa}^{\mu+1,\psi}\left(\frac{1}{\psi'(z)}\frac{d}{dz}\right)\wp(z) = {}^{RL}\Re_{\varkappa}^{\mu,\psi}\wp(z)-\wp(\varkappa)\frac{(\psi(z)-\psi(\varkappa))^{\mu}}{\Gamma(1+\mu)},\ Re(\mu)>0,$$

one has

$$\begin{aligned}
&\left({}^{CAB}D_{\varkappa}^{\varpi,\psi}\ {}^{RLAB}\Re_{\varkappa}^{\varpi,\psi}\wp\right)(z)\\
=\;& {}^{CAB}D_{\varkappa}^{\varpi,\psi}\left(\frac{1-\varpi}{\Lambda(\varpi)}\wp(z)+\frac{\varpi}{\Lambda(\varpi)}\left({}^{RL}\Re_{\varkappa}^{\varpi,\psi}\wp\right)(z)\right)\\
=\;& \frac{1-\varpi}{\Lambda(\varpi)}\left({}^{CAB}D_{\varkappa}^{\varpi,\psi}\wp\right)(z)+\frac{\varpi}{\Lambda(\varpi)}{}^{CAB}D_{\varkappa}^{\varpi,\psi}\left({}^{RL}\Re_{\varkappa}^{\varpi,\psi}\wp\right)(z)\\
=\;& \sum_{j=0}^{\infty}\left(\frac{-\varpi}{1-\varpi}\right)^{j}\ {}^{RL}\Re_{\varkappa}^{j\varpi+1,\psi}\left(\frac{1}{\psi'(z)}\frac{d}{dz}\right)\wp(z)
\end{aligned}$$

$$+\frac{\varpi}{1-\varpi}\sum_{j=0}^{\infty}\left(\frac{-\varpi}{1-\varpi}\right)^{j}\ {}^{RL}\Re_{\varkappa}^{j\varpi+1,\psi}\ \left(\frac{1}{\psi'(z)}\frac{d}{dz}\right)\left({}^{RL}\Re_{\varkappa}^{\varpi,\psi}\wp\right)(z)$$

$$= \sum_{j=0}^{\infty}\left(\frac{-\varpi}{1-\varpi}\right)^{j}\left\{{}^{RL}\Re_{\varkappa}^{j\varpi,\psi}\wp(z)-\frac{\wp(\varkappa)\left[\psi(z)-\psi(\varkappa)\right]^{j\varpi}}{\Gamma(1+j\varpi)}\right\}-\sum_{j=0}^{\infty}\left(\frac{-\varpi}{1-\varpi}\right)^{j+1}\ {}^{RL}\Re_{\varkappa}^{j\varpi+\varpi,\psi}\wp(z)$$

$$= \wp(z)-\sum_{j=0}^{\infty}\left(\frac{-\varpi}{1-\varpi}\right)^{j}\frac{\wp(\varkappa)\left[\psi(z)-\psi(\varkappa)\right]^{j\varpi}}{\Gamma(1+j\varpi)}$$

$$= \wp(z)-\wp(\varkappa)L_{\varpi}\left(\frac{-\varpi}{1-\varpi}(\psi(z)-\psi(\varkappa))^{\varpi}\right).$$

Lemma 4. *Assume that $\wp\in C^{k}[\Im_u,\mathbb{R}]$ and $\psi\in C^{k}[\Im_u,\mathbb{R}^{+}]$. For $\varpi_u\in(k,k+1]$ and $\theta_u=\varpi_u-k$, for $k\geq 0$, $u\geq 1$ and all $\tau\in\Im_u$, the following equations are true:*

(i) $\left({}^{RLAB}D_{\varkappa}^{\varpi_u,\psi}\ {}^{RLAB}\Re_{\varkappa}^{\varpi_u,\psi}\wp\right)(z)=\wp(z).$

(ii) $\left({}^{RLAB}\Re_{\varkappa}^{\varpi_u,\psi}\ {}^{RLAB}D_{\varkappa}^{\varpi_u,\psi}\wp\right)(z)=\wp(z).$

(iii) $\left({}^{CAB}D_{\varkappa}^{\varpi_u,\psi}\ {}^{RLAB}\Re_{\varkappa}^{\varpi_u,\psi}\wp\right)(z)=\wp(z)-\wp(\varkappa)L_{\varpi_u-k}\left(\frac{-(\varpi_u-k)}{1-(\varpi_u-k)}(\psi(z)-\psi(\varkappa))^{\varpi_u-k}\right).$

(iv) $\left({}^{RLAB}\Re_{\varkappa}^{\varpi_u,\psi}\ {}^{CAB}D_{\varkappa}^{\varpi_u,\psi}\wp\right)(z)=\wp(z)-\sum_{m=0}^{k}\frac{\wp_{\psi}^{(m)}(\varkappa)}{m!}(\psi(z)-\psi(\varkappa))^{m}.$

Proof. (i) In light of Definitions 10 and 12 and using Lemmas 1 and 2, for $u\geq 1$, we can write

$$\begin{aligned}\left({}^{RLAB}D_{\varkappa}^{\varpi_u,\psi}\ {}^{RLAB}\Re_{\varkappa}^{\varpi_u,\psi}\wp\right)(z) &= \left(\left(\frac{1}{\psi(z)}\frac{d}{dz}\right)^{k}\ {}^{RLAB}D_{\varkappa}^{\theta_u,\psi}\ {}^{RLAB}\Re_{\varkappa}^{\theta_u,\psi}\ {}^{RL}\Re_{\varkappa}^{k,\psi}\wp\right)(z)\\ &= \left(\left(\frac{1}{\psi(z)}\frac{d}{dz}\right)^{k}\ {}^{RL}\Re_{\varkappa}^{k,\psi}\wp\right)(z)=\wp(z).\end{aligned}$$

