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On Total Double Italian Domination in Graphs

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Abstract. For a simple graph $G = (V(G), E(G))$, a total double Italian dominating function is a function $f : V(G) \to \{0, 1, 2, 3\}$ with properties that every vertex $v \in V(G)$ with $f(v) \in \{0, 1\}$, $\sum_{u \in N[v]} f(u) \geq 3$ and every vertex $v \in V(G)$ with $f(v) \neq 0$ has a neighbor u with $f(u) \neq 0$. The weight of a total double Italian dominating function is the sum $\omega_G(f) = \sum_{v \in V(G)} f(v) \geq 3$ and the minimum weight of all the total double Italian dominating functions on a graph G is the total double Italian domination number, denoted by $\gamma_{tdI}(G)$. In this paper we explore further the concept of total double Italian domination. We characterize graphs G with smaller values for $\gamma_{tdI}(G)$. Also, we characterize the total double Italian dominating function on the join, corona, edge corona, and complementary prism of graphs. Exact values or bounds are also determined for their respective total double Italian domination number.

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Key Words and Phrases: Total double Italian dominating function, Total double Italian domination number.

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1. Introduction

Since its introduction in 2004 by Cockayne et al. [12], Roman domination is one of the most well-studied concepts in graph theory. For a comprehensive understanding of its origins, historical development, and significance in the field along with recent advances, we refer to [1, 2, 10, 13, 16–19, 21, 22, 24, 25]. Building on the foundations of Roman domination, Chellali et al. [13] introduced a broader concept known as Italian domination (also referred as Roman-{2} domination). Meanwhile, Beeler et al. [10] extended the idea even further by developing the notion of double Roman domination, a stronger variant that inspires new research in the field.

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In 2020, Mojdeh et al. [19] introduced the concept of double Italian domination (or Roman {3}-domination) which is an optimization of the double Roman domination. In the same year, Shao et al. [26] initiated the study of total double Italian domination and have shown its relationship to other domination parameters.

This present paper investigates further the total double Italian domination, particularly in graphs under the join, corona, edge corona and complementary prism of graphs.

Throughout this paper, all graphs considered are undirected, finite and simple. See [3, 4, 9] for all the basic graph terminologies that are not defined but used in this paper.

For a graph $G = (V(G), E(G))$, the *open neighborhood* of a vertex $v \in V(G)$, denoted by $N_G(v)$, consists of all the vertices adjacent to v and its *closed neighborhood*, denoted by $N_G[v]$, is the open neighborhood of v together with vertex v. The *degree* of v, denoted by $deg_G(v)$, $deg_G(v) = |N_G(v)|$. The *minimum degree*, $\delta(G)$ of G is the minimum degree among the vertices of G. The *maximum degree* of G, denoted by $\Delta(G)$, is the maximum degree among the vertices of G. For $S \subseteq V(G)$, $N_G(S) = \cup_{v\in S} N_G(v)$ and $N_G[S] = S \cup N_G(S)$.

Let G and H be graphs with disjoint vertex sets. The *join* of graphs G and H is the graph $G + H$ with vertex set $V(G + H) = V(G) \cup V(H)$ and edge set $E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G) \wedge v \in V(H)\}$. The *corona* of G and H, $G \circ H$, is the graph obtained by taking one copy of G and $|V(G)|$ copies of H and then joining the *ith* vertex of G to every vertex of the *ith* copy of H. The *edge corona*, denoted by $G \diamond H$, of G and H is a graph obtained by taking one copy of G and $|E(G)|$ copies of H and joining each of the end vertices u and v of each edge uv of G to every vertex of the copy H^{uv} of H. The *complementary prism*, denoted $G\overline{G}$, is formed from the disjoint union of G and its complement $\overline{G}$ by adding a perfect matching between corresponding vertices of G and $\overline{G}$. The *gluing* of G and H along a common subgraph K is the graph $G \sqcup_K H$ by combining G and H through K. Graphs $C_4 \sqcup_{P_3} C_4$ and $C_4 \sqcup_{K_2} K_3$ are given in Figure 1. We refer to [5] for a detailed information on the gluing of graphs.

A set $S \subseteq V(G)$ is a *dominating set* of G if $N_G[S] = V(G)$. The *domination number* of G, denoted by $\gamma(G)$, is the smallest cardinality of a dominating set of G. A set S of vertices in a graph G is called a *total dominating set* if $N_G(S) = V(G)$. The *total domination number* $\gamma_t(G)$ of G is the minimum cardinality of a total dominating set of vertices in G. A dominating set of G of cardinality $\gamma(G)$ is referred to as γ-*set* of G. A total dominating set of G of cardinality $\gamma_t(G)$ is called a γ_t-*set* of G. We refer to [6, 11, 14, 15, 20] for the fundamental concepts, some recent developments and applications of domination and total domination in graphs.

For a positive integer k, a set $D \subseteq V(G)$ is called a k-*dominating set* if each $v \in V(G) \setminus D$ is adjacent to at least k vertices in D. The k-*domination number* $\gamma_k(G)$ is then defined to be the smallest cardinality of a k-dominating set of G. M. Chellali et al. in [8] presented an outstanding survey of results in k-domination in graphs.

A subset $S \subseteq V(G)$ is a *vertex cover* of G if for every edge $uv \in E(G)$, $u \in S$ or $v \in S$.

The smallest cardinality of a vertex cover is the *vertex cover number* of G, and is denoted by $\beta(G)$. Excellent references for vertex cover include [7, 23].

A *double Italian dominating function* (or $DIDF$) of G is a function $f : V(G) \to \{0,1,2,3\}$ having the property that for every vertex $v \in V(G)$, if $f(v) \in \{0,1\}$, then $\sum_{u \in N[v]} f(u) \geq 3$. The weight of a $DIDF$ is the sum $\omega_G(f) = \sum_{v \in V(G)} f(v)$, and the minimum weight of a $DIDF$ f is the *double Italian domination number*, denoted by $\gamma_{dI}(G)$. A $DIDF$ function $f : V(G) \to \{0,1,2,3\}$ is a *total double Italian dominating function* (or $TDIDF$) of G if for each $v \in V(G)$ with $f(v) \neq 0$, there exists $u \in V(G)$ such that $f(u) \neq 0$ and $uv \in E(G)$. The minimum weight of a $TDIDF$ of G is the *total double Italian domination number* of G, and is denoted by $\gamma_{tdI}(G)$. We write $f \in TDIDF(G)$ to mean that f is $TDIDF$ of G. Any $TDIDF$ f of G with weight $\gamma_{tdI}(G)$ is referred to as γ_{tdI}-function of G.

For a function $f : V(G) \to \{0,1,2,3\}$, let (V_0, V_1, V_2, V_3) be the ordered partition induced by f, where $V_i = \{v \in V(G) : f(v) = i\}$ for $i \in \{0,1,2,3\}$. Then we can write $f = (V_0, V_1, V_2, V_3)$. The weight of f is defined by $\omega_G(f) = |V_1| + 2|V_2| + 3|V_3|$.

More precisely, $f = (V_0, V_1, V_2, V_3) \in TDIDF(G)$ if each of the following holds:

(i) For each $v \in V_0$, at least one of the following holds:

- (a) $|V_1 \cap N_G(v)| \geq 3$;
- (b) $|V_1 \cap N_G(v)| \geq 1$ and $|V_2 \cap N_G(v)| \geq 1$;
- (c) $|V_2 \cap N_G(v)| \geq 2$;
- (d) $|V_3 \cap N_G(v)| \geq 1$.

(ii) For each $v \in V_1$, at least one of the following holds:

- (a) $|V_1 \cap N_G(v)| \geq 2$;
- (b) $|(V_2 \cup V_3) \cap N_G(v)| \geq 1$.

(iii) For each $v \in V_1 \cup V_2 \cup V_3$, $|(V_1 \cup V_2 \cup V_3) \cap N_G(v)| \geq 1$.

