EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 17, No. 4, 2024, 3847-3855
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Degenerate Moments and Expectation of Monomials

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Abstract. The aim of this paper is twofold. Firstly, we obtain expressions of the degenerate moments of a discrete nonnegative integer-valued random variable. Secondly, we get an expression for the expectation of any monomial in discrete nonnegative integer-valued random variables.

2020 Mathematics Subject Classifications: 60-08, 60E0

Key Words and Phrases: Degenerate moments, expectation of monomials, discrete nonnegative integer-valued random variables

1. Introduction

Let X be a discrete nonnegative integer-valued random variable. Then the probability mass function on X is defined by $p_X(x) = P\{X = x\}$. Oftentimes, we omit X from $p_X(x)$ and denote it simply by $p(x)$. This convention applies to other similar situations. The cumulative distribution function on X is given by: for any nonnegative integer a,

$$F_X(a) = P\{X \leq a\} = \sum_{x=0}^{a} p(x) = \sum_{x=0}^{a} P\{X = x\}, \quad \text{(see [8–12, 14–19, 22, 23])}. \tag{1}$$

Let $g(x)$ be a real valued function. Then the expectation of $g(X)$ is given by

$$E\big[g(X)\big] = \sum_{x=0}^{\infty} g(x)p_X(x) = \sum_{x=0}^{\infty} g(x)P\{X = x\}, \quad \text{(see [5, 8–12, 14–19, 22, 23])}. \tag{2}$$

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DOI: https://doi.org/10.29020/nybg.ejpam.v17i4.5604

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The n-th moment of X is defined by

$$E\big[X^n\big] = \sum_{k=0}^{\infty} k^n p(k) = \sum_{k=0}^{\infty} k^n P\{X = k\}, \quad \text{(see [17–19, 22, 23])}. \tag{3}$$

The variance of X is given by

$$\mathrm{Var}(X) = E\big[\big(X - E[X]\big)^2\big] = E\big[X^2\big] - \big(E[X]\big)^2, \quad \text{(see [23])}. \tag{4}$$

Let X and Y be discrete nonnegative integer-valued random variables. Then the joint probability mass function of X and Y is defined by

$$p(x, y) = P\{X = x,\ Y = y\}, \quad \text{(see [4, 23])}. \tag{5}$$

We note that

$$P\{X = x|Y = y\} = \frac{P\{X = x, Y = y\}}{P\{Y = y\}}, \quad \text{(see [23])}. \tag{6}$$

Thus, by (5) and (6), we get

$$p(x, y) = P\{X = x|Y = y\}P\{Y = y\}. \tag{7}$$

Let $p_X(x)$ and $p_Y(y)$ be respectively the probability mass function of X and that of Y. Then we have

$$\begin{aligned} p_X(x) &= \sum_y P\{X = x,\ Y = y\} = \sum_y p(x, y), \\ p_Y(y) &= \sum_x P\{X = x,\ Y = y\} = \sum_x p(x, y). \end{aligned} \tag{8}$$

The joint cumulative distribution function of X and Y is defined by: for any nonnegative integers a and b,

$$F_{X,Y}(a, b) = P\{X \le a,\ Y \le b\} = \sum_{y=0}^{b} \sum_{x=0}^{a} p(x, y). \tag{9}$$

By (7), we get

$$\begin{aligned} F_X(a) &= P\{X \le a\} = P\{X \le a,\ Y \le \infty\} = F_{X,Y}(a, \infty), \\ F_Y(b) &= P\{Y \le b\} = P\{X \le \infty, Y \le b\} = F_{X,Y}(\infty, b). \end{aligned} \tag{10}$$

For any $\lambda \in \mathbb{R}$, the degenerate falling factorial sequence is defined by (see [8, 12, 14, 15, 17, 18, 21, 27])

$$(x)_{0,\lambda} = 1, \quad (x)_{n,\lambda} = x(x - \lambda)(x - 2\lambda) \cdots \big(x - (n - 1)\lambda\big),\ (n \ge 1). \tag{11}$$

