EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 1, Article Number 5608
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Necessary and Sufficient Conditions for the Equivalence of Statistical, Ideal, and Standard Convergence in G-Metric Spaces

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Abstract. This paper investigates the conditions for the equivalence between statistical convergence, ideal convergence, and standard convergence in G-metric spaces. Although statistical and ideal convergence studies have been extensively developed in various settings, no prior research has explicitly explored the relationship between statistical convergence and standard convergence within G-metric spaces. By addressing this gap, we establish necessary and sufficient conditions for the equivalence of these types of convergence in G-metric spaces. Our results contribute to a deeper understanding of the interplay between these convergence notions and extend the theory of convergence in generalized metric spaces.

2020 Mathematics Subject Classifications: 40A35

Key Words and Phrases: Statistical convergence, ideal convergence, standard convergence, G-metric spaces, necessary and sufficient conditions, equivalence theorems.

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1. Introduction

The concept of convergence plays a central role in analysis and its applications, with various types of convergence being developed to generalize the classical notion of pointwise convergence. Among these generalizations, statistical convergence and ideal convergence have attracted considerable attention. The notion of statistical convergence, introduced by Fast [3] and further studied by Šalát [32], provides a probabilistic framework that generalizes classical convergence by considering the density of indices at which a sequence fails to converge to a limit. This approach has proven useful in various applications, from number theory to functional analysis [3, 32].

Ideal convergence, introduced by Kostyrko et al. in [20], extends statistical convergence by incorporating ideals of sets, which are collections of subsets of natural numbers closed

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DOI: https://doi.org/10.29020/nybg.ejpam.v18i1.5608

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under certain set operations. Ideal convergence provides a flexible framework that unifies several known types of convergence, including statistical convergence and convergence with respect to filters [20]. Ideal convergence has been studied in various contexts, including Banach spaces and normed spaces, offering insights into the structure of function spaces and the behaviour of sequences [2].

Despite substantial progress in these areas, research on the relationship between statistical convergence and standard convergence, particularly in the setting of G-metric spaces, remains limited. G-metric spaces, introduced by Mustafa and Sims [27], generalize the notion of a metric space by defining a distance function on triplets of points rather than pairs. This structure has led to the development of various fixed-point theorems and applications in nonlinear analysis [27]. Many mathematicians have conducted research on G-metric spaces. Recent results on G-metric spaces include, among others, [13], [14], [17], [18], [16], [15], [25], [26], [30], [4], [7] and [33]. However, the interaction between different types of convergence, such as statistical and standard convergence ([10], [11],[23], and [29]), in G-metric spaces has not been fully explored.

The concept of convergence, particularly in G-metric spaces, has wide-ranging applications across various fields. In machine learning and data analysis, these results can be applied to understand the stability of algorithms or models operating within complex metric structures, such as G-metric spaces, which better represent non-linear data relationships [35]. Furthermore, in optimization theory, G-metric spaces provide a framework for analyzing iterative algorithms for non-linear problems commonly encountered in dynamic programming [9]. In theoretical physics, G-metric spaces aid in modelling systems with multiple parameter interactions, making them relevant for studying dynamical systems and physical geometries [28]. Finally, in mathematical finance, this approach enhances the modelling of complex market data through statistical and ideal convergence, enabling more robust analysis of economic behaviour [16]. Thus, this study not only deepens the theoretical understanding of convergence in G-metric spaces, but also opens avenues for its application across diverse scientific disciplines.

Existing research has focused mainly on the individual properties of statistical and ideal convergence in G-metric spaces [12, 31]. These studies have established important results concerning the behaviour of sequences and functions in such spaces. However, no prior study has investigated the precise relationship between statistical convergence and standard convergence in G-metric spaces, leaving a significant gap in the literature.

In this paper, we aim to bridge this gap by studying the conditions under which statistical convergence, ideal convergence, and standard convergence are equivalent in G-metric spaces. Our main contribution is the establishment of necessary and sufficient conditions for this equivalence, which provide a comprehensive framework for understanding how these different convergence notions relate to one another. The results presented in this paper extend the theory of convergence in G-metric spaces and offer new insights into the behaviour of sequences in generalized metric structures.

