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Graphical Representation of Zadeh's Max-Min Composition Operator for Two 3-Dimensional Quadratic Fuzzy Numbers

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Abstract. We computed the extended operations for generalized quadratic fuzzy sets and extended quadratic fuzzy numbers from $\mathbb{R}$ to $\mathbb{R}^2$. By defining parametric operations between two α-cuts, which are regions, we derived the parametric operations for two quadratic fuzzy numbers defined on $\mathbb{R}^2$. The outcomes of these parametric operations serve as a generalization of Zadeh's extended algebraic operations. We demonstrated that the results obtained from the parametric operations represent an extension of Zadeh's extended algebraic operations. Additionally, we expanded quadratic fuzzy numbers initially defined in two dimensions to three dimensions and calculated Zadeh's max-min composition operator for two extended three-dimensional quadratic fuzzy numbers. We presented an illustrative example of three-dimensional results along with corresponding graphs.

2020 Mathematics Subject Classifications: 47S40, 03E72

Key Words and Phrases: Graphic representation, parametric operation, 3-dimensional quadraric fuzzy number

1. Introduction

A quadratic fuzzy number expands upon the concept of traditional fuzzy numbers by incorporating quadratic functions to represent the degree of membership of an element in a set. Fuzzy numbers are utilized to model uncertainty and imprecision in various applications, and quadratic fuzzy numbers provide a more flexible representation through the use of quadratic functions. These numbers are commonly employed in decision-making processes, especially in scenarios where there is a need to model and analyze uncertain or imprecise information. They find applications in diverse fields such as optimization, control systems, and decision analysis [4, 7]. It's essential to recognize that various researchers and practitioners may employ slightly different formulations and definitions for quadratic

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fuzzy numbers. Additionally, the specific context of their application can influence how they are defined and utilized.

The membership function of a quadratic fuzzy number is composed of a quadratic function with the maximum value of 1. In contrast, a general quadratic fuzzy set is a quadratic fuzzy set that may not have a maximum value of 1. We calculated the extended operations for generalized quadratic fuzzy sets [9] and expanded the quadratic fuzzy numbers from $\mathbb{R}$ to $\mathbb{R}^2$ [2]. By defining parametric operations between two α-cuts, which are regions, we derived the parametric operations for two quadratic fuzzy numbers defined on $\mathbb{R}^2$. The outcomes of these parametric operations serve as a generalization of Zadeh's extended algebraic operations.

We have shown that the results of parametric operations represent a generalization of Zadeh's max-min composition operations. Furthermore, we expanded the concept of general quadratic fuzzy sets from $\mathbb{R}$ to $\mathbb{R}^2$. We performed calculations for the parametric operations applied to two generalized 2-dimensional quadratic fuzzy sets [8]. Our evidence demonstrates that the parametric operations for two generalized quadratic fuzzy sets defined on $\mathbb{R}^2$ constitute a broader generalization of Zadeh's max-min composition operations for two general quadratic fuzzy sets defined on $\mathbb{R}$ [6].

In this paper, we extend quadratic fuzzy numbers defined in two dimensions to three dimensions and calculate Zadeh's max-min composition operator for two extended three-dimensional quadratic fuzzy numbers. We provide an illustrative example showcasing three-dimensional results and present graphs depicting the example.

2. max-min composition operations of Zadeh for generalized quadratic fuzzy sets defined on $\mathbb{R}$

We start by introducing the α-cut and α-set of the fuzzy set A on $\mathbb{R}$ with the membership function $\mu_A(x)$. An *α-cut* of the fuzzy number A is formally defined as $A_\alpha = \{x \in \mathbb{R} \mid \mu_A(x) \geq \alpha\}$ when $\alpha \in (0, 1]$ and A_0 is determined as the closure of $\{x \in \mathbb{R} \mid \mu_A(x) > 0\}$. For $\alpha \in (0, 1)$, the set $A^\alpha = \{x \in X \mid \mu_A(x) = \alpha\}$ is referred to as the *α-set* of the fuzzy set A, where A^0 represents the boundary of $\{x \in \mathbb{R} \mid \mu_A(x) > 0\}$, and A^1 is equivalent to A_1.

Definition 1. [12] *The extended addition $A(+)B$, extended subtraction $A(-)B$, extended multiplication $A(\cdot)B$, and extended division $A(/)B$ are fuzzy sets with membership functions as follows. For all $x \in A$ and $y \in B$,*

$$\mu_{A(*)B}(z) = \sup_{z=x*y} \min\{\mu_A(x), \mu_B(y)\}, \quad * = +, -, \cdot, /$$

Now, we extend the concept to encompass general quadratic fuzzy sets. A general quadratic fuzzy set is symmetric and may not necessarily attain the maximum value of 1. The membership function graph of a general quadratic fuzzy set exhibits symmetry with respect to a certain line defined by $x = m$.

Definition 2. [9] *A fuzzy set A with a membership function*

$$\mu_A(x) = \begin{cases} 0, & x < x_1,\ x_2 \le x, \\ -a(x-x_1)(x-x_2) = -a(x-m)^2 + p, & x_1 \le x < x_2, \end{cases}$$

where $m = \frac{x_1+x_2}{2}, 0 < a,\ 0 < p \le 1$, *is called a generalized quadratic fuzzy set and denoted by* $[[x_1, p, x_2]]$ *or* $[[a, m, p]]_+$.

Theorem 1. [9] *Let* $A = [[a, m, p]]_+ = [[x_1, p, x_2]]$ *and* $B = [[b, n, q]]_+ = [[x_3, q, x_4]]$ *be generalized quadratic fuzzy sets. Assume that* $p \le q$ *and* $\mu_B(x) \ge p$ *on* $[k_1, k_2]$. *We can then deduce the followings:*
(1) $A(+)B$ *is a fuzzy set with a membership function*

$$\mu_{A(+)B}(x) = \begin{cases} 0 & (x < x_1 + x_3,\ x_2 + x_4 \le x) \\ f_1(x) & (x_1 + x_3 \le x < m + k_1) \\ p & (m + k_1 \le x < m + k_2) \\ f_2(x) & (m + k_2 \le x < x_2 + x_4) \end{cases}$$

where

$$\begin{aligned} f_1(x) =& \frac{1}{a^2 - 2ab + b^2}\Big(-abm(a+b+an+bn) - abn(am+bm \\ &+ an+bn) - ab(p+q) + a^2q + b^2p + 2ab(am+bm+an \\ &+ bn)x - ab(a+b)x^2 + 2ab(m+n-x)\cdot\sqrt{g_1(x)}\Big), \\ f_2(x) =& \frac{1}{a^2 - 2ab + b^2}\Big(-abm(a+b+an+bn) - abn(am+bm \\ &+ an+bn) - ab(p+q) + a^2q + b^2p + 2ab(am+bm+an \\ &+ bn)x - ab(a+b)x^2 - 2ab(m+n-x)\cdot\sqrt{g_1(x)}\Big), \end{aligned}$$

and $g_1(x) = ab(m+n)^2 + (a-b)(p-q) - 2ab(m+n)x + abx^2$.