With the notation in (11), we note that the degenerate exponentials are given by

$$e_\lambda^x(t) = \sum_{n=0}^{\infty} (x)_{n,\lambda} \frac{t^n}{n!}, \quad \text{(see [12, 14–18])}. \tag{12}$$

We see that

$$\lim_{\lambda\to 0}(x)_{n,\lambda}=x^n,\ \lim_{\lambda\to 0}e_\lambda^x(t)=e^{xt}.$$

The generating function of the degenerate moments $E\big[(X)_{n,\lambda}\big]$ of the random variable X is given by

$$E\big[e_\lambda^X(t)\big]=\sum_{n=0}^{\infty}E\big[(X)_{n,\lambda}\big]\frac{t^n}{n!},\quad \text{(see [12, 14–18])}.$$

In Section 1, we recall some necessary facts that are needed throughout this paper. Section 2 contains the main results of this paper. Let X be a discrete nonnegative integer-valued random variable. Then we obtain expressions for the r-th degenerate moment $E\big[(X)_{r,\lambda}\big]$ (see (11)) as infinite series involving the cumulative distribution function F_X (see (1)) in Theorems 2.1 and 2.3. Assume that X,Y are discrete nonnegative integer-valued random variables. In Theorem 2.2, we show that $E[XY]$ is equal to the double sum over x,y of $T(x,y)$, where $T(x,y)=P\{X>x,\ Y>y\}$. In Theorem 2.4, this is generalized to the case of $E[X^{r_1}Y^{r_2}]$, where r_1,r_2 are any positive integers. Let $r_1,r_2,\cdots,r_k$ be positive integers, and let $X_1,X_2,\dots,X_k$ be discrete nonnegative integer-valued random variables. In Theorem 2.5, we get an expression for $E[X_1^{r_1}X_2^{r_2}\cdots X_k^{r_k}]$ as a multiple sum over $x_1,x_2,\dots,x_k$, which involves $T(x_1,x_2,\dots,x_k)$. Here $T(x_1,x_2,\dots,x_k)=P\{X_1>x_1,X_2>x_2,\dots,X_k>x_k\}$.

2. Degenerate moments and expectation of monomials

For $r\in\mathbb{N}$, the r-th degenerate moment of X is given by

$$\begin{aligned}E\big[(X)_{r,\lambda}\big]&=\sum_{x=0}^{\infty}p_X(x)(x)_{r,\lambda}=\sum_{x=1}^{\infty}p_X(x)(x)_{r,\lambda}=\sum_{x=0}^{\infty}p_X(x+1)(x+1)_{r,\lambda} \qquad (13)\\&=\sum_{x=0}^{\infty}p_X(x+1)\sum_{i=0}^{x}\Big((i+1)_{r,\lambda}-(i)_{r,\lambda}\Big)=\sum_{i=0}^{\infty}\Big((i+1)_{r,\lambda}-(i)_{r,\lambda}\Big)\sum_{x=i}^{\infty}p_X(x+1)\\&=\sum_{i=0}^{\infty}\Big((i+1)_{r,\lambda}-(i)_{r,\lambda}\Big)P\{X>i\}=\sum_{i=0}^{\infty}\Big((i+1)_{r,\lambda}-(i)_{r,\lambda}\Big)\Big(1-P\{X\le i\}\Big)\\&=\sum_{i=0}^{\infty}\Big((i+1)_{r,\lambda}-(i)_{r,\lambda}\Big)\Big(1-F_X(i)\Big).\end{aligned}$$

Therefore, by (13), we obtain the following theorem.

