**EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS**
2025, Vol. 18, Issue 1, Article Number 5608
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

# Two-variable Trapezoidal Type Inequalities in Banach Spaces

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**Abstract.** This study presents weighted trapezoidal-type inequalities for the product of two functions. While one function takes its values in the Banach spaces, the other takes values in the complex plane. We employed the technique of integration by parts for Bochner integrals for functions of two variables, along with principles of analysis for functions taking values in the product of Banach spaces, to report our findings. In addition to the extension of previous studies on functions of a single variable, our work generalizes results reported in Dragomir for two functions whose product of their variables lies in Banach spaces. Our results are new and original since they do not yet exist in the literature.

**2020 Mathematics Subject Classifications**: 26D15, 26A45, 26D10, 26B25, 41A55, 46B20, 47A63, 47A99

**Key Words and Phrases**: Weighted trapezoidal inequalities, Banach spaces, Bochner integrals, complex-valued functions

## 1. Introduction

Many fundamental inequalities, of Ostrowski, Lubenov and Hermite-Hadamard types, play a crucial role in the realm of mathematical analysis [2, 12]. In addition to providing bound estimates for the difference between functions and their averages, these inequalities form the basis of many discoveries in mathematical analysis. A broad range of their applications has inspired researchers from many disciplines to generalize and extend the inequalities beyond their classical formulations. Ostrowski's inequality [10], for example, generalizes functions with precise properties including monotonicity, convexity and bounded variation to estimate bounds for deviations between such functions and their averages. These fundamental inequalities can be more relevant in both pure and applied

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DOI: https://doi.org/10.29020/nybg.ejpam.v18i1.5608

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mathematics when their extensions enhance basic mathematical concepts to suitably handle variant types of functions.

The application of these inequalities exists in numerous areas of interest, for example, in applied mathematics, the inequalities are essential tools for solving differential equations, improving bounds of the solutions, estimating integrals and providing accuracy of numerical methods. The inequalities are equally important tools for approximating the error bounds of numerical integration, as well as improving the accuracy of computational methods, like the Simpson's rule and the trapezoidal rule. In addition to simple inequalities involving real-valued functions, others appear in more abstract settings like Banach and Hilbert spaces, two of which, that is, Ostrowski's and Hermite-Hadamard's inequalities play a pivotal role in developing functional analysis and modern mathematical physics. The generalizations of these inequalities in abstract vector spaces can provide a unified technique through which multiple problems existing in different fields of study including approximation theory, optimization and quantum mechanics can be solved.

One of the important integral inequalities is known as the Ostrowski's inequality that involves mappings with bounded derivatives. As emphasized earlier, this inequality plays a significant role in approximation theory and numerical analysis where the bound for the deviation between the value of a function at a particular point and its average can be estimated.

This inequality states that for a differentiable function $f : [a, b] \to \mathbb{R}$ with $|f'(t)| \leq M$ for all $t \in (a, b)$, the following inequality holds for all $x \in [a, b]$:

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \leq M(b-a) \left[ \frac{1}{4} + \left( \frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right]. \tag{1.1}$$

Other inequalities with broad applications in numerical analysis include those of Trapezoidal-types that provide error bounds when evaluating the integral of a function through the average of its values at the boundary points of an interval [7]. This approach provides valuable insights into the formulation of many integral inequalities. In addition, the trapezoidal inequality has been further developed to include convex functions in a study of Pearce and Pecaric[11], whose impacts continues to manifest in pure and applied related disciplines, such as economics. Broader classes of functions and spaces were later emerged generalizing trapezoidal-type inequalities. This can be seen in the work of Cerone et al. [4] where trapezoidal inequalities were extended to provide new bounds for functions of bounded variation.

The trapezoidal-type inequalities involving functions with values in Hilbert spaces present another generalization of such inequalities in more abstract settings Dragmir [5]. This has been recently extended by in more abstract settings to [9] involve functions with values in Banach spaces and operator monotone functions. These generalizations are of great importance in optimization and functional analysis since they can analyze errors generated from numerical integration. Several extensions are noted in many studied, such as those by Budak et al. [3], Alomari [1], and Tseng and Hwangy [13].

The following result is of great importance that gives the refinement and reverse of the weighted trapezoid inequality as follows [6]:

**Theorem 1.** *If $g : [a, b] \to \mathbb{R}$ is Riemann integrable on $[a, b]$ and $f : [a, b] \to \mathbb{R}$ is of bounded variation on $[a, b]$, then*

$$\left|\int_a^b f(t)g(t)dt - f(b)\int_u^b g(t)dt - f(a)\int_a^u g(t)dt\right| \leq \left[\frac{1}{2}(b-a) + \left|u - \frac{a+b}{2}\right|\right] \sup_{t\in[a,b]} |g(t)| \bigvee_a^b (f) \quad (1.2)$$

*for all $u \in [a, b]$, where $\bigvee_a^b(f)$ is the total variation of $f$ on $[a, b]$.*
*In particular, we have the mid-point trapezoid inequality*

$$\left|\int_a^b f(t)g(t)dt - f(b)\int_{\frac{a+b}{2}}^b g(t)dt - f(a)\int_a^{\frac{a+b}{2}} g(t)dt\right| \leq \frac{1}{2}(b-a) \sup_{t\in[a,b]} |g(t)| \bigvee_a^b (f). \quad (1.3)$$

*The constant $1/2$ in (1.3) is sharp in the sense that it cannot be replaced by a smaller quantity.*

The result on trapezoidal type inequalities for operator convex functions for functions in Hilbert spaces $H$ followed by the following definition is proved by Dragomir in [8]:

**Definition 1.** *[8] A continuous function $f : I \to \mathbb{R}$ is operator convex on the interval $I$ if*

$$f\left((1-t)A + tB\right) \leq (1-t) f(A) + t f(B) \quad (1.4)$$

*for all $t \in [0, 1]$, where $A$ and $B$ are selfadjoint operators on a Hilbert space $(H, \langle\cdot,\cdot\rangle)$ with spectra $Sp(A), Sp(B) \subset I$.*

The finding is reported

**Theorem 2.** *Let $f$ be an operator convex function on $I$ and $A, B, A \neq B$, selfadjoint operators on $H$ with $Sp(A), Sp(B) \subset I$. If $f$ is Gâteaux differentiable on $[A, B] := \{(1-t)A + tB, t \in [0, 1]\}$ and $p : [0, 1] \to [0, \infty)$ is Lebesgue integrable and symmetric, namely $p(1-t) = p(t)$ for all $t \in [0, 1]$, then*

$$\begin{aligned} 0 &\leq \left(\int_0^1 p(t)dt\right)\frac{f(A)+f(B)}{2} - \int_0^1 p(t)f((1-t)A+tB)dt \\ &\leq \frac{1}{2}\int_0^1 \left(\frac{1}{2} - \left|t - \frac{1}{2}\right|\right) p(t)dt \left[\nabla f_B(B-A) - \nabla f_A(B-A)\right], \end{aligned} \quad (1.5)$$

*where $\nabla f_C(V)$ is the Gâteaux derivative in $C$ over the direction $V$.*
*In particular, for $p \equiv 1$ we get*

$$0 \leq \frac{f(A)+f(B)}{2} - \int_0^1 f((1-t)A+tB)dt \leq \frac{1}{8}\left[\nabla f_B(B-A) - \nabla f_A(B-A)\right]. \quad (1.6)$$

Dragomir [9] studied the following weighted version of generalized trapezoid inequality involving two functions whose values are within Banach spaces:

**Theorem 3.** *[9]Assume that $\alpha : [a,b] \to \mathbb{C}, Y : [a,b] \to E$ are continuous and $Y$ is strongly differentiable on $(a,b)$, then for all $u \in [a,b]$*

$$\left\| \left(\int_u^b \alpha(s)ds\right) Y(b) + \left(\int_a^u \alpha(s)ds\right) Y(a) - \int_a^b \alpha(t)Y(t)dt \right\| \leq C(\alpha, Y, u), \quad (1.7)$$

*where*

$$C(\alpha, Y, u) := \int_u^b \left(\int_u^t |\alpha(s)|ds\right) \left\|Y'(t)\right\| dt + \int_a^u \left(\int_t^u |\alpha(s)|ds\right) \left\|Y'(t)\right\| dt.$$