(2) $A(-)B$ *is a fuzzy set with a membership function*

$$\mu_{A(-)B}(x) = \begin{cases} 0 & (x < x_1 - x_4, x_2 - x_3 \le x) \\ f_3(x) & (x_1 - x_4 \le x < m - k_2) \\ p & (m - k_2 \le x < m - k_1) \\ f_4(x) & (m - k_1 \le x < x_2 - x_3) \end{cases}$$

where

$$\begin{aligned} f_3(x) =& \frac{1}{a^2 - 2ab + b^2}\Big(-abm(am+bm-an-bn) - abn(an+bn \\ &- am-bm) - ab(p+q) + a^2q + b^2p + 2ab(am+bm-an \\ &- bn)x - ab^2x^2 + 2ab(m-n-x)\cdot\sqrt{g_2(x)}\Big), \end{aligned}$$

$$
\begin{aligned}
f_4(x) =& \frac{1}{a^2-2ab+b^2}\Big(-abm(am+bm-an-bn)-abn(an+bn\\
&-am-bm)-ab(p+q)+a^2q+b^2p+2ab(am+bm-an\\
&-bn)x-ab^2x^2-2ab(m-n-x)\cdot\sqrt{g_2(x)}\Big),
\end{aligned}
$$

and $g_2(x) = ab(m-n)^2+(a-b)(p-q)-2ab(m-n)x+abx^2$.

(3) If $p = q$, $A(\cdot)B$ *is a fuzzy set with a membership function*

$$
\mu_{A(\cdot)B}(x) = \begin{cases} 0 & (x < x_1x_3, \;\; x_2x_4 \le x) \\ f_5(x) & (x_1x_3 \le x < x_2x_4) \end{cases}
$$

where

$$
\begin{aligned}
f_5(x) =& \frac{1}{2}(-am^2-bn^2+2p)-\sqrt{ab}x+\frac{1}{2}\sqrt{g_3(x)},\\
g_3(x) =& -am^2(am^2+3bn^2)-bn^2(bn^2+3am^2)+2(am^2+bn^2\\
&-2p)^2+8p(am^2+bn^2-p)+8abmnx-\frac{1}{8\sqrt{ab}x}\Big\{-8(am^2\\
&+bn^2-2p)^3+8(am^2+bn^2-2p)h_1(x)-16h_2(x)\Big\},\\
h_1(x) =& am^2(am^2+2bn^2)+bn^2(bn^2+2am^2)-6p(am^2+bn^2-p)-4abmnx-2abx^2,\\
h_2(x) =& abm^2n^2(am^2+bn^2-4p)-am^2p(am^2-3p)-bn^2p(bn^2-3p)\\
&-2p^3-2abmn(am^2+bn^2-2p)x+ab(am^2+bn^2+2p)x^2.
\end{aligned}
$$

(4) $A(/)B$ *is a fuzzy set with a membership function*

$$
\mu_{A(/)B}(x) = \begin{cases} 0 & (x < x_1/x_4, \;\; x_2/x_3 \le x) \\ f_6(x) & (x_1/x_4 \le x < m/k_2) \\ p & (m/k_2 \le x < m/k_1) \\ f_7(x) & (m/k_1 \le x < x_2/x_3) \end{cases}
$$

where

$$
\begin{aligned}
f_6(x) =& \frac{1}{b^2-2abx^2+a^2x^4}\Big(-b^2(am^2+p)+2ab^2mnx-ab(am^2+bn^2+p\\
&+q)x^2+2a^2bmnx^3-a^2(bn^2-q)x^4+2abx(m-nx)\cdot\sqrt{g_4(x)}\Big),\\
f_7(x) =& \frac{1}{b^2-2abx^2+a^2x^4}\Big(-b^2(am^2+p)+2ab^2mnx-ab(am^2+bn^2+p\\
&+q)x^2+2a^2bmnx^3-a^2(bn^2-q)x^4-2abx(m-nx)\cdot\sqrt{g_4(x)}\Big),
\end{aligned}
$$

and $g_4(x) = b(am^2-p+q)-2abmnx+a(bn^2+p-q)x^2$.

3. max-min composition operations of Zadeh for quadratic fuzzy numbers defined on $\mathbb{R}^2$

We extended the concept of quadratic fuzzy numbers from $\mathbb{R}$ to $\mathbb{R}^2$, introducing 2-dimensional quadratic fuzzy numbers. Additionally, we formulated parametric operations for two such 2-dimensional quadratic fuzzy numbers by employing region-valued α-cuts in $\mathbb{R}^2$.

Definition 3. [2] *A fuzzy set A with a membership function*

$$\mu_A(x,y) = \begin{cases} 1 - \left(\frac{(x-x_1)^2}{a^2} + \frac{(y-y_1)^2}{b^2}\right), & b^2(x-x_1)^2 + a^2(y-y_1)^2 \le a^2b^2, \\ 0, & \textit{otherwise}, \end{cases}$$

where $a, b > 0$ is reffered to as a 2-dimensional quadratic fuzzy number, denoted by $[a,\ x_1,\ b,\ y_1]^2$.

The α-cut A_α of a 2-dimensional quadratic fuzzy number $A = [a, x_1, b, y_1]^2$ is the interior of an ellipse in an xy-plane, including the boundary

$$\begin{aligned} A_\alpha &= \left\{(x,y) \in \mathbb{R}^2 \;\middle|\; b^2(x-x_1)^2 + a^2(y-y_1)^2 \le a^2b^2(1-\alpha)\right\} \\ &= \left\{(x,y) \in \mathbb{R}^2 \;\middle|\; \frac{(x-x_1)^2}{a^2(1-\alpha)} + \frac{(y-y_1)^2}{b^2(1-\alpha)} \le 1\right\}. \end{aligned}$$

Theorem 2. [3] *Let A be a continuous convex fuzzy number defined on $\mathbb{R}^2$ and $A^\alpha = \{(x,y) \in \mathbb{R}^2 | \mu_A(x,y) = \alpha\}$ be the α-set of A. Then for all $\alpha \in (0,1)$, there exist continuous functions $f_1^\alpha(t)$ and $f_2^\alpha(t)$ defined on $[0, 2\pi]$ such that*

$$A^\alpha = \{(f_1^\alpha(t), f_2^\alpha(t)) \in \mathbb{R}^2 | 0 \le t \le 2\pi\}.$$

