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Langevin Fractional System Driven by Two ψ-Caputo Derivatives With Random Effects

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Abstract. A nonlinear Langevin fractional system involving two ψ-Caputo derivatives with random effects is investigated. First, a random version of Perov's fixed-point theorem in generalized Banach space endowed with the Bielecki-type vector-valued norm is employed to achieve a uniqueness result. Second, the existence result is established using Sadovskii's fixed point principle under fairly general conditions on the nonlinear forcing terms. Finally, our findings are justified through illustrative examples.

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1. Introduction

Fractional calculus and its applications have garnered significant attention from scientists and researchers in recent years, not only in mathematics but also across various scientific disciplines, including physics [20], chemical kinetics [26], fluid dynamics [21], viscoelastic [11], electrochemistry [19], elasticity [4], engineering [29],economics [28], financial systems [22], biology [14], medicine [24], statistics [2], computing image [31], nonlinear heat conduction [8], optimal control [9], etc. Moreover, many cosmic events that classical differential equations cannot describe can be described by fractional differential equations.

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On the other side, Almeida [3] proposed a general definition of Caputo FD with respect to functions which is more flexible, beneficial and play an important role in modeling practical applications, see for instance [10]. Multitude scholars investigate several aspect of the theory [6, 32, 33].

The classical Langevin equation, as proposed in [18], is crucial for demonstrating how particles interact with their surrounding medium and the random forces or fluctuations that lead to their unpredictable motion. Nevertheless, the reliance on the specific relationship between a particle's position and velocity has prompted the development of the fractional Langevin model, aimed at describing anomalous diffusion phenomena [17]. Also, it's important to highlight that certain phenomena are more accurately described by coupled random systems. For example, in epidemiology, the migration of birds from various regions worldwide can introduce infectious diseases. Therefore, the transmission rate of these diseases increases as migratory birds flock together. Moreover, this scenario warrants consideration of the presence of random disturbances. While the above-mentioned motivational models have a great advantage, the difficulty of the corresponding mathematical model may significantly increase, complicating the study of the existence of solutions. Accordingly, exploring the qualitative aspects of ψ-Caputo nonlinear Langevin coupled systems with random effects has become increasingly important.

Recently, the authors in [13, 33] studied theoretically some quantitative aspects for the following problem:

$$\begin{cases} \left({}^{c}\mathfrak{D}_{a^+}^{\vartheta;\psi} + \varpi {}^{c}\mathfrak{D}_{a^+}^{\vartheta-1;\psi}\right) \mathfrak{z}(\xi) = f(\xi, \mathfrak{z}(\xi)), \quad \xi \in [a,b], \\ \mathfrak{z}(a) = \mathfrak{z}'(a) = 0, \end{cases}$$

where $1 < \vartheta < 2$, $\varpi \in \mathbb{R}$, ${}^{c}\mathfrak{D}_{a^+}^{\theta;\psi}$ represents the Caputo fractional derivative FD with respect to ψ of order $\theta \in \{\vartheta, \vartheta-1\}$, $f : [a,b] \times \mathbb{G} \to \mathbb{G}$ is a given function and $\mathbb{X}$ is a Banach space.

O. Zentar *et al.* in [34] investigated the existence of solutions for the following system:

$$\begin{cases} \mathfrak{D}_{0^+}^{\vartheta_1} \mathfrak{z}_1(\xi,\omega) = f_1(\xi, \mathfrak{z}_1(\xi,\omega), \mathfrak{z}_2(\xi,\omega), \omega), \quad \xi \in (0,b], \\ \mathfrak{D}_{0^+}^{\vartheta_2} \mathfrak{z}_2(\xi,\omega) = f_2(\xi, \mathfrak{z}_1(\xi,\omega), \mathfrak{z}_2(\xi,\omega), \omega), \quad \xi \in (0,b], \\ \lim\limits_{\xi\to 0^+} \xi^{1-\vartheta_1} \mathfrak{z}_1(\xi,\omega) = \mathfrak{Z}_3(\omega), \quad \omega \in \Omega, \\ \lim\limits_{\xi\to 0^+} \xi^{1-\vartheta_2} \mathfrak{z}_2(\xi,\omega) = \mathfrak{Z}_4(\omega), \quad \omega \in \Omega, \end{cases}$$

where $\mathfrak{Z}_3, \mathfrak{Z}_4 : \Omega \to \mathbb{G}$ are random variables, $\mathfrak{D}_{0^+}^{\vartheta_i}$ represents the standard Riemann-Liouville FD of order $\vartheta_i \in (0,1]$ for each $i = 1,2$ and $f_i : [0,b] \times \mathbb{G} \times \mathbb{G} \times \Omega \to \mathbb{G}$ are funcions and $(\mathbb{G}, \|\cdot\|)$ is a real separable Banach space.

Motivated by the preceding discussions, this paper presents new qualitative results for the following random coupled Langevin system involving ψ-Caputo FD:

$$\begin{cases} \left({}^{c}\mathfrak{D}_{a^+}^{\vartheta_1;\psi} + \varpi_1 {}^{c}\mathfrak{D}_{a^+}^{\vartheta_1-1;\psi}\right)\mathfrak{z}_1(\xi,\omega) = \mathfrak{f}_1(\xi,\mathfrak{z}_1(\xi,\omega),\mathfrak{z}_2(\xi,\omega),\omega),\ \xi\in\mathfrak{I}:=[a,b], \\ \left({}^{c}\mathfrak{D}_{a^+}^{\vartheta_2;\psi} + \varpi_2 {}^{c}\mathfrak{D}_{a^+}^{\vartheta_2-1;\psi}\right)\mathfrak{z}_2(\xi,\omega) = \mathfrak{f}_2(\xi,\mathfrak{z}_1(\xi,\omega),\mathfrak{z}_2(\xi,\omega),\omega),\ \xi\in\mathfrak{I}:=[a,b], \\ \mathfrak{z}_1(a,\omega) = \mathfrak{z}_1'(a,\omega) = 0, \\ \mathfrak{z}_2(a,\omega) = \mathfrak{z}_2'(a,\omega) = 0, \end{cases} \tag{1}$$

where $1 < \vartheta_i < 2$, $\varpi_i > 0$. ${}^{c}\mathfrak{D}_{a^+}^{\theta_i;\psi}$ (for $i = 1,2$) is the FD with respect to ψ of order $\theta_i \in \{\vartheta_i, \vartheta_i - 1\}$, $\mathfrak{f}_i : \mathfrak{I}\times\mathbb{G}\times\mathbb{G}\to\mathbb{G}$, $(i = 1,2)$ verifying some conditions that will be precised later.

A notable feature of our research is the following:

- We utilize Perov's fixed-point theorem with the Bielecki-type vector-valued norm to establish a new uniqueness criterion.
- We established existence results by applying Sadovskii's fixed-point principle in a random setting, utilizing the measure of noncompactness (MNC) procedure and the a priori estimate technique.
- The obtained findings generalize the results appearing in the existing research, such as in [6, 13, 33].

This research is structured as follows. Section 2 presents some preliminary facts that will be utilized in subsequent sections. The main results are provided in Section 3. Finally, illustrative examples are presented in Section 4.

2. Preliminary Results

Throughout the paper, let $(\mathbb{G}, \|\cdot\|)$ be a separable Banach space, we endow the space $C(\mathfrak{I},\mathbb{G})$ of $\mathbb{G}$-valued continuous functions on $\mathfrak{I}$ with the supnorm

$$\|\mathfrak{u}\|_\infty = \sup_{\xi\in\mathfrak{I}} \|\mathfrak{u}(\xi)\|. \tag{2}$$

$L^1(\mathfrak{I},\mathbb{G})$ denotes the space of Bochner integrable functions $\mathfrak{u}:\mathfrak{I}\to\mathbb{G}$ normed by

$$\|\mathfrak{u}\|_{L^1} = \int_a^b \|\mathfrak{u}(s)\|ds, \quad \text{for all } \mathfrak{u}\in L^1(\mathfrak{I},\mathbb{G}).$$

$L^\infty(\mathfrak{I},\mathbb{R}_+)$ stands for the space all essentially bounded functions normed by

$$\|\mathfrak{u}\|_{L^\infty} = \operatorname*{ess\,sup}_{\xi\in\mathfrak{I}} \|\mathfrak{u}(\xi)\| = \inf\{M > 0;\ \|\mathfrak{u}(\xi)\| \le M \text{ for almost every } \xi\in\mathfrak{I}\}.$$

Set

$$\mathbb{S}^1_+(\mathfrak{I},\mathbb{R}) = \{\psi \ : \ \psi \in C^1(\mathfrak{I},\mathbb{R}) \ \text{ and } \ \psi'(\xi) > 0 \ \text{ for all } \ \xi \in \mathfrak{I}\}.$$

Let $\psi \in \mathbb{S}^1_+(\mathfrak{I},\mathbb{R})$ for $\xi, s \in \mathfrak{I}$, $(s < \xi)$, we define

$$\phi(\xi,s) = \psi(\xi) - \psi(s) \ \text{ and } \ \phi(\xi,s)^{\vartheta} = (\psi(\xi) - \psi(s))^{\vartheta}.$$

Definition 1. *The Mittag-Leffler function is defined as follows:*

$$\mathbb{E}_{\vartheta}(\mathfrak{u}) = \sum_{j=0}^{\infty} \frac{\mathfrak{u}^j}{\Gamma(j\vartheta+1)}, \quad \vartheta > 0.$$

where $\Gamma(\cdot)$ *is the gamma function .*

Definition 2. [3, 16] *Let* $\psi \in \mathbb{S}^1_+(\mathfrak{I},\mathbb{R})$ *and* $\vartheta > 0$*. The* ψ*-fractional integral (FI) of a function* f *of order* ϑ *is defined by*

$$\mathfrak{I}^{\vartheta,\psi}_{a^+} f(\xi) = \frac{1}{\Gamma(\vartheta)} \int_a^{\xi} \phi(t,s)^{\vartheta-1} \psi'(s) f(s) ds, \quad t > a,$$

Lemma 1. [3, 16] *Let* $\vartheta, \gamma > 0$*, then*

$$\mathfrak{I}^{\vartheta;\psi}_{a^+} \ \phi(\xi,a)^{\gamma-1} = \frac{\Gamma(\gamma)}{\Gamma(\vartheta+\gamma)} \phi(\xi,a)^{\vartheta+\gamma-1}.$$

Definition 3. [3] *Let* $n-1 < \vartheta \leq n$ *with* $n \in \mathbb{N}$, $\psi \in \mathbb{S}^1_+(\mathfrak{I},\mathbb{R})$*. The* ψ*-Caputo FDs of a function* f *of order* ϑ *is defined as*

$$\left({}^c\mathfrak{D}^{\vartheta;\psi}_{a^+} f\right)(\xi) = \mathfrak{I}^{n-\vartheta;\psi}_{a^+} \left(\frac{1}{\psi'(\xi)} \frac{d}{d\xi}\right)^n f(\xi).$$