Theorem 1. *Let r be a positive integer, and let X be a discrete nonnegative integer-valued random variable. Then the r-th degenerate moment of X is given by*

$$E\big[(X)_{r,\lambda}\big]=\sum_{i=0}^{\infty}\Big((i+1)_{r,\lambda}-(i)_{r,\lambda}\Big)\Big(1-F_X(i)\Big). \qquad (14)$$

Note that, when $r = 1$, we have

$$E[X] = \sum_{i=0}^{\infty} \Big(1 - F_X(i)\Big) = \sum_{i=0}^{\infty} P\{X > i\}.$$

Assume that X and Y are discrete nonnegative integer-valued random variables with their respective probability density functions $p_X(x)$ and $p_Y(y)$.
Let $T(x, y) = P\{X > x,\ Y > y\}$, and let $p(x, y)$ be the joint probability mass function of X and Y. Now, we observe that

$$\begin{aligned} E[XY] &= \sum_{x=0}^{\infty}\sum_{y=0}^{\infty} xyp(x,y) = \sum_{x=1}^{\infty}\sum_{y=1}^{\infty} xyp(x,y) \\ &= \sum_{x=1}^{\infty}\sum_{y=1}^{\infty}\sum_{i=1}^{x}\sum_{j=1}^{y} p(x,y) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{x=i}^{\infty}\sum_{y=j}^{\infty} p(x,y) \\ &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{x=i+1}^{\infty}\sum_{y=j+1}^{\infty} p(x,y) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} P\{X > i, Y > j\} \\ &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} T(i,j) = \sum_{x=0}^{\infty}\sum_{y=0}^{\infty} T(x,y). \end{aligned} \tag{15}$$

Therefore, by (15), we obtain the following theorem.

Theorem 2. *Let X and Y be discrete nonnegative integer-valued random variables. Then we have*

$$E[XY] = \sum_{x=0}^{\infty}\sum_{y=0}^{\infty} T(x,y),$$

where $T(x, y) = P\{X > x,\ Y > y\}$.

If X and Y are independent, then we note that

$$E[XY] = E[X]E[Y] = \sum_{x=0}^{\infty}\sum_{y=0}^{\infty} xyp_X(x)p_Y(y).$$

Note that

$$\begin{aligned} (i+1)_{r,\lambda} - (i)_{r,\lambda} &= \sum_{j=0}^{r} \binom{r}{j} (i)_{r-j,\lambda}(1)_{j,\lambda} - (i)_{r,\lambda} \\ &= \sum_{j=1}^{r} \binom{r}{j} (i)_{r-j,\lambda}(1)_{j,\lambda}, \quad (r \geq 1). \end{aligned} \tag{16}$$

Thus, by (14) and (16), we get

$$E\big[(X)_{r,\lambda}\big] = \sum_{i=0}^{\infty} \big((i+1)_{r,\lambda} - (i)_{r,\lambda}\big)\big(1 - F_X(i)\big) \tag{17}$$

$$= \sum_{i=0}^{\infty} \sum_{j=1}^{r} \binom{r}{j} (i)_{r-j,\lambda} (1)_{j,\lambda} \big(1 - F_X(i)\big)$$
$$= \sum_{i=0}^{\infty} \sum_{j=0}^{r-1} \binom{r}{j+1} (i)_{r-1-j,\lambda} (1)_{j+1,\lambda} \big(1 - F_X(i)\big).$$

Therefore, by (17), we obtain the following theorem.

Theorem 3. *Let r be a positive integer, and let X be a discrete nonnegative integer-valued random variable. Then the r-th degenerate moment of X is given by*

$$E\big[(X)_{r,\lambda}\big] = \sum_{i=0}^{\infty} \sum_{j=0}^{r-1} \binom{r}{j+1} (i)_{r-1-j,\lambda} (1)_{j+1,\lambda} \big(1 - F_X(i)\big).$$

Assume that X, Y are discrete nonnegative integer-valued random variables. Let r_1, r_2 be positive integers. Then we have