*We also have the bounds*

$$C(\alpha, Y, u) \leq \begin{cases} \int_u^b |\alpha(s)|ds \int_u^b \|Y'(t)\| dt + \left(\int_a^u |\alpha(s)|ds\right) \int_a^u \|Y'(t)\| dt, \\ \left[\int_u^b \left(\int_u^t |\alpha(s)|ds\right)^p dt\right]^{1/p} \left(\int_u^b \|Y'(t)\|^q dt\right)^{1/q} \\ \quad + \left[\int_a^u \left(\int_t^u |\alpha(s)|ds\right)^p dt\right]^{1/p} \left(\int_a^u \|Y'(t)\|^q dt\right)^{1/q}, \\ \int_u^b \left(\int_u^t |\alpha(s)|ds\right) dt \sup_{t\in[u,b]} \|Y'(t)\| \\ \quad + \int_a^u \left(\int_t^u |\alpha(s)|ds\right) dt \sup_{t\in[a,u]} \|Y'(t)\|, \end{cases} \quad (1.8)$$

*where $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.*

Most fundamental inequalities discussed earlier are interconnected with Banach spaces in different areas of mathematical analysis, particularly in functional analysis and approximation theory. Inequalities like Minkowski's and Hölder's inequalities, as well as convexity-based inequalities can be analyzed through the direct product of complete normed vector spaces, such as Banach spaces.

Thus, we consider how the structure of direct product of Banach spaces relates to these inequalities. Suppose that $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are Banach spaces, and let $V = X \times Y$ (also a Banach space with a suitable norm) be their direct product with component-wise operations. Therefore, the common norms for the space $V = X \times Y$ are given as:

$$\|(x,y)\|_{X\times Y} = \|x\|_X + \|y\|_Y,$$

and

$$\|(x,y)\|_{X\times Y} = \max\{\|x\|_X, \|y\|_Y\}$$

or, more generally

$$\|(x,y)\|_{X\times Y} = \left(\|x\|_X^p + \|y\|_Y^p\right)^{\frac{1}{p}},$$

for all $(x,y) \in V = X \times Y$, where $1 \leq p < \infty$.

Whenever an innovative idea in mathematics takes a form of a mathematical result, it becomes an inspiration and motivation to mathematicians. Mathematicians try to improve, refine or generalise results of such innovative notions and seek applications of those results in other fields of mathematical and physical sciences. The main inspiration of our current study are the results of established in Dragomir [9]. The results conducted in this study were proven are the weighted trapezoidal-type inequalities for the product of two functions of one of which takes values in the Banach spaces and the second function takes the values the complex plane. The idea is quite logical since the product of a scalar (value of function that takes values in the complex plane) and a vector (the value of function that takes values in the Banach space) is defined. Dragomir [9] utilized the the integration by parts formula for Bochner integral, properties of norm, Hölder's inequality and techniques of analysis in Banach spaces to obtain his results.

Motivated by the study of Dragomir [9] to prove some similar innovative results to the product of functions, one of which have values in the complex plane and the other have values in the direct product of two Banach spaces together with the norms as defined above. We have also used the integration by parts technique for Bochner integrals for functions of two variables and principles of analysis for functions that take values in the product of Banach spaces to demonstrate the results of this study. The results are original as no such studies have been seen in any previously conducted studies so far and thus our results can be very useful for further exploration of the topic of inequalities for functions with values in product Banach spaces and even in more general abstract spaces.

## 2. Main Results

We present our first results generalized Trapezoidal-type inequalities for a two-variable function within Banach spaces.

**Theorem 4.** *Assume that* $\alpha : [a,b] \times [c,d] \to \mathbb{C}$ *and* $Y : [a,b] \times [c,d] \to E_1 \times E_2$ *are continuous and* $\frac{\partial Y}{\partial t}$, $\frac{\partial Y}{\partial w}$, $\frac{\partial^2 Y}{\partial t \partial w}$ *exist on* $(a,b) \times (c,d)$ *in the norm topology* $\|(x,y)\|_{X\times Y} = \|x\|_X + \|y\|_Y$ *of* $E_1 \times E_2$, $(x,y) \in E_1 \times E_2$, *then we get the following inequalities for all* $(u,v) \in [a,b] \times [c,d]$ *for* $p$, $q > 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$*:*

$$\Bigg\| \left(\int_u^b \int_v^d \alpha(s,r)\, dr ds\right) Y(b,d) + \left(\int_a^u \int_v^d \alpha(s,r)\, dr ds\right) Y(a,d)$$
$$+ \left(\int_u^b \int_c^v \alpha(s,r)\, dr ds\right) Y(b,c) + \left(\int_a^u \int_c^v \alpha(s,r)\, dr ds\right) Y(a,c)$$
$$- \int_a^b Y(t,d) \left(\int_v^d \alpha(t,r)\, dr\right) dt - \int_a^b Y(t,c) \left(\int_c^v \alpha(t,r)\, dr\right) dt$$

$$
\begin{aligned}
&-\int_c^d Y(b,w)\left(\int_u^b \alpha(s,w)\,ds\right)dw-\int_c^d Y(a,w)\left(\int_a^u \alpha(s,w)\,ds\right)dw\\
&\qquad+\int_a^b\int_c^d \alpha(t,w)\,Y(t,w)\,dwdt\Big\|_{E_1\times E_2}\le B(\alpha,Y,u,v)\,, \qquad (2.1)
\end{aligned}
$$

*where*

$$
\begin{aligned}
B(\alpha,Y,u,v) &:= \int_u^b\int_v^d\left(\int_u^t\int_v^w |\alpha(s,r)|\,drds\right)\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2} dwdt\\
&+\int_a^u\int_v^d\left(\int_t^u\int_v^w |\alpha(s,r)|\,drds\right)\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2} dwdt\\
&+\int_u^b\int_c^v\left(\int_u^t\int_w^v |\alpha(s,r)|\,drds\right)\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2} dwdt\\
&+\int_a^u\int_c^v\left(\int_t^u\int_w^v |\alpha(s,r)|\,drds\right)\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2} dwdt. \qquad (2.2)
\end{aligned}
$$

*We also have the following bounds for* $B(\alpha,Y,u,v)$*:*

$$B(\alpha,Y,u,v)$$

$$\leq \begin{cases} \int_u^b \int_v^d |\alpha(s,r)| \, drds \int_u^b \int_v^d \left\| \frac{\partial^2}{\partial w \partial t} Y(t,w) \right\|_{E_1 \times E_2} dwdt \\ + \int_a^u \int_v^d |\alpha(s,r)| \, drds \int_a^u \int_v^d \left\| \frac{\partial^2}{\partial w \partial t} Y(t,w) \right\|_{E_1 \times E_2} dwdt \\ + \int_u^b \int_c^v |\alpha(s,r)| \, drds \int_u^b \int_c^v \left\| \frac{\partial^2}{\partial w \partial t} Y(t,w) \right\|_{E_1 \times E_2} dwdt \\ + \int_a^u \int_c^v |\alpha(s,r)| \, drds \int_a^u \int_c^v \left\| \frac{\partial^2}{\partial w \partial t} Y(t,w) \right\|_{E_1 \times E_2} dwdt, \\ \\ \left[ \int_u^b \int_v^d \left( \int_u^t \int_c^w |\alpha(s,r)| \, drds \right)^p dwdt \right]^{\frac{1}{p}} \left[ \int_u^b \int_v^d \left\| \frac{\partial^2}{\partial w \partial t} Y(t,w) \right\|_{E_1 \times E_2}^q dwdt \right]^{\frac{1}{q}} \\ + \left[ \int_a^u \int_v^d \left( \int_t^u \int_v^w |\alpha(s,r)| \, drds \right)^p dwdt \right]^{\frac{1}{p}} \left[ \int_a^u \int_v^d \left\| \frac{\partial^2}{\partial w \partial t} Y(t,w) \right\|_{E_1 \times E_2}^q dwdt \right]^{\frac{1}{q}} \\ + \left[ \int_u^b \int_c^v \left( \int_u^t \int_w^v |\alpha(s,r)| \, drds \right)^p dwdt \right]^{\frac{1}{p}} \left[ \int_u^b \int_c^v \left\| \frac{\partial^2}{\partial w \partial t} Y(t,w) \right\|_{E_1 \times E_2}^q dwdt \right]^{\frac{1}{q}} \\ + \left[ \int_a^u \int_c^v \left( \int_t^u \int_w^v |\alpha(s,r)| \, drds \right)^p dwdt \right]^{\frac{1}{p}} \left[ \int_a^u \int_c^v \left\| \frac{\partial^2}{\partial w \partial t} Y(t,w) \right\|_{E_1 \times E_2}^q dwdt \right]^{\frac{1}{q}}, \\ \\ \int_u^b \int_v^d \left( \int_u^t \int_v^w |\alpha(s,r)| \, drds \right) dwdt \sup_{(t,w) \in [u,b] \times [v,d]} \left\| \frac{\partial^2}{\partial w \partial t} Y(t,w) \right\|_{E_1 \times E_2} \\ + \int_a^u \int_v^d \left( \int_t^u \int_v^w |\alpha(s,r)| \, drds \right) dwdt \sup_{(t,w) \in [a,u] \times [v,d]} \left\| \frac{\partial^2}{\partial w \partial t} Y(t,w) \right\|_{E_1 \times E_2} \\ + \int_u^b \int_c^v \left( \int_u^t \int_w^v |\alpha(s,r)| \, drds \right) dwdt \sup_{(t,w) \in [u,b] \times [c,v]} \left\| \frac{\partial^2}{\partial w \partial t} Y(t,w) \right\|_{E_1 \times E_2} \\ + \int_a^u \int_c^v \left( \int_t^u \int_w^v |\alpha(s,r)| \, drds \right) dwdt \sup_{(t,w) \in [a,u] \times [c,v]} \left\| \frac{\partial^2}{\partial w \partial t} Y(t,w) \right\|_{E_1 \times E_2}. \end{cases} \tag{2.3}$$