Now, for $\zeta > 0$, we endow the space $C(\mathfrak{I},\mathbb{G})$ by the Bielecky norm

$$\|f\|_B = \sup_{\xi \in \mathfrak{I}} e^{-\zeta\phi(\xi,a)} \|f(\xi)\|. \tag{3}$$

Lemma 2. [27, 30] *The norms* $\|\cdot\|_B$ *defined by (3) and* $\|\cdot\|_\infty$ *are equivalent, i.e; there exist* $c_0 \in (0,\infty)$ *such that*

$$\|\cdot\|_B \leq \|\cdot\|_\infty \leq c_0 \|\cdot\|_B.$$

Lemma 3. [6] *Let* $\vartheta > 1$ *and* $\zeta > 0$*. Then for all* $\xi \in \mathfrak{I}$*, one has*

$$\mathfrak{I}^{\vartheta-1;\psi}_{a^+} e^{\zeta\phi(\xi,a)} \leq \frac{e^{\zeta\phi(\xi,a)}}{\zeta^{\vartheta-1}}.$$

If, $y, v \in \mathbb{R}^n$, $y = (y_1, \ldots, y_n)$, $v = (v_1, \ldots, v_n)$, by $y \leq v$ we mean $y_i \leq v_i$ for all $i = 1, \ldots, n$. Also $|y| = (|y_1|, \ldots, |y_n|)$, $\max(y,v) = (\max(y_1,v_1), \ldots, \max(y_n,v_n))$ and $\mathbb{R}^n_+ = \{y \in \mathbb{R}^n : \ \ y_i > 0\}$. If $c \in \mathbb{R}$, then $y \leq c$ means $y_i \leq c$ for each $i = 1, \ldots, n$.

Definition 4. *Let $\mathbb{F}$ be a nonempty set. By a vector-valued metric on $\mathbb{F}$ we mean a map $d : \mathbb{F} \times \mathbb{F} \to \mathbb{R}^n_+$ with the following properties:*

(i) $d(y, v) \geq 0$ *for all* $y, v \in \mathbb{F}$; *if* $d(y, v) = 0$ *then* $y = v$;

(ii) $d(y, v) = d(v, u)$ *for all* $y, v \in \mathbb{F}$;

(iii) $d(y, v) \leq d(y, u) + d(u, v)$ *for all* $u, v, y \in \mathbb{F}$.

For $d_i,\ i = 1, \dots, n$ are metrics on $\mathbb{F}$, the pair $(\mathbb{F}, d)$ is called a generalized metric space (shortly, GMS) (or a vector-valued metric space) with $d(y, v) := \begin{pmatrix} d_1(y, v) \\ \vdots \\ d_n(y, v) \end{pmatrix}$.

Definition 5. *We call a matrix $\mathcal{M} \in \mathcal{M}_{n\times n}(\mathbb{R})$ of real numbers convergent to zero if its spectral radius $\rho(\mathcal{M}) < 1$. In other words, this means that all the eigenvalues of $\mathcal{M}$ are in the open unit disc i.e. $|\lambda| < 1$, for every $\lambda \in \mathbb{C}$ with $\det(\mathcal{M} - \lambda I) = 0$, where I denote the unit matrix.*

Proposition 1. [23] *Let $\mathcal{M} \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$. The following statements are equivalent:*

(i) $\mathcal{M}^r \to 0$ *when* $r \to \infty$.

(ii) $\mathcal{M}$ *is convergent to zero.*

(iii) The matrix $(I - \mathcal{M})$ *is nonsingular and*

$$(I - \mathcal{M})^{-1} = I + \mathcal{M} + \mathcal{M}^2 + \ldots + \mathcal{M}^k + \ldots .$$

(iv) $(I - \mathcal{M})$ *is nonsingular matrix and* $(I - \mathcal{M})^{-1}$ *has positive elements.*

Let $\mathbb{G}$ be a separable GMS and $(\Omega, \mathcal{F})$ be a measurable space. We denote $\mathcal{B}(\mathbb{G})$ the Borel σ-algebra on $\Omega \times \mathbb{G}$. Therefore, $\mathcal{F} \times \mathcal{B}(\mathbb{G})$ is the smallest σ-algebra on $\Omega \times \mathbb{G}$ which contains all the sets $F \times S$, where $F \in \mathcal{F}$ and $S \in \mathcal{B}(\mathbb{G})$.

Definition 6. *Given two separable GMSs $\mathbb{G}$ and $\mathbb{X}$, a mapping $\mathcal{Q} : \Omega \times \mathbb{G} \to \mathbb{X}$ is called a random operator if $\omega \longmapsto \mathcal{Q}(\omega, \mathfrak{u})$ is measurable for all $\mathfrak{u} \in \mathbb{G}$. The random operator $\mathcal{L}$ on $\mathbb{G}$ will be denoted by*

$$\mathcal{Q}(\mathfrak{u})(\omega) = \mathcal{Q}(\omega, \mathfrak{u}),\ \omega \in \Omega,\ \ \mathfrak{u} \in \mathbb{G}.$$

Definition 7. *The fixed point of a random operator $\mathcal{Q}$ is a measurable function $\mathfrak{u} : \Omega \to \mathbb{G}$ such that*

$$\mathfrak{u}(\omega) = \mathcal{Q}(\omega, \mathfrak{u}(\omega)) \quad \text{for all} \quad \omega \in \Omega.$$

Definition 8. *Let $f : \mathfrak{I} \times \mathbb{G} \times \Omega \to \mathbb{X}$ is called random Carathéodory if the following statements are verified:*

(i) The map $\mathfrak{u} \longmapsto f(\xi, \mathfrak{u}, \omega)$ *is continuous for all* $\xi \in \mathfrak{I}$ *and* $\omega \in \Omega$.

(ii) The map $(\xi, \omega) \longmapsto f(\xi, \mathfrak{u}, \omega)$ *is jointly measurable for all* $\mathfrak{u} \in \mathbb{G}$.

Lemma 4. [25] *Let* $\mathbb{G}$ *be a separable metric space and* $\mathcal{Q} : \Omega \times \mathbb{G} \to \mathbb{G}$ *be a mapping such that* $\mathcal{Q}(\omega, \cdot)$ *is continuous for all* $\omega \in \Omega$ *and* $\mathcal{Q}(\cdot, \mathfrak{u})$ *is measurable for all* $\mathfrak{u} \in \mathbb{G}$ *. Then the map* $(\omega, \mathfrak{u}) \to \mathcal{Q}(\omega, \mathfrak{u})$ *is jointly measurable.*

Definition 9. [12] *Let* $\mathbb{G}$ *be a generalized Banach space and* $(\mathcal{O}, \leq)$ *be a partially ordered set. A map* $\Lambda : \mathcal{P}(\mathbb{G}) \to \mathcal{O} \times \mathcal{O} \times \ldots \times \mathcal{O}$ *is called a generalized MNC on* $\mathbb{G}$, *if*

$$\Lambda(\overline{co}\ \mathcal{O}) = \Lambda(\mathcal{O}) \text{ for every } \mathcal{O} \in \mathcal{P}(\mathbb{G}),$$

where $\Lambda(\mathcal{O}) := \begin{pmatrix} \Lambda_1(\mathcal{O}) \\ \vdots \\ \Lambda_n(\mathcal{O}) \end{pmatrix}$, $\mathcal{P}(\mathbb{G})$ *denotes the family of all bounded subsets of* $\mathbb{G}$ *and* $\overline{co}\mathcal{O}$ *is the closed convex hull of* $\mathcal{O}$.

Definition 10. *The application* Λ *is called:*

(i) Monotone if $\mathcal{O}_0, \mathcal{O}_1 \in \mathcal{P}(\mathbb{G}), \mathcal{O}_0 \subset \mathcal{O}_1$ *implies* $\Lambda(\mathcal{O}_0) \leq \Lambda(\mathcal{O}_1)$.

(ii) Nonsingular if $\Lambda(\{a\} \cup \mathcal{O}) = \Lambda(\mathcal{O})$ *for every* $a \in \mathbb{G}$ *and* $\mathcal{O} \in \mathcal{P}(\mathbb{G})$.

If $\mathcal{O}$ *is a cone in a normed space, we say that the MNC is*

(iii) Regular if the condition $\Lambda(\mathcal{O}) = 0$ *is equivalent to the compactness of* $\overline{\mathcal{O}}$.

The most well-known example of a MNC possessing all previous properties is the Hausdorff MNC defined by:

$$\eta(\mathcal{O}) = \inf \{\epsilon > 0 : \mathcal{O} \text{ has a finite } \epsilon - \text{net}\}.$$

Definition 11. [12] *Let* $\mathbb{X}, \mathbb{Y}$ *be two generalized normed spaces. A continuous map* $G : \mathbb{X} \to \mathbb{Y}$ *is called a* $\mathcal{M}$*-contraction (with respect to the generalized MNC* Λ*) if there exists a matrix* $\mathcal{M} \in \mathcal{M}_{n\times n}(\mathbb{R})$ *converges to zero such that for every* $D \in \mathcal{P}(\mathbb{X})$, *one has*

$$\Lambda(G(D)) \leq \mathcal{M}\Lambda(D).$$

Lemma 5. [15] *If* $\{x_n\}_{n=1}^{+\infty} \subset L^1(\mathfrak{I}, \mathbb{G})$ *satisfies* $\|x_n(\xi)\| \leq \iota(\xi)$ *a.e. on* $\mathfrak{I}$ *for all* $n \geq 1$ *with some* $\iota \in L^1(\mathfrak{I}, \mathbb{R}_+)$. *Then, the function* $\eta(\{x_n(\xi)\}_{n=1}^{+\infty})$ *is integrable and*

$$\eta\left(\left\{\int_0^\xi x_n(s)ds \ : \ n \geq 1\right\}\right) \leq \int_0^\xi \eta(x_n(s) \ : \ n \geq 1)ds. \tag{4}$$

Theorem 1. [7, 25] *Let* $\mathbb{X}$ *be a real separable generalized Banach space and* $(\Omega, \mathcal{G})$ *be a measurable space and* $\mathcal{Q} : \Omega \times \mathbb{X} \to \mathbb{X}$ *a continuous random operator, and let* $\mathcal{M}(\omega) \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ *be a random variable matrix such that for every* $\omega \in \Omega$, *the matrix* $\mathcal{M}(\omega)$ *converges to zero and:*

$$d(\mathcal{Q}(\omega, \mathfrak{z}_1), \mathcal{Q}(\omega, \mathfrak{z}_2)) \leq \mathcal{M}(\omega)d(\mathfrak{z}_1, \mathfrak{z}_2), \quad \text{for each } \mathfrak{z}_1, \mathfrak{z}_2 \in \mathbb{X} \text{ and } \omega \in \Omega.$$

Then, there exists a unique random fixed point of $\mathcal{Q}$.