$$\begin{aligned} E\big[X^{r_1} Y^{r_2}\big] &= \sum_{x=0}^{\infty} \sum_{y=0}^{\infty} x^{r_1} y^{r_2} p(x,y) = \sum_{x=1}^{\infty} \sum_{y=1}^{\infty} x^{r_1} y^{r_2} p(x,y) \qquad (18) \\ &= \sum_{x=0}^{\infty} \sum_{y=0}^{\infty} (x+1)^{r_1} (y+1)^{r_2} p(x+1, y+1) \\ &= \sum_{x=0}^{\infty} \sum_{y=0}^{\infty} \sum_{i=0}^{x} \big((i+1)^{r_1} - i^{r_1}\big) \sum_{j=0}^{y} \big((j+1)^{r_2} - j^{r_2}\big) p(x+1, y+1) \\ &= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \big((i+1)^{r_1} - i^{r_1}\big) \big((j+1)^{r_2} - j^{r_2}\big) \sum_{x=i}^{\infty} \sum_{y=j}^{\infty} p(x+1, y+1) \\ &= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \Big((i+1)^{r_1} - i^{r_1}\Big) \Big((j+1)^{r_2} - j^{r_2}\Big) P\{X > i,\ Y > j\} \\ &= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \Big((i+1)^{r_1} - i^{r_1}\Big) \Big((j+1)^{r_2} - j^{r_2}\Big) T(i,j) \\ &= \sum_{x=0}^{\infty} \sum_{y=0}^{\infty} \Big((x+1)^{r_1} - x^{r_1}\Big) \Big((y+1)^{r_2} - y^{r_2}\Big) T(x,y). \end{aligned}$$

Therefore, by (18), we obtain the following theorem.

Theorem 4. *Let r_1, r_2 be positive integers, and let X, Y be discrete nonnegative integer-valued random variables. Then we have*

$$E\Big[X^{r_1} Y^{r_2}\Big] = \sum_{x=0}^{\infty} \sum_{y=0}^{\infty} \Big((x+1)^{r_1} - x^{r_1}\Big) \Big((y+1)^{r_2} - y^{r_2}\Big) T(x,y),$$

where $T(x,y) = P\{X > x,\ Y > y\}$.

Assume that $X_1, X_2, \ldots, X_k$ are discrete nonnegative integer-valued random variables. The joint probability mass function of $X_1, X_2, \ldots, X_k$ is defined by

$$p(x_1, x_2, \ldots, x_k) = P\{X_1 = x_1, X_2 = x_2, \ldots, X_k = x_k\}. \tag{19}$$

The joint cumulative distribution function of $X_1, \ldots, X_k$ is given by

$$F_{X_1,X_2,\ldots,X_k}(a_1, a_2, \ldots, a_k) = P\{X_1 \le a_1, X_2 \le a_2, \ldots, X_k \le a_k\}. \tag{20}$$

Let

$$T(x_1, x_2, \ldots, x_k) = P\{X_1 > x_1, X_2 > x_2, \ldots, X_k > x_k\}. \tag{21}$$

For $r_1, r_2, \ldots, r_k \in \mathbb{N}$, we have

$$\begin{aligned}
E\Big[X_1^{r_1} X_2^{r_2} \cdots X_k^{r_k}\Big] &= \sum_{x_1=0}^{\infty}\sum_{x_2=0}^{\infty}\cdots\sum_{x_k=0}^{\infty} x_1^{r_1} x_2^{r_2}\cdots x_k^{r_k} p(x_1, x_2, \ldots, x_k) \qquad (22)\\
&= \sum_{x_1=1}^{\infty}\sum_{x_2=1}^{\infty}\cdots\sum_{x_k=1}^{\infty} x_1^{r_1} x_2^{r_2}\cdots x_k^{r_k} p(x_1, x_2, \ldots, x_k)\\
&= \sum_{x_1=0}^{\infty}\sum_{x_2=0}^{\infty}\cdots\sum_{x_k=0}^{\infty} (x_1+1)^{r_1}\cdots(x_k+1)^{r_k} p(x_1+1, \cdots, x_k+1)\\
&= \sum_{x_1=0}^{\infty}\sum_{x_2=0}^{\infty}\cdots\sum_{x_r=0}^{\infty}\sum_{i_1=0}^{x_1}\Big((i_1+1)^{r_1} - i_1^{r_1}\Big)\cdots\sum_{i_k=0}^{x_k}\Big((i_k+1)^{r_k} - i_k^{r_k}\Big)p(x_1+1, \cdots, x_k+1)\\
&= \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\cdots\sum_{i_k=0}^{\infty}\prod_{j=1}^{k}\Big((i_j+1)^{r_j} - i_j^{r_j}\Big)\sum_{x_1=i_1}^{\infty}\sum_{x_2=i_2}^{\infty}\cdots\sum_{x_k=i_k}^{\infty} p(x_1+1, x_2+1, \cdots, x_k+1)\\
&= \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\cdots\sum_{i_k=0}^{\infty}\prod_{j=1}^{k}\Big((i_j+1)^{r_j} - i_j^{r_j}\Big)P\{X_1 > i_1, X_2 > i_2, \ldots, X_k > i_k\}\\
&= \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\cdots\sum_{i_k=0}^{\infty}\prod_{j=1}^{k}\Big((i_j+1)^{r_j} - i_j^{r_j}\Big)T\big(i_1, i_2, \ldots, i_k\big)\\
&= \sum_{x_1=0}^{\infty}\sum_{x_2=0}^{\infty}\cdots\sum_{x_k=0}^{\infty}\prod_{j=1}^{k}\Big((x_j+1)^{r_j} - x_j^{r_j}\Big)T(x_1, x_2, \ldots, x_k).
\end{aligned}$$