*Proof.* Let $(u,v) \in [a,b] \times [c,d]$. Using the integration by parts formula for Bochner integral first with respect to $w$ and then with respect to $t$, we have

$$\begin{aligned} \int_a^b \int_c^d \bigg[ & \int_a^t \int_c^w \alpha(s,r) \, drds - \int_a^t \int_c^v \alpha(s,r) \, drds \\ & - \int_a^u \int_c^w \alpha(s,r) \, drds + \int_a^u \int_c^v \alpha(s,r) \, drds \bigg] \frac{\partial^2}{\partial w \partial t} Y(t,w) \, dwdt \\ & = \left( \int_u^b \int_v^d \alpha(s,r) \, drds \right) Y(b,d) - \left( \int_u^b \int_c^v \alpha(s,r) \, drds \right) Y(b,c) \\ & - \left( \int_a^u \int_v^d \alpha(s,r) \, drds \right) Y(a,d) + \left( \int_a^u \int_c^v \alpha(s,r) \, drds \right) Y(a,c) \\ & - \int_a^b Y(t,d) \left( \int_v^d \alpha(t,r) \, dr \right) dt - \int_a^b Y(t,c) \left( \int_c^v \alpha(t,r) \, dr \right) dt \\ & - \int_c^d Y(b,w) \left( \int_u^b \alpha(s,w) \, ds \right) dw - \int_c^d Y(a,w) \left( \int_a^u \alpha(s,w) \, ds \right) dw \end{aligned}$$

$$+\int_{a}^{b}\int_{c}^{d}\alpha\left(t,w\right)Y\left(t,w\right)dwdt. \quad (2.4)$$

On the other hand, we can observe that for all $(u,v)\in[a,b]\times[c,d]$, we have

$$\begin{aligned}
&\int_{a}^{b}\int_{c}^{d}\left[\int_{a}^{t}\int_{c}^{w}\alpha\left(s,r\right)drds-\int_{a}^{t}\int_{c}^{v}\alpha\left(s,r\right)drds\right.\\
&\left.-\int_{a}^{u}\int_{c}^{w}\alpha\left(s,r\right)drds+\int_{a}^{u}\int_{c}^{v}\alpha\left(s,r\right)drds\right]\frac{\partial^{2}}{\partial w\partial t}Y\left(t,w\right)dwdt\\
&=\int_{a}^{u}\int_{c}^{d}\left[\int_{a}^{t}\int_{c}^{w}\alpha\left(s,r\right)drds-\int_{a}^{t}\int_{c}^{v}\alpha\left(s,r\right)drds\right.\\
&\left.-\int_{a}^{u}\int_{c}^{w}\alpha\left(s,r\right)drds+\int_{a}^{u}\int_{c}^{v}\alpha\left(s,r\right)drds\right]\frac{\partial^{2}}{\partial w\partial t}Y\left(t,w\right)dwdt\\
&+\int_{u}^{b}\int_{c}^{d}\left[\int_{a}^{t}\int_{c}^{w}\alpha\left(s,r\right)drds-\int_{a}^{t}\int_{c}^{v}\alpha\left(s,r\right)drds\right.\\
&\left.-\int_{a}^{u}\int_{c}^{w}\alpha\left(s,r\right)drds+\int_{a}^{u}\int_{c}^{v}\alpha\left(s,r\right)drds\right]\frac{\partial^{2}}{\partial w\partial t}Y\left(t,w\right)dwdt\\
&=\int_{a}^{u}\int_{c}^{v}\left[\int_{a}^{t}\int_{c}^{w}\alpha\left(s,r\right)drds-\int_{a}^{t}\int_{c}^{v}\alpha\left(s,r\right)drds\right.\\
&\left.-\int_{a}^{u}\int_{c}^{w}\alpha\left(s,r\right)drds+\int_{a}^{u}\int_{c}^{v}\alpha\left(s,r\right)drds\right]\frac{\partial^{2}}{\partial w\partial t}Y\left(t,w\right)dwdt\\
&+\int_{a}^{u}\int_{v}^{d}\left[\int_{a}^{t}\int_{c}^{w}\alpha\left(s,r\right)drds-\int_{a}^{t}\int_{c}^{v}\alpha\left(s,r\right)drds\right.\\
&\left.-\int_{a}^{u}\int_{c}^{w}\alpha\left(s,r\right)drds+\int_{a}^{u}\int_{c}^{v}\alpha\left(s,r\right)drds\right]\frac{\partial^{2}}{\partial w\partial t}Y\left(t,w\right)dwdt\\
&+\int_{u}^{b}\int_{c}^{v}\left[\int_{a}^{t}\int_{c}^{w}\alpha\left(s,r\right)drds-\int_{a}^{t}\int_{c}^{v}\alpha\left(s,r\right)drds\right.\\
&\left.-\int_{a}^{u}\int_{c}^{w}\alpha\left(s,r\right)drds+\int_{a}^{u}\int_{c}^{v}\alpha\left(s,r\right)drds\right]\frac{\partial^{2}}{\partial w\partial t}Y\left(t,w\right)dwdt\\
&+\int_{u}^{b}\int_{v}^{d}\left[\int_{a}^{t}\int_{c}^{w}\alpha\left(s,r\right)drds-\int_{a}^{t}\int_{c}^{v}\alpha\left(s,r\right)drds\right.\\
&\left.-\int_{a}^{u}\int_{c}^{w}\alpha\left(s,r\right)drds+\int_{a}^{u}\int_{c}^{v}\alpha\left(s,r\right)drds\right]\frac{\partial^{2}}{\partial w\partial t}Y\left(t,w\right)dwdt\\
&=\int_{u}^{b}\int_{v}^{d}\left(\int_{u}^{t}\int_{v}^{w}\alpha\left(s,r\right)drds\right)\frac{\partial^{2}}{\partial w\partial t}Y\left(t,w\right)dwdt\\
&-\int_{a}^{u}\int_{v}^{d}\left(\int_{t}^{u}\int_{v}^{w}\alpha\left(s,r\right)drds\right)\frac{\partial^{2}}{\partial w\partial t}Y\left(t,w\right)dwdt\\
&-\int_{u}^{b}\int_{c}^{v}\left(\int_{u}^{t}\int_{w}^{v}\alpha\left(s,r\right)drds\right)\frac{\partial^{2}}{\partial w\partial t}Y\left(t,w\right)dwdt
\end{aligned}$$

$$+\int_a^u\int_c^v\left(\int_t^u\int_w^v \alpha(s,r)\,drds\right)\frac{\partial^2}{\partial w\partial t}Y(t,w)\,dwdt. \quad (2.5)$$