Theorem 2. [7, 25] *Let $\mathbb{G}$ be a separable generalized Banach space, and let $\mathfrak{Q} : \Omega \times \mathbb{G} \to \mathbb{G}$ be a condensing continuous random operator. Then either of the following holds:*

(i) The random equation $\mathfrak{Q}(\omega, \mathfrak{z}) = \mathfrak{z}$ has a random solution, i.e., there is a measurable function $\mathfrak{z} : \Omega \to \mathbb{G}$ such that $\mathfrak{Q}(\omega, \mathfrak{z}(\omega)) = \mathfrak{z}(\omega)$ for all $\omega \in \Omega$,
or

(ii) The set

$$\mathbb{W} = \{\mathfrak{z} : \Omega \to \mathbb{G} \text{ is measurable } \kappa(\omega)\mathfrak{Q}(\omega, \mathfrak{z}) = \mathfrak{z}\}$$

is unbounded for some measurable function $\kappa : \Omega \to \mathbb{G}$ with $\mu(\omega) \in (0, 1)$ on Ω.

Lemma 6. [5, Corollary 2.1.] *Let $\alpha_l > 0$, $l = \overline{1, n}$, $n \in \mathbb{N}$ and $\psi \in \mathbb{S}^1_+(\mathfrak{I}, \mathbb{R})$. Assume that*

(i) The functions g_l are the bounded and monotonic increasing functions on $[a, b)$,

(ii) $\mathfrak{s}$ and $\mathfrak{u}$ are nonnegative functions locally integrable on $[a, b)$.

(iii) $\mathfrak{u}(\xi)$ is a nondecreasing function for $\xi \in [a, b)$,

If

$$\mathfrak{s}(\xi) \leq \mathfrak{u}(\xi) + \sum_{l=1}^{n} g_l(\xi) \int_a^\xi \phi(\xi, s)^{\alpha_l - 1} \mathfrak{s}(s)\psi'(s)ds, \tag{5}$$

then

$$\mathfrak{s}(\xi) \quad \leq \mathfrak{u}(\xi) \sum_{l=0}^{n} \mathbb{E}_{\alpha_l} \left(g_l(\xi)\Gamma(\alpha_l)\phi(\xi, a)^{\alpha_l} \right).$$

Lemma 7. *Let $\psi \in \mathbb{S}^1_+(\mathfrak{I}, \mathbb{R})$, $\gamma > 0$, $1 < \vartheta_i < 2$, $\varpi_i > 0$ and a constant random variable $\varrho_{i,j} : \Omega \to [0, \infty)$, $i, j = 1, 2$. Then*

$$\aleph_{i,j}(\gamma, \omega) := \sup_{\xi \in \mathfrak{I}} \frac{4e^{\varpi_i \phi(b,a)} \varrho_{i,j}(\omega)}{\Gamma(\vartheta_i - 1)} \int_a^\xi \psi'(s)\phi(s, a)^{\vartheta_i - 1} e^{-\gamma(\xi - s)} ds \xrightarrow[\gamma \to +\infty]{} 0, \; i, j = 1, 2, \tag{6}$$

Proof. From

$$\phi(\cdot, a)^{\vartheta_i - 1}\psi'(\cdot) \in L^1(\mathfrak{I}, \mathbb{R}), \quad i = 1, 2.$$

So, there exists $\hbar \in C(\mathfrak{I}, \mathbb{R})$ such that

$$\int_a^\xi \left| \phi(s, a)^{\vartheta_i - 1}\psi'(s) - \hbar(s) \right| ds < \frac{1}{2}\epsilon.$$

Hence

$$\begin{aligned}
&\left| \int_a^\xi \phi(s, a)^{\vartheta_i - 1}\psi'(s)e^{-\gamma(\xi - s)} ds \right| \\
&\qquad \leq \int_a^\xi \left| \phi(s, a)^{\vartheta_i - 1}\psi'(s) - \hbar(s) \right| e^{-\gamma(\xi - s)} ds + \int_a^\xi |\hbar(s)| e^{-\gamma(\xi - s)} ds \\
&\qquad \leq \frac{\epsilon}{2} + \frac{1 - e^{-\gamma(b-a)}}{\gamma} \|\hbar\|_\infty, \quad i = 1, 2,
\end{aligned}$$

Consequently,

$$\int_a^\xi \psi'(s)\phi(s,a)^{\vartheta_i-1}e^{-\gamma(\xi-s)}ds \longrightarrow 0 \text{ as } \gamma \longrightarrow +\infty, \quad i=1,2.$$

This completes the proof of the Lemma.

3. Main results

Our first result establishes the existence and uniqueness result for the system (1), where Perov's fixed-point principle is applied.

Theorem 3. *Suppose that*

(A1) The functions $\mathfrak{f}_i$ are random Carathéodory on $\mathfrak{I}\times\mathbb{G}\times\mathbb{G}\times\Omega$.

(A2) There exists random variables $\Phi_{i,j}:\Omega\to(0,\infty)$; $i,j=1,2$ such that:

$$\|\mathfrak{f}_i(\xi,u_1,u_2,\omega)-\mathfrak{f}_i(\xi,v_1,v_2,\omega)\|\le\Phi_{i,1}(\omega)\|u_1-v_1\|+\Phi_{i,2}(\omega)\|u_2-v_2\|,\ i=1,2,$$

for $u_1,u_2,v_1,v_2\in\mathbb{G}$, $(\xi,\omega)\in\mathfrak{I}\times\Omega$.

Then, system (1) admits a unique random solution.

Proof. Firstly, endowing the product Banach space $\mathbb{J}=C(\mathfrak{I},\mathbb{G})\times C(\mathfrak{I},\mathbb{G})$ by the vector-norm

$$\|(\mathfrak{z}_1,\mathfrak{z}_2)\|_{\mathbb{J}}=\begin{pmatrix}\|\mathfrak{z}_1\|_\infty\\ \|\mathfrak{z}_2\|_\infty\end{pmatrix}. \tag{7}$$

Next, according to [13, Theorem 3.1], system (1) is equivalent to the operator equation $\mathcal{H}(\mathfrak{z}_1,\mathfrak{z}_2,\omega)=(\mathfrak{z}_1,\mathfrak{z}_2)$ where $\mathcal{H}:\mathbb{J}\times\Omega\to\mathbb{J}$ be the operator given by:

$$\mathcal{H}(\mathfrak{z}_1(\xi,\omega),\mathfrak{z}_2(\xi,\omega),\omega)=\left(\mathcal{H}_1(\mathfrak{z}_1(\xi,\omega),\mathfrak{z}_2(\xi,\omega),\omega),\mathcal{H}_2(\mathfrak{z}_1(\xi,\omega),\mathfrak{z}_2(\xi,\omega),\omega)\right). \tag{8}$$

where,

$$\begin{aligned}&\mathcal{H}_i(\mathfrak{z}_1(\xi,\omega),\mathfrak{z}_2(\xi,\omega),\omega)\\&=(\vartheta_i-1)\int_a^\xi e^{-\varpi_i\phi(\xi,s)}\left(\int_a^s\frac{\psi'(\tau)\phi(s,\tau)^{\vartheta_i-2}}{\Gamma(\vartheta_i-1)}\mathfrak{f}_i(\tau,\mathfrak{z}_1(\tau,\omega),\mathfrak{z}_2(\tau,\omega),\omega)d\tau\right)\psi'(s)ds,\ i=1,2.\end{aligned} \tag{9}$$

Since the function $\mathfrak{f}_i$, $i=1,2$ are absolutely continuous for all $\omega\in\Omega$ and $\xi\in\mathfrak{I}$, then $(\mathfrak{z}_1,\mathfrak{z}_2)$ is a random solution for the problem (1) if and only if $(\mathfrak{z}_1,\mathfrak{z}_2)=(\mathcal{H}(\mathfrak{z}_1,\mathfrak{z}_2))(\xi,\omega)$.

We need to demonstrate that the operator $\mathcal{H}$ is a contraction mapping on $\mathbb{J}$ using Bielecki's vector-norm.

Step 1. $\mathcal{H}$ is a random operator on $\mathbb{J}$.

Using (A1), the functions $\omega \to \mathfrak{f}_i(\xi, \mathfrak{z}_1, \mathfrak{z}_2, \omega)$ are measurable for $i = 1, 2$. In view of Lemma 4, the products

$$\phi(s,\tau)^{\vartheta_i - 2}\mathfrak{f}_i(\tau, \mathfrak{z}_1(\tau,\omega), \mathfrak{z}_2(\tau,\omega), \omega), \quad i = 1, 2,$$

are again measurable. Further, the integral is a limit of a finite sum of measurable functions, therefore, the maps

$$\omega \to \mathcal{H}_i(\mathfrak{z}_1(\xi,\omega), \mathfrak{z}_2(\xi,\omega), \omega), \quad i = 1, 2,$$

are measurable. Accordingly, $\mathcal{H}$ is a random operator on $\mathbb{J} \times \Omega$ into $\mathbb{J}$.

Step 2. $\mathcal{H}$ is a contraction mapping on $\mathbb{J}$.

For any $\omega \in \Omega$ and each $(\mathfrak{z}_1, \mathfrak{z}_2), (\mathfrak{r}_1, \mathfrak{r}_2) \in \mathbb{J}$, using (A2), we can get

$$\begin{aligned}
&\|\mathcal{H}_i(\mathfrak{z}_1(\xi,\omega), \mathfrak{z}_2(\xi,\omega), \omega) - \mathcal{H}_i(\mathfrak{r}_1(\xi,\omega), \mathfrak{r}_2(\xi,\omega), \omega)\| \\
&\le (\vartheta_i - 1)\int_a^\xi \psi'(s)e^{-\varpi_i\phi(\xi,s)}\int_a^s \frac{\psi'(\tau)\phi(s,\tau)^{\vartheta_i-2}}{\Gamma(\vartheta_i-1)}\|\mathfrak{f}_i(\tau, \mathfrak{z}_1(\tau,\omega), \mathfrak{z}_2(\tau,\omega), \omega) \\
&\quad -\mathfrak{f}_i(\tau, \mathfrak{r}_1(\tau,\omega), \mathfrak{r}_2(\tau,\omega), \omega)\|d\tau ds \\
&\le \sum_{j=1}^{2}(\vartheta_i - 1)\int_a^\xi \psi'(s)e^{-\varpi_i\phi(\xi,s)}\int_a^s \frac{\psi'(\tau)\phi(s,\tau)^{\vartheta_i-2}}{\Gamma(\vartheta_i-1)}\Phi_{i,j}(\omega)\times \\
&\quad \|\mathfrak{z}_j(\tau,\omega) - \mathfrak{r}_j(\tau,\omega)\|d\tau ds, \quad i = 1, 2.
\end{aligned}$$

which, by (3), can be written as

$$\begin{aligned}
&\|\mathcal{H}_i(\mathfrak{z}_1(\xi,\omega), \mathfrak{z}_2(\xi,\omega), \omega) - \mathcal{H}_i(\mathfrak{r}_1(\xi,\omega), \mathfrak{r}_2(\xi,\omega), \omega)\| \\
&\le \sum_{j=1}^{2}(\vartheta_i - 1)\Phi_{i,j}(\omega)\|\mathfrak{z}_j(\cdot,\omega) - \mathfrak{r}_j(\cdot,\omega)\|_B \times \\
&\quad \int_a^\xi \psi'(s)e^{-\varpi_i\phi(\xi,s)}\int_a^s \frac{\psi'(\tau)\phi(s,\tau)^{\vartheta_i-2}}{\Gamma(\vartheta_i-1)}e^{\zeta\phi(\tau,a)}d\tau ds, \quad i = 1, 2.
\end{aligned}$$