2. Preliminary results

Proposition 1. [26] *If G is a connected graph of order $n \geq 2$, then $\gamma_{tdI}(G) \geq 3$ and $\gamma_{tdI}(G) = 3$ if and only if G has at least two vertices of degree $\Delta(G) = n - 1$.*

Proposition 2. [26] *If G has only one vertex of degree $\Delta(G) = n - 1$, then $\gamma_{tdI}(G) = 4$.*

Theorem 1. [26] *If G is a graph with $\delta(G) = \delta \geq 2$, then $\gamma_{tdI}(G) \leq |V(G)| + 2 - \delta$, and this bound is sharp.*

Proposition 3. *Let G be a nontrivial connected graph of order $n \geq 4$. Then $\gamma_{tdI}(G) = 4$ if and only if one of the following holds:*

(i) *G has exactly one vertex of degree $\Delta(G) = n - 1$;*

(ii) *$\gamma(G) \geq 2$ and G has a 3-dominating set D with $|D| = 4$ and $\delta(\langle D\rangle) \geq 2$.*

Proof: Suppose that $\gamma_{tdI}(G) = 4$. If $\gamma(G) = 1$, then by Proposition 1 and Proposition 2, G contains exactly one vertex of degree $n - 1$, and (i) holds. Suppose that $\gamma(G) \geq 2$. Let $f = (V_0, V_1, V_2, V_3)$ be a γ_{tdI}-function of G. Since $\gamma(G) \neq 1$, $V_2 = V_3 = \varnothing$ and $|V_1| = 4$. Put $D = V_1$. Then $\delta(\langle D\rangle) \geq 2$. If $V_0 = \varnothing$, then $G = C_4$. If $V_0 \neq \varnothing$, then $3 \leq |N_G(v) \cap V_1| \leq 4$ for every $v \in V_0$. In any case, D is a 3-dominating set of G. Thus, (ii) holds.

Conversely, if (i) holds, then the desired result follows from Proposition 2. Suppose (ii) holds. By Proposition 1 and Proposition 2, $\gamma_{tdI}(G) \geq 4$. On the other hand, since $f = (V(G) \setminus D, D, \varnothing, \varnothing)$ is a total double Italian dominating function on G, $\gamma_{tdI}(G) \leq |D| = 4$. Therefore, $\gamma_{tdI}(G) = 4$. ■

In Statement (ii) of Proposition 3, it is not necessary that $\gamma_3(G) = 4$. To see this, note that if $G = C_4 + \overline{K_3}$, then $\gamma_{tdI}(G) = 4$ while $\gamma_3(G) = 3$.

It can be verified that if G is connected of order $n = 4$, then $\gamma_{tdI}(G) \neq 5$.

Proposition 4. *Let G be a nontrivial connected graph of order $n \geq 5$. Then $\gamma_{tdI}(G) = 5$ if and only if one of the following holds:*

(i) *$G \in \{C_5, K_{2,3}, K_3 \sqcup_{K_2} C_4, \overline{K_2} + (K_1 \cup K_2)\}$ (see Figure 1);*

(ii) *$n > 5$, $\gamma(G) \geq 2$, G does not contain a 3-dominating set D with $|D| = 4$ and $\delta(\langle D\rangle) \geq 2$, and one of the following holds:*

- (a) *G contains a 3-dominating set D with $|D| = 5$ and $\delta(\langle D\rangle) \geq 2$;*
- (b) *G contains a 2-dominating set D with $|D| = 3$ and $\langle D\rangle$ is connected.*
- (c) *G contains a 2-dominating set D with $|D| = 4$ such that there exists $v \in D$ for which $uv \in E(G)$ for all $u \in V(G) \setminus D$ with $|N_G(u) \cap D| = 2$. Moreover, $\langle D\rangle$ is connected and $xv \in E(G)$ for every $x \in D \setminus \{v\}$ with $|N_G(x) \cap D| = 1$.*

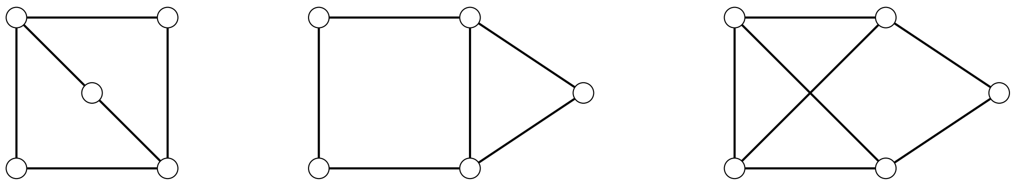
Proof: Suppose that $\gamma_{tdI}(G) = 5$. By Proposition 1 and Proposition 3, $\gamma(G) \geq 2$ and G does not contain a 3-dominating set D with $|D| = 4$ and $\delta(\langle D\rangle) \geq 2$. If $n = 5$, then $G \in \{C_5, K_{2,3}, K_3 \sqcup_{K_2} C_4, \overline{K_2} + (K_1 \cup K_2)\}$. Assume that $n > 5$. Let $f = (V_0, V_1, V_2, V_3)$ be a γ_{tdI}function of G. By Proposition 1 and Proposition 3, $V_3 = \varnothing$. Consider the following cases:

Case 1: Suppose that $|V_1| = 5$ and $|V_2| = 0$. Since $n > 5$, $V_0 \neq \varnothing$. Then $3 \leq |N_G(u) \cap V_1| \leq 5$ for every $u \in V_0$. Hence, $D = V_1$ is a 3-dominating set on G. Since $|N_G(v) \cap V_1| \geq 2$ for every $v \in V_1$, $\delta(\langle D\rangle) \geq 2$ and (ii)(a) holds.

Case 2: Suppose that $|V_1| = 1$ and $|V_2| = 2$. Since $n > 5$, $V_0 \neq \varnothing$. Thus $D = V_1 \cup V_2$ is a 2-dominating set of G. Moreover, since $f \in TIDF(G)$, $\langle D\rangle$ is connected. Therefore, (ii)(b) holds.

Case 3: Suppose that $|V_1| = 3$ and $|V_2| = 1$. Put $V_2 = \{v\}$. Then for each $u \in V_0$, either $|N(u) \cap V_1| = 3$ or $1 \leq |N(u) \cap V_1| \leq 2$ and $uv \in E(G)$. Hence, $|N(u) \cap (V_1 \cup V_2)| \geq 2$. Thus, $D = V_1 \cup V_2$ is a 2-dominating set of G. Observe that, if $|N(u) \cap (V_1 \cup V_2)| = 2$, then $uv \in E(G)$. By the definition of f, $\langle V_1 \cup V_2 \rangle$ has no isolated vertex. If $u' \in V_1$ such that $d_{\langle S \rangle}(u') = 1$, then $u'v \in E(G)$. Thus, $(ii)(c)$ holds.

Conversely, if $G \in \{C_5, K_{2,3}, K_3 \sqcup_{K_2} C_4, \overline{K_2} + (K_1 \cup K_2)\}$, then $\gamma_{tdI}(G) = 5$. Now, suppose that $n > 5$, $\gamma(G) \geq 2$, G does not contain a 3-dominating set D for which $|D| = 4$ and $\delta(\langle D \rangle) \geq 2$. Then by Proposition 3, $\gamma_{tdI}(G) \geq 5$. Assume that $(ii)(a)$ holds. Let $V_0 = V(G) \setminus D$, $V_1 = D$, $V_2 = \varnothing$, and $V_3 = \varnothing$. Then $f = (V_0, V_1, V_2, V_3)$ is a $TDIDF$ on G with $\omega_G(f) = 5$. This means that $\gamma_{tdI}(G) = 5$. Assume $(ii)(b)$ holds. Take $x \in D$. Let $V_0 = V(G) \setminus D$, $V_1 = \{x\}$, $V_2 = D \setminus \{x\}$, and $V_3 = \varnothing$. Then $f = (V_0, V_1, V_2, V_3)$ is a $TDIDF$ on G with $\omega_G(f) = 5$. Hence, $\gamma_{tdI}(G) = 5$. Assume $(ii)(c)$ holds. Let $v \in D$. Then $f = (V_0, V_1, V_2, V_3)$ where $V_0 = V(G) \setminus D$, $V_1 = D \setminus \{v\}$, $V_2 = \{v\}$, and $V_3 = \varnothing$ is a $TDIDF$ on G with $\omega_G(f) = 5$. Thus, $\gamma_{tdI}(G) = 5$. ■



$C_4 \sqcup_{P_3} C_4 = K_{3,2}$ $K_3 \sqcup_{K_2} C_4$ $\overline{K_2} + (K_1 \cup K_2)$

Figure 1: Examples of graphs G with $\gamma_{tdI}(G) = 5$

3. On the join of graphs

In this section, we denote by $f|_G$ the restriction of the function f on the subgraph G of a graph H. The following proposition characterizes all $TDIDF$ on the join of two nontrivial connected graphs.