By utilizing (2.4) and (2.5), we otain the following equality:

$$\begin{aligned}
&\left(\int_u^b\int_v^d \alpha(s,r)\,drds\right)Y(b,d)-\left(\int_u^b\int_c^v \alpha(s,r)\,drds\right)Y(b,c)\\
&-\left(\int_a^u\int_v^d \alpha(s,r)\,drds\right)Y(a,d)+\left(\int_a^u\int_c^v \alpha(s,r)\,drds\right)Y(a,c)\\
&-\int_a^b Y(t,d)\left(\int_v^d \alpha(t,r)\,dr\right)dt-\int_a^b Y(t,c)\left(\int_c^v \alpha(t,r)\,dr\right)dt\\
&-\int_c^d Y(b,w)\left(\int_u^b \alpha(s,w)\,ds\right)dw-\int_c^d Y(a,w)\left(\int_a^u \alpha(s,w)\,ds\right)dw\\
&+\int_a^b\int_c^d \alpha(t,w)\,Y(t,w)\,dwdt\\
&=\int_u^b\int_v^d\left(\int_u^t\int_v^w \alpha(s,r)\,drds\right)\frac{\partial^2}{\partial w\partial t}Y(t,w)\,dwdt\\
&-\int_a^u\int_v^d\left(\int_t^u\int_v^w \alpha(s,r)\,drds\right)\frac{\partial^2}{\partial w\partial t}Y(t,w)\,dwdt\\
&-\int_u^b\int_c^v\left(\int_u^t\int_w^v \alpha(s,r)\,drds\right)\frac{\partial^2}{\partial w\partial t}Y(t,w)\,dwdt\\
&+\int_a^u\int_c^v\left(\int_t^u\int_w^v \alpha(s,r)\,drds\right)\frac{\partial^2}{\partial w\partial t}Y(t,w)\,dwdt. \quad (2.6)
\end{aligned}$$

By taking the norm $\|\cdot\|_{E_1\times E_2}$ on both sides (2.6), we get

$$\begin{aligned}
&\Bigg\|\left(\int_u^b\int_v^d \alpha(s,r)\,drds\right)Y(b,d)-\left(\int_u^b\int_c^v \alpha(s,r)\,drds\right)Y(b,c)\\
&-\left(\int_a^u\int_v^d \alpha(s,r)\,drds\right)Y(a,d)+\left(\int_a^u\int_c^v \alpha(s,r)\,drds\right)Y(a,c)\\
&-\int_a^b Y(t,d)\left(\int_v^d \alpha(t,r)\,dr\right)dt-\int_a^b Y(t,c)\left(\int_c^v \alpha(t,r)\,dr\right)dt\\
&-\int_c^d Y(b,w)\left(\int_u^b \alpha(s,w)\,ds\right)dw-\int_c^d Y(a,w)\left(\int_a^u \alpha(s,w)\,ds\right)dw\\
&+\int_a^b\int_c^d \alpha(t,w)\,Y(t,w)\,dwdt\Bigg\|_{E_1\times E_2}\\
&\le\int_u^b\int_v^d\left(\int_u^t\int_v^w |\alpha(s,r)|\,drds\right)\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}dwdt
\end{aligned}$$

$$\int_a^u\int_v^d\left(\int_t^u\int_v^w|\alpha(s,r)|\,drds\right)\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}dwdt$$
$$+\int_u^b\int_c^v\left(\int_u^t\int_w^v|\alpha(s,r)|\,drds\right)\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}dwdt$$
$$+\int_a^u\int_c^v\left(\int_t^u\int_w^v|\alpha(s,r)|\,drds\right)\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}dwdt$$
$$:=B(\alpha,Y,u,v). \quad (2.7)$$

**Corollary 1.** *With the assumptions of Theorem 4, we get*

$$\Bigg\|\left(\int_u^b\int_v^d\alpha(s,r)\,drds\right)Y(b,d)+\left(\int_a^u\int_v^d\alpha(s,r)\,drds\right)Y(a,d)$$
$$+\left(\int_u^b\int_c^v\alpha(s,r)\,drds\right)Y(b,c)+\left(\int_a^u\int_c^v\alpha(s,r)\,drds\right)Y(a,c)$$
$$-\int_a^bY(t,d)\left(\int_v^d\alpha(t,r)\,dr\right)dt-\int_a^bY(t,c)\left(\int_c^v\alpha(t,r)\,dr\right)dt$$
$$-\int_c^dY(b,w)\left(\int_u^b\alpha(s,w)\,ds\right)dw-\int_c^dY(a,w)\left(\int_a^u\alpha(s,w)\,ds\right)dw$$
$$+\int_a^b\int_c^d\alpha(t,w)Y(t,w)\,dwdt\Bigg\|_{E_1\times E_2}$$
$$\le\int_u^b\int_v^d|\alpha(s,r)|\,drds\int_u^b\int_v^d\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}dwdt$$
$$+\int_a^u\int_v^d|\alpha(s,r)|\,drds\int_a^u\int_v^d\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}dwdt$$
$$+\int_u^b\int_c^v|\alpha(s,r)|\,drds\int_u^b\int_c^v\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}dwdt$$
$$+\int_a^u\int_c^v|\alpha(s,r)|\,drds\int_a^u\int_c^v\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}dwdt,$$
$$\le\begin{cases}\max\Big\{\int_u^b\int_v^d|\alpha(s,r)|\,drds,\int_a^u\int_v^d|\alpha(s,r)|\,drds\\ \quad+\int_u^b\int_c^v|\alpha(s,r)|\,drds,\int_a^u\int_c^v|\alpha(s,r)|\,drds\Big\}\\ \quad\times\int_a^b\int_c^d\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}dwdt,\\ \\ \max\Big\{\int_u^b\int_v^d\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}dwdt,\int_a^u\int_v^d\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}dwdt\\ \quad+\int_u^b\int_c^v\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}dwdt,\int_a^u\int_c^v\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}dwdt\Big\}\\ \quad\times\int_a^b\int_c^d|\alpha(s,r)|\,drds\end{cases}$$

$$\leq \int_a^b \int_c^d |\alpha(s,r)|\, drds \int_a^b \int_c^d \frac{\partial^2}{\partial w \partial t} \|Y(t,w)\|_{E_1\times E_2}\, dwdt, \quad (2.8)$$

*where* $(u,v) \in [a,b]\times[c,d]$.

**Corollary 2.** *With the assumptions of Theorem 4, we have*

$$\begin{aligned}
&\Bigg\| \left(\int_u^b \int_v^d \alpha(s,r)\, drds\right) Y(b,d) - \left(\int_u^b \int_c^v \alpha(s,r)\, drds\right) Y(b,c) \\
&\quad - \left(\int_a^u \int_v^d \alpha(s,r)\, drds\right) Y(a,d) + \left(\int_a^u \int_c^v \alpha(s,r)\, drds\right) Y(a,c) \\
&\quad - \int_a^b Y(t,d) \left(\int_v^d \alpha(t,r)\, dr\right) dt - \int_a^b Y(t,c) \left(\int_c^v \alpha(t,r)\, dr\right) dt \\
&\quad - \int_c^d Y(b,w) \left(\int_u^b \alpha(s,w)\, ds\right) dw - \int_c^d Y(a,w) \left(\int_a^u \alpha(s,w)\, ds\right) dw \\
&\quad + \int_a^b \int_c^d \alpha(t,w)\, Y(t,w)\, dwdt \Bigg\|_{E_1\times E_2} \\
&\leq \frac{1}{4} \int_a^b \int_c^d |\alpha(s,r)|\, drds \int_a^b \int_c^d \left\| \frac{\partial^2}{\partial w \partial t} Y(t,w) \right\|_{E_1\times E_2} dwdt \quad (2.9)
\end{aligned}$$

*for all* $m,n \in (a,b)\times(c,d)$.