Definition 4. [3] *Let A and B be convex fuzzy numbers defined on $\mathbb{R}^2$ and*

$$\begin{aligned} A^\alpha &= \{(f_1^\alpha(t), f_2^\alpha(t)) \in \mathbb{R}^2 | 0 \le t \le 2\pi\}, \\ B^\alpha &= \{(g_1^\alpha(t), g_2^\alpha(t)) \in \mathbb{R}^2 | 0 \le t \le 2\pi\} \end{aligned}$$

be the α-sets of A and B, respectively. For $\alpha \in (0,1)$, the parametric addition, parametric subtraction, parametric multiplication, and parametric division are fuzzy numbers that have their α-sets as follows:

(1) parametric addition $A(+)_pB$:

$$(A(+)_pB)^\alpha = \{(f_1^\alpha(t) + g_1^\alpha(t), f_2^\alpha(t) + g_2^\alpha(t)) \in \mathbb{R}^2 | 0 \le t \le 2\pi\}$$

(2) parametric subtraction $A(-)_pB$:

$$(A(-)_pB)^\alpha = \{(x_\alpha(t), y_\alpha(t)) \in \mathbb{R}^2 | 0 \le t \le 2\pi\},$$

where

$$x_\alpha(t) = \begin{cases} f_1^\alpha(t) - g_1^\alpha(t+\pi), & \textit{if } 0 \le t \le \pi \\ f_1^\alpha(t) - g_1^\alpha(t-\pi), & \textit{if } \pi \le t \le 2\pi \end{cases}$$

and

$$y_\alpha(t) = \begin{cases} f_2^\alpha(t) - g_2^\alpha(t+\pi), & \textit{if } 0 \le t \le \pi \\ f_2^\alpha(t) - g_2^\alpha(t-\pi), & \textit{if } \pi \le t \le 2\pi \end{cases}$$

(3) parametric multiplication $A(\cdot)_pB$:

$$(A(\cdot)_pB)^\alpha = \{(f_1^\alpha(t) \cdot g_1^\alpha(t), f_2^\alpha(t) \cdot g_2^\alpha(t)) \in \mathbb{R}^2 | 0 \le t \le 2\pi\}$$

(4) parametric division $A(/)_pB$:

$$(A(/)_pB)^\alpha = \{(x_\alpha(t), y_\alpha(t)) \in \mathbb{R}^2 | 0 \le t \le 2\pi\},$$

where

$$x_\alpha(t) = \frac{f_1^\alpha(t)}{g_1^\alpha(t+\pi)} \quad (0 \le t \le \pi), \quad x_\alpha(t) = \frac{f_1^\alpha(t)}{g_1^\alpha(t-\pi)} \quad (\pi \le t \le 2\pi)$$

and

$$y_\alpha(t) = \frac{f_2^\alpha(t)}{g_2^\alpha(t+\pi)} \quad (0 \le t \le \pi), \quad y_\alpha(t) = \frac{f_2^\alpha(t)}{g_2^\alpha(t-\pi)} \quad (\pi \le t \le 2\pi)$$

For $\alpha = 0$ and $\alpha = 1$, $(A(*)_pB)^0 = \lim_{\alpha\to 0^+}(A(*)_pB)^\alpha$ and $(A(*)_pB)^1 = \lim_{\alpha\to 1^-}(A(*)_pB)^\alpha$, where $* = +,\ -,\ \cdot,\ /$.

Theorem 3. [3] *Let $A = [a_1,\ x_1,\ b_1,\ y_1]^2$ and $B = [a_2,\ x_2,\ b_2,\ y_2]^2$ be two 2-dimensional quadratic fuzzy numbers. Subsequently, the following results hold:*

(1) $A(+)_pB = \Big[a_1 + a_2,\ x_1 + x_2,\ b_1 + b_2,\ y_1 + y_2\Big]^2$

(2) $A(-)_pB = \Big[a_1 + a_2,\ x_1 - x_2,\ b_1 + b_2,\ y_1 - y_2\Big]^2$

(3) $(A(\cdot)_pB)^\alpha = \{(x_\alpha(t), y_\alpha(t)) \mid 0 \le t \le 2\pi\}$, *where*

$$x_\alpha(t) = x_1x_2 + (x_1a_2 + x_2a_1)\sqrt{1-\alpha}\cos t + a_1a_2(1-\alpha)\cos^2 t$$

and

$$y_\alpha(t) = y_1y_2 + (y_1b_2 + y_2b_1)\sqrt{1-\alpha}\sin t + b_1b_2(1-\alpha)\sin^2 t.$$

(4) $(A(/)_pB)^\alpha = \{(x_\alpha(t), y_\alpha(t)) \mid 0 \le t \le 2\pi\}$, *where*

$$x_\alpha(t) = \frac{x_1 + a_1\sqrt{1-\alpha}\cos t}{x_2 - a_2\sqrt{1-\alpha}\cos t} \quad \textit{and} \quad y_\alpha(t) = \frac{y_1 + b_1\sqrt{1-\alpha}\sin t}{y_2 - b_2\sqrt{1-\alpha}\sin t}.$$

Therefore, $A(+)_pB$ and $A(-)_pB$ become 2-dimensional quadratic fuzzy numbers, whereas $A(\cdot)_pB$ and $A(/)_pB$ do not qualify as 2-dimensional quadratic fuzzy numbers.

Example 1. [3] *Consider* $A = [6,\ 3,\ 8,\ 5]^2$ *and* $B = [4,\ 2,\ 5,\ 3]^2$*. Subsequently, the following observations hold:*

(1) $A(+)_pB = [10,\ 5,\ 13,\ 8]^2$

(2) $A(-)_pB = [10,\ 1,\ 13,\ 2]^2$

(3) $(A(\cdot)_pB)^\alpha = \{(x_\alpha(t), y_\alpha(t)) \mid 0 \le t \le 2\pi\}$, *where*

$$x_\alpha(t) = 6 + 24\sqrt{1-\alpha}\cos t + 24(1-\alpha)\cos^2 t$$

and

$$y_\alpha(t) = 15 + 49\sqrt{1-\alpha}\sin t + 40(1-\alpha)\sin^2 t.$$

(4) $(A(/)_pB)^\alpha = \{(x_\alpha(t), y_\alpha(t)) \mid 0 \le t \le 2\pi\}$, *where*

$$x_\alpha(t) = \frac{3 + 6\sqrt{1-\alpha}\cos t}{2 - 4\sqrt{1-\alpha}\cos t} \quad \text{and} \quad y_\alpha(t) = \frac{5 + 8\sqrt{1-\alpha}\sin t}{3 - 5\sqrt{1-\alpha}\sin t}.$$

Thus $A(+)_pB$ *and* $A(-)_pB$ *become 2-dimensional quadratic fuzzy numbers, but* $A(\cdot)_pB$ *and* $A(/)_pB$ *are not 2-dimensional quadratic fuzzy numbers.*

4. max-min composition operations of Zadeh for quadratic fuzzy numbers defined on $\mathbb{R}^3$

We extended the concept of quadratic fuzzy numbers from $\mathbb{R}^2$ to $\mathbb{R}^3$, thereby introducing 3-dimensional quadratic fuzzy numbers. Our objective is to formulate parametric operations between two such 3-dimensional quadratic fuzzy numbers. In $\mathbb{R}^3$, α-cuts take the form of cubics, which makes the traditional calculation methods between α-cuts infeasible. Therefore, we adopted a novel approach to reinterpret the existing method and apply it to cubic-valued α-cuts in $\mathbb{R}^3$.