Proposition 5. *Let G and H be nontrivial connected graphs. Then $f = (V_0, V_1, V_2, V_3)$ is a TDIDF on $(G + H)$ if and only if one of the following holds:*

(i) $f|_G \in TDIDF(G)$;

(ii) $f|_H \in TDIDF(H)$;

(iii) $f|_G \notin TDIDF(G)$, $f|_H \notin TDIDF(H)$, *and each of the following holds:*

- *(a) For every* $v \in V_0 \cup V_1$,
  - (1) $\omega_H(f|_H) \geq 3 - f|_G(N_G[v])$, *whenever* $v \in V(G)$ *and* $f|_G(N_G[v]) < 3$;
  - (2) $\omega_G(f|_G) \geq 3 - f|_H(N_H[v])$, *whenever* $v \in V(H)$ *and* $f|_H(N_H[v]) < 3$.
- *(b) For every* $v \in V_1 \cup V_2 \cup V_3$,

(1) $(V_1 \cup V_2 \cup V_3) \cap V(H) \neq \varnothing$, *whenever* $v \in V(G)$ *and* $N_G(v) \subseteq V_0$;

(2) $(V_1 \cup V_2 \cup V_3) \cap V(G) \neq \varnothing$, *whenever* $v \in V(H)$ *and* $N_H(v) \subseteq V_0$.

Proof: Let $f = (V_0, V_1, V_2, V_3)$ be a function on $V(G + H)$. Assume that (i) holds for f. Let $v \in (V_0 \cup V_1) \cap V(G + H)$. If $v \in V(G)$, then

$$3 \leq f|_G(N_G[v]) = \sum_{x \in N_G[v]} f(x) \leq \sum_{x \in N_{G+H}[v]} f(x) = f(N_{G+H}[v]).$$

Suppose that $v \in V(H)$. Since $f|_G$ is a $TDIDF$ on G, Proposition 1 implies that $\omega_G(f|_G) \geq 3$. Hence,

$$f(N_{G+H}[v]) = \sum_{x \in N_{G+H}[v]} f(x) = \sum_{x \in N_{G+H}[v] \setminus V(G)} f(x) + \sum_{x \in V(G)} f(x) \geq 3.$$

Now, let $v \in V_1 \cup V_2 \cup V_3$. If $v \in V(G)$, then since $f|_G \in TDIDF(G)$, there exists $u \in ((V_1 \cup V_2 \cup V_3) \cap V(G)) \setminus \{v\}$ such that $vu \in E(G) \subseteq E(G + H)$. If $v \in V(H)$, then $\varnothing \neq (V_1 \cup V_2 \cup V_3) \cap V(G) \subseteq N_{G+H}(v)$. Thus, $f \in TDIDF(G + H)$. Similarly, if condition (ii) holds for f, then $f \in TDIDF(G + H)$. Suppose (iii) holds. Let $v \in V_0 \cup V_1$. If $v \in V(G)$ such that $f|_G(N_G[v]) < 3$, then by $(iii)(a)$, $\omega_H(f|_H) \geq 3 - f|_G(N_G[v])$. Since $f(N_{G+H}[v]) = f|_G(N_G[v]) + \omega_H(f|_H)$, $f(N_{G+H}[v]) \geq 3$. Similarly, if $v \in V(H)$ with $f|_H(N_H[v]) < 3$ then $f(N_{G+H}[v]) \geq 3$. Since v is arbitrary, $f(N_{G+H}[v]) \geq 3$ for each $v \in V_0 \cup V_1$. Let $u \in V_1 \cup V_2 \cup V_3$. If $u \in V(G)$ with $N_G(u) \subseteq V_0$, then by $(iii)(b)$, $(V_1 \cup V_2 \cup V_3) \cap V(H) \neq \varnothing$. This means that $N_{G+H}(u) \cap (V_1 \cup V_2 \cup V_3) \neq \varnothing$. Similarly, $N_{G+H}(u) \cap (V_1 \cup V_2 \cup V_3) \neq \varnothing$ for each $u \in V(H)$ with $N_H(u) \subseteq V_0$. Thus, $\langle V_1 \cup V_2 \cup V_3 \rangle$ has no isolated vertex. Therefore, $f \in TDIDF(G + H)$.

Conversely, suppose that $f \in TDIDF(G + H)$. Suppose neither (i) nor (ii) holds for f, i.e., $f|_G \notin TDIDF(G+H)$ and $f|_H \notin TDIDF(G+H)$. Since $f|_G \notin TDIDF(G+H)$, either there exists $v \in [(V_0 \cup V_1) \cap V(G)]$ with $f|_G(N_G[v]) < 3$ or $\langle (V_1 \cup V_2 \cup V_3) \cap V(G) \rangle$ has an isolated vertex or both. Assume that there exists $v \in [(V_0 \cup V_1) \cap V(G)]$ with $f|_G(N_G[v]) < 3$. Since $f \in TDIDF(G+H)$, $\omega_H(f|_H) \geq 3 - f|_G(N_G[v])$. Thus, $(iii)(a(1))$ holds. Similarly, $(iii)(a(2))$ follows. On the other hand, assume that $\langle (V_1 \cup V_2 \cup V_3) \cap V(G) \rangle$ has an isolated vertex. Let $u \in [(V_1 \cup V_2 \cup V_3) \cap V(G)]$ such that $N_G(v) \subseteq V_0$. Since $\langle V_1 \cup V_2 \cup V_3 \rangle$ is isolated vertex-free, $(V_1 \cup V_2 \cup V_3) \cap V(H) \neq \varnothing$. Thus, $(iii)(b(1))$ holds. Similarly, $(iii)(b(2))$ follows. ■

Corollary 1. *Let G and H be nontrivial connected graphs. Then*

$$3 \leq \gamma_{tdI}(G + H) \leq \min\{6, \gamma_{tdI}(G), \gamma_{tdI}(H)\}. \tag{1}$$

Proof: The lower bound follows immediately from Propositon 1. To show the upperbound, take $u \in V(G)$ and $v \in V(H)$. Then $f = (V(G) \setminus \{u, v\}, \varnothing, \varnothing, \{u, v\})$ is a $TDIDF$ on $G + H$, with $\omega_{G+H}(f) = 6$. Thus, $\gamma_{tdI}(G + H) \leq 6$. Let $f = (V_0, V_1, V_2, V_3)$ be a γ_{tdI}-function of G. By Proposition 5, $g = (V_0 \cup V(H), V_1 \cup V(H), V_2 \cup V(H), V_3 \cup V(H))$ is a $TDIDF$ on G with $\omega_{G+H}(f) = \omega_G(f)$. This means that $\gamma_{tdI}(G+H) \leq \gamma_{tdI}(G)$. Similarly, $\gamma_{tdI}(G + H) \leq \gamma_{tdI}(H)$. ■

Corollary 2. *Let G and H be nontrivial connected graphs. If there exists a γ_{tdI}-function $f = (V_0, V_1, V_2, V_3)$ of $G+H$ such that either $f|_G$ is a $TDIDF$ on G or $f|_H$ is a $TDIDF$ on H then $\gamma_{tdI}(G+H) = \min\{\gamma_{tdI}(G), \gamma_{tdI}(H)\}$.*

Proof: By Corollary 1, $\gamma_{tdI}(G+H) \leq \min\{\gamma_{tdI}(G), \gamma_{tdI}(H)\}$. Assume, WLOG, $f|_G = (V_0 \cap V(G), V_1 \cap V(G), V_2 \cap V(G), V_3 \cap V(G))$ is a $TDIDF$ on G. Then $\gamma_{tdI}(G) \leq \omega_G(f|_G) \leq \omega_{G+H}(f)$. This implies that $\gamma_{tdI}(G+H) \geq \min\{\gamma_{tdI}(G), \gamma_{tdI}(H)\}$. Therefore, $\gamma_{tdI}(G+H) = \min\{\gamma_{tdI}(G), \gamma_{tdI}(H)\}$. ■

Proposition 6. *Let G and H be nontrivial connected graphs of orders n and m, respectively. Then $\gamma_{tdI}(G+H) = 3$ if and only if one of the following holds:*

(i) $\gamma_{tdI}(G) = 3$;

(ii) $\gamma_{tdI}(H) = 3$;

(iii) *G and H each contains at least one vertex of degree $n-1$ and $m-1$, respectively.*

Proof: Suppose that $\gamma_{tdI}(G+H) = 3$. By Proposition 1, $G+H$ has at least two vertices of degree $n+m-1$. Take $u, v \in V(G+H)$ for which $d_{G+H}(u) = n+m-1$ and $d_{G+H}(v) = n+m-1$. If $u, v \in V(G)$, then both u and v have degree $n-1$. By Proposition 1, (i) holds. Similarly, if $u, v \in V(H)$, then (ii) holds. If $u \in V(G)$ and $v \in V(H)$, then (iii) holds.