By Lemma 3, one obtains

$$\begin{aligned}
&\|\mathcal{H}_i(\mathfrak{z}_1(\xi,\omega), \mathfrak{z}_2(\xi,\omega), \omega) - \mathcal{H}_i(\mathfrak{r}_1(\xi,\omega), \mathfrak{r}_2(\xi,\omega), \omega)\| \\
&\le \sum_{j=1}^{2}(\vartheta_i - 1)\Phi_{i,j}(\omega)\|\mathfrak{z}_j(\cdot,\omega) - \mathfrak{r}_j(\cdot,\omega)\|_B \int_a^\xi \psi'(s)\frac{e^{-\varpi_i\phi(\xi,s)}e^{\zeta\phi(s,a)}}{\zeta^{\vartheta_i-1}}ds, \\
&\le \sum_{j=1}^{2}(\vartheta_i - 1)\Phi_{i,j}(\omega)\|\mathfrak{z}_j(\cdot,\omega) - \mathfrak{r}_j(\cdot,\omega)\|_B \frac{e^{-\varpi_i\psi(\xi)-\zeta\psi(a)}}{(\zeta+\varpi_i)\zeta^{\vartheta_i-1}}\int_a^\xi \psi'(s)(\zeta+\varpi_i)e^{(\zeta+\varpi_i)\psi(s)}ds \\
&= \sum_{j=1}^{2}\frac{(\vartheta_i - 1)e^{-\varpi_i\psi(\xi)-\zeta\psi(a)}}{(\zeta+\varpi_i)\zeta^{\vartheta_i-1}}\left[e^{(\zeta+\varpi_i)\psi(\xi)} - e^{(\zeta+\varpi_i)\psi(a)}\right]\Phi_{i,j}(\omega)\|\mathfrak{z}_j(\cdot,\omega) - \mathfrak{r}_j(\cdot,\omega)\|_B, \quad i = 1, 2.
\end{aligned}$$

By $e^{(\varpi_i+\zeta)\psi(\xi)} - e^{(\varpi_i+\zeta)\psi(a)} \leq e^{(\varpi_i+\zeta)\psi(\xi)}$ and $e^{-\varpi_i\psi(\xi)-\zeta\psi(a)} \leq e^{-(\varpi_i+\zeta)\psi(a)}$, we get

$$\begin{aligned}&\|\mathcal{H}_i(\mathfrak{z}_1(\xi,\omega),\mathfrak{z}_2(\xi,\omega),\omega) - \mathcal{H}_i(\mathfrak{r}_1(\xi,\omega),\mathfrak{r}_2(\xi,\omega),\omega)\|\\ &\quad\leq \sum_{j=1}^{2} \frac{(\vartheta_i - 1)e^{(\varpi_i+\zeta)\phi(\xi,a)}}{(\zeta+\varpi_i)\zeta^{\vartheta_i-1}}\Phi_{i,j}(\omega)\|\mathfrak{z}_j(\cdot,\omega) - \mathfrak{r}_j(\cdot,\omega)\|_B, \quad i = 1,2.\end{aligned}$$

Hence

$$\begin{aligned}&\|\mathcal{H}_i(\mathfrak{z}_1(\cdot,\omega),\mathfrak{z}_2(\cdot,\omega),\omega) - \mathcal{H}_i(\mathfrak{r}_1(\cdot,\omega),\mathfrak{r}_2(\cdot,\omega),\omega)\|_B\\ &\quad\leq \sum_{j=1}^{2} \frac{(\vartheta_i - 1)e^{\varpi_i\phi(\xi,a)}}{(\zeta+\varpi_i)\zeta^{\vartheta_i-1}}\Phi_{i,j}(\omega)\|\mathfrak{z}_j(\cdot,\omega) - \mathfrak{r}_j(\cdot,\omega)\|_B, \quad i = 1,2.\end{aligned}$$

Therefore, we have

$$d((\mathcal{H}(\mathfrak{z}_1,\mathfrak{z}_2))(\cdot,\omega),(\mathcal{H}(\mathfrak{r}_1,\mathfrak{r}_2))(\cdot,\omega)) \leq N_\zeta(\omega)d\left((\mathfrak{z}_1(\cdot,\omega),\mathfrak{z}_2(\cdot,\omega)),(\mathfrak{r}_1(\cdot,\omega),\mathfrak{r}_2(\cdot,\omega))\right),$$

where:

$$N_\zeta(\omega) = \begin{pmatrix} \dfrac{(\vartheta_1-1)e^{\varpi_1\phi(b,a)}}{(\zeta+\varpi_1)\zeta^{\vartheta_1-1}}\Phi_{1,1}(\omega) & \dfrac{(\vartheta_1-1)e^{\varpi_1\phi(b,a)}}{(\zeta+\varpi_1)\zeta^{\vartheta_1-1}}\Phi_{1,2}(\omega) \\ \\ \dfrac{(\vartheta_2-1)e^{\varpi_2\phi(b,a)}}{(\zeta+\varpi_2)\zeta^{\vartheta_2-1}}\Phi_{2,1}(\omega) & \dfrac{(\vartheta_2-1)e^{\varpi_2\phi(b,a)}}{(\zeta+\varpi_2)\zeta^{\vartheta_2-1}}\Phi_{2,2}(\omega) \end{pmatrix},$$

and

$$d\left((\mathfrak{z}_1(\cdot,\omega),\mathfrak{z}_2(\cdot,\omega)),(\mathfrak{r}_1(\cdot,\omega),\mathfrak{r}_2(\cdot,\omega))\right) = \begin{pmatrix} \|\mathfrak{z}_1(\cdot,\omega) - \mathfrak{r}_1(\cdot,\omega)\|_{\mathfrak{B}} \\ \|\mathfrak{z}_2(\cdot,\omega) - \mathfrak{r}_2(\cdot,\omega)\|_{\mathfrak{B}} \end{pmatrix}.$$

Choosing $\zeta > 0$ large enough, the matrix $N_\zeta(\omega)$ converges to zero. Then, according to Theorem 1, $\mathcal{H}$ possesses a unique random fixed-point, serving as the unique random solution to system (1).

Our second result investigates the existence result for the system (1), employing Theorem 2 as a tool.

Theorem 4. *Suppose that*

(A1) The functions $\mathfrak{f}_i$ *are random Carathéodory on* $\mathfrak{I} \times \mathbb{G} \times \mathbb{G} \times \Omega$.

(A3) There exist $\Psi_i : \mathfrak{I} \times \Omega \to L^\infty(\mathfrak{I}, \mathbb{R}_+)$, $i = 1,2$ *such that*

$$\|\mathfrak{f}_i(\xi,u_1,u_2,\omega)\| \leq \Psi_i(\xi,\omega)(1 + \|u_1\| + \|u_2\|), \quad i = 1,2,$$

for all $(\xi,u_1,u_2,\omega) \in \mathfrak{I} \times \mathbb{G}^2 \times \Omega$.

(A4) There exists a constant random variable $\varrho_{i,j} : \Omega \to [0,\infty)$, $i,j = 1,2$ *such that for each* $U^j \subset \mathcal{P}(C(\mathfrak{I},\mathbb{O}))$,

$$\Lambda(\mathfrak{f}_i(\xi,U^1,U^2,\omega)) \leq \sum_{j=1}^{2} \varrho_{i,j}(\omega)\Lambda(U^j(\xi)), \ \textit{for all } (\xi,\omega) \in \mathfrak{I} \times \Omega.$$

Then, the system (1) possesses at least one random solution.

For easy computations, let $\Psi_i^* = \sup_{\omega\in\Omega} \|\Psi_i(\cdot,\omega)\|_{L^\infty}, \quad i=1,2.$

Proof. For $R>0$, consider a closed ball

$$\mathbb{B}_R = \{(\mathfrak{z}_1,\mathfrak{z}_2)\in\mathbb{J} \ : \ \|\mathfrak{z}_i(\cdot,\omega)\|_\infty < R, i=1,2\}. \tag{10}$$

The proof of Theorem 4 will proceed through several steps.

Step 1. $\mathcal{H}$ transforms bounded sets into bounded sets in $\mathbb{J}$.

Let $(\mathfrak{z}_1,\mathfrak{z}_2)\in\mathbb{B}_R$ and $\xi\in\mathfrak{I}$, then for $i=1,2$ we have :

$$\begin{aligned}&\|\mathcal{H}_i(\mathfrak{z}_1(\xi,\omega),\mathfrak{z}_2(\xi,\omega),\omega)\|\\ &\leq(\vartheta_i-1)e^{-\varpi_i\phi(\xi,a)}\int_a^\xi e^{\varpi_i\phi(s,a)}\left(\int_a^s\frac{\psi'(\tau)\phi(s,\tau)^{\vartheta_i-2}}{\Gamma(\vartheta_i-1)}\|\mathfrak{f}_i(\tau,\mathfrak{z}_1(\tau,\omega),\mathfrak{z}_2\tau,\omega))\|d\tau\right)\psi'(s)ds\end{aligned}$$

By using hypothesis (A3), for each $\xi\in\mathfrak{I}$, we have

$$\begin{aligned}\|\mathfrak{f}_i(\tau,\mathfrak{z}_1(\tau,\omega),\mathfrak{z}_2\tau,\omega))\| &\leq \Psi_i(\tau,\omega)(1+\|\mathfrak{z}_1(\tau,\omega)\|+\|\mathfrak{z}_2(\tau,\omega)\|)\\ &\leq\|\Psi_i(\cdot,\omega)\|_{L^\infty}(1+\|\mathfrak{z}_1(\cdot,\omega)\|_\infty+\|\mathfrak{z}_1(\cdot,\omega)\|_\infty),\quad i=1,2.\end{aligned} \tag{11}$$

So, by the fact $e^{-\varpi_i\phi(\xi,a)}\leq 1$ for $\xi\in\mathfrak{I}$ and using (11) to gathere Lemma 1 , we get

$$\begin{aligned}&\|\mathcal{H}_i(\mathfrak{z}_1(\xi,\omega),\mathfrak{z}_2(\xi,\omega),\omega)\|\\ &\leq(\vartheta_i-1)\|\Psi_i(\cdot,\omega)\|_{L^\infty}(1+\|\mathfrak{z}_1(\cdot,\omega)\|_\infty+\|\mathfrak{z}_1(\cdot,\omega)\|_\infty)\int_a^\xi e^{\varpi_i\phi(s,a)}\int_a^s\frac{\psi'(\tau)\phi(s,\tau)^{\vartheta_i-2}}{\Gamma(\vartheta_i-1)}d\tau\psi'(s)ds\\ &\leq(1+2R)\|\Psi_i(\cdot,\omega)\|_{L^\infty}\int_a^\xi e^{\varpi_i\phi(s,a)}\frac{\phi(s,a)^{\vartheta_i-1}}{\Gamma(\vartheta_i-1)}\psi'(s)ds\\ &\leq(1+2R)\|\Psi_i(\cdot,\omega)\|_{L^\infty}e^{\varpi_i\phi(b,a)}\frac{\phi(\xi,a)^{\vartheta_i}}{\vartheta_i\Gamma(\vartheta_i-1)},\quad i=1,2.\end{aligned}$$