(ii) According to Definitions 10 and 12, for $u\geq 1$, we have

$$\left({}^{RLAB}\Re_{\varkappa}^{\varpi_u,\psi}\ {}^{RLAB}D_{\varkappa}^{\varpi_u,\psi}\wp\right)(z)$$

$$= \frac{k+1-\varpi_u}{\Lambda(\varpi_u-k)}{}^{RL}\Re_{\varkappa}^{k,\psi}\left({}^{RLAB}D_{\varkappa}^{\varpi_u,\psi}\wp(z)\right)+\frac{\varpi_u-k}{\Lambda(\varpi_u-k)}{}^{RL}\Re_{\varkappa}^{\varpi_u,\psi}\left({}^{RLAB}D_{\varkappa}^{\varpi_u,\psi}\wp(z)\right)$$

$$= \Re_{\varkappa}^{k,\psi}\left(\frac{1}{\psi'(z)}\frac{d}{dz}\right)^{k+1}\sum_{j=0}^{\infty}\left(\frac{-(\varpi_u-k)}{k+1-\varpi_u}\right)^{j}\int_{\varkappa}^{z}\psi'(r)\left(\frac{\psi(z)-\psi(r)}{\Gamma(1+j(\varpi_u-k))}\right)^{j(\varpi_u-k)}\wp(r)\,dr$$

$$+\frac{(\varpi_u-k)}{(k+1-\varpi_u)}{}^{RL}\Re_{\varkappa}^{\varpi_u,\psi}\left(\frac{1}{\psi'(z)}\frac{d}{dz}\right)^{k+1}\sum_{j=0}^{\infty}\left(\frac{-(\varpi_u-k)}{k+1-\varpi_u}\right)^{j}$$

$$\times \int_{\varkappa}^{z} \psi'(r) \left(\frac{\psi(z)-\psi(r)}{\Gamma(1+j(\varpi_u - k))} \right)^{j(\varpi_u - k)} \wp(r)\, dr,$$

which implies that

$$\begin{aligned}
& \left({}^{RLAB}\Re_{\varkappa}^{\varpi_u,\psi} \; {}^{RLAB}D_{\varkappa}^{\varpi_u,\psi} \wp \right)(z) \\
= & \sum_{j=0}^{\infty} \left(\frac{-(\varpi_u - k)}{k+1-\varpi_u} \right)^{j} \Re_{\varkappa}^{k,\psi} \left(\frac{1}{\psi'(z)} \frac{d}{dz} \right)^{k+1} {}^{RL}\Re_{\varkappa}^{j(\varpi_u-k)+1,\psi} \wp(z) \\
& - \sum_{j=0}^{\infty} \left(\frac{-(\varpi_u - k)}{k+1-\varpi_u} \right)^{j+1} \Re_{\varkappa}^{\varpi_u,\psi} \left(\frac{1}{\psi'(z)} \frac{d}{dz} \right)^{k+1} {}^{RL}\Re_{\varkappa}^{j(\varpi_u-k)+1,\psi} \wp(z) \\
= & \sum_{j=0}^{\infty} \left(\frac{-(\varpi_u - k)}{k+1-\varpi_u} \right)^{j} {}^{RL}\Re_{\varkappa}^{j(\varpi_u-k)+1,\psi} \wp(z) \\
& - \sum_{j=0}^{\infty} \left(\frac{-(\varpi_u - k)}{k+1-\varpi_u} \right)^{j+1} {}^{RL}\Re_{\varkappa}^{j(\varpi_u-k)+(\varpi_u-k),\psi} \wp(z) \\
= & \wp(z).
\end{aligned}$$

(iii) Using Definitions 10, 12, Lemmas 1 and 3, for $u \geq 1$, one has

$$\begin{aligned}
& \left({}^{CAB}D_{\varkappa}^{\varpi_u,\psi} \; {}^{RLAB}\Re_{\varkappa}^{\varpi_u,\psi} \wp \right)(z) \\
= & \left({}^{CAB}D_{\varkappa}^{\theta_u,\psi} \left(\frac{1}{\psi(z)} \frac{d}{dz} \right)^{k} {}^{RL}\Re_{\varkappa}^{k,\psi} \; {}^{RLAB}\Re_{\varkappa}^{\theta_u,\psi} \wp \right)(z) \\
= & \left({}^{CAB}D_{\varkappa}^{\theta_u,\psi} \; {}^{RLAB}\Re_{\varkappa}^{\theta_u,\psi} \wp \right)(z) \\
= & \wp(z) - \wp(\varkappa) L_{\theta_u} \left(\frac{-\theta_u}{1-\theta_u} (\psi(z)-\psi(\varkappa))^{\theta_u} \right) \\
= & \wp(z) - \wp(\varkappa) L_{\varpi_u - k} \left(\frac{-(\varpi_u - k)}{1-(\varpi_u - k)} (\psi(z)-\psi(\varkappa))^{\varpi_u - k} \right).
\end{aligned}$$

(iv) Based on Definitions 10, 12 and Lemma 3, for $u \geq 1$, we get

$$\begin{aligned}
& \left({}^{RLAB}\Re_{\varkappa}^{\varpi_u,\psi} \; {}^{CAB}D_{\varkappa}^{\varpi_u,\psi} \wp \right)(z) \\
= & \left({}^{RL}\Re_{\varkappa}^{k,\psi} \; {}^{RLAB}\Re_{\varkappa}^{\theta_u,\psi} \; {}^{CAB}D_{\varkappa}^{\theta_u,\psi} \wp_{\psi}^{(k)} \right)(z) = {}^{RL}\Re_{\varkappa}^{k,\psi} \left(\wp_{\psi}^{(k)}(z) - \wp_{\psi}^{(k)}(\varkappa) \right) \\
= & \wp(z) - \sum_{m=0}^{k-1} \frac{\wp_{\psi}^{(m)}(\varkappa)}{m!} (\psi(z)-\psi(r))^{m} - \frac{\wp_{\psi}^{(k)}(\varkappa)}{k!} (\psi(z)-\psi(r))^{k} \\
= & \wp(z) - \sum_{m=0}^{k} \frac{\wp_{\psi}^{(m)}(\varkappa)}{m!} (\psi(z)-\psi(\varkappa))^{m}.
\end{aligned}$$

Lemma 5. *Assume that* $\wp \in C^k(\Im_u, \mathbb{R})$ *and* $\psi \in C^k(\Im_u, \mathbb{R}^+)$ *with* $\psi'(z) \neq 0$*. For* $\varpi_u \in (k, k+1]$, $\theta_u = \varpi_u - k$, $\lambda \geq k+1$ *and* $\zeta \geq 0$*, for* $k \geq 0$, $u \geq 1$*, the relations below are true:*

(i) ${}^{RLAB}\Re_{\varkappa}^{\varpi_u,\psi}\left(\wp(z)-\wp(\varkappa)\right)^{\zeta} = \frac{(k+1-\varpi_u)\Gamma(1+\zeta)(\wp(z)-\wp(\varkappa))^{\zeta+k}}{\Lambda(\varpi_u-k)\Gamma(1+\zeta+k)} + \frac{(\varpi_u-k)\Gamma(1+\zeta)(\wp(z)-\wp(\varkappa))^{\zeta+\varpi_u}}{\Lambda(\varpi_u-k)\Gamma(1+\zeta+k)}$.