Definition 5. *A fuzzy set A with a membership function*

$$\mu_A(x,y,z) = \begin{cases} 1 - \left(\frac{(x-x_1)^2}{a^2} + \frac{(y-y_1)^2}{b^2} + \frac{(z-z_1)^2}{c^2}\right), & \text{if } \ b^2c^2(x-x_1)^2 + c^2a^2(y-y_1)^2 \\ & \quad +a^2b^2(z-z_1)^2 \le a^2b^2c^2, \\ 0, & \text{otherwise}, \end{cases}$$

where $a, b, c > 0$ *is called the 3-dimensional quadratic fuzzy number and denoted by* $[a, x_1, b, y_1, c, z_1]^3$.

Note that $\mu_A(x,y)$ forms a cone in $\mathbb{R}^2$, but we can not determine the shape of $\mu_A(x,y,z)$ in $\mathbb{R}^3$. The α-cut A_α of a 3-dimensional quadratic fuzzy number $A = [a, x_1, b, y_1, c, z_1]^3$ is defined as the following set

$$\begin{aligned} A_\alpha &= \left\{(x,y,z) \in \mathbb{R}^3 \;\middle|\; \frac{(x-x_1)^2}{a^2} + \frac{(y-y_1)^2}{b^2} + \frac{(z-z_1)^2}{c^2} \le 1-\alpha\right\} \\ &= \left\{(x,y,z) \in \mathbb{R}^3 \;\middle|\; \frac{(x-x_1)^2}{a^2(1-\alpha)} + \frac{(y-y_1)^2}{b^2(1-\alpha)} + \frac{(z-z_1)^2}{c^2(1-\alpha)} \le 1\right\}. \end{aligned}$$

Definition 6. *A 3-dimensional fuzzy number, A defined on $\mathbb{R}^3$, is termed a convex fuzzy number if, for all $\alpha \in (0,1)$, the α-cuts*

$$A_\alpha = \{(x,y,z) \in \mathbb{R}^3 | \mu_A(x,y,z) \geq \alpha\}$$

represent convex subsets in $\mathbb{R}^3$.

Theorem 4. [10] *Let A be a continuous convex fuzzy number defined on $\mathbb{R}^3$, and $A^\alpha = \{(x,y,z) \in \mathbb{R}^3 | \mu_A(x,y,z) = \alpha\}$ be the α-set of A. Then, for all $\alpha \in (0,1)$, there exist continuous functions $f_1^\alpha(s), f_2^\alpha(s,t)$, and $f_3^\alpha(s,t)(0 \leq s \leq 2\pi, 0 \leq t \leq \frac{\pi}{2})$ such that*

$$A^\alpha = \{(f_1^\alpha(s), f_2^\alpha(s,t), f_3^\alpha(s,t)) \in \mathbb{R}^3 | 0 \leq s \leq 2\pi, -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}\}.$$

Definition 7. *Let A and B are two continuous convex fuzzy numbers defined on $\mathbb{R}^3$ and*

$$\begin{aligned} A^\alpha &= \{(x,y,z) \in \mathbb{R}^3 | \mu_A(x,y,z) = \alpha\} \\ &= \{(f_1^\alpha(s), f_2^\alpha(s,t), f_3^\alpha(s,t)) \in \mathbb{R}^3 | 0 \leq s \leq 2\pi, -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}\}, \\ B^\alpha &= \{(x,y,z) \in \mathbb{R}^3 | \mu_B(x,y,z) = \alpha\} \\ &= \{(g_1^\alpha(s), g_2^\alpha(s,t), g_3^\alpha(s,t)) \in \mathbb{R}^3 | 0 \leq s \leq 2\pi, -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}\} \end{aligned}$$

be the α-sets of A and B, respectively. For $\alpha \in (0,1)$, we define the parametric addition, parametric subtraction, parametric multiplication, and parametric division of two fuzzy numbers A and B as fuzzy numbers with α-sets as follows:

(1) parametric addition $A(+)_pB$:

$$\begin{aligned} (A(+)_pB)^\alpha = \{(f_1^\alpha(s) + g_1^\alpha(s),\ f_2^\alpha(s,t) + g_2^\alpha(s,t), f_3^\alpha(s,t) + g_3^\alpha(s,t)) \in \mathbb{R}^3 | \\ 0 \leq s \leq 2\pi, -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}\} \end{aligned}$$

(2) parametric subtraction $A(-)_pB$:

$$\begin{aligned} (A(-)_pB)^\alpha = \{(&f_1^\alpha(s) - g_1^\alpha(s+\pi),\ f_2^\alpha(s,t) - g_2^\alpha(s+\pi,t), \\ &f_3^\alpha(s,t) - g_3^\alpha(s+\pi,t)) \in \mathbb{R}^3 | 0 \leq s \leq \pi, -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}\}, \end{aligned}$$

$$\begin{aligned} (A(-)_pB)^\alpha = \{(&f_1^\alpha(s) - g_1^\alpha(s-\pi),\ f_2^\alpha(s,t) - g_2^\alpha(s-\pi,t), \\ &f_3^\alpha(s,t) - g_3^\alpha(s-\pi,t)) \in \mathbb{R}^3 | \pi \leq s \leq 2\pi, -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}\} \end{aligned}$$

(3) parametric multiplication $A(\cdot)_pB$:

$$\begin{aligned} (A(\cdot)_pB)^\alpha = \{(f_1^\alpha(s) \cdot g_1^\alpha(s),\ f_2^\alpha(s,t) \cdot g_2^\alpha(s,t),\ f_3^\alpha(s,t) \cdot g_3^\alpha(s,t)) \in \mathbb{R}^3 | \\ 0 \leq s \leq 2\pi, -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}\} \end{aligned}$$

(4) parametric division $A(/)_pB$:

$$(A(/)_pB)^\alpha = \{(\frac{f_1^\alpha(s)}{g_1^\alpha(s+\pi)}, \frac{f_2^\alpha(s,t)}{g_2^\alpha(s+\pi,t)}, \frac{f_3^\alpha(s,t)}{g_3^\alpha(s+\pi,t)}) \in \mathbb{R}^3 | \\ 0 \leq s \leq \pi, -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}\},$$

$$(A(/)_pB)^\alpha = \{(\frac{f_1^\alpha(s)}{g_1^\alpha(s-\pi)}, \frac{f_2^\alpha(s,t)}{g_2^\alpha(s-\pi,t)}, \frac{f_3^\alpha(s,t)}{g_3^\alpha(s-\pi,t)}) \in \mathbb{R}^3 | \\ \pi \leq s \leq 2\pi, -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}\}$$

For $\alpha = 0$ and $\alpha = 1$, $(A(*)_pB)^0 = \lim_{\alpha\to 0^+}(A(*)_pB)^\alpha$ and $(A(*)_pB)^1 = \lim_{\alpha\to 1^-}(A(*)_pB)^\alpha$, where $* = +,\ -,\ \cdot,\ /$.