Conversely, if (i) or (ii) holds, then Equation (1) in Corollary 1 yields $\gamma_{tdI}(G+H) = 3$. Suppose (iii) holds. Then $G+H$ contains at least two vertices of maximum degree $\Delta(G+H) = (m+n)-1$. By Proposition 1, $\gamma_{tdI}(G+H) = 3$. ■

Proposition 7. *Let G and H be nontrivial connected graphs of orders n and m, respectively. Then $\gamma_{tdI}(G+H) = 4$ if and only if one of the following holds:*

(i) *$\Delta(H) \leq m-2$ and G has exactly one vertex of degree $n-1$;*

(ii) *$\Delta(G) \leq n-2$ and H has exactly one vertex of degree $m-1$;*

(iii) *$\gamma(G) = 2$ and $\gamma(H) = 2$.*

(iv) *$\gamma(G) \geq 2$ and $\gamma(H) \geq 2$ and one of the following holds:*

- *(a)* *G or H has a 3-dominating set D with $|D| = 4$ and $\delta(\langle D\rangle) \geq 2$;*
- *(b)* *G or H has a 2-dominating set D such that $|D| = 3$ and $\langle D\rangle$ is connected;*

Proof: Assume that $\gamma_{tdI}(G+H) = 4$. If $G+H$ has exactly one vertex v for which $deg_{G+H}(v) = m+n-1$, then (i) or (ii) holds. Otherwise, by Proposition 3, $\gamma(G) \geq 2$, $\gamma(H) \geq 2$ and $G+H$ has a 3-dominating set D with $|D| = 4$ and $\delta(\langle D\rangle) \geq 2$. Let $D_G = D \cap V(G)$ and $D_H = D \cap V(H)$. Suppose first that $|D_G| = 2 = |D_H|$. For each $x \in V(G) \setminus D_G$, since D is a 3-dominating set of $G+H$, there exists $u \in D_G$ for which $ux \in E(G)$. Thus, D_G is a dominating set of G. Since $\gamma(G) \geq 2$, $\gamma(G) = |D_G| = 2$

Similarly, $\gamma(H) = 2$. Thus, (iii) holds. Next, if $D \subseteq V(G)$ or $D \subseteq V(H)$, then $(iv)(a)$ holds. Finally, WLOG suppose that $|D_G| = 3$ and $|D_H| = 1$. Clearly, D_G is a 2-dominating set of G. Since $\delta(\langle D \rangle) \geq 2$, $\langle D_G \rangle$ is connected and $(iv)(b)$ holds.

Conversely, note that each of the conditions implies that one of the conditions in Proposition 3 is satisfied for $G + H$. Thus, $\gamma_{tdI}(G + H) = 4$. ■

In view of Proposition 4, if G and H are nontrivial graphs of orders n and m, respectively with $n+m = 5$, then $\gamma_{tdI}(G+H) = 5$ if and only if $G+H \in \{K_{2,3}, \overline{K_2}+(K_1 \cup K_2)\}$.

Proposition 8. *Let G and H be nontrivial graphs of orders n and m, respectively, such that $m + n > 5$. Then $\gamma_{tdI}(G + H) = 5$ if and only if each of the following holds:*

- (i) *$\gamma(G) \geq 2$ and $\gamma(H) \geq 2$, but $\gamma(G)$ and $\gamma(H)$ cannot be both equal to 2;*
- (ii) *Neither G nor H contains a 2-dominating set D for which $|D| = 3$ and $\langle D \rangle$ is connected;*
- (iii) *Neither G nor H has a 3-dominating set D with $|D| = 4$ and $\delta(\langle D \rangle) \geq 2$;*
- (iv) *One of the following holds:*
  - (a) $\min\{\gamma_{tdI}(G), \gamma_{tdI}(H)\} = 5$;
  - (b) *G or H has a 2-dominating set D for which $|D| = 4$ and $\langle D \rangle$ is connected;*
  - (c) *G or H has a dominating set D with $|D| = 3$;*
  - (d) $\gamma(G) = 2$ *and* $\gamma(H) \geq 3$;
  - (e) $\gamma(H) = 2$ *and* $\gamma(G) \geq 3$.

Proof: Suppose that $\gamma_{tdI}(G + H) = 5$. By Proposition 6, $\gamma(G) \geq 2$ and $\gamma(H) \geq 2$. Moreover, by Proposition 7, $\gamma(G)$ and $\gamma(H)$ cannot be both equal to 2; neither G nor H contains a 2-dominating set D for which $|D| = 3$ and $\langle D \rangle$ is connected; and neither G nor H has a 3-dominating set D with $|D| = 4$ and $\delta(\langle D \rangle) \geq 2$. Now we claim that $\alpha = \min\{\gamma_{tdI}(G), \gamma_{tdI}(H)\} \geq 5$. Suppose not. Then $\alpha = 4$. WLOG, assume $\gamma_{tdI}(G) = 4$. By Proposition 3, G has a 3-dominating set D with $|D| = 4$ and $\delta(\langle D \rangle) \geq 2$, a contradiction. This establishes our claim. If $\alpha = 5$, then $(iv)(a)$ holds.

Suppose that $\alpha > 5$. In view of Proposition 4, one of the following cases holds:

Case 1: $G + H$ contains a 3-dominating set D with $|D| = 5$ and $\delta(\langle D \rangle) \geq 2$. Let $D_G = D \cap V(G)$ and $D_H = D \cap V(H)$. Since $\alpha > 5$, $1 \leq |D_G| \leq 4$ and $1 \leq |D_H| \leq 4$. If either $|D_G| = 4$ or $|D_H| = 4$, then $(iv)(b)$ holds. If either $|D_G| = 3$ or $|D_H| = 3$, then $(iv)(c)$ holds.

Case 2: $G + H$ contains a 2-dominating set D with $|D| = 3$ and $\langle D \rangle$ is connected. If $D_G = D \cap V(G)$ and $D_H = D \cap V(H)$, then $1 \leq |D_G| \leq 2$ and $1 \leq |D_H| \leq 2$. If $|D_G| = 2$ and $|D_H| = 1$, then D_G is a dominating set of G and $\gamma(G) = 2$. By (i), $\gamma(H) \geq 3$, and $(iv)(d)$ holds. Similarly, if $|D_G| = 1$ and $|D_H| = 2$, then $(iv)(e)$ holds.

Case 3: $G+H$ contains a 2-dominating set D with $|D|=4$ such that there exists $v\in D$ for which $uv\in E(G+H)$ for all $u\in V(G+H)\setminus D$ with $|N_{G+H}(u)\cap D|=2$. Moreover, $\langle D\rangle$ is connected and $xv\in E(G+H)$ for every $x\in D\setminus\{v\}$ with $|N_{G+H}(x)\cap D|=1$. If $D_G=D\cap V(G)$ and $D_H=D\cap V(H)$, then $1\leq|D_G|\leq 3$ and $1\leq|D_H|\leq 3$. If $|D_G|=3$ and $|D_H|=1$, then whether $v\in V(G)$ or $v\in V(H)$, D_G is a dominating set of G. Similarly, if $|D_H|=3$ and $|D_G|=1$, then D_H is a dominating set of H. This implies $(ii)(c)$. Suppose that $|D_G|=2=|D_H|$. Then either $\gamma(G)=2$ and $\gamma(H)\geq 3$ or $\gamma(H)=2$ and $\gamma(G)\geq 3$.

Conversely, assume that statements (i)-(iii) hold. Suppose that $(iv)(a)$ holds. WLOG, assume $\gamma_{tdI}(G)=5$. In view of Proposition 5, $\gamma_{tdI}(G+H)\leq\gamma_{tdI}(G)=5$. Since statements (i)-(iii) hold, Proposition 6 and Proposition 7 imply that $\gamma_{tdI}(G+H)\geq 5$. Suppose that $(iv)(b)$ holds, say G has a 2-dominating set D for which $|D|=4$ and $\langle D\rangle$ is connected. Pick $v\in V(H)$. Then $D\cup\{v\}$ is a 3-dominating set of $G+H$ with $\delta(\langle D\cup\{v\}\rangle)\geq 2$. By Proposition 4, $\gamma_{tdI}(G+H)=5$. Suppose that $(iv)(c)$ holds, say G has a dominating set D with $|D|=3$. Choose $v\in V(H)$. Then $D\cup\{v\}$ satisfies Proposition 4$(ii)(c)$. Thus, $\gamma_{tdI}(G+H)=5$. Suppose that $(iv)(d)$ holds. Then $D\cup\{v\}$ is a 2-dominating set of cardinality 3 and $\langle D\cup\{v\}\rangle$ is connected. Thus, $\gamma_{tdI}(G+H)=5$. Similarly, if $(iv)(e)$ holds, then $\gamma_{tdI}(G+H)=5$. ■

In view of the above results, in particular, if $\gamma(G)\geq 5$ and $\gamma(H)\geq 5$, then $\gamma_{tdI}(G+H)=6$.