Hence

$$\|\mathcal{H}_i(\mathfrak{z}_1(\cdot,\omega),\mathfrak{z}_2(\cdot,\omega),\omega)\|_\infty\leq(1+2R)\|\Psi_i(\cdot,\omega)\|_{L^\infty}e^{\varpi_i\phi(b,a)}\frac{\phi(b,a)^{\vartheta_i}}{\vartheta_i\Gamma(\vartheta_i-1)},\quad i=1,2.$$

This implies that:

$$\begin{aligned}\|\mathcal{H}(\mathfrak{z}_1(\cdot,\omega),\mathfrak{z}_2(\cdot,\omega),\omega)\|_{\mathbb{J}} &=\|\mathcal{H}_1(\mathfrak{z}_1(\cdot,\omega),\mathfrak{z}_2(\cdot,\omega),\omega)\|_\infty+\|\mathcal{H}_2(\mathfrak{z}_1(\cdot,\omega),\mathfrak{z}_2(\cdot,\omega),\omega)\|_\infty\\ &\leq\sum_{i=1}^2(1+2R)\|\Psi_i(\cdot,\omega)\|_{L^\infty}e^{\varpi_i\phi(b,a)}\frac{\phi(b,a)^{\vartheta_i}}{\vartheta_i\Gamma(\vartheta_i-1)}.\end{aligned}$$

This shows that $\mathcal{H}$ transforms bounded sets into bounded sets in $\mathbb{J}$.

Step 2. $\mathcal{H}$ is continuous.

Let $\{\mathfrak{z}_{1,n}, \mathfrak{z}_{2,n}\}$ be a sequence satisfying $\{\mathfrak{z}_{1,n}, \mathfrak{s}_{z,n}\} \to (\mathfrak{z}_1, \mathfrak{z}_2)$ in $\mathbb{B}_R$ as $n \to \infty$. For each $(\xi, \omega) \in \mathfrak{I} \times \Omega$, making use of (A1), we easily have

$$\|\mathfrak{f}_i(\tau, \mathfrak{z}_{1,n}(\tau,\omega), \mathfrak{z}_{2,n}(\tau,\omega), \omega) - \mathfrak{f}_i(\tau, \mathfrak{z}_1(\tau,\omega), \mathfrak{z}_2(\tau,\omega), \omega)\| \to 0, \text{ as } \; n \longrightarrow \infty, \, i = 1, 2.$$

Next, in view of (A3), one gets

$$\begin{aligned}
&\|\mathfrak{f}_i(\tau, \mathfrak{z}_{1,n}(\tau,\omega), \mathfrak{z}_{2,n}(\tau,\omega), \omega) - \mathfrak{f}_i(\tau, \mathfrak{z}_1(\tau,\omega), \mathfrak{z}_2(\tau,\omega), \omega)\| \\
&\quad \le \|\mathfrak{f}_i(\tau, \mathfrak{z}_{1,n}(\tau,\omega), \mathfrak{z}_{2,n}(\tau,\omega), \omega)\| + \|\mathfrak{f}_i(\tau, \mathfrak{z}_{1,n}(\tau,\omega), \mathfrak{z}_{2,n}(\tau,\omega), \omega)\| \\
&\quad \le 2\Psi_i(\tau,\omega)\left(1 + \|\mathfrak{z}_1(\tau,\omega)\| + \|\mathfrak{z}_2(\tau,\omega)\|\right) \\
&\quad \le 2(1+2R)\Psi_i(\tau,\omega), \quad i = 1, 2.
\end{aligned}$$

Since, the functions $\tau \mapsto \dfrac{\psi'(\tau)\phi(s,\tau)^{\vartheta_i-2}}{\Gamma(\vartheta_i-1)}\Psi_i(\tau,\omega)$ and $s \mapsto \dfrac{\psi'(s)\phi(s,a)^{\vartheta_i-1}}{\Gamma(\vartheta_i)}\Psi_i(s,\omega)$, $i = 1, 2$ are Lebesgue integrable over $[a,s]$ (resp. $[a,\xi]$). Then it follows from the Lebesgue dominated convergence theorem that

$$\begin{aligned}
&\|\mathcal{H}_i(\mathfrak{z}_{1,n}(\xi,\omega), \mathfrak{z}_{2,n}(\xi,\omega), \omega) - \mathcal{H}_i(\mathfrak{z}_1(\xi,\omega), \mathfrak{z}_2(\xi,\omega), \omega)\| \\
&\le (\vartheta_i - 1)e^{\varpi_i\phi(b,a)} \int_a^\xi \int_a^s \frac{\psi'(\tau)\phi(s,\tau)^{\vartheta_i-2}}{\Gamma(\vartheta_i-1)} \times \\
&\quad \|\mathfrak{f}_i(\tau, \mathfrak{z}_{1,n}(\tau,\omega), \mathfrak{z}_{2,n}(\tau,\omega), \omega) - \mathfrak{f}_i(\tau, \mathfrak{z}_1(\tau,\omega), \mathfrak{z}_2(\tau,\omega), \omega)\| d\tau \psi'(s) ds \\
&\xrightarrow[n\to\infty]{} 0, \;\text{ for all } \xi \in \mathfrak{I}, \, i = 1, 2.
\end{aligned}$$

Therefore,

$$\|\mathcal{H}_i(\cdot, \mathfrak{z}_{1,n}(\cdot,\omega), \mathfrak{z}_{2,n}(\cdot,\omega), \omega) - \mathcal{H}_i(\cdot, \mathfrak{z}_1(\cdot,\omega), \mathfrak{z}_2(\cdot,\omega), \omega)\|_\infty \xrightarrow[n\to\infty]{} 0, \quad i = 1, 2.$$

Accordingly, the operator $\mathcal{H}(\cdot,\cdot)$ is continuous.

Step 3. $\mathcal{H}(\mathbb{B}_R)$ is equicontinuous.

For any $\xi_1, \xi_2 \in \mathfrak{I}$ with $\xi_1 < \xi_2$ and $(\mathfrak{z}_1, \mathfrak{z}_2) \in \mathbb{B}_R$, we obtain

$$\|\mathcal{H}_i(\mathfrak{z}_1(\xi_2,\omega), \mathfrak{z}_2(\xi_2,\omega), \omega) - \mathcal{H}_i(\mathfrak{z}_1(\xi_1,\omega), \mathfrak{z}_2(\xi_1,\omega), \omega)\| \;\le J_{i,1} + J_{i,2}, \quad i = 1, 2,$$

where

$$J_{i,1} = \frac{(\vartheta_i - 1)}{e^{\varpi_i\phi(\xi_2,a)}} \int_{\xi_1}^{\xi_2} \psi'(s)e^{\varpi_i\phi(s,a)} \int_a^s \frac{\psi'(\tau)\phi(s,\tau)^{\vartheta_i-2}}{\Gamma(\vartheta_i-1)} \|\mathfrak{f}_i(\tau, \mathfrak{z}_1(\tau,\omega), \mathfrak{z}_2(\tau,\omega), \omega)\| d\tau ds,$$

and

$$J_{i,2}=(\vartheta_i-1)\int_a^{\xi_1}\psi'(s)\left|e^{-\varpi_i\phi(\xi_2,s)}-e^{-\varpi_i\phi(\xi_1,s)}\right|\left\|\left(\mathcal{I}_{a^+}^{\vartheta_i-1;\psi}\mathfrak{f}_i(\tau,\mathfrak{z}_1(\tau,\omega),\mathfrak{z}_2(\tau,\omega),\omega\right)(s)\right\|ds,$$

From (A3) and using (11) and the fact $e^{-\varpi_i\phi(\xi_2,a)}\leq 1$ and Lemma 1, we get

$$\begin{aligned}J_{i,1}&\leq(1+2R)(\vartheta_i-1)\|\Psi_i(\cdot,\omega)\|_{L^\infty}\int_{\xi_1}^{\xi_2}e^{\varpi_i\phi(s,a)}\int_a^s\frac{\psi'(\tau)\phi(s,\tau)^{\vartheta_i-2}}{\Gamma(\vartheta_i-1)}d\tau\psi'(s)ds\\&\leq(1+2R)\|\Psi_i(\cdot,\omega)\|_{L^\infty}\int_{\xi_1}^{\xi_2}e^{\varpi_i\phi(s,a)}\frac{\psi'(s)\phi(s,a)^{\vartheta_i-1}}{\Gamma(\vartheta_i-1)}ds\\&\leq(1+2R)\|\Psi_i(\cdot,\omega)\|_{L^\infty}e^{\varpi_i\phi(b,a)}\left[\frac{\phi(\xi_2,a)^{\vartheta_i}-\phi(\xi_1,a)^{\vartheta_i}}{\vartheta_i\Gamma(\vartheta_i-1)}\right],\quad i=1,2.\end{aligned}$$

Thus,

$$J_{i,1}\longrightarrow 0\quad\text{when}\quad\xi_2\longrightarrow\xi_1,\quad i=1,2.\tag{12}$$

On the other side,

$$\begin{aligned}J_{i,2}&=(\vartheta_i-1)\left(e^{-\varpi_i\phi(\xi_1)}-e^{-\varpi_i\phi(\xi_2)}\right)\times\\&\quad\int_a^{\xi_1}e^{\varpi_i\phi(s)}\left\|\left(\mathcal{I}_{a^+}^{\vartheta_i-1;\psi}\mathfrak{f}_i(\tau,\mathfrak{z}_1(\tau,\omega),\mathfrak{z}_2(\tau,\omega),\omega\right)(s)\right\|\psi'(s)ds,\quad i=1,2.\end{aligned}$$

Thus,

$$J_{i,2}\longrightarrow 0\quad\text{when}\quad\xi_2\longrightarrow\xi_1,\quad i=1,2.\tag{13}$$

From (12) and (13), we get

$$\|\mathcal{H}_i(\mathfrak{z}_1(\xi_2,\omega),\mathfrak{z}_2(\xi_2,\omega),\omega)-\mathcal{H}_i(\mathfrak{z}_1(\xi_1,\omega),\mathfrak{z}_2(\xi_1,\omega),\omega)\|\xrightarrow[\xi_2\to\xi_1]{}0,\quad i=1,2.$$

This proves that, $\mathcal{H}(\mathbb{B}_R)$ is equicontinuous.