Theorem 5. *Let $A = [a_1, x_1, b_1, y_1, c_1, z_1]^3$ and $B = [a_2, x_2, b_2, y_2, c_2, z_2]^3$ be two 3-dimensional quadratic fuzzy numbers. Subsequently, the following results hold:*

(1) $A(+)_pB = \Big[a_1 + a_2, x_1 + x_2, b_1 + b_2, y_1 + y_2, c_1 + c_2, z_1 + z_2\Big]^3$

(2) $A(-)_pB = \Big[a_1 + a_2, x_1 - x_2, b_1 + b_2, y_1 - y_2, c_1 + c_2, z_1 - z_2\Big]^3$

(3) $(A(\cdot)_pB)^\alpha = \{(x_\alpha(s), y_\alpha(s,t), z_\alpha(s,t)) \in \mathbb{R}^3 \mid 0 \leq s \leq 2\pi, -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}\}$, *where*

$$x_\alpha(s) = x_1x_2 + (x_1a_2 + x_2a_1)\sqrt{1-\alpha}\cos s + a_1a_2(1-\alpha)\cos^2 s,$$

$$y_\alpha(s,t) = y_1y_2 + (y_1b_2 + y_2b_1)\sqrt{1-\alpha}\sin s\cos t + b_1b_2(1-\alpha)\sin^2 s\cos^2 t$$

and

$$z_\alpha(s,t) = z_1z_2 + (z_1c_2 + z_2c_1)\sqrt{1-\alpha}\sin s\sin t + c_1c_2(1-\alpha)\sin^2 s\sin^2 t.$$

(4) $(A(/)_pB)^\alpha = \{(x_\alpha(s), y_\alpha(s,t), z_\alpha(s,t)) \in \mathbb{R}^3 \mid 0 \leq s \leq 2\pi, -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}\}$, *where*

$$x_\alpha(s) = \frac{x_1 + a_1\sqrt{1-\alpha}\cos s}{x_2 - a_2\sqrt{1-\alpha}\cos s} \qquad y_\alpha(s,t) = \frac{y_1 + b_1\sqrt{1-\alpha}\sin s\cos t}{y_2 - b_2\sqrt{1-\alpha}\sin s\cos t}$$

and

$$z_\alpha(s,t) = \frac{z_1 + c_1\sqrt{1-\alpha}\sin s\sin t}{z_2 - c_2\sqrt{1-\alpha}\sin s\sin t}.$$

Therefore, $A(+)_pB$ and $A(-)_pB$ become 3-dimensional quadratic fuzzy numbers, while $A(\cdot)_pB$ and $A(/)_pB$ do not qualify as 3-dimensional quadratic fuzzy numbers.

Proof. Since A and B are continuous convex fuzzy numbers defined on $\mathbb{R}^3$, by Theorem 4, there exists $f_1^\alpha(s), g_1^\alpha(s), f_i^\alpha(s,t), g_i^\alpha(s,t)$ $(i = 2, 3)$ such that

$$A^\alpha = \{(f_1^\alpha(s), f_2^\alpha(s,t), f_3^\alpha(s,t)) \in \mathbb{R}^3 | 0 \leq s \leq 2\pi, -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}\},$$

and

$$B^\alpha = \{(g_1^\alpha(s), g_2^\alpha(s,t), g_3^\alpha(s,t)) \in \mathbb{R}^3 | 0 \le s \le 2\pi, -\frac{\pi}{2} \le t \le \frac{\pi}{2}\}.$$

Since $A = [a_1, x_1, b_1, y_1, c_1, z_1]^3$ and $B = [a_2, x_2, b_2, y_2, c_2, z_2]^3$, we have

$$f_1^\alpha(s) = x_1 + a_1\sqrt{1-\alpha}\cos s, \quad f_2^\alpha(s,t) = y_1 + b_1\sqrt{1-\alpha}\sin s\cos t$$
$$f_3^\alpha(s,t) = z_1 + c_1\sqrt{1-\alpha}\sin s\sin t$$

and

$$g_1^\alpha(s) = x_2 + a_2\sqrt{1-\alpha}\cos s, \quad g_2^\alpha(s,t) = y_2 + b_2\sqrt{1-\alpha}\sin s\cos t$$
$$g_3^\alpha(s,t) = z_2 + c_2\sqrt{1-\alpha}\sin s\sin t.$$

(1) Since

$$f_1^\alpha(s) + g_1^\alpha(s) = x_1 + x_2 + (a_1 + a_2)\sqrt{1-\alpha}\cos s$$

$$f_2^\alpha(s,t) + g_2^\alpha(s,t) = y_1 + y_2 + (b_1 + b_2)\sqrt{1-\alpha}\sin s\cos t$$

and

$$f_3^\alpha(s,t) + g_3^\alpha(s,t) = z_1 + z_2 + (c_1 + c_2)\sqrt{1-\alpha}\sin s\sin t$$

we have

$$(A(+)_pB)^\alpha = \Big\{(x,y,z) \in \mathbb{R}^3 \Big| \frac{(x-x_1-x_2)^2}{(a_1+a_2)^2(1-\alpha)} + \frac{(y-y_1-y_2)^2}{(b_1+b_2)^2(1-\alpha)} + \frac{(z-z_1-z_2)^2}{(c_1+c_2)^2(1-\alpha)} = 1\Big\}.$$

Thus

$$A(+)_pB = \Big[a_1 + a_2,\ x_1 + x_2,\ b_1 + b_2,\ y_1 + y_2,\ c_1 + c_2,\ z_1 + z_2\Big]^3.$$

(2) If $0 \le s \le \pi$,

$$f_1^\alpha(s) - g_1^\alpha(s+\pi) = x_1 - x_2 + (a_1 + a_2)\sqrt{1-\alpha}\cos s$$

$$f_2^\alpha(s,t) - g_2^\alpha(s+\pi,t) = y_1 - y_2 + (b_1 + b_2)\sqrt{1-\alpha}\sin s\cos t$$

and

$$f_3^\alpha(s,t) - g_3^\alpha(s+\pi,t) = z_1 - z_2 + (c_1 + c_2)\sqrt{1-\alpha}\sin s\sin t.$$