4. On the corona of graphs

Let G and H be connected graphs. We adapt the notation H^v used in [16] to denote the copy of H whose vertices is joined to $v\in V(G)$.

Proposition 9. *Let G be a nontrivial connected graph and H be any graph without isolated vertices, and let $f=(V_0,V_1,V_2,V_3)$ be a function on $V(G\circ H)$. Then $f\in TDIDF(G\circ H)$ if and only if each of the following holds:*

(i) For each $v\in V_0\cap V(G)$, $f|_{H^v}\in TDIDF(H^v)$;

(ii) For each $v\in V_1\cap V(G)$, $f(N_{H^v}[u])\geq 2$ for all $u\in(V_0\cup V_1)\cap V(H^v)$;

(iii) For each $v\in V_2\cap V(G)$, $f(N_{H^v}(u))\geq 1$ for all $u\in V_0\cap V(H^v)$;

(iv) For each $v\in V_3\cap V(G)$ for which $N_G(v)\subseteq V_0$, $f(V(H^v))\geq 1$.

Proof: Note first that H^v admits a $TDIDF$ for every $v\in V(G)$. Assume $f\in TDIDF(G\circ H)$. For each $v\in V(G)$,

$$N_{G\circ H}[u]=\{v\}\cup N_{H^v}[u] \tag{2}$$

so that

$$f(N_{G\circ H}[u])=f(v)+f(N_{H^v}[u]) \tag{3}$$

for all $u \in V(H^v)$. It follows from Equation (2) and Equation (3) that if $v \in V_0$, then $f|_{H^v}(N_{H^v}[u]) = f(N_{G \circ H}[u]) \geq 3$ for all $u \in (V_0 \cup V_1) \cap V(H^v)$ and $N_{G \circ H}(u) \cap [V_1 \cup V_2 \cup V_3] = N_{H^v}(u) \cap [(V_1 \cup V_2 \cup V_3) \cap V(H^v)]$ for all $u \in (V_1 \cup V_2 \cup V_3) \cap V(H^v)$. Thus, $f|_{H^v} \in TDIDF(H^v)$ and (i) holds. Similarly, (ii) and (iii) follow immediately from Equation (3). Statement (iv) follows from the fact that $\langle V_1 \cup V_2 \cup V_3 \rangle$ has no isolated vertex.

Conversely, suppose conditions (i) - (iv) hold for f. First, let $u \in V_0$, and let $v \in V(G)$ for which $u \in V(H^v + v)$. Suppose that $v \in V_0$. If $u \neq v$, then by (i), $f(N_{G \circ H}[u]) = f|_{H^v}(N_{H^v}[u]) \geq 3$. Suppose that $u = v$. Statement (i) implies that $f(N_{G \circ H}[u]) \geq f|_{H^v}(V(H^v)) \geq 3$. Suppose that $v \in V(G) \setminus V_0$. Then $u \in V_0 \cap V(H^v)$ and $v \in N_{G \circ H}[u]$. If $v \in V_3$, then $f(N_{G \circ H}[u]) \geq f(v) = 3$. If $v \in V_1 \cup V_2$, then by (ii) and (iii), $f(N_{G \circ H}[u]) = f(v) + f(N_{H^v}[u]) \geq 3$. Similar arguments show that if $u \in V_1$, then $f(N_{G \circ H}[u]) \geq 3$.

Now, let $v \in V_1 \cup V_2 \cup V_3$. Consider the following cases:

Case 1: Suppose $v \in V(G)$. If $V_0 \cap V(H^v) = \varnothing$, then for each $u \in V(H^v)$, $u \in V_1 \cup V_2 \cup V_3$ and $uv \in E(G \circ H)$. Suppose that $V_0 \cap V(H^v) \neq \varnothing$, say $w \in V_0 \cap V(H^v)$. If $v \in V_1 \cup V_2$, then by Conditions (ii) and (iii), there exists $u \in [(V_1 \cup V_2 \cup V_3) \cap V(H^v)]$ for which $uw \in E(H^v)$. Incidentally, $uv \in E(G \circ H)$. Suppose that $v \in V_3$. Then either there exists $u \in [(V_1 \cup V_2 \cup V_3) \cap N_G(v)]$ or, by Condition (iv), there exists $u \in V(H^v)$ for which $f(u) \geq 1$. In this case, $uv \in E(G \circ H)$.

Case 2: Suppose $v \in V(H^x)$ for some $x \in V(G)$. If $x \in V_1 \cup V_2 \cup V_3$, then we are done. If $x \in V_0$, then since $f|_{H^x} \in TDIDF(H^x)$ (Condition (i)), there exists $u \in [(V_1 \cup V_2 \cup V_3) \cap V(H^v)]$ for which $uv \in E(H^x)$, which means $uv \in E(G \circ H)$. ■

Corollary 3. *Let G be a nontrivial connected graph of order n, and let H be any graph. Then $\gamma_{tdI}(G \circ H) = 3n$.*

Proof: Since

$$f = (\cup_{x \in V(G)} V(H^x), \varnothing, \varnothing, V(G)) \in TDIDF(G \circ H),$$

$\gamma_{tdI}(G \circ H) \leq 3n$. To get the other inequality, first, suppose that H has an isolated vertex, say x. For each $v \in V(G)$, let x^v denote the isolated vertex of H^v being identified with x. Let $f = (V_0, V_1, V_2, V_3)$ be a γ_{tdI}-function of $G \circ H$. Since x^v is an endvertex in $H^v + v$, $f(V(H^v + v)) \geq f(v) + f(x^v) \geq 3$ for all $v \in V(G)$. This yields

$$\gamma_{tdI}(G \circ H) \geq \sum_{v \in V(G)} f(V(H^v + v) \geq 3n.$$

Next, suppose that H has no isolated vertices, and let $f = (V_0, V_1, V_2, V_3)$ be a γ_{tdI}-function of $G \circ H$. Let $v \in V(G)$. Clearly, if $v \in V_3$, then $f(V(H^v + v)) \geq 3$. If $v \in V_0$, then $f|_{H^v} \in TDIDF(H^v)$ by Proposition 9 (i). Thus, $f(V(H^v + v)) = f|_{H^v}(V(H^v)) \geq 3$. Suppose that $v \in V_1 \cup V_2$. If $V_0 \cap V(H^v) = \varnothing$, then since $|V(H^v)| \geq 2$, $f(V(H^v)) \geq 2$ so that $f(V(H^v+v)) = f(v) + f(V(H^v)) \geq 3$. On the other hand, if $V_0 \cap V(H^v) \neq \varnothing$, say $u \in V_0 \cap V(H^v)$, then Proposition 9(ii) and Proposition 9(iii) yield

$$f(V(H^v+v)) \geq f(v) + f(N_{H^v}[u]) \geq 3.$$

Therefore,

$$\gamma_{tdI}(G \circ H) = \omega_{G\circ H}(f) = \sum_{v\in V(G)} f(V(H^v+v)) \geq 3n. \qquad \blacksquare$$

5. On the edge corona of graphs

We adapt the following notations from [21]. Given graphs G and H, we write H^{uv} to denote the copy of H that is being joined with the end vertices of the edge $uv \in E(G)$ in the edge corona $G \diamond H$. Moreover, we denote by $H^{uv}+uv$ the subgraph of $G \diamond H$ corresponding to the join $H^{uv} + \langle u, v\rangle$, $u, v \in V(G)$. For $f = (V_0, V_1, V_2, V_3)$ on $V(G \diamond H)$, we write for each $i, j \in \{0, 1, 2, 3\}$,

$$E_{ij} = \{uv \in E(G) : \text{either } u \in V_i \text{ and } v \in V_j \text{ or } u \in V_j \text{ and } v \in V_i\}.$$

Proposition 10. *Let G be nontrivial connected graph and H be any graph without isolated vertices. Let $f = (V_0, V_1, V_2, V_3)$ be a function on $V(G)$. Then $f \in TDIDF(G \diamond H)$ if and only if each of the following holds:*

- (i) *For each $uv \in E_{00}$, $f|_{H^{uv}} \in TDIDF(H^{uv})$;*
- (ii) *For each $uv \in E_{01}$, $f(N_{H^{uv}}[w]) \geq 2$ for all $w \in (V_0 \cup V_1) \cap V(H^{uv})$;*
- (iii) *For each $uv \in E_{11} \cup E_{02}$, $f(N_{H^{uv}}[w]) \geq 1$ for all $w \in V_0 \cap V(H^{uv})$;*
- (iv) *For each $uv \in E_{03}$ with $v \in V_3$ and $N_G(v) \subseteq V_0$, we have $V(H^{uv}) \setminus V_0 \neq \varnothing$.*

Proof: Since H has no isolated vertices, H^{uv} admits a $TDIDF$ for each $uv \in E(G)$. If $f \in TDIDF(G \diamond H)$, then properties (i)-(iii) follow immediately from the fact that for each $uv \in E(G)$, $f(N_{G\diamond H}[w]) = f(u) + f(v) + f(N_{H^{uv}}[w])$ for all $w \in V(H^{uv})$. While property (iv) is clear from the definition of f.