This paper is organized as follows. In Section 2, we provide preliminary definitions and review key results concerning the statistical and ideal convergence in G-metric spaces. Section 3 presents the main theorems, including the necessary and sufficient conditions for the equivalence of statistical, ideal, and standard convergence in G-metric spaces. Finally, in Section 4, we conclude with a discussion of the implications of our results and possible directions for future research.

2. Preliminary Definition

Before proceeding with the main discussion, we need to establish several definitions of the key concepts that will be explored throughout this research. In the following section, we will provide the definitions of G-metric spaces and the various types of convergence within G-metric spaces.

Definition 1. *[27] Let X be a non-empty set. A Function $G : X \times X \times X \to \mathbb{R}^+$ is called a G-metric if, for all $x, y, z \in X$ the following conditions are satisfied:*

(i) $G(x, y, z) = 0$ *if and only if* $x = y = z$

(ii) $G(x, x, y) > 0$ *for* $x \neq y$

(iii) $G(x, x, y) \leq G(x, y, z)$, *for* $z \neq y$

(iv) $G(x, y, z) = G(x, z, y) = G(y, z, x) = G(y, x, z) = G(z, x, y) = G(z, y, x)$

(v) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ *for any* $a \in X$

A Set X equipped with the function G is called a G-metric space and is denoted by (X, G).

Definition 2. *[6] Let (X, G) be a G-metric space and (x_n) be a sequence in X. The sequence (x_n) is said to converge to $x \in X$ if $\lim\limits_{n,m\to+\infty} G(x, x_n, x_m) = 0$ means that for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $G(x, x_n, x_m) < \varepsilon$ for all $n, m \geq n_0$. x In this case, x is called the limit of the sequence (x_n) denoted by $x_n \to x$ or $\lim\limits_{n\to+\infty} x_n = x$.*

Definition 3. *[6] Let (X, G) be a G-metric space and (x_n) be a sequence in X. The sequence (x_n) is said to be a Cauchy sequence in the G-metric space if for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $k, n, m \geq n_0, G(x_k, x_n, x_m) < \varepsilon$.*

Theorem 1. *[6] Let (x_n) be a sequence in the G-metric space (X, G). If the sequence (x_n) converges to $x \in \mathbb{R}$, then (x_n) is a Cauchy sequence.*

Definition 4. *[1] Let (x_n) be a sequence in the G-metric space (X, G). The sequence (x_n) is said to converge statistically to x in the G-metric space if, for every real number $\varepsilon > 0$, we have*

$$\lim_{n\to+\infty} \left(\frac{2}{n^2} |\{(n_1, n_2) \in N^2 : n_1, n_2 \leq n, G(x, x_{n_1}, x_{n_2}) \geq \varepsilon\}| \right) = 0$$

and this is denoted as

$$Gs - \lim(x_n) = x$$

Definition 5. *[22] Let (X,G) be a G-metric space and (x_n) a sequence in X. The sequence (x_n) is said to be a statistical Cauchy sequence in the G-metric space if, for every $\varepsilon > 0$, there exists $i \in \mathbb{N}$ such that*

$$\lim_{n\to+\infty}\left(\frac{2}{n^2}|\{(n_1,n_2)\in\mathbb{N}^2 : n_1,n_2 \le n, G(x_i,x_{n_1},x_{n_2}) \ge \varepsilon\}|\right) = 0$$

Let $\mathcal{I}^2 \subset 2^{N^2}$ be a nontrivial ideal on $\mathbb{N}^2$, where $A \in \mathcal{I}^2$ and $A = \{(n_1,n_2) : n_1,n_2 \in N)\}$.

Definition 6. *[22] Let $\mathcal{I}^2$ be an ideal. Let (x_n) be a sequence in the G-metric space (X,G). The sequence (x_n) is said to be ideally convergent to x if, for every real number $\varepsilon > 0$, the set $\{(n_1,n_2)\in\mathbb{N}^2 : G(x,x_{n_1},x_{n_2}) \ge \varepsilon\} \in \mathcal{I}^2$. and this is denoted as*

$$GI - \lim(x_n) = x$$

Definition 7. *[22] Let (X,G) be a G-metric space and $\mathcal{I}^2$ an ideal. Let (x_n) be a sequence in X. The sequence (x_n) is said to be an ideal Cauchy sequence in the G-metric space if, for every $\varepsilon > 0$, there exists $i \in \mathbb{N}$ such that $\{(n_1,n_2)\in\mathbb{N}^2 : G(x_i,x_{n_1},x_{n_2}) \ge \varepsilon\} \in \mathcal{I}^2$.*

3. Main Results

In this section, we discuss the relationship between standard convergence, statistical convergence, and ideal convergence in the G-metric space $(\mathbb{R},G)$.