In the case of $\pi \le s \le 2\pi$, we have

$$f_1^\alpha(s) - g_1^\alpha(s-\pi) = f_1^\alpha(s) - g_1^\alpha(s+\pi)$$

$$f_2^\alpha(s,t) - g_2^\alpha(s-\pi,t) = f_2^\alpha(s,t) - g_2^\alpha(s+\pi,t)$$

and

$$f_3^\alpha(s,t) - g_3^\alpha(s-\pi,t) = f_3^\alpha(s,t) - g_3^\alpha(s+\pi,t).$$

Thus

$$(A(-)_pB)^\alpha = \Big\{(x,y,z) \in \mathbb{R}^3 \Big| \frac{(x-x_1+x_2)^2}{(a_1+a_2)^2(1-\alpha)} + \frac{(y-y_1+y_2)^2}{(b_1+b_2)^2(1-\alpha)} + \frac{(z-z_1+z_2)^2}{(c_1+c_2)^2(1-\alpha)} = 1\Big\},$$

i.e.,

$$A(-)_pB = \Big[a_1+a_2,\ x_1-x_2,\ b_1+b_2,\ y_1-y_2,\ c_1+c_2,\ z_1-z_2\Big]^3.$$

(3) Let $(A(\cdot)_pB)^\alpha = \{(x_\alpha(s), y_\alpha(s,t), z_\alpha(s,t)) \mid 0 \le s \le 2\pi, -\frac{\pi}{2} \le t \le \frac{\pi}{2}\}$. Since

$$f_1^\alpha(s) = x_1 + a_1\sqrt{1-\alpha}\cos s, \quad f_2^\alpha(s,t) = y_1 + b_1\sqrt{1-\alpha}\sin s\cos t,$$
$$f_3^\alpha(s,t) = z_1 + c_1\sqrt{1-\alpha}\sin s\sin t$$

and

$$g_1^\alpha(s) = x_2 + a_2\sqrt{1-\alpha}\cos s, \quad g_2^\alpha(s,t) = y_2 + b_2\sqrt{1-\alpha}\sin s\cos t,$$
$$g_3^\alpha(s,t) = z_2 + c_2\sqrt{1-\alpha}\sin s\sin t,$$

we have

$$x_\alpha(s) = f_1^\alpha(s)\cdot g_1^\alpha(s) = x_1x_2 + (x_1a_2 + x_2a_1)\sqrt{1-\alpha}\cos s + a_1a_2(1-\alpha)\cos^2 s,$$
$$y_\alpha(s,t) = f_2^\alpha(s,t)\cdot g_2^\alpha(s,t) = y_1y_2 + (y_1b_2 + y_2b_1)\sqrt{1-\alpha}\sin s\cos t + b_1b_2(1-\alpha)\sin^2 s\cos^2 t,$$
$$z_\alpha(s,t) = f_3^\alpha(s,t)\cdot g_3^\alpha(s,t) = z_1z_2 + (z_1c_2 + z_2c_1)\sqrt{1-\alpha}\sin s\sin t + c_1c_2(1-\alpha)\sin^2 s\sin^2 t.$$

(4) Let $(A(/)_pB)^\alpha = \{(x_\alpha(s), y_\alpha(s,t), z_\alpha(s,t)) \mid 0 \le s \le 2\pi, -\frac{\pi}{2} \le t \le \frac{\pi}{2}\}$. Similarly, we have

$$x_\alpha(s) = \frac{x_1 + a_1\sqrt{1-\alpha}\cos s}{x_2 - a_2\sqrt{1-\alpha}\cos s} \quad y_\alpha(s,t) = \frac{y_1 + b_1\sqrt{1-\alpha}\sin s\cos t}{y_2 - b_2\sqrt{1-\alpha}\sin s\cos t},$$
$$z_\alpha(s,t) = \frac{z_1 + c_1\sqrt{1-\alpha}\sin s\sin t}{z_2 - c_2\sqrt{1-\alpha}\sin s\sin t}.$$

The proof is complete.

Example 2. *Consider $A = [6,\ 3,\ 8,\ 5,\ 4,\ 7]^3$ and $B = [4,\ 2,\ 5,\ 3,\ 6,\ 4]^3$. Subsequently, the following observations hold:*

(1) $A(+)_pB = [10,\ 5,\ 13,\ 8,\ 10,\ 11]^3$

(2) $A(-)_pB = [10,\ 1,\ 13,\ 2,\ 10,\ 3]^3$

(3) $(A(\cdot)_pB)^\alpha = \{(x_\alpha(s), y_\alpha(s,t), z_\alpha(s,t)) \mid 0 \le s \le 2\pi, -\frac{\pi}{2} \le t \le \frac{\pi}{2}\}$*, where*

$$x_\alpha(s) = 6 + 24\sqrt{1-\alpha}\cos s + 24(1-\alpha)\cos^2 s,$$
$$y_\alpha(s,t) = 15 + 49\sqrt{1-\alpha}\sin s\cos t + 40(1-\alpha)\sin^2 s\cos^2 t$$

and

$$z_\alpha(s,t) = 28 + 58\sqrt{1-\alpha}\sin s\sin t + 24(1-\alpha)\sin^2 s\sin^2 t.$$

(4) $(A(/)_pB)^\alpha = \{(x_\alpha(s), y_\alpha(s,t), z_\alpha(s,t)) \mid 0 \le s \le 2\pi, -\frac{\pi}{2} \le t \le \frac{\pi}{2}\}$*, where*

$$x_\alpha(s) = \frac{3 + 6\sqrt{1-\alpha}\cos s}{2 - 4\sqrt{1-\alpha}\cos s} \quad y_\alpha(s,t) = \frac{5 + 8\sqrt{1-\alpha}\sin s\cos t}{3 - 5\sqrt{1-\alpha}\sin s\cos t}$$

and

$$z_\alpha(s,t) = \frac{7 + 4\sqrt{1-\alpha}\sin s\sin t}{4 - 6\sqrt{1-\alpha}\sin s\sin t}.$$

Thus $A(+)_pB$ *and* $A(-)_pB$ *become 3-dimensional quadratic fuzzy numbers, but* $A(\cdot)_pB$ *and* $A(/)_pB$ *are not 3-dimensional quadratic fuzzy numbers.*

The membership function of the 3-dimensional quadratic fuzzy number is a function defined on $\mathbb{R}^3$ with values in $[0,1]$. In the case of the 3-dimensional quadratic fuzzy numbers $A = [6,\ 3,\ 8,\ 5,\ 4,\ 7]^3$ and $B = [4,\ 2,\ 5,\ 3,\ 6,\ 4]^3$, we depict the values of the membership function using colors, as illustrating in Figure 1 and Figure 2.