Conversely, suppose that conditions (i)-(iv) hold for f. Let $w \in V_0$ and $uv \in E(G)$ for which $w \in V(H^{uv}+uv)$. Clearly, if $uv \in [E_{03} \cup E_{12} \cup E_{13} \cup E_{22} \cup E_{23} \cup E_{33}]$, then $f(N_{G\diamond H}[w]) \geq 3$. We proceed with the following cases:

Case 1: Suppose that $uv \in E_{00}$. Then $f|_{H^{uv}} \in DIDF(H^{uv})$ by (i). If $w = u$ or $w = v$, then

$$f(N_{G\diamond H}[w]) \geq f|_{H^{uv}}(V(H^{uv})) \geq 3.$$

If $u \neq w \neq v$, then

$$f(N_{G\diamond H}[w]) = f|_{H^{uv}}(N_{H^{uv}}[w]) \geq 3.$$

Case 2: Suppose that $uv \in E_{01} \cup E_{02}$, and assume $u \in V_0$. If $V_0 \cap V(H^{uv}) = \varnothing$, then $w = u$ and since H^{uv} has no isolated vertices, $f(V(H^{uv}) \geq 2$. Thus,

$$f(N_{G\diamond H}[w]) \geq f(v) + f(V(H^{uv})) \geq 3.$$

Suppose that $V_0 \cap V(H^{uv}) \neq \varnothing$, say $z \in V_0 \cap V(H^{uv})$. If $w = u$, then by (ii) and (iii),

$$f(N_{G\diamond H}[w]) \geq f(v) + f(N_{H^{uv}}[z]) \geq 3.$$

If $w \in V(H^{uv})$, then

$$f(N_{G\diamond H}[w]) \geq f(v) + f(N_{H^{uv}}[w]) \geq 3.$$

Case 3: Suppose that $uv \in E_{11}$. Then $w \in V(H^{uv})$ and by (iii),

$$f(N_{G\diamond H}[w]) \geq f(u) + f(v) + f(N_{H^{uv})}[w]) \geq 3.$$

Following similar arguments, $f(N_{G\diamond H}[w]) \geq 3$ for all $w \in V_1$.

Finally, let $v \in V_1 \cup V_2 \cup V_3$. First, suppose that $v \in V(G)$. If $N_G(v) \nsubseteq V_0$, then there exists $x \in V_1 \cup V_2 \cup V_3$ such that $xv \in E(G \diamond H)$. Suppose that $N_G(v) \subseteq V_0$, and let $u \in N_G(v)$. Then $uv \in E_{01} \cup E_{02} \cup E_{03}$. In view of conditions (ii)-(iv), $f(V(H^{uv}) \geq 1$. Thus, there exists $u \in [(V_1 \cup V_2 \cup V_3) \cap V(H^{uv})]$ for which $uv \in E(G \diamond H)$.

Next, suppose that $v \in V(H^{xy})$ for some $xy \in E(G)$. If $x \notin V_0$, then $x \in V_1 \cup V_2 \cup V_3$ with $xv \in E(G \diamond H)$. Similarly, if $y \notin V_0$, then $y \in V_1 \cup V_2 \cup V_3$ with $yv \in E(G \diamond H)$. Suppose that $xy \in E_{00}$. By (i), $f|_{H^{xy}} \in TDIDF(H^{uv})$ so that there exists $u \in V(H^{xy})$ for which $f(u) = f|_{H^{xy}}(u) > 0$ and $uv \in E(G \diamond H)$.

The argument above implies that $f \in TDIDF(G \diamond H)$. ■

For the purpose of the next result, we define for any $D \subseteq V(G)$,

$$E(G, D) = \{uv \in E(G) : \text{both } u, v \notin D\}.$$

Let $G = P_5 = [v_1, v_2, v_3, v_4, v_5]$. If $S_1 = \{v_1, v_4\}$ and $S_2 = \{v_1, v_2\}$, then $E(G, S_1) = \varnothing$ and $E(G, S_2) = \{v_3v_4, v_4v_5\}$.

Corollary 4. *Let G be a nontrivial connected graph of order n. Then for any graph H,*

$$3 \leq \gamma_{tdI}(G \diamond H) \leq n + \beta(G).$$

Moreover, if H has no isolated vertices, then

$$3 \leq \gamma_{tdI}(G \diamond H) \leq \min\{n + \beta(G), \theta(G \diamond H)\}$$

where

$$\theta(G \diamond H) = 3\gamma_t(G) + \gamma_{tdI}(H) \min\{|E(G, S)| : S \text{ is a } \gamma_t\text{-set of } G\},$$

and these bounds are sharp.

Proof: The lower bound follows immediately from Proposition 1. Let $S \subseteq V(G)$ be a β-set of G. Then $f = (V_0, V_1, V_2, V_3) \in TDIDF(G \diamond H)$, where $V_0 = \cup_{uv \in E(G)} V(H^{uv})$, $V_1 = V(G) \setminus S$, $V_2 = S$ and $V_3 = \varnothing$. Thus,

$$\gamma_{tdI}(G \diamond H) \leq 2|S| + n - |S| = n + \beta(G).$$

Assume that H has no isolated vertices. Then H^{uv} admits a $TDIDF$. Let S be a γ_t-set of G. Suppose $f_{uv} = (V_0^{uv}, V_1^{uv}, V_2^{uv}, V_3^{uv})$ is a γ_{tdI}-function of H^{uv} for each $uv \in E(G, S)$. Define $f = (V_0, V_1, V_2, V_3)$ on $G \diamond H$, where

$$\begin{aligned} V_0 &= (V(G) \setminus S) \cup \left(\cup_{uv \in E(G,S)} V_0^{uv}\right) \cup \left(\cup_{uv \in E(G) \setminus E(G,S)} V(H^{uv})\right), \\ V_1 &= \cup_{uv \in E(G,S)} V_1^{uv}, \\ V_2 &= \cup_{uv \in E(G,S)} V_2^{uv}, \quad \text{and} \\ V_3 &= S \cup \left(\cup_{uv \in E(G,S)} V_3^{uv}\right). \end{aligned}$$

Note that $E_{01} = E_{02} = E_{11} = \varnothing$ and $(V_1 \cup V_2) \cap V(G) = \varnothing$. Moreover, $V_3 \cap V(G) = S$ is a γ_t-set of G so that $N_G(x) \nsubseteq V_0$ for each $x \in V_3 \cap V(G)$. Hence, we only need to satisfy condition (i) in Proposition 10. Let $uv \in E_{00}$. Then $f|_{H^{uv}} = f_{uv} \in TDIDF(H^{uv})$. THus, $f \in TDIDF(G \diamond H)$ with $\omega_{G \diamond H}(g) = 3\gamma(G) + \gamma_{tdI}(H)|E(G, S)|$. It follows that, $\gamma_{tdI}(G \diamond H) \leq \theta(G \diamond H)$.