(ii) ${}^{CAB}D_{\varkappa}^{\varpi_u,\psi}\left(\wp(z)-\wp(\varkappa)\right)^{\lambda} = \frac{\Lambda(\varpi_u-k)}{k+1-\varpi_u}\sum_{j=0}^{\infty}\left(\frac{-(\varpi_u-k)}{k+1-\varpi_u}\right)^j \frac{\Gamma(1+\lambda)(\wp(z)-\wp(\varkappa))^{j(\varpi_u-k)+\lambda-k}}{\Gamma(j(\varpi_u-k)+\lambda-k+1)}$.

(iii) ${}^{CAB}D_{\varkappa}^{\varpi_u,\psi}\left(\wp(z)-\wp(\varkappa)\right)^{\sigma} = 0$, $\sigma = 0, 1, ..., k$.

(iv) $\left({}^{RLAB}\Re_{\varkappa}^{\varpi_u,\psi}1\right)(z) = \frac{(k+1-\varpi_u)(\wp(z)-\wp(\varkappa))^{k}}{\Lambda(\varpi_u-k)\Gamma(k+1)} + \frac{(\varpi_u-k)(\wp(z)-\wp(\varkappa))^{\varpi_u}}{\Lambda(\varpi_u-k)\Gamma(1+k)}$.

(v) $\left({}^{CAB}D_{\varkappa}^{\varpi_u,\psi}1\right)(z) = 0$.

Proof. (i) Using Definition 12 and Lemma 3, for $u \geq 1$, we have

$$\begin{aligned}
&{}^{RLAB}\Re_{\varkappa}^{\varpi_u,\psi}\left(\wp(z)-\wp(\varkappa)\right)^{\zeta}\\
= &\frac{k+1-\varpi_u}{\Lambda(\varpi_u-k)}{}^{RL}\Re_{\varkappa}^{k,\psi}\left(\wp(z)-\wp(\varkappa)\right)^{\zeta} + \frac{\varpi_u-k}{\Lambda(\varpi_u-k)}{}^{RL}\Re_{\varkappa}^{\varpi_u,\psi}\left(\wp(z)-\wp(\varkappa)\right)^{\zeta}\\
= &\frac{k+1-\varpi_u}{\Lambda(\varpi_u-k)}\frac{\Gamma(1+\zeta)}{\Gamma(1+\zeta+k)}\left(\wp(z)-\wp(\varkappa)\right)^{\zeta+k}\\
&+\frac{\varpi_u-k}{\Lambda(\varpi_u-k)}\frac{\Gamma(1+\zeta)}{\Gamma(1+\zeta+k)}\left(\wp(z)-\wp(\varkappa)\right)^{\zeta+\varpi_u}.
\end{aligned}$$

(ii) From Definitions 3, 11 and Lemma 3, it follows that for $u \geq 1$,

$$\begin{aligned}
&{}^{CAB}D_{\varkappa}^{\varpi_u,\psi}\left(\wp(z)-\wp(\varkappa)\right)^{\lambda}\\
= &{}^{CAB}D_{\varkappa}^{\theta_u,\psi}\left(\frac{1}{\psi'(z)}\frac{d}{dz}\right)^k\left(\wp(z)-\wp(\varkappa)\right)^{\lambda}\\
= &{}^{CAB}D_{\varkappa}^{\theta_u,\psi}\frac{\Gamma(1+\lambda)}{\Gamma(\lambda-k+1)}\left(\wp(z)-\wp(\varkappa)\right)^{\lambda-k}\\
= &\frac{\Lambda(\theta_u)}{1-\theta_u}\int_{\varkappa}^{z}\psi'(r)\sum_{j=0}^{\infty}\left(\frac{-\theta_u}{1-\theta_u}\right)^j\frac{\Gamma(1+\lambda)\left(\wp(r)-\wp(\varkappa)\right)^{\lambda-(k+1)}}{\Gamma(\lambda-k)\Gamma(j\theta_u+1)}\left(\wp(z)-\wp(r)\right)^{j\theta_u}dr\\
= &\frac{\Gamma(1+\lambda)\Lambda(\theta_u)}{\Gamma(\lambda-k)(1-\theta_u)}\sum_{j=0}^{\infty}\left(\frac{-\theta_u}{1-\theta_u}\right)^j{}^{RL}\Re_{\varkappa}^{j\theta_u+1,\psi}\left(\wp(z)-\wp(\varkappa)\right)^{\lambda-(k+1)}\\
= &\frac{\Lambda(\theta_u)}{1-\theta_u}\sum_{j=0}^{\infty}\left(\frac{-\theta_u}{1-\theta_u}\right)^j\frac{\Gamma(1+\lambda)}{\Gamma(j\theta_u+\lambda-k+1)}\left(\wp(z)-\wp(\varkappa)\right)^{j\theta_u+\lambda-k}\\
= &\frac{\Lambda(\varpi_u-k)}{k+1-\varpi_u}\sum_{j=0}^{\infty}\left(\frac{-(\varpi_u-k)}{k+1-\varpi_u}\right)^j\frac{\Gamma(1+\lambda)\left(\wp(z)-\wp(\varkappa)\right)^{j(\varpi_u-k)+\lambda-k}}{\Gamma(j(\varpi_u-k)+\lambda-k+1)}.
\end{aligned}$$

(iii) From Definitions 7 and 11, one has

$$\begin{aligned} & {}^{CAB}D_{\varkappa}^{\varpi_u,\psi}\left(\wp\left(z\right)-\wp\left(\varkappa\right)\right)^{\sigma} \\ = & {}^{CAB}D_{\varkappa}^{\theta_u,\psi}\left(\frac{1}{\psi'(z)}\frac{d}{dz}\right)^{k}\left(\wp\left(z\right)-\wp\left(\varkappa\right)\right)^{\sigma} \\ = & {}^{CAB}D_{\varkappa}^{\theta_u,\psi}\frac{\Gamma(1+\sigma)}{\Gamma(\lambda-\sigma+1)}\left(\wp\left(z\right)-\wp\left(\varkappa\right)\right)^{\sigma-k} \\ = & \frac{\Lambda(\theta_u)}{1-\theta_u}\int_{\varkappa}^{z}L_{\theta_u}\left(\frac{-\theta_u}{1-\theta_u}\left(\wp\left(z\right)-\wp\left(r\right)\right)^{\theta_u}\right)\frac{\Gamma(1+\sigma)}{\Gamma(\lambda-\sigma+1)}\left(\wp\left(r\right)-\wp\left(\varkappa\right)\right)^{\sigma-k}dr \\ = & 0. \end{aligned}$$

Taking $\zeta=\sigma=0$ in portions (i) and (iii), we conclude (iv) and (v), respectively.