Theorem 2. *[1] If a sequence converges to x in a G-metric space, then the sequence also statistically converges to x in the G-metric space.*

Proof. Let (x_n) be a sequence that converges to x in a G-metric space. This means that for every real number $\varepsilon > 0$, there exists an index $j \in \mathbb{N}$ such that for every $n,m \ge j$, we have $G(x,x_n,x_m) < \varepsilon$. If we form a set, it will take the following form:

$$A(j) = \{(n,m)\in\mathbb{N}^2 : n,m \ge j, G(x,x_n,x_m) < \varepsilon\}$$

It is clear that because there exists $j \in \mathbb{N}$ such that for all $n,m \ge j$ then $G(x,x_n,x_m) < \varepsilon$. Thus, $|\{n : (n,m)\in\mathbb{N}^2, G(x,x_n,x_m) \ge \varepsilon\}| or |\{m : (n,m)\in\mathbb{N}^2, G(x,x_n,x_m) \ge \varepsilon\}|$ is at most $j-1$, so:

$$\begin{aligned}\lim_{n\to+\infty}\left(\frac{2}{n^2}|\{(n,m)\in\mathbb{N}^2 : n,m \ge j, G(x,x_n,x_m) < \varepsilon\}|\right) &= \lim_{n\to+\infty}(\frac{2(j-1)n}{n^2}) \\ &= \lim_{n\to+\infty}\frac{2(j-1)}{n}) = 0\end{aligned}$$

Thus, it is proven that the sequence (x_n) statistically converges to x in the G-metric space.

Example 1. *Consider the G-metric space* $(\mathbb{R}, G)$*, where for all* $x, y, z \in \mathbb{R}$*, the G-metric is defined as:*

$$G(x, y, z) = max\{|x - y| + |x - z| + |y - z|\}.$$

The sequence $\left(\frac{1}{n+1}\right)$ *is statistically convergent to* 0 *in the G-metric space.*

We can investigate whether the sequence $\left(\frac{1}{n+1}\right)$ also converges to 0 in the G-metric space. According to Theorem 2, the sequence $\left(\frac{1}{n+1}\right)$ is statistically convergent to 0 in the G-metric space. The proof is given as follows:

$$\begin{aligned} G(x, x_n, x_m) &= max\{|x - x_n|, |x - x_m|, |x_n - x_m|\} \\ &= max\left\{\left|0 - \frac{1}{n+1}\right|, \left|0 - \frac{1}{m+1}\right|, \left|\frac{1}{n+1} - \frac{1}{m+1}\right|\right\} \\ &\geq max\left\{\frac{1}{n+1}, \frac{1}{m+1}, \frac{1}{n+1} - \frac{1}{m+1}\right\} \\ &\geq max\left\{\frac{1}{n+1}, \frac{1}{m+1}\right\} \end{aligned}$$

Let k be the largest integer less than or equal to $\frac{1}{\varepsilon - 1}$, for any $\varepsilon > 0, \varepsilon \in \mathbb{R}$. Then:

$$\begin{aligned} \lim_{n \to +\infty} \left(\frac{2}{n^2}|\{(n_1, n_2) \in \mathbb{N}^2 : n_1, n_2 \leq n, G(x_i, x_{n_1}, x_{n_2}) \geq \varepsilon\}|\right) &= \lim_{n \to +\infty} (\frac{2}{n^2}|\{(1, 1), (1, 2), (1, \\ &\quad 3), \ldots, (k, 1), (k, 2), \ldots\}|) \\ &\leq \lim_{n \to +\infty} (\frac{2nk}{n^2}) \\ &\leq 2k \lim_{n \to +\infty} (\frac{1}{n}) \\ &= 2k.0 = 0 \end{aligned}$$

Next, we investigate whether a statistically convergent sequence is also a standard convergent sequence. A statistically convergent sequence in the G-metric space is not always a standard convergent sequence in the G-metric space. The following theorem provides the necessary condition for a sequence that is statistically convergent in a G-metric space to also be a standard convergent sequence in the same space.