For the sharpness, note first that for the left-hand side, $\gamma_{tdI}(P_2 \diamond H) = 3$ for any H. For the right-hand side, consider the following graphs. If $G = C_4$ of order $n = 4$, then for any H, $\gamma_{tdI}(G \diamond H) = 6 = n + \beta(G)$. On the other hand, if G is the graph in Figure 2, then for any H with $\gamma_{tdI}(H) = 3$, $\gamma_{tdI}(G \diamond H) = 15 = \theta(G \diamond H)$. ■

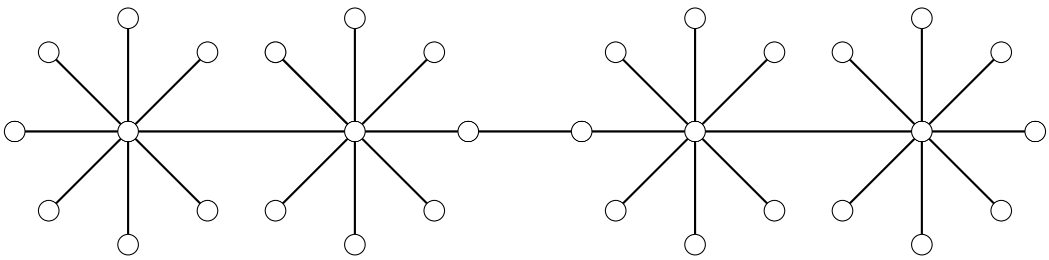


Figure 2: Example of a graph G for which $\gamma_{tdI}(G \diamond H) = \theta(G \diamond H)$

Strict inequality in Corollary 4 may also be attained. Consider, for example, the star graph $G = K_{1,7}$. For any graph H without isolated vertices,

$$\gamma_{tdI}(G \diamond H) = 4 < 6 = \min\{n + \beta(G), \theta(G \diamond H)\}.$$

6. On the complementary prism of graphs

Proposition 11. *Let G be any graph. Then* (*i*) *$\gamma_{tdI}(G\overline{G}) = 3$ if and only if $G = K_1$;*

(*ii*) *$\gamma_{tdI}(G\overline{G}) \geq 6$ whenever G is nontrivial, and this lower bound is sharp.*

Proof: Statement (*i*) follows immediately from Proposition 1. Assume G is nontrivial. Suppose that $\gamma_{tdI}(G\overline{G}) = 4$. Then $\gamma(G\overline{G}) \neq 1$. By Proposition 3, $\gamma(G\overline{G}) \geq 2$ and $G\overline{G}$ has a 3-dominating set D with $|D| = 4$ and $\delta(\langle D\rangle) \geq 2$. If $|D \cap V(G)| \geq 3$, then there exists $u \in D \cap V(G)$ for which $\overline{u} \notin D$. For this u, $|D \cap N_{G\overline{G}}(\overline{u})| \leq 2$, a contradiction. A similar contradiction is attained if $|D \cap V(\overline{G})| \geq 3$. Suppose that $|D \cap V(G)| = 2 = |D \cap V(\overline{G})|$. Since $\delta(\langle D\rangle) \geq 2$, $\langle D\rangle = C_4$, which is impossible. Thus, $\gamma_{tdI}(G\overline{G}) \neq 4$. Suppose that $\gamma_{tdI}(G\overline{G}) = 5$. In view of Proposition 4, it suffices to consider the following cases:

Case 1: $G\overline{G}$ has a 3-dominating set D with $|D| = 5$ and $\delta(\langle D\rangle) \geq 2$. If $D \subseteq V(G)$ and $u \in D$, then $|D \cap N_{G\overline{G}(\overline{u})}| \leq 1$, a contradiction. Similarly, a contradiction is attained if $D \subseteq V(\overline{G})$. Since $\delta(\langle D\rangle) \geq 2$, $|D \cap V(G)| \neq 4$ and $|D \cap V(G)| \neq 4$. Assume, WLOG, that $|D \cap V(G)| = 3$ and $|D \cap V(\overline{G})| = 2$. Since $\delta(\langle D\rangle) \geq 2$, there exist $u, v \in D \cap V(G)$ such that $D \cap V(\overline{G}) = \{\overline{u}, \overline{v}\}$ and $\overline{u}\,\overline{v} \in E(\overline{G})$. If $w \in (D \cap V(G)) \setminus \{u, v\}$, then $\overline{w} \notin D$ and $|D \cap N_{G\overline{G}}(\overline{w})| = 1$, a contradiction.

Case 2: $G\overline{G}$ has a 2-dominating set D with $|D| = 3$ and $\langle D\rangle$ is connected. Following a similar argument, $D \cap V(G) \neq \varnothing$ and $D \cap V(\overline{G}) \neq \varnothing$. Assume $|D \cap V(G)| = 2$ and $|D \cap V(G)| = 1$, say $D \cap V(G) = \{u, v\}$ and $D \cap V(\overline{G}) = \{w\}$. Since $\langle D\rangle$ is connected, either $\overline{u} = w$ and $D \cap N_{G\overline{G}}(\overline{v}) = \{v\}$ or $\overline{v} = w$ and $D \cap N_{G\overline{G}}(\overline{u}) = \{u\}$, which is impossible.

Case 3: $G\overline{G}$ contains a 2-dominating set D with $|D| = 4$ such that there exists $v \in D$ for which $uv \in E(G\overline{G})$ for all $u \in V(G\overline{G}) \setminus D$ with $|N_{G\overline{G}}(u) \cap D| = 2$. Moreover, $\langle D\rangle$ is connected and $xv \in E(G\overline{G})$ for every $x \in D \setminus \{v\}$ with $|N_{G\overline{G}}(x) \cap D| = 1$. If $|D \cap V(G)| \geq 3$, then since $\langle D\rangle$ is connected, there exists $u \in D \cap V(G)$ such that $\overline{u} \notin D$ and $|N_{G\overline{G}(\overline{u})} \cap D| = 1$, a contradiction. Thus, $|D \cap V(G)| = 2$ and $|D \cap V(\overline{G})| = 2$. Assume, $v \in D \cap V(G)$. Let $D \cap V(G) = \{x, v\}$. Since $\langle D\rangle$ is connected, $D = \{x, v, \overline{x}, \overline{v}\}$, $\overline{x}v \notin E(G\overline{G})$, and $N_{G\overline{G}}(\overline{x}) \cap D = \{x\}$. This is a contradiction.

The above contradictions imply that $\gamma_{tdI}(G\overline{G}) \neq 5$.

Finally, observe that if $G = P_2$, then $\gamma_{tdI}(G\overline{G}) = \gamma_{tdI}(P_4) = 6$, showing that the bound provided in (*ii*) is sharp. ■

The following proposition is clear.

Proposition 12. *Let G be a nontrivial connected graph. Then $f = (V_0, V_1, V_2, V_3)$ is a TDIDF on $G\overline{G}$ if and only if each of the following holds:*

(*i*) *For each $v \in (V_0 \cup V_1) \cap V(G)$, either $f|_G(N_G[v]) \geq 3$ or $f|_G(N_G[v]) < 3$ and $3 - f|_G(N_G[v]) \leq f(\overline{v})$;*

(*ii*) *For each $\overline{v} \in (V_0 \cup V_1) \cap V(\overline{G})$, either $f|_{\overline{G}}(N_{\overline{G}}[v]) \geq 3$ or $f|_{\overline{G}}(N_{\overline{G}}[\overline{v}]) < 3$ and $3 - f|_{\overline{G}}(N_{\overline{G}}[\overline{v}]) \leq f(v)$;*

(*iii*) *For each $v \in V(G) \cap (V_1 \cup V_2 \cup V_3)$, $\overline{v} \notin V_0$ whenever $N_G(v) \subseteq V_0$;*

(iv) For each $\overline{v} \in V(\overline{G}) \cap (V_1 \cup V_2 \cup V_3)$*,* $v \notin V_0$ *whenever* $N_{\overline{G}}(\overline{v}) \subseteq V_0$*.*

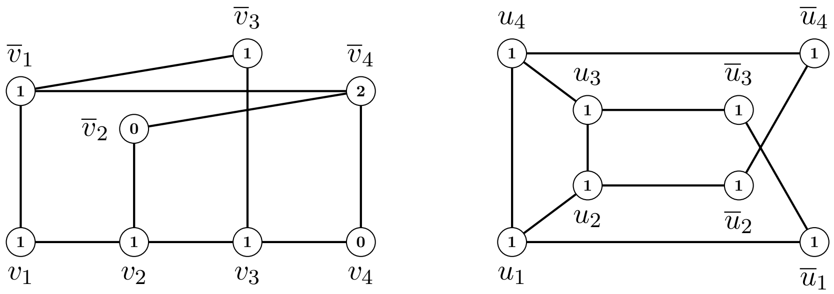


Figure 3: The complementary prisms $P_4\overline{P}_4$ and $C_4\overline{C}_4$

Proposition 13. *Let* G *be a nontrivial connected graph of order* n*. Then*

$$6 \leq \gamma_{tdI}(G\overline{G}) \leq 3n. \tag{4}$$

Moreover, if $\gamma(G) \neq 1$ *and* $\gamma(\overline{G}) \neq 1$*, then*

$$1 + \max\{\gamma_{tdI}(G), \gamma_{tdI}(\overline{G})\} \leq \gamma_{tdI}(G\overline{G}) \leq 2n. \tag{5}$$

These bounds are sharp.