4. Generalizing Gronwall's inequality

This part will begin with the following generalization of Gronwall's inequality.

Lemma 6. *[27] Assume that the function $\psi\in C^{1}(\Im_u,\mathbb{R}^{+})$ is increasing with $\psi'(z)\neq 0$, for each $z\in\Im$ and $\varpi>0$. Let $\ell(z)$ be a nonnegative and nondecreasing function (NNF, for abbreviate), $\hbar(z)$ be a nonnegative function locally integrable (NFLI, for short) on $\Im$ and ϱ be a NFLI on $\Im$. If the inequality*

$$\varrho\left(z\right)\leq\hbar(z)+\ell(z)\int_{\varkappa}^{z}L_{\theta_u}\psi'(r)\left(\psi\left(z\right)-\psi\left(r\right)\right)^{\varpi-1}\varrho\left(r\right)dr,\ z\in\Im$$

holds, then

$$\varrho\left(z\right)\leq\hbar(z)+\int_{\varkappa}^{z}\sum_{j=1}^{\infty}\frac{\left[\ell(z)\Gamma(\varpi)\right]^{j}}{\Gamma(j\varpi)}\psi'(r)\left(\psi\left(z\right)-\psi\left(r\right)\right)^{j\varpi-1}\hbar\left(r\right)dr,$$

for every $z\in\Im$.

Lemma 7. *[27] Assume that all requirements of Lemma 6 are true, if the function $\hbar(z)$ is nondecreasing on $\Im$, one has*

$$\varrho\left(z\right)\leq\hbar(z)L_{\varpi}\left[\ell(z)\Gamma(\varpi)\left(\psi\left(z\right)-\psi\left(\varkappa\right)\right)^{\varpi}\right],\ z\in\Im.$$

In this role, we will present a novel Gronwall inequality within the context of the $\psi-RL-AB$ fractional operator.

Lemma 8. *Suppose that the function* $\psi \in C^1(\Im_u, \mathbb{R}^+)$ *is increasing with* $\psi'(z) \neq 0$, *for each* $z \in \Im$ *and* $\varpi_u = \varpi \in (0,1]$, *for* $u \geq 1$. *Assume that* $\Theta(z) = \frac{U(z)\Lambda(\varpi)}{\Lambda(\varpi)-(1-\varpi)G(z)}$ *is a NFLI on* $\Im$, $\Xi(z) = \frac{\varpi G(z)}{\Lambda(\varpi)-(1-\varpi)G(z)}$ *is a NNF and* ϱ *is a NFLI on* $\Im$, *such that*

$$\varrho(z) \leq U(z) + G(z)^{RLAB}\Re_{\varkappa}^{\varpi,\psi}\varrho(z), \; z \in \Im. \tag{1}$$

Then, for each $z \in \Im$, *we have*

$$\varrho(z) \leq \Theta(z) + \int_{\varkappa}^{z} \sum_{j=1}^{\infty} \frac{[\Xi(z))]^j}{\Gamma(j\varpi)} \psi'(r)(\psi(z) - \psi(r))^{j\varpi-1}\Theta(r)\,dr.$$

Proof. Utilizing (1), Definitions 1 and 8, one has

$$\begin{aligned}
\varrho(z) &\leq U(z) + G(z)\left({}^{RLAB}\Re_{\varkappa}^{\varpi,\psi}\varrho\right)(z) \\
&\leq U(z) + G(z)\left[\frac{1-\varpi}{\Lambda(\varpi)}\varrho(z) + \frac{\varpi}{\Lambda(\varpi)}\frac{1}{\Gamma(\varpi)}\int_{\varkappa}^{z}(\psi(z)-\psi(r))^{\omega-1}\psi'(r)\varrho(r)\,dr\right].
\end{aligned}$$

Therefore,

$$\begin{aligned}
\varrho(z) \leq\; & \frac{U(z)\Lambda(\varpi)}{\Lambda(\varpi)-(1-\varpi)G(z)} \\
& + \frac{\varpi G(z)}{\Lambda(\varpi)-(1-\varpi)G(z)}\frac{1}{\Gamma(\varpi)}\int_{\varkappa}^{z}(\psi(z)-\psi(r))^{\varpi-1}\psi'(r)\varrho(r)\,dr
\end{aligned}$$

Lemma 6 allows us to obtain

$$\begin{aligned}
\varrho(z) \leq\; & \frac{U(z)\Lambda(\varpi)}{\Lambda(\varpi)-(1-\varpi)G(z)} \\
& + \int_{\varkappa}^{z}\sum_{j=1}^{\infty}\frac{1}{\Gamma(j\varpi)}\left(\frac{\varpi G(z)}{\Lambda(\varpi)-(1-\varpi)G(z)}\right)^j \frac{U(r)\Lambda(\varpi)(\psi(z)-\psi(r))^{j\varpi-1}\psi'(r)}{\Lambda(\varpi)-(1-\varpi)G(z)}dr \\
\leq\; & \Theta(z) + \int_{\varkappa}^{z}\sum_{j=1}^{\infty}\frac{[\Xi(z))]^j}{\Gamma(j\varpi)}\psi'(r)(\psi(z)-\psi(r))^{j\varpi-1}\Theta(r)\,dr.
\end{aligned}$$

Corollary 1. *In light of assumptions of Lemma 8, if the function* $\Theta(z)$ *is nondecreasing on* $\Im$, *then, we get*

$$\varrho(z) \leq \Theta(z) L_{\varpi}\left[\Xi(z)(\psi(z) - \psi(\varkappa))^{\varpi}\right], \; z \in \Im.$$

Proof. Based on Lemma 8, one can write

$$\begin{aligned}\varrho(z) &\leq \frac{U(z)\Lambda(\varpi)}{\Lambda(\varpi)-(1-\varpi)G(z)} L_{\varpi}\left(\frac{\varpi G(z)\left(\psi(z)-\psi(r)\right)^{\varpi}}{\Lambda(\varpi)-(1-\varpi)G(z)}\right) \\ &\leq \Theta(z) L_{\varpi}\left[\Xi(z)\left(\psi(z)-\psi(\varkappa)\right)^{\varpi}\right].\end{aligned}$$

A novel Gronwall inequality in the context of the $\psi-$KAB fractional operator will be concluded here.