Therefore, by (22), we obtain the following theorem.

Theorem 5. *Let $r_1, r_2, \cdots, r_k$ be positive integers, and let $X_1, X_2, \ldots, X_k$ be discrete nonnegative integer-valued random variables. Then the expectation of the monomial $X_1^{r_1} X_2^{r_2} \cdots X_k^{r_k}$ is given by*

$$E\Big[X_1^{r_1} X_2^{r_2} \cdots X_k^{r_k}\Big] = \sum_{x_1=0}^{\infty}\sum_{x_2=0}^{\infty}\cdots\sum_{x_k=0}^{\infty}\prod_{j=1}^{k}\Big((x_j+1)^{r_j} - x_j^{r_j}\Big)T(x_1, x_2, \ldots, x_k),$$

where $T(x_1, x_2, \ldots, x_k) = P\{X_1 > x_1, X_2 > x_2, \ldots, X_k > x_k\}$.

3. Conclusion

In recent years, many researchers have investigated various stuffs from probabilistic perspectives and obtained quite a few interesting results (see [1–27] and the references therein).

Let X be a discrete nonnegative integer-valued random variable. Then we showed that the r-th degenerate moment of X is given by

$$E\big[(X)_{r,\lambda}\big] = \sum_{i=0}^{\infty}\Big((i+1)_{r,\lambda} - (i)_{r,\lambda}\Big)\Big(1 - F_X(i)\Big)$$
$$= \sum_{i=0}^{\infty}\sum_{j=0}^{r-1}\binom{r}{j+1}(i)_{r-1-j,\lambda}(1)_{j+1,\lambda}\big(1 - F_X(i)\big).$$

Let $r_1, r_2, \cdots, r_k$ be positive integers, and let $X_1, X_2, \dots, X_k$ be discrete nonnegative integer-valued random variables. Then we proved that the expectation of the monomial $X_1^{r_1}X_2^{r_2}\cdots X_k^{r_k}$ in $X_1, X_2, \dots, X_k$ is given by

$$E\Big[X_1^{r_1}X_2^{r_2}\cdots X_k^{r_k}\Big] = \sum_{x_1=0}^{\infty}\sum_{x_2=0}^{\infty}\cdots\sum_{x_k=0}^{\infty}\prod_{j=1}^{k}\Big((x_j+1)^{r_j} - x_j^{r_j}\Big)T(x_1, x_2, \dots, x_k),$$

where $T(x_1, x_2, \dots, x_k) = P\{X_1 > x_1, X_2 > x_2, \dots, X_k > x_k\}$.

Indeed, we derived this first for $k = 2$ and $r_1 = 1, r_2 = 1$, then for $k = 2$ with r_1, r_2 any positive integers, and finally for the general case of any k and any $r_1, r_2, \cdots, r_k$. Certain interesting applications of these results will be treated in a forthcoming paper.

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