Theorem 3. *Given a sequence* (x_n) *that is statistically convergent to* x *in a G-metric space, if* $|A| = |\{n : (n, m) \in \mathbb{N}^2, G(x, x_n, x_m) \geq \varepsilon\}| < +\infty$ *or* $|B| = |\{m : (n, m) \in \mathbb{N}^2, G(x, x_n, x_m) \geq \varepsilon\}| < +\infty$ *then the sequence is standard convergent to* x *in the G-metric space.*

Proof. The sequence (x_n) is statistically convergent to x, meaning that for every real number $\varepsilon > 0$, the following holds:

$$\lim_{n \to +\infty} (\frac{2}{n^2}|\{(n_1, n_2) \in \mathbb{N}^2 : n_1, n_2 \leq n : G(x, x_{n_1}, x_{n_2}) \geq \varepsilon\}|) = 0$$

If $|A| = |\{n : (n,m) \in \mathbb{N}^2, G(x, x_n, x_m) \geq \varepsilon\}| < +\infty$, then there exists a $\sup(A) + 1 \in \mathbb{N}$ such that for every $n, m \geq \sup(A) + 1, G(x, x_n, x_m) < \varepsilon$. In other words, the sequence (x_n) is convergent to x in the G-metric space. Similary, if $|B| = |\{m : (n,m) \in \mathbb{N}^2, G(x, x_n, x_m) \geq \varepsilon\}| < +\infty$, then there exists a $\sup(B) + 1 \in \mathbb{N}$ such that for every $n, m \geq \sup(B) + 1, G(x, x_n, x_m) < \varepsilon$, and hence the sequence (x_n) is standard convergent to x in the G-metric space.

Example 2. *Consider the G-metric space $(\mathbb{R}, G)$, where for all $x, y, z \in \mathbb{R}$, the G-metric is defined as:*

$$G(x, y, z) = |x - y| + |x - z| + |y - z|.$$

The sequence $(x_n) = \left(\frac{(-1)^n}{n}\right)$ converges to 0 in the G-metric space.

It will be proven that the sequence $(x_n) = \left(\frac{(-1)^n}{n}\right)$ statistically converges to 0 in a G-metric space. Let j be the largest natural number less than or equal to $\frac{2}{\varepsilon}$ for every $\varepsilon > 0, \varepsilon \in \mathbb{R}$, then

$$\begin{aligned}
\lim_{n \to +\infty} \left(\frac{2}{n^2} |\{(n_1, n_2) \in \mathbb{N}^2 : n_1, n_2 \leq n, G(x_i, x_{n_1}, x_{n_2}) \geq \varepsilon\}| \right) &= \lim_{n \to +\infty} (\frac{2}{n^2} |\{(1,1), (1,2), (1, \\
&\quad 3), \dots, (j,1), (j,2), \dots\}|) \\
&\leq \lim_{n \to +\infty} (\frac{2nj}{n^2}) \\
&\leq 2j \lim_{n \to +\infty} (\frac{1}{n}) \\
&= 2j.0 = 0
\end{aligned}$$

Theorem 4. *Given a sequence (x_n) that converges statistically to x in a G-metric space, if the sequence (x_n) is monotonic, then (x_n) converges ordinarily to x in the G-metric space.*

Proof. The sequence (x_n) converging statistically to x means that for every real number $\varepsilon > 0$, the following holds:

$$\lim_{n \to +\infty} \left(\frac{2}{n^2} |\{(n_1, n_2) \in \mathbb{N}^2 : n_1, n_2 \leq n, G(x, x_{n_1}, x_{n_2}) \geq \varepsilon\}| \right) = 0$$

A Sequence (x_n) is monotonic increasing if $x_1 \leq x_2 \leq \cdots \leq x_n \leq x_{n+1} \leq \dots$, and monotonic decreasing if $x_1 \geq x_2 \geq \dots x_n \geq x_{n+1} \geq \dots$. If for some $(m,n) \in \mathbb{N}^2, G(x, x_m, x_n) \geq \varepsilon$, then for every $i < n$ and $k < m$, $G(x, x_i, x_k) \geq \varepsilon$ because (x_n) converges statistically. Additionally, since (x_n) converges statistically, we have $|A| = |\{n : (n,m) \in \mathbb{N}^2, G(x, x_n, x_m) \geq \varepsilon\}| < +\infty$, let $|A| = z$ or $|B| = |\{m : (n,m) \in \mathbb{N}^2, G(x, x_n, x_m) \geq \varepsilon\}| < +\infty$, let $|B| = y$. If $|A| = |\{n : (n,m) \in \mathbb{N}^2, G(x, x_n, x_m) \geq \varepsilon\}| < +\infty$, then there exists $z + 1 \in \mathbb{N}$ such that for all $n, m \geq z + 1, G(x, x_n, x_m) < \varepsilon$, or in other words, the sequence (x_n) converges ordinarily to x in the G-metric space. If