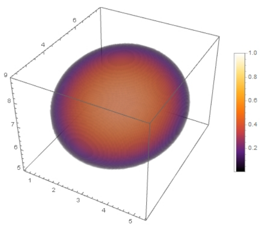

Figure 1: A

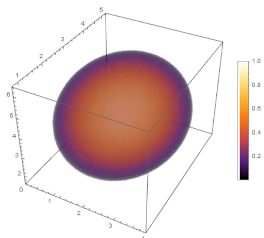

Figure 2: B

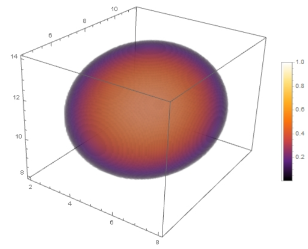

Figure 3: $A(+)_pB$

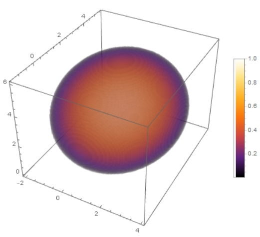

Figure 4: $A(-)_pB$

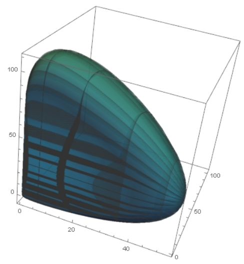

Figure 5: $A(\cdot)_pB$

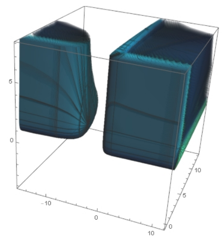

Figure 6: $A(/)_pB$

The results of the example are depicted in Figures 3 to 6. The membership function values for points within $A(+)_pB$ and $A(-)_pB$, when intersected by a plane, are demonstrated in Figures 7 to 12 and Figures 13 to 21, respectively. Although all graphs appear similar in shape, a closer look at the bar graph showing the function values next to each figure uncovers notable differences. It becomes clear that the function values are not consistent, unlike what is typically seen in one or two dimensions.

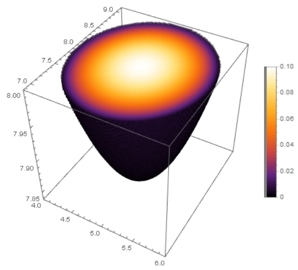

Fig. 7: $A(+)_pBz \leq 8$

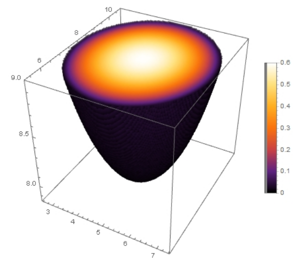

Fig. 8: $A(+)_pBz \leq 9$

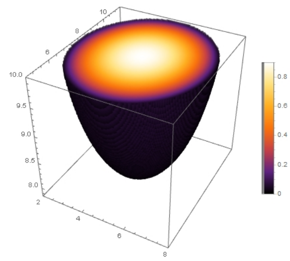

Fig. 9: $A(+)_pBz \leq 10$

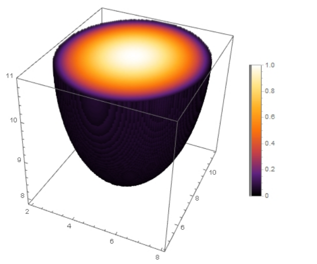

Fig. 10: $A(+)_pBz \leq 11$

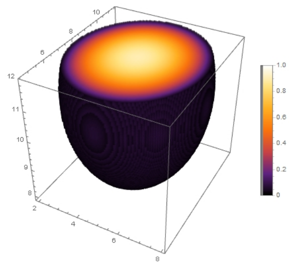

Fig. 11: $A(+)_pBz \leq 12$

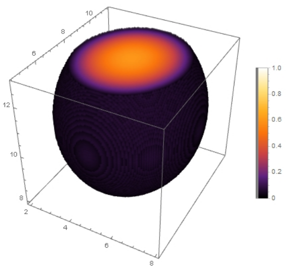

Fig. 12: $A(+)_pBz \leq 13$

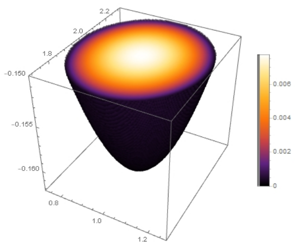

Fig. 13: $A(-)_pBz \leq -0.15$

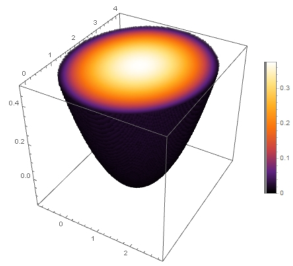

Fig. 14: $A(-)_pBz \leq 0.5$

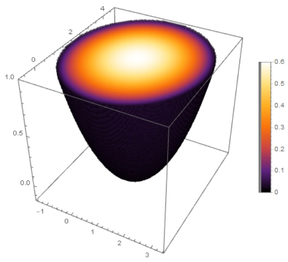

Fig. 15: $A(-)_pBz \leq 1$

5. Conclusion

We are broadening the scope of quadratic fuzzy numbers from a two-dimensional space $\mathbb{R}^2$ to a three-dimensional space $\mathbb{R}^3$. By establishing parametric operations between two α-cuts, which are subsets of $\mathbb{R}^3$, we are able to formulate parametric operations for two quadratic fuzzy numbers within the $\mathbb{R}^3$ space. The significance of this dimensional expansion lies in its incorporation of Zadeh's defined max-min operation in two dimensions [10]. Moreover, as long as the computations of these operations remain consistent, this dimensional expansion is expected to further the research in fractional programming in the future [1].

In [2], the results of quadratic fuzzy numbers in two dimensions have been detailed. When extended to three dimensions, the operations $A(+)B$ and $A(-)B$ evolve into 3-dimensional quadratic fuzzy numbers. However, this transformation does not apply to $A(\cdot)B$ and $A(/)B$. The inherent well-structured nature of $A(+)B$ and $A(-)B$ allows them to be utilized in a wide range of fields without needing any alterations. Conversely, by modifying the forms of $A(\cdot)B$ and $A(/)B$, they can be adapted for use in various applications.