Step 4. $\mathcal{H}$ is $\Theta_{\mathbb{J}}$-condensing.

First, for every $U^1\times U^2\subset\mathcal{P}(\mathbb{J})$, we define the MNC as

$$\Theta_{\mathbb{J}}(U^1\times U^2)=\begin{pmatrix}\Theta(U^1)\\\Theta(U^2)\end{pmatrix},\tag{14}$$

where

$$\Theta(U^i)=\sup_{\xi\in\mathfrak{I}}e^{-\gamma\xi}\Lambda(U^i(\xi));\quad\gamma>0,\,i=1,2,\tag{15}$$

The MNC $\Theta_{\mathbb{J}}$ is well defined and gives a semiadditive, monotone, nonsingular and regular MNC in $\mathbb{J}$.

Secondly, let $U^1 \times U^2 \subset \mathcal{P}(\mathbb{J})$ be such that

$$\Theta_{\mathbb{J}}\left(\mathcal{H}_i(U^1 \times U^2)\right) \geq \Theta_{\mathbb{J}}(U^1 \times U^2), \quad i = 1, 2. \tag{16}$$

We will show that (16) implies the relative compactness of $U^1 \times U^2$. There exists a countable set $\{(\mathfrak{Z}_{1,n}, \mathfrak{Z}_{2,n})\}_{n=1}^{\infty}$ such that

$$\mathfrak{Z}_{i,n}(\xi, \omega) \quad = \mathcal{H}_i\left(\{\mathfrak{z}_{1,n}(\xi, \omega), \mathfrak{z}_{2,n}(\xi, \omega), \omega\}\right), \; i = 1, 2,$$

where $\{(\mathfrak{z}_{1,n}, \mathfrak{z}_{2,n})\}_{n=1}^{\infty} \subset \mathbb{J}$. From the properties of the MNC, one gets (for $i = 1, 2$)

$$\begin{aligned} &\Theta(\{\mathfrak{Z}_{i,n}\}_{n=1}^{\infty}) \\ &\leq \Theta\left(\left\{(\vartheta_i - 1)e^{\varpi_i \phi(b,a)} \int_a^{\xi} \int_a^{s} \frac{\psi'(\tau)\phi(s,\tau)^{\vartheta_i - 2}}{\Gamma(\vartheta_i - 1)} \mathfrak{f}_i(\tau, \mathfrak{z}_{1,n}(\tau, \omega), \mathfrak{z}_{2,n}(\tau, \omega)) d\tau \psi'(s) ds\right\}_{n=1}^{+\infty}\right). \end{aligned} \tag{17}$$

Now, we will find an estimate for $\Theta(\{\mathfrak{Z}_{i,n}\}_{n=1}^{\infty})$, $i = 1, 2$. By using (A4), for all $\xi \in \mathfrak{I}$ and $\tau \leq s \leq \xi$, one has

$$\begin{aligned} &\Lambda\left(\left\{\frac{\psi'(\tau)\phi(s,\tau)^{\vartheta_i - 2}}{\Gamma(\vartheta_i - 1)} \mathfrak{f}_i(\tau, \mathfrak{z}_{1,n}(\tau, \omega), \mathfrak{z}_{2,n}(\tau, \omega))\right\}_{n=1}^{+\infty}\right) \\ &\quad \leq \frac{\psi'(\tau)\phi(s,\tau)^{\vartheta_i - 2}}{\Gamma(\vartheta_i - 1)} \sum_{j=1}^{2} \varrho_{i,j}(\omega) \Lambda(\{\mathfrak{z}_{j,n}(\tau, \omega)\}_{n=1}^{+\infty}) \\ &\quad \leq \sum_{j=1}^{2} \frac{\psi'(\tau)\phi(s,\tau)^{\vartheta_i - 2}}{\Gamma(\vartheta_i - 1)} \varrho_{i,j}(\omega) e^{\gamma\tau} \sup_{a \leq \tau \leq s} e^{-\gamma\tau} \Lambda(\{\mathfrak{z}_{j,n}(\tau, \omega)\}_{n=1}^{+\infty}) \\ &\quad \leq \sum_{j=1}^{2} \frac{\psi'(\tau)\phi(s,\tau)^{\vartheta_i - 2}}{\Gamma(\vartheta_i - 1)} \varrho_{i,j}(\omega) e^{\gamma\tau} \Theta(\{\mathfrak{z}_{j,n}(\cdot, \omega)\}_{n=1}^{+\infty}). \end{aligned}$$

Then, applying Lemma 5, we get for all $\xi \in \mathfrak{I}$, $s \in [a, \xi]$ and $\tau \leq s$,

$$\begin{aligned} &\Lambda\left(\left\{(\vartheta_i - 1)e^{\varpi_i \phi(b,a)} \int_a^{\xi} \int_a^{s} \frac{\psi'(\tau)\phi(s,\tau)^{\vartheta_i - 2}}{\Gamma(\vartheta_i - 1)} \mathfrak{f}_i(\tau, \mathfrak{z}_{1,n}(\tau, \omega), \mathfrak{z}_{2,n}(\tau, \omega)) d\tau \psi'(s) ds\right\}_{n=1}^{+\infty}\right) \\ &\leq \sum_{j=1}^{2} \Theta(\{\mathfrak{z}_{j,n}(\cdot, \omega)\}_{n=1}^{+\infty}) \varrho_{i,j}(\omega) \frac{4(\vartheta_i - 1)e^{\varpi_i \phi(b,a)}}{\Gamma(\vartheta_i - 1)} \int_a^{\xi} \int_a^{s} \psi'(\tau)\phi(s,\tau)^{\vartheta_i - 2} e^{\gamma\tau} d\tau \psi'(s) ds \\ &\leq \sum_{j=1}^{2} \Theta(\{\mathfrak{z}_{j,n}(\cdot, \omega)\}_{n=1}^{+\infty}) \varrho_{i,j}(\omega) \frac{4(\vartheta_i - 1)e^{\varpi_i \phi(b,a)}}{\Gamma(\vartheta_i - 1)} \int_a^{\xi} \psi'(s) e^{\gamma s} \int_a^{s} \psi'(\tau)\phi(s,\tau)^{\vartheta_i - 2} d\tau ds \\ &\leq \sum_{j=1}^{2} \Theta(\{\mathfrak{z}_{j,n}(\cdot, \omega)\}_{n=1}^{+\infty}) \varrho_{i,j}(\omega) \frac{4e^{\varpi_i \phi(b,a)}}{\Gamma(\vartheta_i - 1)} \int_a^{\xi} \psi'(s) e^{\gamma s} \phi(s,a)^{\vartheta_i - 1} ds. \end{aligned}$$

Multiplying both sides by $e^{-\gamma\xi}$ and taking $\sup_{\xi\in\mathfrak{I}}$, one obtains

$$\sup_{\xi\in\mathfrak{I}} e^{-\gamma\xi}\Lambda\left(\left\{(\vartheta_i-1)e^{\varpi_i\phi(b,a)}\int_a^{\xi}\int_a^{s}\frac{\psi'(\tau)\phi(s,\tau)^{\vartheta_i-2}}{\Gamma(\vartheta_i-1)}\mathfrak{f}_i(\tau,\mathfrak{z}_{1,n}(\tau,\omega),\mathfrak{z}_{2,n}(\tau,\omega))d\tau\psi'(s)ds\right\}_{n=1}^{+\infty}\right)$$
$$\leq\sum_{j=1}^{2}\Theta(\{\mathfrak{z}_{j,n}(\cdot,\omega)\}_{n=1}^{+\infty})\aleph_{i,j}(\gamma,\omega).$$

where $\aleph_{i,j}(\gamma,\omega)$, $i,j=1,2$ are defined in (6).

Hence,

$$\Theta\left(\left\{(\vartheta_i-1)e^{\varpi_i\phi(b,a)}\int_a^{\xi}\int_a^{s}\frac{\psi'(\tau)\phi(s,\tau)^{\vartheta_i-2}}{\Gamma(\vartheta_i-1)}\mathfrak{f}_i(\tau,\mathfrak{z}_{1,n}(\tau,\omega),\mathfrak{z}_{2,n}(\tau,\omega))d\tau\psi'(s)ds\right\}_{n=1}^{+\infty}\right)$$
$$\leq\sum_{r=1}^{2}\Theta\left(\{\mathfrak{z}_{j,n}(\cdot,\omega)\}_{n=1}^{\infty}\right)\aleph_{i,j}(\gamma,\omega),\quad i=1,2. \tag{18}$$

Next, by (17) and (18), we derive

$$\Theta(\{\mathfrak{Z}_{i,n}\}_{n=1}^{\infty})\leq\sum_{r=1}^{2}\Theta\left(\{\mathfrak{z}_{j,n}(\cdot,\omega)\}_{n=1}^{\infty}\right)\aleph_{i,j}(\gamma,\omega),\quad i=1,2,$$

which implies

$$\begin{aligned}\Theta_{\mathbb{J}}(\mathcal{H}(\{\mathfrak{z}_{1,n}(\cdot,\omega),\mathfrak{z}_{2,n}(\cdot,\omega),\omega\}_{n=1}^{+\infty}) &= \begin{pmatrix}\Theta(\mathcal{H}_1(\{\mathfrak{z}_{1,n}(\cdot,\omega),\mathfrak{z}_{2,n}(\cdot,\omega),\omega\}_{n=1}^{+\infty})\\ \Theta(\mathcal{H}_2(\{\mathfrak{z}_{1,n}(\cdot,\omega),\mathfrak{z}_{2,n}(\cdot,\omega),\omega\}_{n=1}^{+\infty})\end{pmatrix}\\ &\leq \Xi_\gamma(\omega)\begin{pmatrix}\Theta\left(\{\mathfrak{z}_{1,n}(\cdot,\omega)\}_{n=1}^{\infty}\right)\\ \Theta\left(\{\mathfrak{z}_{2,n}(\cdot,\omega)\}_{n=1}^{\infty}\right)\end{pmatrix},\end{aligned}$$

where

$$\Xi_\gamma(\omega)=\begin{pmatrix}\aleph_{1,1}(\gamma,\omega) & \aleph_{1,2}(\gamma,\omega)\\ \\ \aleph_{2,1}(\gamma,\omega) & \aleph_{2,2}(\gamma,\omega)\end{pmatrix}.$$

By Lemma 7, one can choose γ such that the spectral radius $\rho(\Xi_\gamma(\omega))<1$, therefore

$$\Theta(\mathcal{H}_i(\{\mathfrak{z}_{1,n}(\cdot,\omega),\mathfrak{z}_{2,n}(\cdot,\omega),\omega\}_{n=1}^{+\infty})=0,\quad i=1,2.$$

This implies that

$$\Theta(\mathcal{H}_i(\{\mathfrak{z}_{1,n}(\xi,\omega),\mathfrak{z}_{2,n}(\xi,\omega),\omega\}_{n=1}^{+\infty})=0,\ \text{for}\ \xi\in\mathfrak{I},\ i=1,2.$$

Finally,

$$\Theta_{\mathbb{J}}(U^1 \times U^2) = (0,0),$$

which proves the compactness of the set $\overline{U^1 \times U^2}$.