Corollary 2. *Let $\varpi_u = \varpi > 0$, for $u \geq 1$. Assume that $\Theta(z) = \frac{U(z)\Lambda(\varpi)}{\Lambda(\varpi)-(1-\varpi)G(z)}$ is a NFLI on $\Im$, $\Xi(z) = \frac{\varpi G(z)}{\Lambda(\varpi)-(1-\varpi)G(z)}$ is a NNF and ϱ is a NFLI on $\Im$, such that*

$$\varrho(z) \leq U(z) + G(z)^{KAB}\Re_{\varkappa}^{\varpi,\eta}\varrho(z), \ z \in \Im,$$

Then, for each $z \in \Im$, we get

$$\varrho(z) \leq \Theta(z) + \int_{\varkappa}^{z} \sum_{j=1}^{\infty} \frac{\eta^{1-j\varpi} r^{\eta-1}\left[\Xi(z))\right]^{j}}{\Gamma(j\varpi)} \left(z^{\eta}-r^{\eta}\right)^{j\varpi-1} \Theta(r)\, dr.$$

Proof. The proof follows immediately by taking $\psi(z) = \frac{z^{\eta}}{\eta}$ in Lemma 8.

Corollary 3. *Via the assumptions of Corollary 2, if the function $\Theta(z)$ is nondecreasing on $\Im$, then, we get*

$$\varrho(z) \leq \Theta(z) L_{\varpi}\left[\Xi(z)\left(\frac{z^{\eta}-r^{\eta}}{\eta}\right)^{\varpi}\right], \ z \in \Im.$$

Proof. Taking $\psi(z) = \frac{z^{\eta}}{\eta}$ in Corollary 1, we get the proof.

5. Solving a fractional differential equation

This part is devoted to presenting the existence and uniqueness of solution to the initial FDE below:

$$\begin{cases} {}^{CAB}D_{\varkappa}^{\varpi_u,\psi}\varrho(z) = \phi(z,\varrho(z)), \ z \in \Im, \ u \geq 1, \\ \varrho_{\psi}^{(i)}(\varkappa) = \gamma_i, \ i = 0,1,...,k, \end{cases} \tag{2}$$

where ${}^{CAB}D_{\varkappa}^{\varpi_u,\psi}$ is the ϖ_uth left-sided $\psi-$ABC fractional derivative such that $\varpi_u \in (k, k+1]$, $\gamma_i \in \mathbb{R}$ $(i \geq 0)$ are constants, $\phi : \Im \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $\psi : \Im \to \mathbb{R}$ is an increasing function with $\psi'(z) \in C^k(\Im_u, \mathbb{R}^+)$ and $\psi'(z) \neq 0$, for all $z \in \Im$ and $\varrho(z) \in C^k(\Im_u, \mathbb{R})$ is a recognized function in which $\varrho_{\psi}^{(i)}(z) = \left(\frac{1}{\psi'(z)}\frac{d}{dz}\right)^{i}\varrho(z)$ such that $\varrho_{\psi}^{(0)}(z) = \varrho(z)$.

In fact, by utilizing Lemma 4 in conjunction with the initial FDE (2) and the ϖ_uth left-sided ψ–RL-AB fractional integral operator on both sides of (2), we get

$$\varrho\left(z\right)=\sum_{i=0}^{k}\frac{\gamma_i}{i!}\left(\psi(z)-\psi(\varkappa)\right)^i+^{RLAB}\Re_{\varkappa}^{\varpi_u,\psi}\phi(z,\varrho(z)). \tag{3}$$

Now, we shall apply Picard's iterative technique [19] to demonstrate the existence and uniqueness of the solution to Problem (2).

Theorem 1. *Assume that the assertions below hold:*

(i) there exists a constant $T>0$ such that

$$\sup_{z\in\Im}\left|\phi(z,\varrho_0(z))\right|\leq T,$$

(ii) there exists a constant $P>0$ such that

$$\left|\phi(z,\varrho_1)-\phi(z,\varrho_2)\right|\leq P\left|\varrho_1-\varrho_2\right|,\ \textit{for all } z\in\Im,\ \varrho_1,\varrho_2\in C^k(\Im_u,\mathbb{R}).$$

(iii) we have the inequality

$$P\left(\frac{\left(k+1-\varpi_u\right)\left(\psi(\upsilon)-\psi(\varkappa)\right)^k}{\Lambda(\varpi_u-k)\Gamma(k+1)}+\frac{\left(\varpi_u-k\right)\left(\psi(\upsilon)-\psi(\varkappa)\right)^{\varpi_u}}{\Lambda(\varpi_u-k)\Gamma(\varpi_u+1)}\right)<1. \tag{4}$$

Then, the problem (2) has a unique solution on $\Im$.

Proof. It is evident that the solution to system (2) is the same as the solution to the FIE (3). Set

$$\varrho_0\left(z\right)=\sum_{i=0}^{k}\frac{\gamma_i}{i!}\left(\psi(z)-\psi(\varkappa)\right)^i, \tag{5}$$

and

$$\varrho_s\left(z\right)=\sum_{i=0}^{k}\frac{\gamma_i}{i!}\left(\psi(z)-\psi(\varkappa)\right)^i+^{RLAB}\Re_{\varkappa}^{\varpi_u,\psi}\phi(z,\varrho_{s-1}(z)),\ s\in\mathbb{N}. \tag{6}$$

Clearly, the series $\varrho_0\left(z\right)+\sum\limits_{m=0}^{\infty}\left(\varrho_m-\varrho_{m-1}\right)$ has a partial sum

$$\varrho_s\left(z\right)=\varrho_0\left(z\right)+\sum_{m=0}^{s}\left(\varrho_m-\varrho_{m-1}\right).$$

We want to show that the sequence $\{\varrho_s\left(z\right)\}$ converges to $\varrho\left(z\right)$. By a mathematical induction, for all $z\in[\varkappa,\upsilon]$, we can write

$$\left\|\varrho_s\left(z\right)-\varrho_{s-1}\left(z\right)\right\|$$

$$\leq \quad TP^{s-1}\left(\frac{(k+1-\varpi_u)\left(\psi(\upsilon)-\psi(\varkappa)\right)^k}{\Lambda(\varpi_u-k)\Gamma(k+1)}+\frac{(\varpi_u-k)\left(\psi(\upsilon)-\psi(\varkappa)\right)^{\varpi_u}}{\Lambda(\varpi_u-k)\Gamma(\varpi_u+1)}\right)^s, \ s\in\mathbb{N}. \quad (7)$$