**Proposition 1.** *Suppose that* $\alpha : [a,b]\times[c,d] \to \mathbb{C}$ *and* $Y : [a,b]\times[c,d] \to E_1 \times E_2$ *are continuous and* $\frac{\partial Y}{\partial t}$, $\frac{\partial Y}{\partial w}$, $\frac{\partial^2 Y}{\partial t \partial w}$ *exist on* $(a,b)\times(c,d)$ *in the norm topology* $\|(x,y)\|_{X\times Y} = \|x\|_X + \|y\|_Y$ *of* $E_1\times E_2$, $(x,y)\in E_1\times E_2$, *then we have the following inequalities:*

$$\begin{aligned}
&\Bigg\| \left(\int_u^b \int_v^d \alpha(s,r)\, drds\right) Y(b,d) + \left(\int_a^u \int_v^d \alpha(s,r)\, drds\right) Y(a,d) \\
&\quad + \left(\int_u^b \int_c^v \alpha(s,r)\, drds\right) Y(b,c) + \left(\int_a^u \int_c^v \alpha(s,r)\, drds\right) Y(a,c) \\
&\quad - \int_a^b Y(t,d) \left(\int_v^d \alpha(t,r)\, dr\right) dt - \int_a^b Y(t,c) \left(\int_c^v \alpha(t,r)\, dr\right) dt \\
&\quad - \int_c^d Y(b,w) \left(\int_u^b \alpha(s,w)\, ds\right) dw - \int_c^d Y(a,w) \left(\int_a^u \alpha(s,w)\, ds\right) dw \\
&\quad + \int_a^b \int_c^d \alpha(t,w)\, Y(t,w)\, dwdt \Bigg\|_{E_1\times E_2} \\
&\leq \Bigg[ \int_u^b \int_v^d (b-t)(d-w)|\alpha(t,w)|dwdt + \int_a^u \int_v^d (t-a)(d-w)|\alpha(t,w)|dwdt \\
&\quad + \int_u^b \int_c^v (b-t)(w-c)|\alpha(t,w)|dwdt + \int_a^u \int_c^v (t-a)(w-c)|\alpha(t,w)|dwdt \Bigg]
\end{aligned}$$

$$\times \sup_{(t,w)\in[a,b]\times[c,d]} \left\| \frac{\partial^2}{\partial w\partial t} Y(t,w) \right\|_{E_1\times E_2} \tag{2.10}$$

*for all* $(u,v)\in[a,b]\times[c,d]$.

*Proof.* Using inequality (2.3), we obtain

$$\begin{aligned}
&\Bigg\| \left(\int_u^b \int_v^d \alpha(s,r)\,drds\right) Y(b,d) + \left(\int_a^u \int_v^d \alpha(s,r)\,drds\right) Y(a,d) \\
&\quad + \left(\int_u^b \int_c^v \alpha(s,r)\,drds\right) Y(b,c) + \left(\int_a^u \int_c^v \alpha(s,r)\,drds\right) Y(a,c) \\
&\quad - \int_a^b Y(t,d)\left(\int_v^d \alpha(t,r)\,dr\right)dt - \int_a^b Y(t,c)\left(\int_c^v \alpha(t,r)\,dr\right)dt \\
&\quad - \int_c^d Y(b,w)\left(\int_u^b \alpha(s,w)\,ds\right)dw - \int_c^d Y(a,w)\left(\int_a^u \alpha(s,w)\,ds\right)dw \\
&\quad + \int_a^b \int_c^d \alpha(t,w)\,Y(t,w)\,dwdt \Bigg\|_{E_1\times E_2} \\
&\le \int_u^b \int_v^d \left(\int_u^t \int_v^w |\alpha(s,r)|\,drds\right)dwdt \sup_{(t,w)\in[u,b]\times[v,d]} \left\| \frac{\partial^2}{\partial w\partial t} Y(t,w) \right\|_{E_1\times E_2} \\
&\quad + \int_a^u \int_v^d \left(\int_t^u \int_v^w |\alpha(s,r)|\,drds\right)dwdt \sup_{(t,w)\in[a,u]\times[v,d]} \left\| \frac{\partial^2}{\partial w\partial t} Y(t,w) \right\|_{E_1\times E_2} \\
&\quad + \int_u^b \int_c^v \left(\int_u^t \int_w^v |\alpha(s,r)|\,drds\right)dwdt \sup_{(t,w)\in[u,b]\times[c,v]} \left\| \frac{\partial^2}{\partial w\partial t} Y(t,w) \right\|_{E_1\times E_2} \\
&\quad + \int_a^u \int_c^v \left(\int_t^u \int_w^v |\alpha(s,r)|\,drds\right)dwdt \sup_{(t,w)\in[a,u]\times[c,v]} \left\| \frac{\partial^2}{\partial w\partial t} Y(t,w) \right\|_{E_1\times E_2} \\
&\le \sup_{(t,w)\in[a,b]\times[c,d]} \left\| \frac{\partial^2}{\partial w\partial t} Y(t,w) \right\|_{E_1\times E_2} \Bigg[\int_u^b \int_v^d \left(\int_u^t \int_v^w |\alpha(s,r)|drds\right)dwdt \\
&\quad + \int_a^u \int_v^d \left(\int_t^u \int_v^w |\alpha(s,r)|drds\right)dwdt + \int_u^b \int_c^v \left(\int_u^t \int_w^v |\alpha(s,r)|drds\right)dwdt \\
&\quad + \int_a^u \int_c^v \left(\int_t^u \int_w^v |\alpha(s,r)|drds\right)dwdt\Bigg].
\end{aligned} \tag{2.11}$$

Using integration by parts, we have for $u,w\in[a,b]\times[c,d]$ that

$$\begin{aligned}
&\int_u^b \int_v^d \left(\int_u^t \int_v^w |\alpha(s,r)|drds\right)dwdt \\
&\quad = \int_u^b \left[\int_v^d \left(\int_u^t \int_v^w |\alpha(s,r)|drds\right)dw\right]dt \\
&\quad = \int_u^b d\left(\int_u^t \int_v^d |\alpha(s,r)|drds\right) - \int_u^b \left(\int_v^d w\left(\int_u^t |\alpha(s,w)|ds\right)dw\right)dt
\end{aligned}$$

$$
\begin{aligned}
&= bd\int_u^b\int_v^d |\alpha(s,r)|drds - d\int_u^b\int_v^d t|\alpha(t,r)|drdt \\
&- b\int_v^d w\int_u^b |\alpha(s,w)|dsdw + \int_u^b\int_v^d tw|\alpha(t,w)|dwdt \\
&= bd\int_u^b\int_v^d |\alpha(t,w)|dwdt - d\int_u^b\int_v^d t|\alpha(t,w)|dwdt \\
&- b\int_u^b\int_v^d w|\alpha(t,w)|dwdt + \int_u^b\int_v^d tw|\alpha(t,w)|dwdt \\
&= \int_u^b\int_v^d (b-t)(d-w)|\alpha(t,w)|dwdt.
\end{aligned}
$$

Similarly, we can obtain the following equalities:

$$
\int_a^u\int_v^d \left(\int_t^u\int_v^w |\alpha(s,r)|drds\right) dwdt = \int_a^u\int_v^d (t-a)(d-w)|\alpha(t,w)|dwdt,
$$

$$
\int_u^b\int_c^v \left(\int_u^t\int_w^v |\alpha(s,r)|drds\right) dwdt = \int_u^b\int_c^v (b-t)(w-c)|\alpha(t,w)|dwdt
$$

and

$$
\int_a^u\int_c^v \left(\int_t^u\int_w^v |\alpha(s,r)|drds\right) dwdt = \int_a^u\int_c^v (t-a)(w-c)|\alpha(t,w)|dwdt.
$$

Substituting the above results into (2.11) completes the proof.