$|B| = |\{m : (n,m) \in \mathbb{N}^2, G(x, x_n, x_m) \geq \varepsilon\}| < +\infty$, then there exists $y+1 \in \mathbb{N}$ such that for all $n, m \geq y+1, G(x, x_n, x_m) < \varepsilon$, or in other words, the sequence (x_n) converges ordinarily to x in the G-metric space.

Example 3. *Given a G-metric space, $(\mathbb{R}, G)$, and for every $x, y, z \in \mathbb{R}$, the following condition holds:*

$$G(x, y, z) = max\{|x-y| + |x-z| + |y-z|\}$$

The sequence $(x_n) = \left(\frac{n}{n+1}\right)$ converges to 1 in the G-metric space.

Clearly, (x_n) is an increasing sequence since $\frac{n}{n+1} \leq \frac{n+1}{n+2}$. Next, it will be proven that the sequence (x_n) statistically converges to 1 in the G-metric space. Let i be the greatest integer less than or equal to $\frac{1}{\varepsilon} - 1$ for every $\varepsilon > 0, \varepsilon \in \mathbb{R}$, then

$$\begin{aligned} \lim_{n\to+\infty}\left(\frac{2}{n^2}|\{(n_1, n_2) \in \mathbb{N}^2 : n_1, n_2 \leq n, G(x_i, x_{n_1}, x_{n_2}) \geq \varepsilon\}|\right) &= \lim_{n\to+\infty}(\frac{2}{n^2}|\{(1,1),(1,2),(1, \\ &\quad 3),\ldots,(i,1),(i,2),\ldots\}|) \\ &\leq \lim_{n\to+\infty}(\frac{2ni}{n^2}) \\ &\leq 2i \lim_{n\to+\infty}(\frac{1}{n}) \\ &= 2i.0 = 0 \end{aligned}$$

The statement $(x_n) = \left(\frac{n}{n+1}\right)$ is statistically convergent to 1 in the G-metric space" has been proven. Since $(x_n) = \left(\frac{n}{n+1}\right)$ is statistically convergent to 1 in the G-metric space and is a monotonic sequence, by Theorem 4, $(x_n) = \left(\frac{n}{n+1}\right)$ converges to 1 in the G-metric space.

Theorem 5. *[19] Given an admissible ideal $\mathcal{I}^2$. If a sequence converges to x in a G-metric space, then the sequence converges ideally to x in the G-metric space with the ideal $\mathcal{I}^2$.*

Proof. Let (x_n) be a sequence that converges to x in the G-metric space $(\mathbb{R}, G)$. This means that for every real number $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for every natural number $n, m \geq n_0$, we have $G(x, x_n, x_m) < \varepsilon$. If we define a set, it will form the following set:

$$A(n_0) = \left\{(n,m) \in \mathbb{N}^2 : n, m \geq n_0, G(x, x_n, x_m) < \varepsilon\right\}$$

It is clear that

$$|A(n_0)| = +\infty$$

Since there exists $n_0 \in \mathbb{N}$ such that for every $n, m \geq n_0, G(x, x_n, x_m) < \varepsilon$, the number of $n, m \in \mathbb{N}$ that satisfy $G(x, x_n, x_m) \geq \varepsilon$ is at most j^2, or in other words, it is finite. Thus, $A(\varepsilon) \in \mathcal{I}^2$ Therefore, the sequence (x_n) is ideally convergent to x.

We know that the sequence $(x_n) = \left(\frac{1}{n}\right)$ is ideally convergent to 0 in the G-metric space with one of the admissible ideals, and it also converges ordinarily to 0 in the G-metric space.