Step 5. The set $\mathbb{W}$ (see Theorem 2 (2)) is bounded.

Let $(\mathfrak{z}_1, \mathfrak{z}_2) \in \mathbb{J}$ and $(\mathfrak{z}_1, \mathfrak{z}_2) = \kappa(\omega)\mathcal{H}(\mathfrak{z}_1, \mathfrak{z}_2)$ for some $\kappa(\omega) \in (0,1)$. Then, by the fact $e^{-\varrho\chi(\xi,a)} \le 1$ for all $\xi \in \mathfrak{I}$, we obtain

$$\begin{aligned}
\mathfrak{z}_i(\xi,\omega) &= \kappa(\omega)\Bigg[(\vartheta_i - 1)e^{-\varpi_i\phi(\xi,a)} \int_a^\xi e^{\varpi_i\phi(s,a)} \int_a^s \frac{\psi'(\tau)\phi(s,\tau)^{\vartheta_i-2}}{\Gamma(\vartheta_i - 1)} \times \\
&\qquad \mathfrak{f}_i(\tau, \mathfrak{z}_1(\tau,\omega), \mathfrak{z}_2(\tau,\omega), \omega) d\tau \psi'(s) ds\Bigg] \\
&\le \frac{(\vartheta_i - 1)}{e^{-\varpi_i\phi(b,a)}} \int_a^\xi \int_a^s \frac{\psi'(\tau)\phi(s,\tau)^{\vartheta_i-2}}{\Gamma(\vartheta_i - 1)} \mathfrak{f}_i(\tau, \mathfrak{z}_1(\tau,\omega), \mathfrak{z}_2(\tau,\omega), \omega) d\tau \psi'(s) ds, \quad i = 1,2.
\end{aligned}$$

Using Fubini's Theorem, we have

$$\begin{aligned}
\|\mathfrak{z}_i(\xi,\omega)\| &\le \frac{(\vartheta_i - 1)}{e^{-\varpi_i\phi(b,a)}} \int_a^\xi \|\mathfrak{f}_i(\tau, \mathfrak{z}_1(\tau,\omega), \mathfrak{z}_2(\tau,\omega), \omega)\| \int_\tau^\xi \frac{\psi'(s)\phi(s,\tau)^{\vartheta_i-2}}{\Gamma(\vartheta_i - 1)} ds \psi'(\tau) d\tau \\
&\le \frac{(\vartheta_i - 1)}{e^{-\varpi_i\phi(b,a)}\Gamma(\vartheta_i)} \int_a^\xi \psi'(\tau)\phi(\xi,\tau)^{\vartheta_i-1} \|\mathfrak{f}_i(\tau, \mathfrak{z}_1(\tau,\omega), \mathfrak{z}_2(\tau,\omega), \omega)\| d\tau, \quad i = 1,2.
\end{aligned}$$

Using (A3), we get

$$\begin{aligned}
\|\mathfrak{z}_i(\xi,\omega)\| &\le \frac{(\vartheta_i - 1)}{e^{-\varpi_i\phi(b,a)}\Gamma(\vartheta_i)} \int_a^\xi \psi'(\tau)\phi(\xi,\tau)^{\vartheta_i-1} \Psi_i(\tau,\omega)(1 + \|\mathfrak{z}_1(\tau,\omega)\| + \|\mathfrak{z}_2(\tau,\omega)\|) d\tau \\
&\le \frac{(\vartheta_i - 1)}{e^{-\varpi_i\phi(b,a)}\Gamma(\vartheta_i)} \int_a^\xi \psi'(\tau)\phi(\xi,\tau)^{\vartheta_i-1} \Psi_i(\tau,\omega)(\|\mathfrak{z}_1(\tau,\omega)\| + \|\mathfrak{z}_2(\tau,\omega)\|) d\tau \\
&\quad + \frac{(\vartheta_i - 1)\Psi_i(b,\omega)}{e^{-\varpi_i\phi(b,a)}\vartheta_i\Gamma(\vartheta_i)} \phi(\xi,a)^{\vartheta_i}, \quad i = 1,2.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\|\mathfrak{z}_1(\xi,\omega)\| + \|\mathfrak{z}_2(\xi,\omega)\| \\
&\le \varsigma(\xi) + \frac{(\vartheta_1 - 1)}{e^{-\varpi_1\phi(b,a)}\Gamma(\vartheta_1)} \int_a^\xi \psi'(\tau)\phi(\xi,\tau)^{\vartheta_1-1} \Psi_1(\tau,\omega)(\|\mathfrak{z}_1(\tau,\omega)\| + \|\mathfrak{z}_2(\tau,\omega)\|) d\tau \\
&\quad + \frac{(\vartheta_2 - 1)}{e^{-\varpi_2\phi(b,a)}\Gamma(\vartheta_2)} \int_a^\xi \psi'(\tau)\phi(\xi,\tau)^{\vartheta_2-1} \Psi_2(\tau,\omega)(\|\mathfrak{z}_1(\tau,\omega)\| + \|\mathfrak{z}_2(\tau,\omega)\|) d\tau
\end{aligned}$$

where

$$\varsigma(\xi) := \frac{(\vartheta_1-1)\Psi_1(b,\omega)}{e^{-\varpi_1\phi(b,a)}\vartheta_1\Gamma(\vartheta_1)}\phi(\xi,a)^{\vartheta_1} + \frac{(\vartheta_2-1)\Psi_2(b,\omega)}{e^{-\varpi_2\phi(b,a)}\vartheta_2\Gamma(\vartheta_2)}\phi(\xi,a)^{\vartheta_2}$$

Applying Lemma 6, we obtain

$$\|\mathfrak{z}_1(\xi,\omega)\| + \|\mathfrak{z}_2(\xi,\omega)\| \le \varsigma(b)\sum_{j=0}^{2}\mathbb{E}_{\vartheta_j}\left((\vartheta_j-1)e^{\varpi_j\phi(b,a)}\phi(b,a)^{\vartheta_j}\right) := \mathbf{D}.$$

Hence

$$\|(\mathfrak{z}_1(\cdot,\omega),\mathfrak{z}_2(\cdot,\omega))\|_{\mathbb{J}} \le \widehat{\mathbf{D}} := \begin{pmatrix}\mathbf{D}\\ \mathbf{D}\end{pmatrix}$$

Which achieves the desired estimate. Therefore, Theorem 2 ensures the existence of a random solution for the system (1).

4. Examples

Let $\Omega = (-\infty, 0)$ be endowed with the usual σ-algebra consisting of Lebesgue measurable subsets of $(-\infty, 0)$. Consider the separable Banach space

$$\mathbb{G} = c_0 = \{\mathfrak{s} = (\mathfrak{s}^1, \mathfrak{s}^2, \cdots, \mathfrak{s}^n, \cdots) \ : \ \mathfrak{s}^n \to 0 \ \text{ as } \ n\to\infty\}$$

endowed with

$$\|\mathfrak{s}\|_{\mathbb{G}} = \sup_{n\ge 1}|\mathfrak{s}^n|.$$

Example 1: Illustration of Theorem 3.

Let us take $\varpi_i = ...,\ i = 1,2.$

For $(\xi,\omega)\in\mathfrak{I}\times\Omega$, consider the nonlinear functions $\mathfrak{f}_i,\ i=1,2$ be defined by

$$\begin{cases} \mathfrak{f}_1(\xi,\mathfrak{z}_1(\xi,\omega),\mathfrak{z}_2(\xi,\omega),\omega) = \left\{\dfrac{\mathfrak{z}^{1,n}(\xi,\omega)}{|\omega|(1+|\omega|)} + \dfrac{\sin(\mathfrak{z}^{2,n}(\xi,\omega))}{1+|\omega|^2}\right\}_{n\ge1}, \\ \mathfrak{f}_2(\xi,\mathfrak{z}_1(\xi,\omega),\mathfrak{z}_2(\xi,\omega),\omega) = \left\{\dfrac{\arctan|\mathfrak{s}^{1,n}(\xi,\omega)|}{1+|\omega|} + \dfrac{e^{-|\omega|}\mathfrak{z}^{2,n}(\xi,\omega)}{1+|\mathfrak{z}^{2,n}(\xi,\omega)|}\right\}_{n\ge1} \end{cases} \tag{19}$$

Firstly, we easily see that, the functions $\mathfrak{f}_i,\ i=1,2$, satisfy (A1). Secondly, we can check that

$$\begin{aligned} &\|\mathfrak{f}_1(\xi,\mathfrak{z}_1(\xi,\omega),\mathfrak{z}_2(\xi,\omega),\omega) - \mathfrak{f}_1(\xi,\mathfrak{r}_1(\xi,\omega),\mathfrak{r}_2(\xi,\omega),\omega)\| \\ &\quad\le \frac{1}{|\omega|(1+|\omega|)}\|\mathfrak{z}^{1,n}(\xi,\omega)-\mathfrak{r}^{1,n}(\xi,\omega)\| + \frac{1}{1+|\omega|^2}\|\mathfrak{z}^{2,n}(\xi,\omega)-\mathfrak{r}^{2,n}(\xi,\omega)\|, \end{aligned}$$

and

$$\begin{aligned} &\|\mathfrak{f}_2(\xi,\mathfrak{z}_1(\xi,\omega),\mathfrak{z}_2(\xi,\omega),\omega) - \mathfrak{f}_2(\xi,\mathfrak{r}_1(\xi,\omega),\mathfrak{r}_2(\xi,\omega),\omega)\| \\ &\quad\le \frac{1}{1+|\omega|}\|\mathfrak{z}^{1,n}(\xi,\omega)-\mathfrak{r}^{1,n}(\xi,\omega)\| + \frac{1}{e^{|\omega|}}\|\mathfrak{z}^{2,n}(\xi,\omega)-\mathfrak{r}^{2,n}(\xi,\omega)\|. \end{aligned}$$

So, the hypotheses (A2) holds with

$$\Phi_{1,1}(\omega) = \frac{1}{|\omega|(1+|\omega|)}, \quad \Phi_{1,2}(\omega) = \frac{1}{1+|\omega|^2}, \quad \text{for all } \omega \in \Omega.$$

$$\Phi_{2,1}(\omega) = \frac{1}{1+|\omega|}, \quad \Phi_{2,2}(\omega) = \frac{1}{e^{|\omega|}}, \quad \text{for all } \omega \in \Omega.$$

An application of Theorem 3, we deduce that system (1) with (19) has a unique random solution $(\mathfrak{z}_1, \mathfrak{z}_2)$.

Example 2: Illustration of Theorem 4.