Based on (5) and (6) and Lemma 5 (iv), one has

$$\begin{aligned}\left\|\varrho_1-\varrho_0\right\| &= \sup_{z\in\Im}\left|{}^{RLAB}\Re_{\varkappa}^{\varpi_u,\psi}\phi(z,\varrho_0(z))\right|\\ &\leq T\left(\frac{(k+1-\varpi_u)\left(\psi(\upsilon)-\psi(\varkappa)\right)^k}{\Lambda(\varpi_u-k)\Gamma(k+1)}+\frac{(\varpi_u-k)\left(\psi(\upsilon)-\psi(\varkappa)\right)^{\varpi_u}}{\Lambda(\varpi_u-k)\Gamma(\varpi_u+1)}\right).\end{aligned}$$

Hence, for $s=1$, the inequality (7) is true. After that, consider the inequality (7) is satisfied when $s=n$. Therefore,

$$\begin{aligned}\left\|\varrho_{n+1}-\varrho_n\right\| &= \sup_{z\in\Im}\left|{}^{RLAB}\Re_{\varkappa}^{\varpi_u,\psi}\phi(z,\varrho_n(z))-{}^{RLAB}\Re_{\varkappa}^{\varpi_u,\psi}\phi(z,\varrho_{n-1}(z))\right|\\ &= \sup_{z\in\Im}\left|{}^{RLAB}\Re_{\varkappa}^{\varpi_u,\psi}\left[\phi(z,\varrho_n(z))-\phi(z,\varrho_{n-1}(z))\right]\right|\\ &\leq {}^{RLAB}\Re_{\varkappa}^{\varpi_u,\psi}\left(P\left\|\varrho_n(z)-\varrho_{n-1}(z)\right\|\right)\\ &\leq {}^{RLAB}\Re_{\varkappa}^{\varpi_u,\psi}\left(TP^n\left(\frac{(k+1-\varpi_u)\left(\psi(\upsilon)-\psi(\varkappa)\right)^k}{\Lambda(\varpi_u-k)\Gamma(k+1)}+\frac{(\varpi_u-k)\left(\psi(\upsilon)-\psi(\varkappa)\right)^{\varpi_u}}{\Lambda(\varpi_u-k)\Gamma(\varpi_u+1)}\right)^n\right)\\ &\leq TP^{(s+1)-1}\left(\frac{(k+1-\varpi_u)\left(\psi(\upsilon)-\psi(\varkappa)\right)^k}{\Lambda(\varpi_u-k)\Gamma(k+1)}+\frac{(\varpi_u-k)\left(\psi(\upsilon)-\psi(\varkappa)\right)^{\varpi_u}}{\Lambda(\varpi_u-k)\Gamma(\varpi_u+1)}\right)^{n+1}.\end{aligned}$$

Hence, the inequality (7) is fulfilled for $s=n+1$. Then inequality (7) is true for every $s\in\mathbb{N}$ and all $z\in[\varkappa,\upsilon]$. Thus, one can write

$$\sum_{s=1}^{\infty}\left\|\varrho_s(z)-\varrho_{s-1}(z)\right\|\leq\sum_{s=1}^{\infty}TP^{s-1}\left(\frac{(k+1-\varpi_u)\left(\psi(\upsilon)-\psi(\varkappa)\right)^k}{\Lambda(\varpi_u-k)\Gamma(k+1)}+\frac{(\varpi_u-k)\left(\psi(\upsilon)-\psi(\varkappa)\right)^{\varpi_u}}{\Lambda(\varpi_u-k)\Gamma(\varpi_u+1)}\right)^s.$$

The series on the right side of the aforementioned inequality is convergent as a result of assumption (4), and so $\sum_{s=1}^{\infty}\left\|\varrho_s-\varrho_{s-1}\right\|$ is also convergent that shows that $\varrho_0+\sum_{s=1}^{\infty}\left\|\varrho_s-\varrho_{s-1}\right\|$ converges.

Put

$$\varrho=\varrho_0+\sum_{s=1}^{\infty}\left\|\varrho_s-\varrho_{s-1}\right\|,$$

it follows that

$$\left\|\varrho_s-\varrho\right\|\to 0, \ \text{as } s\to\infty. \quad (8)$$

This indicates that the solution to problem (2) exists. From (8), we get

$$\left\|\phi(.,\varrho_{s-1}(.))-\phi(.,\varrho(.))\right\|\leq P\left|\varrho_{s-1}-\varrho\right|\to 0, \ \text{as } s\to\infty.$$

Thus,

$$\lim_{s\to\infty}\phi(z,\varrho_{s-1}(z))=\phi(z,\varrho(z)). \tag{9}$$

As $s\to\infty$ in (6) and applying (9), we have

$$\varrho(z)=\sum_{i=0}^{k}\frac{\gamma_i}{i!}\left(\psi(z)-\psi(\varkappa)\right)^i+{}^{RLAB}\,\Re_{\varkappa}^{\varpi_u,\psi}\phi(z,\varrho(z)),$$

which is a solution of the initial FDE (2).

Finally, for the uniqueness, assume that $\widehat{\varrho}$ is another solution to Problem (2). Thus, we get

$$\begin{aligned}\|\varrho-\widehat{\varrho}\| &= \sup_{z\in\Im}\left|{}^{RLAB}\Re_{\varkappa}^{\varpi_u,\psi}\phi(z,\varrho(z))-{}^{RLAB}\,\Re_{\varkappa}^{\varpi_u,\psi}\phi(z,\widehat{\varrho}(z))\right|\\ &= \sup_{z\in\Im}\left|{}^{RLAB}\Re_{\varkappa}^{\varpi_u,\psi}\left[\phi(z,\varrho(z))-\phi(z,\widehat{\varrho}(z))\right]\right|\\ &\le {}^{RLAB}\Re_{\varkappa}^{\varpi_u,\psi}\left(P\left\|\varrho(z)-\widehat{\varrho}(z)\right\|\right)\\ &\le p\left(\frac{(k+1-\varpi_u)\left(\psi(\upsilon)-\psi(\varkappa)\right)^k}{\Lambda(\varpi_u-k)\Gamma(k+1)}+\frac{(\varpi_u-k)\left(\psi(\upsilon)-\psi(\varkappa)\right)^{\varpi_u}}{\Lambda(\varpi_u-k)\Gamma(\varpi_u+1)}\right)\|\varrho-\widehat{\varrho}\|.\end{aligned}$$

In light of (4), we conclude that $\|\varrho-\widehat{\varrho}\|$, that is, $\varrho(z)=\widehat{\varrho}(z)$. This completes the proof.