Theorem 6. *Let (x_n) be a sequence that ideally converges to L in a G-metric space with the ideal $\mathcal{I}^2$. If for every $A \in \mathcal{I}^2$, $|A| < \lim\limits_{n\to+\infty} \left(n^2\right)$, then the sequence (x_n) statistically converges to L in the G-metric space.*

Proof. It is known that the sequence (x_n) is ideally convergent to L, which means that for every $\varepsilon > 0$, we have $A_\varepsilon = (n, m) \in \mathbb{N}^2 : G\left(L, x_n, x_m\right) \geq \varepsilon \in \mathcal{I}^2$. Since $|A| < \lim\limits_{n\to+\infty} \left(n^2\right)$ for every $A \in \mathcal{I}^2$ it follows that $|A_\varepsilon| < +\infty$. Let $|A_\varepsilon| = \lim\limits_{n\to+\infty} (nj)$, *with* $j \in \mathbb{N}$, so that for every $\varepsilon > 0$, the following holds:

$$\begin{aligned}
\lim_{n\to+\infty} \left(\frac{2}{n^2}|\{(n,m) \in \mathbb{N}^2 : n, m \geq n, G(0, x_n, x_m) < \varepsilon\}|\right) &< \lim_{n\to+\infty} (\frac{nj}{n^2}) \\
&< \lim_{n\to+\infty} (\frac{j}{n}) \\
&= 0
\end{aligned}$$

So, the sequence (x_n) statistically converges to L.

Example 4. *It will be proven that the sequence $\left(\frac{1}{n}\right)$ statistically converges to 0 in the G-metric space.*

It has been known that the sequence $\left(\frac{1}{n}\right)$ converges ideally to 0 in the G-metric space with the ideal $\mathcal{I} = A \subset \mathbb{N}^2 : |A| < \lim\limits_{n\to+\infty} \left(n^2\right)$. It will also be shown that the sequence $\left(\frac{1}{n}\right)$ statistically converges to 0 in the G-metric space.

Preliminary Analysis

$$\begin{aligned}
G\left(x, x_{n_1}, x_{n_2}\right) &= \left|0 - \frac{1}{n_1}\right| + \left|\frac{1}{n_1} - \frac{1}{n_2}\right| + \left|\frac{1}{n_2} - 0\right| \\
&= \frac{1}{n_1} + \frac{1}{n_2} + \left|\frac{1}{n_1} - \frac{1}{n_2}\right| \\
&\geq \frac{1}{n_1} + \frac{1}{n_2} + \left(\left|\frac{1}{n_1}\right| - \left|\frac{1}{n_2}\right|\right) \\
&\geq \frac{2}{n_1}
\end{aligned}$$

To ensure $\frac{2}{n_1} \geq \varepsilon$, then n_1 must be a natural number k such that $k \leq \frac{2}{\varepsilon}$. Therefore, for every real number $\varepsilon > 0$, the following holds:

$$\begin{aligned}\lim_{n\to+\infty}\left(\frac{2}{n^2}|\{(n_1,n_2)\in\mathbb{N}^2 : n_1,n_2\leq n, G(x_i,x_{n_1},x_{n_2})\geq\varepsilon\}|\right) &= \lim_{n\to+\infty}(\frac{2}{n^2}|\{(1,1),(1,2),(1, \\ &\quad 3),\ldots,(k,1),(k,2),\ldots\}|) \\ &\leq \lim_{n\to+\infty}(\frac{2nk}{n^2}) \\ &\leq 2k\lim_{n\to+\infty}(\frac{1}{n}) \\ &= 2k.0 = 0\end{aligned}$$

Thus, the sequence $\left(\frac{1}{n}\right)$ is also statistically proven to converge to 0 in the G-metric space.

Theorem 7. *[19] Given a sequence (x_n) that statistically converges to x in the G-metric space and the ideal $\mathcal{I}^2$, where $\mathcal{I}^2 = A : \delta(A) = 0$, the sequence (x_n) ideally converges to x in the G-metric space.*

Proof. A sequence (x_n) is said to statistically converge to x in a G-metric space if

$$\lim_{n\to+\infty}\left(\frac{2}{n^2}\left|(n_1,n_2)\in\mathbb{N}^2 : n_1,n_2\leq n, G\left(x,x_{n_1},x_{n_2}\right)\geq\varepsilon\right|\right) = 0$$

Thus, the set $\{(n_1,n_2)\in\mathbb{N}^2 : G\left(x,x_{n_1},x_{n_2}\right)\geq\varepsilon\}$ has asymptotic density 0. This means $(x_n)\in\mathcal{I}^2$, or equivalently, (x_n) converges ideally to x in the G-metric space.