**Proposition 2.** *Suppose that $\alpha : [a,b]\times[c,d] \to \mathbb{C}$ and $Y : [a,b]\times[c,d] \to E_1 \times E_2$ are as defined in Theorem 4 and all the assumptions of are satisified Theorem 4, then we have the following inequalities hold for all $(u,v) \in [a,b]\times[c,d]$:*

$$
\begin{aligned}
&\Bigg\| \left(\int_u^b\int_v^d \alpha(s,r)\,drds\right) Y(b,d) + \left(\int_a^u\int_v^d \alpha(s,r)\,drds\right) Y(a,d) \\
&+ \left(\int_u^b\int_c^v \alpha(s,r)\,drds\right) Y(b,c) + \left(\int_a^u\int_c^v \alpha(s,r)\,drds\right) Y(a,c) \\
&- \int_a^b Y(t,d)\left(\int_v^d \alpha(t,r)\,dr\right) dt - \int_a^b Y(t,c)\left(\int_c^v \alpha(t,r)\,dr\right) dt \\
&- \int_c^d Y(b,w)\left(\int_u^b \alpha(s,w)\,ds\right) dw - \int_c^d Y(a,w)\left(\int_a^u \alpha(s,w)\,ds\right) dw \\
&+ \int_a^b\int_c^d \alpha(t,w)\,Y(t,w)\,dwdt \Bigg\|_{E_1\times E_2} \\
&\le \sup_{(t,w)\in[a,b]\times[c,d]} \left\| \frac{\partial^2}{\partial w\partial t} Y(t,w)\right\|_{E_1\times E_2}
\end{aligned}
$$

$$\times\begin{cases} (b-u)(d-v)\int_u^b\int_v^d|\alpha(t,w)|dwdt+(u-u)(d-v)\int_a^u\int_v^d|\alpha(t,w)|dwdt \\ +(b-u)(v-c)\int_u^b\int_c^v|\alpha(t,w)|dwdt+(u-a)(v-c)\int_a^u\int_c^v|\alpha(t,w)|dwdt, \\ \\ \frac{[(b-u)(d-v)]^{1+\frac{1}{q}}}{(q+1)^{\frac{2}{q}}}\left(\int_u^b\int_v^d|\alpha(t,w)|^p dwdt\right)^{\frac{1}{p}}+\frac{[(u-a)(d-v)]^{1+\frac{1}{q}}}{(q+1)^{\frac{2}{q}}}\left(\int_a^u\int_v^d|\alpha(t,w)|^p dwdt\right)^{\frac{1}{p}} \\ +\frac{[(b-u)(v-c)]^{1+\frac{1}{q}}}{(q+1)^{\frac{2}{q}}}\left(\int_u^b\int_c^v|\alpha(t,w)|^p dwdt\right)^{\frac{1}{p}}+\frac{[(u-a)(v-c)]^{1+\frac{1}{q}}}{(q+1)^{\frac{2}{q}}}\left(\int_a^u\int_c^v|\alpha(t,w)|^p dwdt\right)^{\frac{1}{p}}, \\ \\ \frac{(b-u)^2(d-v)^2}{4}\sup_{(t,w)\in[u,b]\times[v,d]}|\alpha(t,s)|+\frac{(u-a)^2(d-v)^2}{4}\sup_{(t,w)\in[a,u]\times[v,d]}|\alpha(t,s)| \\ +\frac{(b-u)^2(v-c)^2}{4}\sup_{(t,w)\in[u,b]\times[c,v]}|\alpha(t,s)|+\frac{(u-a)^2(v-c)^2}{4}\sup_{(t,w)\in[a,u]\times[c,v]}|\alpha(t,s)|, \end{cases} \tag{2.12}$$

*where* $p,q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$.

*Proof.* Applying Hölder's integral inequality and properties of supremum, we obtain

$$\int_u^b\int_v^d(b-t)(d-w)|\alpha(t,w)|dwdt$$
$$\leq\begin{cases} \sup_{(t,w)\in[u,b]\times[v,d]}[(b-t)(d-w)]\int_u^b\int_v^d|\alpha(t,w)|dwdt, \\ \\ \left(\int_u^b\int_v^d[(b-t)(d-w)]^q\,dwdt\right)^{\frac{1}{q}}\left(\int_u^b\int_v^d|\alpha(t,w)|^p dwdt\right)^{\frac{1}{p}}, \\ \\ \sup_{(t,w)\in[u,b]\times[v,d]}|\alpha(t,w)|\int_u^b\int_v^d(b-t)(d-w)dwdt, \end{cases} \tag{2.13}$$

where $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$.
The inequalities (2.12) can be re-written as follows:

$$\int_u^b\int_v^d(b-t)(d-w)|\alpha(t,w)|dwdt$$
$$\leq\begin{cases} [(b-u)(d-v)]\int_u^b\int_v^d|\alpha(t,w)|dwdt, \\ \\ \frac{[(b-u)(d-v)]^{1+\frac{1}{q}}}{(q+1)^{\frac{2}{q}}}\left(\int_u^b\int_v^d|\alpha(t,w)|^p dwdt\right)^{\frac{1}{p}}, \\ \\ \frac{(b-u)^2(d-v)^2}{4}\sup_{(t,w)\in[u,b]\times[v,d]}|\alpha(t,s)|, \end{cases} \tag{2.14}$$

where $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$.
Similarly, we also observe that the following inequalities hold:

$$\int_a^u \int_v^d (t-a)(d-w)|\alpha(t,w)|dwdt$$
$$\leq \begin{cases} [(u-a)(d-v)] \int_a^u \int_v^d |\alpha(t,w)|dwdt, \\ \frac{[(u-a)(d-v)]^{1+\frac{1}{q}}}{(q+1)^{\frac{2}{q}}} \left(\int_a^u \int_v^d |\alpha(t,w)|^p dwdt\right)^{\frac{1}{p}}, \\ \frac{(u-a)^2(d-v)^2}{4} \sup_{(t,w)\in[a,u]\times[v,d]} |\alpha(t,s)|, \end{cases} \tag{2.15}$$

$$\int_u^b \int_c^v (b-t)(w-c)|\alpha(t,w)|dwdt$$
$$\leq \begin{cases} [(b-u)(v-c)] \int_u^b \int_c^v |\alpha(t,w)|dwdt, \\ \frac{[(b-u)(v-c)]^{1+\frac{1}{q}}}{(q+1)^{\frac{2}{q}}} \left(\int_u^b \int_c^v |\alpha(t,w)|^p dwdt\right)^{\frac{1}{p}}, \\ \frac{(b-u)^2(v-c)^2}{4} \sup_{(t,w)\in[u,b]\times[c,v]} |\alpha(t,s)|, \end{cases} \tag{2.16}$$

and

$$\int_a^u \int_c^v (t-a)(w-c)|\alpha(t,w)|dwdt$$
$$\leq \begin{cases} [(u-a)(v-c)] \int_a^u \int_c^v |\alpha(t,w)|dwdt, \\ \frac{[(u-a)(v-c)]^{1+\frac{1}{q}}}{(q+1)^{\frac{2}{q}}} \left(\int_a^u \int_c^v |\alpha(t,w)|^p dwdt\right)^{\frac{1}{p}}, \\ \frac{(u-a)^2(v-c)^2}{4} \sup_{(t,w)\in[a,u]\times[c,v]} |\alpha(t,s)|, \end{cases} \tag{2.17}$$

where $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.
Using (2.14)-(2.17), we get the inequalities (2.12).

**Proposition 3.** *Suppose that $\alpha : [a,b] \times [c,d] \to \mathbb{C}$ and $Y : [a,b] \times [c,d] \to E_1 \times E_2$ are as defined in Theorem 4 and all the assumptions of are satisified Theorem 4, then we have the following inequalities hold for all $(u,v) \in [a,b] \times [c,d]$:*

$$\Bigg\| \left(\int_u^b \int_v^d \alpha(s,r)\, drds\right) Y(b,d) + \left(\int_a^u \int_v^d \alpha(s,r)\, drds\right) Y(a,d)$$
$$+ \left(\int_u^b \int_c^v \alpha(s,r)\, drds\right) Y(b,c) + \left(\int_a^u \int_c^v \alpha(s,r)\, drds\right) Y(a,c)$$

$$
\begin{aligned}
&-\int_a^b Y(t,d)\left(\int_v^d \alpha(t,r)\,dr\right)dt-\int_a^b Y(t,c)\left(\int_c^v \alpha(t,r)\,dr\right)dt\\
&-\int_c^d Y(b,w)\left(\int_u^b \alpha(s,w)\,ds\right)dw-\int_c^d Y(a,w)\left(\int_a^u \alpha(s,w)\,ds\right)dw\\
&+\int_a^b\int_c^d \alpha(t,w)\,Y(t,w)\,dwdt\Bigg\|_{E_1\times E_2}\\
&\leq \sup_{(t,w)\in[a,b]\times[c,d]}\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}\\
&\times\begin{cases}
\left\{\frac{1}{2}(b-a)+\left|u-\frac{a+b}{2}\right|\right\}\left\{\frac{1}{2}(d-c)+\left|v-\frac{c+d}{2}\right|\right\}\int_a^b\int_c^d|\alpha(t,w)|dwdt,\\
\frac{\left[[(b-u)(d-v)]^{q+1}+[(u-a)(d-v)]^{q+1}+[(b-u)(v-c)]^{q+1}+[(u-a)(v-c)]^{q+1}\right]^{\frac{1}{q}}}{(q+1)^{\frac{2}{q}}}\\
\qquad\times\left(\int_a^b\int_c^d|\alpha(t,w)|^p dwdt\right)^{\frac{1}{p}},\\
\left\{\frac{1}{4}(b-a)+\left(u-\frac{a+b}{2}\right)^2\right\}\left\{\frac{1}{4}(d-c)+\left(v-\frac{c+d}{2}\right)^2\right\}\\
\qquad\times\sup_{(t,w)\in[a,b]\times[c,d]}|\alpha(t,s)|,
\end{cases}
\end{aligned}\tag{2.18}
$$

*where* $p,q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$.