6. Supportive examples

In this part, we support our results by the following examples:

Example 1. *Consider the following initial FDE:*

$$\begin{cases} {}^{CAB}D_1^{\varpi_u,\psi}\varrho(z)=\cos\left(z^2\right)-\frac{\varrho(z)}{\frac{1}{4}-\varrho(z)}, & z\in[1,3],\\ \varrho(1)=1, \quad \varrho'_{\psi}(1)=1, \end{cases} \tag{10}$$

where $\psi(z)=\ln(z)$ *and*

$$\varpi_u(\tau)=\begin{cases}0.98, & \tau\in[1,2],\\ 1.21, & \tau\in(2,3],\end{cases} \quad \textit{for } u\ge 1.$$

The requirements of Theorem 1 shall be examined as follows:

$$\begin{aligned}|\phi(z,\varrho_1)-\phi(z,\varrho_2)| &= \left|\cos\left(z^2\right)-\frac{\varrho_1(z)}{\frac{1}{4}-\varrho_1(z)}-\cos\left(z^2\right)+\frac{\varrho_2(z)}{\frac{1}{4}-\varrho(z)}\right|\\ &= \left|\frac{\varrho_1(z)}{\frac{1}{4}-\varrho_1(z)}-\frac{\varrho_2(z)}{\frac{1}{4}-\varrho(z)}\right|\\ &\le \frac{1}{4}\left|\varrho_1(z)-\varrho_2(z)\right|.\end{aligned}$$

Then, $P=\frac{1}{4}>0$. *If we take* $\Lambda(\varpi_u-k)=1$, *then, we have the following cases:*

(a) If $\tau \in [1,2]$, one has $k=1$, and

$$P\left(\frac{(k+1-\varpi_u)\left(\psi(\upsilon)-\psi(\varkappa)\right)^k}{\Lambda(\varpi_u-k)\Gamma(k+1)}+\frac{(\varpi_u-k)\left(\psi(\upsilon)-\psi(\varkappa)\right)^{\varpi_u}}{\Lambda(\varpi_u-k)\Gamma(\varpi_u+1)}\right)\approx 0.171925<1.$$

(b) If $\tau \in (2,3]$, we have $k=3$, and

$$P\left(\frac{(k+1-\varpi_u)\left(\psi(\upsilon)-\psi(\varkappa)\right)^k}{\Lambda(\varpi_u-k)\Gamma(k+1)}+\frac{(\varpi_u-k)\left(\psi(\upsilon)-\psi(\varkappa)\right)^{\varpi_u}}{\Lambda(\varpi_u-k)\Gamma(\varpi_u+1)}\right)\approx 0.568509<1.$$

Therefore, all axioms of Theorem 1 are fulfilled. Hence, there exists a unique solution to Problem (10).

Example 2. *Consider the following initial FDE:*

$$\begin{cases} {}^{CAB}D_0^{\varpi_u,\psi}\varrho(z)=z^3-\varrho(z),\ z\in[0,2], \\ \varrho(0)=0,\quad \varrho'_\psi(0)=0,\ \varrho''_\psi(0)=2 \end{cases} \tag{11}$$

where $\psi(z)=z$ and

$$\varpi_u(\tau)=\begin{cases} 0.2, & \tau\in[\frac{1}{2},1], \\ 0.4, & \tau\in(1,2], \end{cases}\quad \text{for } u\geq 1.$$

Assume that problem (11) has an exact solution $\varrho(z)=z^3$.

The conditions of Theorem 1 shall be checked as follows:

$$\begin{aligned} \left|\phi(z,\varrho_1)-\phi(z,\varrho_2)\right| &= \left|z^3-\varrho_1(z)-z^3+\varrho_2(z)\right| \\ &\leq \left|\varrho_1(z)-\varrho_2(z)\right|. \end{aligned}$$

Then, $P=1>0$. If we take $\Lambda(\varpi_u-k)=1$, then, we have the following cases:

(a) If $\tau\in[\frac{1}{2},1]$, one has $k=1$, and

$$P\left(\frac{(k+1-\varpi_u)\left(\psi(\upsilon)-\psi(\varkappa)\right)^k}{\Lambda(\varpi_u-k)\Gamma(k+1)}+\frac{(\varpi_u-k)\left(\psi(\upsilon)-\psi(\varkappa)\right)^{\varpi_u}}{\Lambda(\varpi_u-k)\Gamma(\varpi_u+1)}\right)\approx 0.645318<1.$$

Example 3. *If $\tau\in(1,2]$, we have $k=2$, and*

$$P\left(\frac{(k+1-\varpi_u)\left(\psi(\upsilon)-\psi(\varkappa)\right)^k}{\Lambda(\varpi_u-k)\Gamma(k+1)}+\frac{(\varpi_u-k)\left(\psi(\upsilon)-\psi(\varkappa)\right)^{\varpi_u}}{\Lambda(\varpi_u-k)\Gamma(\varpi_u+1)}\right)\approx 0.735214<1.$$

Hence, all assumptions of Theorem 1 are fulfilled. Therefore, the problem (11) possesses a unique solution.

Next, using Picard's iterative method, we will compute the solution of system (11) as follows:

$$\varrho_s(z)=\varrho_0(z)+{}^{CAB}D_0^{\varpi_u,\psi}\left(z^3-\varrho_{s-1}(z)\right),\ \varrho_0(z)=z^3.$$

Then

$$\begin{aligned}
\varrho_1(z) &= z^3 + {}^{CAB}D_0^{\varpi_u,\psi}\left(z^3 - \varrho_0(z)\right) = z^3, \\
\varrho_2(z) &= z^3 + {}^{CAB}D_0^{\varpi_u,\psi}\left(z^3 - \varrho_1(z)\right) = z^3, \\
\varrho_3(z) &= z^3, \\
&\vdots \\
\varrho_k(z) &= z^3.
\end{aligned}$$

This corresponds to the exact solution.

7. Conclusion and open problems

This paper delves into the application of AB fractional operators with higher-variable orders using increasing functions. We explore the qualitative properties of these operators and demonstrate the existence of a unique solution for an initial FDE using Picard's iteration method. Our findings are supported by two illustrative examples. We also introduce a generalized Gronwall inequality within the framework of AB fractional integrals. This study contributes to the advancement of fractional calculus and its potential applications. Future research will focus on applying these extended operators to real-world dynamic systems, investigating new properties and inequalities associated with them, and exploring the corresponding right-sided fractional operators for RL-AB, ABC, and KAB.