Theorem 8. *[1] If (x_n) statistically converges in a G-metric space, then (x_n) is a statistically Cauchy sequence in the G-metric space.*

Proof. Let $\varepsilon > 0$ be any real number. Suppose $Gs - lim(x_n) = x$. Since ε is a real number and $\varepsilon > 0$, then $\frac{\varepsilon}{6}$ is also a real number and greater than 0. Therefore, the set $\{(n_1,n_2)\in\mathbb{N}^2 : G\left(x,x_{n_1},x_{n_2}\right)\geq\frac{\varepsilon}{6}\}$ has asymptotic density 0. Let $m\in\mathbb{N}$ be chosen such that $G\left(x,x_{n_1},x_m\right)\geq\frac{\varepsilon}{6}$. Then

$$\begin{aligned}G\left(x_m,x_{n_1},x_{n_2}\right) &\leq G\left(x_m,x,x\right) + G\left(x,x_{n_1},x\right) + G\left(x,x,x_{n_2}\right) \\ &\leq 2(G\left(x,x_{n_1},x_{n_2}\right) + G\left(x,x_{n_1},x_{n_2}\right) + G\left(x,x_{n_1},x_m\right)) \\ &< 2(\frac{\varepsilon}{6}+\frac{\varepsilon}{6}+\frac{\varepsilon}{6}) = \varepsilon\end{aligned}$$

Thus, the set $\{(n_1,n_2)\in\mathbb{N}^2 : G\left(x_m,x_{n_1},x_{n_2}\right)\geq\varepsilon\}$ has asymptotic density 0, meaning that (x_n) is an is a statistically Cauchy sequence.

Theorem 9. *[19] If (x_n) ideally converges in a G-metric space, then (x_n) is an ideal Cauchy sequence in the G-metric space.*

Proof. Let $\varepsilon > 0$ be any real number. Suppose $GI - lim(x_n) = x$. Since ε is a real number and $\varepsilon > 0$, then $\frac{\varepsilon}{6}$ is also a real number and greater than 0. Therefore, the set $\{(n_1, n_2) \in \mathbb{N}^2 : G(x, x_{n_1}, x_{n_2}) \geq \frac{\varepsilon}{6}\} \in I$. Let $m \in \mathbb{N}$ be chosen such that $G(x, x_{n_1}, x_m) \geq \frac{\varepsilon}{6}$. Then

$$\begin{aligned} G(x_m, x_{n_1}, x_{n_2}) &\leq G(x_m, x, x) + G(x, x_{n_1}, x) + G(x, x, x_{n_2}) \\ &\leq 2(G(x, x_{n_1}, x_{n_2}) + G(x, x_{n_1}, x_{n_2}) + G(x, x_{n_1}, x_m)) \\ &< 2(\frac{\varepsilon}{6} + \frac{\varepsilon}{6} + \frac{\varepsilon}{6}) = \varepsilon \end{aligned}$$

Thus, the set $\{(n_1, n_2) \in \mathbb{N}^2 : G(x_m, x_{n_1}, x_{n_2}) \geq \varepsilon\} \in \mathcal{I}$, which means (x_n) is an ideal Cauchy sequence.

Theorems 1, 2, 5, 8, and 9 yield the following corollaries:

Corollary 1. *If the sequence (x_n) is a Cauchy sequence in a G-metric space, then the sequence (x_n) is statistically Cauchy in the G-metric space.*

Proof. Based on Theorem 1, it can be observed that a normally convergent sequence in a G-metric space is a Cauchy sequence in the G-metric space. According to Theorem 2, if a sequence is normally convergent in a G-metric space, then it is statistically convergent in the G-metric space. Theorem 8 states that a statistically convergent sequence in a G-metric space is a statistically Cauchy sequence in the G-metric space. From these three theorems, it can be concluded that a Cauchy sequence in a G-metric space is a statistically Cauchy sequence in the G-metric space.

Corollary 2. *Given an admissible ideal $\mathcal{I}$, if the sequence (x_n) is a Cauchy sequence, then the sequence (x_n) is an Ideal Cauchy sequence.*

Proof. According to Theorem 1, a normally convergent sequence in a G-metric space is a Cauchy sequence in the G-metric space. Based on Theorem 5, if a sequence is normally convergent in a G-metric space, then it is Ideal convergent in the G-metric space with respect to the admissible ideal. Theorem 9 states that an Ideal convergent sequence in a G-metric space is an Ideal Cauchy sequence in the G-metric space. From these three theorems, it can be concluded that a Cauchy sequence in a G-metric space.