Proof: The left-hand inequality in Inequality 4 is a reiteration of Proposition 11. By Proposition 12, the function $f = (V(\overline{G}), \varnothing, \varnothing, V(G)) \in TDIDF(G\overline{G})$. Thus, $\gamma_{tdI}(G\overline{G}) \leq 3|V_3| = 3n$ and the Inequality 4 holds.

Now, suppose that $\gamma(G) \neq 1$ and $\gamma(\overline{G}) \neq 1$. Let $V_0 = \varnothing = V_2 = V_3$ and $V_1 = V(G) \cup V(\overline{G})$, and let $f = (V_0, V_1, V_2, V_3)$. For each $v \in V(G)$, since v is not an isolated vertex, there exists $u \in V(G)$ for which $uv \in E(G)$. Thus, $\{v, u\} \subseteq N_G[v]$ so that $f(N_G[v]) \geq 2$. If $f(N_G[v]) = 2$, then $3 - f(N_G[v]) = 1 \leq f(\overline{v})$. Thus, Condition (i) in Proposition 12 holds. Similarly, since $\overline{G}$ has no isolated vertex, Proposition 12(ii) holds. Since $V_0 = \varnothing$, Conditions (iii) and (iv) of Proposition 12 also hold. Thus, $f \in TDIDF(G\overline{G})$. Therefore,

$$\gamma_{tdI}(G\overline{G}) \leq 2n.$$

WLOG assume that $\gamma_{tdI}(G) \geq \gamma_{tdI}(\overline{G})$. Let f be a γ_{tdI}-function of $G\overline{G}$. If $V(\overline{G}) \subseteq V_0$, then $V(G) \subseteq V_3$ and $\omega_{G\overline{G}}(f) = 3|V(G)|$. However, by Theorem 1, $\gamma_{tdI}(G\overline{G}) \leq |V(G\overline{G})| + 2 - \delta(G\overline{G})$. Since G and $\overline{G}$ have no isolated vertices, the least possible value of $\delta(G\overline{G})$ is 2. Hence,

$$\gamma_{tdI}(G\overline{G}) \leq 2|V(G)| < 3|V(G)|,$$

a contradiction. Suppose $f|_G$ is a $TDIDF$ on G. Since $V(\overline{G}) \nsubseteq V_0$, $(V_1 \cup V_2 \cup V_3) \cap V(\overline{G}) \neq \varnothing$ and $\omega(f|_{\overline{G}}) \geq 1$. Thus,

$$\omega(f|_G) + 1 \leq \omega(f|_G) + \omega(f|_{\overline{G}}) = \omega_{G\overline{G}}(f)$$

Suppose $f|_G$ is not a $TDIDF$ on G. Let $A = \{v \in V_0 \cap V(G) : 0 \leq f(N_G(v)) \leq 1\}$, $B = \{v \in V(G) \cap V_0 : f(N_G(v)) = 2\}$, $C = \{v \in V(G) \cap V_1 : 0 \leq f(N_G(v)) \leq 1\}$, $D = \{v \in V(G) \cap (V_2 \cup V_3) : N_G(v) \subseteq V_0 \setminus B\}$. Now, let $X \subseteq N_G(D) \cap V_0$ be the smallest set that dominates D. Then $|X| \leq |D|$. Define a function g on $V(G)$ as follows:

$$g(x) = \begin{cases} 0, & \text{if } x \in (V(G) \cap V_0) \setminus (A \cup B \cup X); \\ 1, & \text{if } x \in [(V(G) \cap V_1) \setminus C] \cup B \cup X; \\ 2, & \text{if } x \in (V(G) \cap V_2) \cup A \cup C; \\ 3, & \text{if } x \in (V(G) \cap V_3), \end{cases}$$

Then g is a $TDIDF$ on G with

$$\begin{aligned} \omega_G(g) &= |V(G) \cap V_1| - |C| + |B \cup X| + 2|V(G) \cap V_2| + 2|A| + 2|C| \\ &\quad + 3|V(G) \cap V_3| \\ &= |V(G) \cap V_1| + 2|V(G) \cap V_2| + 3|V(G) \cap V_3| + 2|A| + |C| + |B \cup X| \\ &= \omega_G(f|_G) + 2|A| + |C| + |B \cup X|. \end{aligned}$$

We claim that $\omega_G(f|_G) - 1 \leq \omega_{G\overline{G}}(f)$. Put $A_1 = \{v \in V_0 \cap V(G) : f(N_G(v)) = 0\}$, $A_2 = \{v \in V_0 \cap V(G) : f(N_G(v)) = 1\}$, $C_1 = \{v \in V(G) \cap V_1 : f(N_G(v)) = 0\}$, and $C_2 = \{v \in V(G) \cap V_1 : f(N_G(v)) = 1\}$. Then $A_1 \cup A_2 = A$ and $C_1 \cup C_2 = C$. Now, we denote by $\overline{S} = \{\overline{v} \in V(\overline{G}) : v \in S\}$ for each $S \subseteq V(G)$. Note that, $\overline{A}_1 \subseteq V(\overline{G}) \cap V_3$, $\overline{A}_2 \subseteq V(\overline{G}) \cap (V_2 \cup V_3)$, $\overline{B} \subseteq V(\overline{G}) \cap (V_1 \cup V_2 \cup V_3)$, $\overline{C_1} \subseteq V(\overline{G}) \cap (V_2 \cup V_3)$, $\overline{C_2} \subseteq V(\overline{G}) \cap (V_1 \cup V_2 \cup V_3)$, and $\overline{D} \subseteq V(\overline{G}) \cap (V_1 \cup V_2 \cup V_3)$. WLOG, assume that $\overline{A}_2 \subseteq V(\overline{G}) \cap V_2$, $\overline{B} \subseteq V(\overline{G}) \cap V_1$, $\overline{C_1} \subseteq V(\overline{G}) \cap V_2$, $\overline{C_2} \subseteq V(\overline{G}) \cap V_1$, and $\overline{D} \subseteq V(\overline{G}) \cap V_1$. Then $\overline{C}_2 \cap \overline{D} = \varnothing$, $\overline{C}_2 \cap \overline{B} = \varnothing$ and $\overline{A}_2 \cap \overline{C}_1 = \varnothing$. Thus,

$$\begin{aligned} \omega_{\overline{G}}(f|_{\overline{G}}) &= |V(\overline{G}) \cap V_1| + 2|V(\overline{G}) \cap V_2| + 3|V(\overline{G}) \cap V_3| \\ &\geq 3|\overline{A}_1| + 2|\overline{A}_2| + +2|\overline{C}_1| + |\overline{C}_2| + |\overline{B} \cup \overline{D}| \\ &= 2|A| + |C| + |A_1| + |C_1| + |B \cup D|. \end{aligned}$$

Since $|X| \leq |D|$, $\omega_{\overline{G}}(f|_{\overline{G}}) \geq 2|A| + |C| + |B \cup X| + 1$. Hence,

$$\begin{aligned} \gamma_{tdI}(G\overline{G}) &= \omega_{G\overline{G}}(f) \\ &= \omega_G(f|_G) + \omega_{\overline{G}}(f|_{\overline{G}}) \\ &\geq \omega_G(f|_G) + 2|A| + |C| + |B \cup X| + 1 \\ &= \omega_G(g) + 1. \end{aligned}$$

Therefore, $\gamma_{tdI}(G\overline{G}) \geq \gamma_{tdI}(G) + 1$.

If $G = K_n$, then $\gamma_{tdI}(G\overline{G}) = 3n$. If $G = P_4$, then $\overline{G} = P_4$ and $\gamma_{tdI}(G\overline{G}) = 1 + \max\{\gamma_{tdI}(G), \gamma_{tdI}(\overline{G})\}$. And if $G = C_n$ on $n = 4$ vertices, then $\gamma_{tdI}(G\overline{G}) = 2n$. Therefore, the inequalities in Inequality 4 and Inequality 5 are sharp. ■

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