For $(\varsigma, \omega) \in \mathfrak{I} \times \Omega$ and $\mathfrak{s}_i = \{\mathfrak{s}^{i,n}\}_n \in c_0$, consider the nonlinear forcing terms,

$$\begin{cases} \mathfrak{f}_1(\xi, \mathfrak{z}_1(\xi,\omega), \mathfrak{z}_2(\xi,\omega), \omega) \\ \quad = \dfrac{2\sin(\omega/5)}{\pi} \arctan(\xi) \left\{ \sin|\mathfrak{z}^{1,n}(\xi,\omega)| + \log_e(|\mathfrak{z}^{2,n}(\xi,\omega)| + 1) + 5^{-n} \right\}_{n\geq 1} \\ \mathfrak{f}_2(\xi, \mathfrak{z}_1(\xi,\omega), \mathfrak{z}_2(\xi,\omega), \omega) \quad = \dfrac{|\omega|(e^{2\xi}-1)}{(1+|\omega|)(e^{\xi}+1)} \left\{ \arctan(|\mathfrak{z}^{1,n}(\xi,\omega)|) + |\mathfrak{z}^{2,n}(\xi,\omega)| + \pi^{-n} \right\}_{n\geq 1} \end{cases} \tag{20}$$

Obviously, $\mathfrak{f}_i$, $(i = 1, 2)$ satisfy hypothesis (A1).

To illustrate (A3), let $\xi \in \mathfrak{I}$ and $\mathfrak{z}_i = \{\mathfrak{z}^{i,n}\}_n \in U \subset c_0$, $i = 1, 2$. Then

$$\|\mathfrak{f}_1(\xi, \mathfrak{z}_1(\xi,\omega), \mathfrak{z}_2(\xi,\omega), \omega)\| \leq \frac{2\sin(\omega/5)}{\pi} \arctan(\xi) \Big(\|\mathfrak{z}^{1,n}(\xi,\omega)\| + \|\mathfrak{z}^{2,n}(\xi,\omega)\| + 1 \Big), \tag{21}$$

and

$$\|\mathfrak{f}_2(\xi, \mathfrak{z}_1(\xi,\omega), \mathfrak{z}_2(\xi,\omega), \omega)\| \leq \frac{|\omega|(e^{2\xi}-1)}{(1+|\omega|)(e^{\xi}+1)} \Big(\|\mathfrak{z}^{1,n}(\xi,\omega)\| + \|\mathfrak{z}^{2,n}(\xi,\omega)\| + 1 \Big), \tag{22}$$

Therefore, (H3) is verified with

$$\Psi_1(\xi,\omega) = \frac{2\sin(\omega/5)}{\pi} \arctan(\xi) \quad \text{and} \quad \Psi_2(\xi,\omega) = \frac{|\omega|(e^{2\xi}-1)}{(1+|\omega|)(e^{\xi}+1)} \quad \text{for all} \quad (\xi,\omega) \in \mathfrak{I}\times\Omega.$$

Next, hypothesis (A4) is satisfied. Indeed, we recall that the Hausdorff MNC Θ in $(c_0, \|\cdot\|_{c_0})$ can be computed by means of the formula

$$\Theta(U) = \lim_{n\to\infty} \sup_{\mathfrak{z}\in U} \|(\mathrm{I} - \mathrm{P}_n)\,\mathfrak{z}\|_\infty ,$$

where $U \in \mathcal{P}(c_0)$, P_n represents the projection onto the linear span of the first n vectors in the standard basis (see [1]).

Using (21) and (22) (see also Example in [32]), we get

$$\Theta\left(\mathfrak{f}_i(\xi, U^1, U^2)\right) \leq \varrho_{i,1}(\omega)\Theta(U^1) + \varrho_{i,2}(\omega)\Theta(U^2), \quad \text{for all } (\xi, \omega) \in \mathfrak{I} \times \Omega.$$

where

$$\varrho_{1,1}(\omega) = \varrho_{1,2}(\omega) = \sin(\omega/5), \quad \varrho_{2,1}(\omega) = \varrho_{2,2}(\omega) = \frac{|\omega|}{1+|\omega|}, \quad \text{for all } \omega \in \Omega.$$

The conclusion of Teorem 4 implies that problem (1) with (20) has at least one solution $(\mathfrak{z}_1, \mathfrak{z}_2)$.

5. Conclusion

The fractional Langevin system is a crucial mathematical model for describing the random motion of particles. Consequently, we investigated a class of ψ-Caputo Langevin systems with random effects in a generalized separable Banach space. By employing the Bielecki-type vector-valued norm, we established a new uniqueness criterion. Additionally, we imposed rather mild assumptions to obtain a new existence result by utilizing a recent random version of Sadovski's fixed-point theorem. As a result, numerous findings in the literature can be recovered through our results.

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References

[1] R R Akhmerov, M I Kamenskii, A S Potapov, A E Rodkina, and B N Sadovskii. *Measures of noncompactness and condensing operators.* Birkhäuser Verlag, Basel-Boston-Berlin, 1992.

[2] I Alhribat and M H Samuh. Generating statistical distributions using fractional differential equations. *Jordan Journal of Mathematics and Statistics*, 16(2):379–396, 2023.

[3] R Almeida. A Caputo fractional derivative of a function with respect to another function. *Commun. Nonlinear Sci. Numer. Simul.*, 44:460–481, 2017.

[4] G Alotta, M Di Paola, and F P Pinnola. An unified formulation of strong non-local elasticity with fractional order calculus. *Commun. Nonlinear Sci. Numer. Simul.*, 57:Meccanica, 2022.

[5] J Alzabut, Y Adjabi, W Sudsutad, and M Rehman. New generalizations for Gronwall type inequalities involving a ϕ-fractional operator and their applications. *AIMS Mathematics*, 6(5):5053–5077, 2021.

[6] Z Baitiche, C Derbazi, and M Matar. Ulam-stability results for a new form of nonlinear fractional Langevin differential equations involving two fractional orders in the ψ–Caputo sense. *Applicable Anal.*, 101(14):4866–4881, 2022.

[7] A Baliki, J Nieto, A Ouahab, and M Sinacer. Random semilinear system of differential equations with impulses. *Fixed Point Theory Appl.*, 2017:29 p, 2017. No 27.

[8] R Dhayal, M Malik, and S Abbas. Nonlinear heat conduction equations with memory: Physical meaning and analytical results. *J. Math. Phys.*, 58:063501, 2017.

[9] R Dhayal, M Malik, and S Abbas. Solvability and optimal controls of noninstantaneous impulsive stochastic fractional differential equation of order $q \in (1, 2)$. *Stochastics*, 93(5):780–802, 2021.

[10] H Fu, G Wu, G Yang, and L Huang. Continuous time random walk to a general fractional Fokker–Planck equation on fractal media. *Eur. Phys. J. Spec. Top.*, 230:3927–3933, 2021.

[11] A Giusti. On infinite order differential operators in fractional viscoelasticity. *Fract. Calc. Appl. Anal.*, 20(4):854–867, 2017.

[12] J R Graef, J Henderson, and A Ouahab. *Topological Methods for Differential Equations and Inclusions.* CRC Press, Boca Raton, FL, 2018.

[13] M Abu Hammad, O Zentar, S Alshorm, M Ziane, and I Zitouni. Theoretical analysis of a class of φ-Caputo fractional differential equations in Banach space. *AIMS Mathematics*, 9(3):6411–6423, 2024.

[14] C Ionescu, A Lopes, D Copot, J A T Machado, and J H T Bates. The role of fractional calculus in modeling biological phenomena: A review. *Commun. Nonlinear Sci. Numer. Simul.*, 51:141–159, 2017.

[15] M Kamenskii, V Obukhovskii, and P Zecca. *Condensing Multivalued Maps and Semilinear Differential Inclusions in Banach Spaces.* De Gruyter, Berlin, 2001.

[16] A A Kilbas, H M Srivastava, and J J Trujillo. *Theory and applications of fractional differential equations.* North-Holland Mathematics Studies, Elsevier, Amsterdam, Netherlands, 2006.

[17] V Kobelev and E Romanov. Fractional Langevin equation to describe anomalous diffusion. *Prog. Theor. Phys. Suppl.*, 139:470–476, 2000.

[18] P Langevin. Sur la théorie du mouvement brownien. *C. R.*, 146:530–533, 1908.

[19] K B Oldham. Fractional differential equations in electrochemistry. *Adv. Eng. Softw.*, 41(1):9–12, 2010.

[20] R Ozarslan, E Bas, D Baleanu, and B Acay. Fractional physical problems including wind-influenced projectile motion with Mittag-Leffler kernel. *AIMS Mathematics*, 5(1):467–481, 2020.

[21] J C Prajapati, A D Patel, K N Pathak, and A K Shukla. Fractional calculus approach in the study of instability phenomenon in fluid dynamics. *Palest. J. Math.*, 1(2):95–103, 2012.

[22] Z U Rehman, S Boulaaras, R Jan, I Ahmad, and S Bahramand. Computational analysis of financial system through non-integer derivative. *Journal of Computational Science*, 45:102204, 2024.

[23] I Rus. *Generalized contractions and applications.* Cluj University Press, Cluj-Napoca,

2001.

[24] B Shiria and D Baleanu. Numerical solution of some fractional dynamical systems in medicine involving non-singular kernel with vector order. *Results in Nonlinear Analysis*, 2(4):160–168, 2019.

[25] M L Sinacer, J J Nieto, and A Ouahab. Random fixed point theorem in generalized Banach space and applications. *Random Oper. Stoch. Equ.*, 24(2):93–112, 2016.

[26] J Singh, D Kumar, and D Baleanu. On the analysis of chemical kinetics system pertaining to a fractional derivative with Mittag-Leffler type kernel. *Chaos.*, 27:103113, 2017.

[27] J Sousa and E Oliveira. Existence, uniqueness, estimation and continuous dependence of the solutions of a nonlinear integral and an integrodifferential equations of fractional order. *ArXiv Preprint*, 1806.01441, 2018.

[28] A Traore and N Sene. Model of economic growth in the context of fractional derivative. *Alex. Eng. J.*, 59(6):4843–4850, 2024.

[29] V V Uchaikin. *Fractional Derivatives for Physicists and Engineers.* Springer, Berlin, 2013.

[30] J Vanterler and C Sousa. Existence results and continuity dependence of solutions for fractional equations. *Differ. Equ. Appl.*, 12:377–396, 2020.

[31] X Yang, J Zeng, C Xu, L Peng, and J Alsultan. Modeling of fractional differential equation in cloud computing image fusion algorithm. *Appl. Math. Nonlinear Sci.*, 8(1):1125–1134, 2023.

[32] O Zentar, M Ziane, and M Al Horani. Theoretical study of a φ-Hilfer fractional differential system in Banach spaces. *Can. Math. Bull.*, 67(3):742–759, 2024.

[33] O Zentar, M Ziane, M Al Horani, and I Zitouni. Theoretical study of a class of ζ-Caputo fractional differential equations in a Banach space. *J. Appl. Anal. Comput.*, 14(5):2808–2821, 2024.

[34] O Zentar, M Ziane, and S Khelifa. Coupled fractional differential systems with random effects in Banach spaces. *Random Oper. Stoch. Equ.*, 29(4):251–263, 2021.