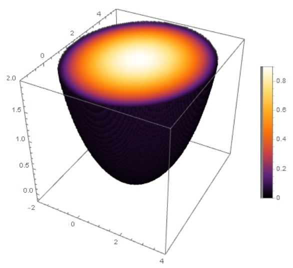

Fig. 16: $A(-)_pBz \leq 2$

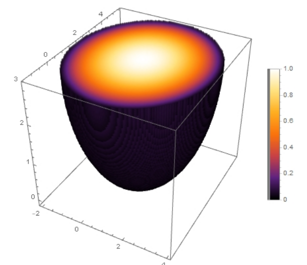

Fig. 17: $A(-)_pBz \leq 3$

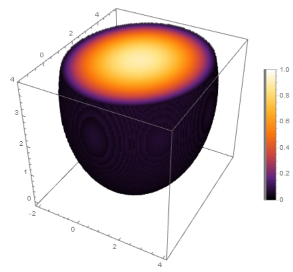

Fig. 18: $A(-)_pBz \leq 4$

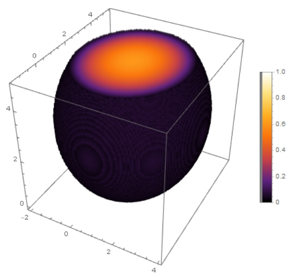

Fig. 19: $A(-)_pBz \leq 5$

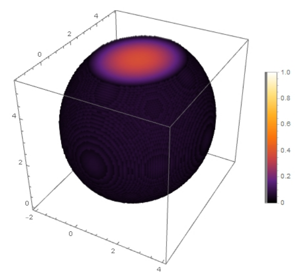

Fig. 20: $A(-)_pBz \leq 5.5$

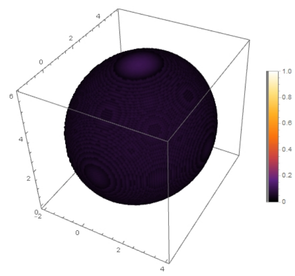

Fig. 21: $A(-)_pBz \leq 6$

This result can be applied to demonstrate that the 3-dimensional case is a generalization of the 2-dimensional case. While there have been various attempts to expand the dimension, no studies have successfully achieved expansion while preserving Zadeh's results for 1 and 2-dimensional quadratic fuzzy numbers. This paper aims to contribute to the advancement of applications of quadratic fuzzy numbers by extending their dimension. The application scope of this paper includes solving the 3-dimensional flow shop scheduling problem and quadratic fuzzy equations [5, 11], with expectations of further applications across numerous fields.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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Appendix

The Mathematica commands to obtain the above graphs are as follows.

(Figure 1)

```
DensityPlot3D[1 - ((x - 3)^2/6 + (y - 5)^2/8 + (z - 7)^2/4), {x, y, z} in
Ellipsoid[{3, 5, 7}, {Sqrt[6], Sqrt[8], 2}], PlotPoints -> 100, ColorFunct
ion -> "SunsetColors", OpacityFunction -> 0.05, BoxRatios -> {Sqrt[6], Sqr
t[8], 2}, PlotLegends -> Automatic]
```

(Figure 3)

```
DensityPlot3D[1 - ((x - 5)^2/10 + (y - 8)^2/13 + (z - 11)^2/10), {x, y, z}
in Ellipsoid[{5, 8, 11}, {Sqrt[10], Sqrt[13], Sqrt[10]}], PlotPoints -> 10
0, ColorFunction -> "SunsetColors", OpacityFunction -> 0.05, BoxRatios ->
{Sqrt[10], Sqrt[13], Sqrt[10]}, PlotLegends -> Automatic]
```

(Figure 5)

```
g[a_] := ParametricPlot3D[{6 + 24 Sqrt[1 - a] Cos[s] + 24 (1 - a) (Cos[s])
^2, 15 + 49 Sqrt[1 - a]  Sin[s] Cos[t] + 40 (1 - a) (Sin[s])^2 (Cos[t])^2,
28 + 58 Sqrt[1 - a]  Sin[s] Sin[t] + 24(1 - a) (Sin[s])^2 (Sin[t])^2}, {s,
0, 2 Pi}, {t, -Pi/2, Pi/2}, PlotStyle -> Directive[RGBColor[0.2, 0.5 + a/2
, 0.5 + a/2], Opacity[0.3]], BoxRatios -> {1, 1, 1}];
tg = Table[g[i], {i, 0, 1.0, 0.01}];
Show[tg]
```

(Figure 7)

```
reg1 = ImplicitRegion[0 <= (x - 5)^2/10 + (y - 8)^2/13 + (z - 11)^2/10 <=
1 && z <= 8, {x, y, z}];DensityPlot3D[1 - ((x - 5)^2/10 + (y - 8)^2/13 + (
z - 11)^2/10), {x, y, z} in reg1, PlotPoints -> 100, ColorFunction -> "Sun
setColors", OpacityFunction -> 1, BoxRatios -> {Sqrt[10], Sqrt[13], Sqrt[1
0]}, PlotLegends -> Automatic]
```

(Figure 10)

```
reg1 = ImplicitRegion[0 <= (x - 5)^2/10 + (y - 8)^2/13 + (z - 11)^2/10 <=
1 && z <= 11, {x, y, z}];DensityPlot3D[1 - ((x - 5)^2/10 + (y - 8)^2/13 +
(z - 11)^2/10), {x, y, z} in reg1, PlotPoints -> 100, ColorFunction -> "Su
nsetColors", OpacityFunction -> 1, BoxRatios -> {Sqrt[10], Sqrt[13], Sqrt[
10]}, PlotLegends -> Automatic]
```

(Figure 13)

```
reg1 = ImplicitRegion[0 <= (x - 1)^2/10 + (y - 2)^2/13 + (z - 3)^2/10 <= 1
&& z <= -0.15, {x, y, z}];DensityPlot3D[1 - ((x - 1)^2/10+ (y - 2)^2/13 +
(z - 3)^2/10),{x, y, z} in reg1, PlotPoints -> 100,  ColorFunction -> "Sun
setColors", OpacityFunction -> 1, BoxRatios -> {Sqrt[10], Sqrt[13], Sqrt[1
0]}, PlotLegends -> Automatic]
```

(Figure 16)

```
reg1 = ImplicitRegion[0 <= (x - 1)^2/10 + (y - 2)^2/13 + (z - 3)^2/10 <= 1
&& z <= 2, {x, y, z}];DensityPlot3D[1 - ((x - 1)^2/10+ (y - 2)^2/13 + (z -
3)^2/10),{x, y, z} in reg1, PlotPoints -> 100,  ColorFunction -> "SunsetCo
lors", OpacityFunction -> 1, BoxRatios -> {Sqrt[10], Sqrt[13], Sqrt[10]},
PlotLegends -> Automatic]
```

(Figure 19)

```
reg1 = ImplicitRegion[0 <= (x - 1)^2/10 + (y - 2)^2/13 + (z - 3)^2/10 <= 1
&& z <= 5, {x, y, z}];DensityPlot3D[1 - ((x - 1)^2/10+ (y - 2)^2/13 + (z -
3)^2/10),{x, y, z} in reg1, PlotPoints -> 100,  ColorFunction -> "SunsetCo
lors", OpacityFunction -> 1, BoxRatios -> {Sqrt[10], Sqrt[13], Sqrt[10]},
PlotLegends -> Automatic]
```