*Proof.* By using the properties of maximum, from the first part of inequalities in (2.12), we get the following inequalitity:

$$
\begin{aligned}
&[(b-u)(d-v)]\int_u^b\int_v^d|\alpha(t,w)|dwdt+[(u-a)(d-v)]\int_a^u\int_v^d|\alpha(t,w)|dwdt\\
&\quad+[(b-u)(v-c)]\int_u^b\int_c^v|\alpha(t,w)|dwdt+[(u-a)(v-c)]\int_a^u\int_c^v|\alpha(t,w)|dwdt\\
&\leq\max\{b-u,u-a\}\left[(d-v)\int_u^b\int_v^d|\alpha(t,w)|dwdt+(d-v)\int_a^u\int_v^d|\alpha(t,w)|dwdt\right]\\
&\quad+\max\{b-u,u-a\}\left[(v-c)\int_u^b\int_c^v|\alpha(t,w)|dwdt+(v-c)\int_a^u\int_c^v|\alpha(t,w)|dwdt\right]\\
&=\max\{b-u,u-a\}\left\{(d-v)\int_a^b\int_v^d|\alpha(t,w)|dwdt+(v-c)\int_a^b\int_c^v|\alpha(t,w)|dwdt\right\}\\
&\leq\max\{b-u,u-a\}\max\{d-v,v-c\}\int_a^b\int_c^d|\alpha(t,w)|dwdt.
\end{aligned}\tag{2.19}
$$

Since

$$
\max\{x-z,y-z\}=\frac{1}{2}(y-x)+\left|z-\frac{x+y}{2}\right|,
$$

for any real numbers $x$, $y$ and $z$. Thus, we get from (2.19)

$$\begin{aligned}
&[(b-u)(d-v)]\int_u^b\int_v^d|\alpha(t,w)|dwdt+[(u-a)(d-v)]\int_a^u\int_v^d|\alpha(t,w)|dwdt\\
&\quad+[(b-u)(v-c)]\int_u^b\int_c^v|\alpha(t,w)|dwdt+[(u-a)(v-c)]\int_a^u\int_c^v|\alpha(t,w)|dwdt\\
&\quad\le\left\{\frac{1}{2}(b-a)+\left|u-\frac{a+b}{2}\right|\right\}\left\{\frac{1}{2}(d-c)+\left|v-\frac{c+d}{2}\right|\right\}\int_a^b\int_c^d|\alpha(t,w)|dwdt. \quad (2.20)
\end{aligned}$$

Hence, the first part in the inequalities (2.18) is proved.
By the elementary inequality

$$x_1x_2+x_3x_4+x_5x_6+x_7x_8\le\left(x_1^p+x_3^p+x_5^p+x_7^p\right)^{\frac{1}{p}}\left(x_2^q+x_4^q+x_6^q+x_8^q\right)^{\frac{1}{q}},$$

for $x_1,x_2,x_3,x_3,x_5,x_6,x_7,x_8\ge 0$, $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, we obtain the following inequality, from the second part of the inequality (2.12):

$$\begin{aligned}
&[(b-u)(d-v)]^{1+\frac{1}{q}}\left(\int_u^b\int_v^d|\alpha(t,w)|^pdwdt\right)^{\frac{1}{p}}\\
&\quad+[(u-a)(d-v)]^{1+\frac{1}{q}}\left(\int_a^u\int_v^d|\alpha(t,w)|^pdwdt\right)^{\frac{1}{p}}\\
&\quad+[(b-u)(v-c)]^{1+\frac{1}{q}}\left(\int_u^b\int_c^v|\alpha(t,w)|^pdwdt\right)^{\frac{1}{p}}\\
&\quad+[(u-a)(v-c)]^{1+\frac{1}{q}}\left(\int_a^u\int_c^v|\alpha(t,w)|^pdwdt\right)^{\frac{1}{p}}\le\big[(b-u)(d-v)]^{q+1}\\
&\quad+[(u-a)(d-v)]^{q+1}+[(b-u)(v-c)]^{q+1}+[(u-a)(v-c)]^{q+1}\big]^{\frac{1}{q}}\\
&\quad\times\left[\int_u^b\int_v^d|\alpha(t,w)|^pdwdt+\int_a^u\int_v^d|\alpha(t,w)|^pdwdt+\int_u^b\int_c^v|\alpha(t,w)|^pdwdt\right.\\
&\quad\left.+\int_a^u\int_c^v|\alpha(t,w)|^pdwdt\right]^{\frac{1}{p}}\le\big[(b-u)(d-v)]^{q+1}+[(u-a)(d-v)]^{q+1}\\
&\quad+[(b-u)(v-c)]^{q+1}+[(u-a)(v-c)]^{q+1}\big]^{\frac{1}{q}}\left(\int_a^b\int_c^d|\alpha(t,w)|^pdwdt\right)^{\frac{1}{p}}, \quad (2.21)
\end{aligned}$$

where $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$.
Thus, the second part in the inequalities (2.18) is established.
Now, from the last part of the inequalities in (2.12), we observe that the following inequality holds:

$$\frac{(b-u)^2(d-v)^2}{4}\sup_{(t,w)\in[u,b]\times[v,d]}|\alpha(t,s)|+\frac{(u-a)^2(d-v)^2}{4}\sup_{(t,w)\in[a,u]\times[v,d]}|\alpha(t,s)|$$

$$+\frac{(b-u)^2(v-c)^2}{4}\sup_{(t,w)\in[u,b]\times[c,v]}|\alpha(t,s)|+\frac{(u-a)^2(v-c)^2}{4}\sup_{(t,w)\in[a,u]\times[c,v]}|\alpha(t,s)|$$
$$\leq\left[\frac{(b-u)^2(d-v)^2}{4}+\frac{(u-a)^2(d-v)^2}{4}+\frac{(b-u)^2(v-c)^2}{4}+\frac{(u-a)^2(v-c)^2}{4}\right]$$
$$\times\sup_{(t,w)\in[a,b]\times[c,d]}|\alpha(t,s)|$$
$$=\left\{\frac{1}{4}(b-a)+\left(u-\frac{a+b}{2}\right)^2\right\}\left\{\frac{1}{4}(d-c)+\left(v-\frac{c+d}{2}\right)^2\right\}. \quad (2.22)$$

Thus, the third part in the inequalities (2.18) is established.