Theorem 10. *Given (x_n) is a sequence of real numbers with an admissible ideal $\mathcal{I}^2$. If (x_n) is Ideal convergent to L in the G-metric space, then there exists a sequence (y_n) that is statistically convergent to L in the G-metric space, such that $(|xn - yn|)$ is Ideal convergent to 0 in the G-metric space.*

Proof. Let $\varepsilon > 0$ be a given real number. The sequence (x_n) is Ideal convergent to x, which means that for each such ε, the set $(n_1, n_2) \in \mathbb{N}^2 : G(x, x_{n_1}, x_{n_2}) \geq \varepsilon \in \mathcal{I}^2$. The sequence (y_n) is statistically convergent to x, which means that for each such ε, the following holds:

$$\lim_{n \to +\infty} \left(\frac{2}{n^2} \left| (n_1, n_2) \in \mathbb{N}^2 : n_1, n_2 \leq n : G(x, y_{n_1}, y_{n_2}) \geq \varepsilon \right| \right) = 0$$

Let $\{(n_1, n_2) \in \mathbb{N}2 : n1, n2 \leq n : G(x, y_{n1}, y_{n2}) \geq \varepsilon\} = A_\varepsilon$. If the sequence (y_n) satisfies $A_\varepsilon \in \mathcal{I}$ for each given ε, then

$$G(x, x_{n_1}, x_{n_2}) - G(x, y_{n_1}, y_{n_2}) = G(0, x_{n_1} - y_{n_1}, x_{n_2} - y_{n_2})$$

Thus, the set $\{(n_1, n_2) \in \mathbb{N}^2 : n_1, n_2 \leq n : G\left(0, x_{n_1} - y_{n_1}, x_{n_2} - y_{n_2}\right) \geq \varepsilon\} \in \mathcal{I}$. Therefore, the sequence $(|x_n - y_n|)$ is Ideal convergent to 0.

Theorem 11. *Let $\mathcal{I}$ be a non-trivial ideal on $\mathbb{N}^2$. If the sequence of real numbers (x_n) ideal converges to L in the metric-G space, then there exists a subsequence (x_{n_k}) of (x_n) that converges to x in the usual sense in the metric-G space, and there exists a subsequence (x_{m_k}) that converges statistically to L in the metric-G space.*

Proof. Let $\varepsilon > 0$ be an arbitrary real number. The fact that (x_n) ideal converges to L means that for any ε, the set $(n, m) \in \mathbb{N}^2 : G(L, x_n, x_m) \geq \varepsilon \in \mathcal{I}$. If we select a subsequence (x_{n_k}) of (x_n) such that its members satisfy $G(L, x_n, x_m) < \varepsilon$, then clearly (x_{n_k}) converges to L. Similarly, we can choose a subsequence (x_{m_k}) of (x_n) such that

$$\lim_{n \to +\infty} \left(\frac{2}{n^2} |\{(n_1, n_2) \in \mathbb{N}^2 : n_1, n_2 \leq n, G(x_i, x_{n_1}, x_{n_2}) \geq \varepsilon\}| \right)$$

which implies that the sequence converges statistically to L.

4. Conclusion

In this study, several significant theorems regarding the convergence properties of sequences in G-metric spaces have been established. The results demonstrate a strong connection between different types of convergence—statistical, ideal, and standard—within the framework of G-metric spaces. Overall, this research enhances our understanding of the behaviour of sequences in G-metric spaces and the interplay between different convergence concepts, paving the way for further exploration in this area of metric space theory.

Future research directions could include extending these results to more generalized metric spaces, such as cone G-metric spaces or fuzzy G-metric spaces, to explore whether similar equivalences and relationships hold. Another promising avenue would be to investigate the applications of these convergence properties in solving fixed point problems or optimization problems, where G-metric spaces often provide a natural framework. Additionally, studying the implications of these results in functional analysis or dynamical systems may yield new insights into stability and convergence behaviours in broader mathematical contexts. Finally, integrating these findings with probabilistic and stochastic settings could open new paths for exploring the role of convergence in real-world applications, such as data science or machine learning.

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