8. Abbreviations

AB→Atangana-Baleanu
ML→Mittag-Leffler
FDE→fractional differential equations
ABC→Atangana-Baleanu-Caputo
w.r.t.→with respect to
RL→Riemann-Liouville
ABK→Atangana-Baleanu-Kashuri
RAB→Riemann-Atangana-Baleanu
RL-AB→Riemann-Liouville-tangana-Baleanu
NNF→nonnegative and nondecreasing function
NFLI→nonnegative function locally integrable
FIE→fractional integral equation

Conflicts of interest

The authors declare that they have no conflicts of interest.

Author's contributions

All authors contributed equally and significantly in writing this article.

Funding

This work was supported in part by the Basque Government under Grant IT1555-22.

Acknowledgments

The authors thank the Basque Government for Grant IT1555-22 and to MICIU/AEI/ 10.13039/501100011033 and ERDF/E for Grants PID2021-123543OB-C21 and PID2021-123543OB-C22.

References

[1] S Abbas, M Benchohra, and G M N'Guérékata. *Topics in Fractional Differential Equations.* Springer, Berlin, 2012.

[2] T Abdeljawad. A lyapunov type inequality for fractional operators with nonsingular mittag-leffler kernel. *J. Inequal. Appl.*, 130:1–11, 2017.

[3] T. Abdeljawad, S. T. M. Thabet, I. Kedim, M. I. Ayari, and A. Khan. A higher-order extension of atangana–baleanu fractional operators with respect to another function and a gronwall-type inequality. *Boundary Value Problems*, 2023:49, 2023.

[4] R Almeida, A B Malinowska, and M T T Monteiro. Fractional differential equations with a caputo derivative with respect to a kernel function and their applications. *Math. Methods Appl. Sci.*, 41(1):336–352, 2018.

[5] A Atangana and D Baleanu. New fractional derivatives with non-local and non-singular kernel: theory and application to heat transfer model. *Therm. Sci.*, 20(2):763–769, 2016.

[6] M I Ayari and S T M Thabet. Qualitative properties and approximate solutions of thermostat fractional dynamics system via a nonsingular kernel operator. *Arab J. Math. Sci.*, https://doi.org/10.1108/AJMS-06-2022-0147, 2023.

[7] M Dalir and M Bashour. Applications of fractional calculus. *Math. Sci.*, 4(21):1021–1032, 2010.

[8] A Fernandez and D Baleanu. Differintegration with respect to functions in fractional models involving mittag-leffler functions. In https://doi.org/10.2139/ssrn.3275746, editor, *In Proceedings of International Conference on Fractional Differentiation and Its Applications (ICFDA).*, 2018.

[9] H A Hammad, H Aydi, H. Işık, and M De la Sen. Stability and controllability study for mixed integral fractional delay dynamic systems endowed with impulsive effects on time scales. *AIMS Math.*, 8(3):6913–6941, 2023.

[10] H A Hammad, H Aydi, and M Zayed. On the qualitative evaluation of the variable-order coupled boundary value problems with a fractional delay. *J. Inequal. Appl.*, 2023:105, 2023.

[11] H A Hammad and M De la Sen. Solutions of fractional differential type equations by fixed point techniques for multivalued contractions. *Complexity.*, 2021:5730853, 2021.

[12] H A Hammad and M De la Sen. Stability and controllability study for mixed integral fractional delay dynamic systems endowed with impulsive effects on time scales. *Fractal Fract.*, 7:92, 2023.

[13] H A Hammad, M De la Sen, and H Aydi. Generalized dynamic process for an extended multivalued fcontraction in metric-like spaces with applications. *Alexandria Engineering Journal*, 59(5):3817–3825, 2020.

[14] H A Hammad, R A Rashwan, A Nafea, M E Samei, and M De la Sen. Stability and existence of solutions for a tripled problem of fractional hybrid delay differential equations. *Symmetry.*, 14:2579, 2022.

[15] H A Hammad, R A Rashwan, A Nafea, M E Samei, and S Noeiaghdam. Stability analysis for a tripled system of fractional pantograph differential equations with nonlocal conditions. *J. Vib. Control.*, 30(3-4)):632–647, 2024.

[16] R Hilfer. *Applications of Fractional Calculus in Physics.* Singapore, World Scientific, 2000.

[17] A Kashuri. Hermite-hadamard type inequalities for the abk-fractional integrals. *J. Comput. Anal. Appl.*, 29:309–326, 2021.

[18] A A Kilbas, H M Srivastava, and J J Trujillo. *Theory and Applications of Fractional Differential Equations. North-Holland Mathematics Studies.* Elsevier, Amsterdam, 2006.

[19] R Lyons, A Vatsala, and R Chiquet. Picard's iterative method for caputo fractional differential equations with numerical results. *Mathematics.*, 5(4):65, 2017.

[20] P O Mohammed and T Abdeljawad. Integral inequalities for a fractional operator of a function with respect to another function with nonsingular kernel. *Adv. Differ. Equ.*, 2020:363, 2020.

[21] M Rahimy. Applications of fractional differential equations. *Appl. Math. Sci.*, 4(50):2453–2461, 2010.

[22] S K Samko, A A Kilbas, and O I. Marichev. *Fractional Integrals and Derivatives: Theory and Applications.* Gordon and Breach Science, Switzerland, 1993.

[23] K Shah, T Abdeljawad, M B Jeelani, and M A Alqudah. Spectral analysis of variable-order multi-terms fractional differential equations. *Open Phy.*, 21(1):20230136, 2023.

[24] K Shah, G Ali, K J Ansari, M Meganathan, and B Abdalla. On qualitative analysis of boundary value problem of variable order fractional delay differential equations. *Bound. Value Probl.*, 1023:55, 2023.

[25] K Shah, H Naz, M Sarwar, and T Abdeljawad. On spectral numerical method for variable-order partial differential equations. *AIMS Math.*, 7(6):10422–10438, 2022.

[26] M Sher, K Shah, and J Rassias. On qualitative theory of fractional order delay evolution equation via the prior estimate method. *Math. Methods Appl. Sci.*, 43(10):6464–6475, 2020.

[27] J V C Sousa and E C Oliveira. A gronwall inequality and the cauchy-type problem by means of $\psi-$hilfer operators. *Differ. Equ. Appl.*, 11:87–106, 2019.

[28] V Stojiljković. Hermite hadamard type inequalities involving $(k-p)$ fractional operator with $(\alpha, h-m)-p$ convexity. *Eur. J. Pure Appl. Math.*, 1616):503–522, 2023.