**Proposition 4.** *Suppose that $\alpha:[a,b]\times[c,d]\to\mathbb{C}$ and $Y:[a,b]\times[c,d]\to E_1\times E_2$ are as defined in Theorem 4 and all the assumptions of are satisified Theorem 4, then we have the following inequalities hold for all $(u,v)\in[a,b]\times[c,d]$:*

$$\Bigg\|\left(\int_u^b\int_v^d\alpha(s,r)\,drds\right)Y(b,d)+\left(\int_a^u\int_v^d\alpha(s,r)\,drds\right)Y(a,d)$$
$$+\left(\int_u^b\int_c^v\alpha(s,r)\,drds\right)Y(b,c)+\left(\int_a^u\int_c^v\alpha(s,r)\,drds\right)Y(a,c)$$
$$-\int_a^b Y(t,d)\left(\int_v^d\alpha(t,r)\,dr\right)dt-\int_a^b Y(t,c)\left(\int_c^v\alpha(t,r)\,dr\right)dt$$
$$-\int_c^d Y(b,w)\left(\int_u^b\alpha(s,w)\,ds\right)dw-\int_c^d Y(a,w)\left(\int_a^u\alpha(s,w)\,ds\right)dw$$
$$-\int_a^b\int_c^d\alpha(t,w)\,Y(t,w)\,dwdt\Bigg\|_{E_1\times E_2}$$
$$\leq\Bigg[(b-u)(d-v)\left(\int_u^b\int_v^d|\alpha(s,r)|\,drds\right)^p+(u-a)(d-v)\left(\int_a^u\int_v^d|\alpha(s,r)|\,drds\right)^p$$
$$+(b-u)(v-c)\left(\int_u^b\int_c^v|\alpha(s,r)|\,drds\right)^p+(u-a)(v-c)\left(\int_a^u\int_c^v|\alpha(s,r)|\,drds\right)^p\Bigg]^{\frac{1}{p}}$$
$$\times\left(\int_a^b\int_c^d\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}^q dwdt\right)^{\frac{1}{q}}$$
$$\leq[(b-a)(d-c)]^{\frac{1}{p}}\Bigg[\left(\int_u^b\int_v^d|\alpha(s,r)|\,drds\right)^p+\left(\int_a^u\int_v^d|\alpha(s,r)|\,drds\right)^p$$
$$+\left(\int_u^b\int_c^v|\alpha(s,r)|\,drds\right)^p+\left(\int_a^u\int_c^v|\alpha(s,r)|\,drds\right)^p\Bigg]^{\frac{1}{p}}$$

$$\times\left(\int_a^b\int_c^d\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}^q dwdt\right)^{\frac{1}{q}}. \quad (2.23)$$

*where* $p,q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$.

*Proof.* By the elementary inequality

$$x_1x_2+x_3x_4+x_5x_6+x_7x_8\le\left(x_1^p+x_3^p+x_5^p+x_7^p\right)^{\frac{1}{p}}\left(x_2^q+x_4^q+x_6^q+x_8^q\right)^{\frac{1}{q}},$$

for $x_1,x_2,x_3,x_3,x_5,x_6,x_7,x_8\ge 0$, $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, we obtain the following inequality, from the second part of the inequality (2.3):

$$\begin{aligned}
&\left[\int_u^b\int_v^d\left(\int_u^t\int_c^w|\alpha(s,r)|\,drds\right)^p dwdt\right]^{\frac{1}{p}}\left[\int_u^b\int_v^d\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}^q dwdt\right]^{\frac{1}{q}}\\
&+\left[\int_a^u\int_v^d\left(\int_t^u\int_v^w|\alpha(s,r)|\,drds\right)^p dwdt\right]^{\frac{1}{p}}\left[\int_a^u\int_v^d\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}^q dwdt\right]^{\frac{1}{q}}\\
&+\left[\int_u^b\int_c^v\left(\int_u^t\int_w^v|\alpha(s,r)|\,drds\right)^p dwdt\right]^{\frac{1}{p}}\left[\int_u^b\int_c^v\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}^q dwdt\right]^{\frac{1}{q}}\\
&+\left[\int_a^u\int_c^v\left(\int_t^u\int_w^v|\alpha(s,r)|\,drds\right)^p dwdt\right]^{\frac{1}{p}}\left[\int_a^u\int_c^v\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}^q dwdt\right]^{\frac{1}{q}}\\
&\le\left[\int_u^b\int_v^d\left(\int_u^t\int_c^w|\alpha(s,r)|\,drds\right)^p dwdt+\int_a^u\int_v^d\left(\int_t^u\int_v^w|\alpha(s,r)|\,drds\right)^p dwdt\right.\\
&\left.+\int_u^b\int_c^v\left(\int_u^t\int_w^v|\alpha(s,r)|\,drds\right)^p dwdt+\int_a^u\int_c^v\left(\int_t^u\int_w^v|\alpha(s,r)|\,drds\right)^p dwdt\right]^{\frac{1}{p}}\\
&\quad\times\left[\int_u^b\int_v^d\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}^q dwdt+\int_a^u\int_v^d\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}^q dwdt\right.\\
&\quad\left.+\int_u^b\int_c^v\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}^q dwdt+\int_a^u\int_c^v\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}^q dwdt\right]\\
&\le\left[\int_u^b\int_v^d\left(\int_u^b\int_c^d|\alpha(s,r)|\,drds\right)^p dwdt+\int_a^u\int_v^d\left(\int_a^u\int_v^d|\alpha(s,r)|\,drds\right)^p dwdt\right.\\
&\left.+\int_u^b\int_c^v\left(\int_u^b\int_c^v|\alpha(s,r)|\,drds\right)^p dwdt+\int_a^u\int_c^v\left(\int_a^u\int_c^v|\alpha(s,r)|\,drds\right)^p dwdt\right]^{\frac{1}{p}}\\
&\qquad\times\left(\int_a^b\int_c^d\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}^q dwdt\right)^{\frac{1}{q}},
\end{aligned}$$

$$= \Bigg[(b-u)(d-v)\left(\int_u^b\int_v^d |\alpha(s,r)|\,drds\right)^p + (u-a)(d-v)\left(\int_a^u\int_v^d |\alpha(s,r)|\,drds\right)^p$$
$$+ (b-u)(v-c)\left(\int_u^b\int_c^v |\alpha(s,r)|\,drds\right)^p + (u-a)(v-c)\left(\int_a^u\int_c^v |\alpha(s,r)|\,drds\right)^p\Bigg]^{\frac{1}{p}}$$
$$\times\left(\int_a^b\int_c^d \left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}^q dwdt\right)^{\frac{1}{q}}$$
$$\leq [(b-a)(d-c)]^{\frac{1}{p}}\Bigg[\left(\int_u^b\int_c^d |\alpha(s,r)|\,drds\right)^p + \left(\int_a^u\int_v^d |\alpha(s,r)|\,drds\right)^p$$
$$+\left(\int_u^b\int_c^v |\alpha(s,r)|\,drds\right)^p + \left(\int_a^u\int_c^v |\alpha(s,r)|\,drds\right)^p\Bigg]^{\frac{1}{p}}$$
$$\times\left(\int_a^b\int_c^d \left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}^q dwdt\right)^{\frac{1}{q}}. \quad (2.24)$$

Hence the result is established.

**Remark 1.** *If$(m,n)\in(a,b)\times(c,d)$ such that the following equalities hold:*

$$\int_m^b\int_n^d |\alpha(s,r)|\,drds = \int_a^m\int_n^d |\alpha(s,r)|\,drds = \int_m^b\int_c^n |\alpha(s,r)|\,drds$$
$$= \int_a^m\int_c^n |\alpha(s,r)|\,drds = \frac{1}{4}\int_a^b\int_c^d |\alpha(s,r)|\,drds. \quad (2.25)$$

*Then from (2.23), we can get the following inequality:*

$$\Bigg\|\left(\int_u^b\int_v^d \alpha(s,r)\,drds\right)Y(b,d) + \left(\int_a^u\int_v^d \alpha(s,r)\,drds\right)Y(a,d)$$
$$+\left(\int_u^b\int_c^v \alpha(s,r)\,drds\right)Y(b,c) + \left(\int_a^u\int_c^v \alpha(s,r)\,drds\right)Y(a,c)$$
$$-\int_a^b Y(t,d)\left(\int_v^d \alpha(t,r)\,dr\right)dt - \int_a^b Y(t,c)\left(\int_c^v \alpha(t,r)\,dr\right)dt$$
$$-\int_c^d Y(b,w)\left(\int_u^b \alpha(s,w)\,ds\right)dw - \int_c^d Y(a,w)\left(\int_a^u \alpha(s,w)\,ds\right)dw$$
$$-\int_a^b\int_c^d \alpha(t,w)\,Y(t,w)\,dwdt\Bigg\|_{E_1\times E_2}$$
$$\leq \frac{[(b-a)(d-c)]^{\frac{1}{p}}}{4}\int_a^b\int_c^d |\alpha(s,r)|\,drds$$

$$\times\left(\int_a^b\int_c^d\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}^q dwdt\right)^{\frac{1}{q}}. \quad (2.26)$$

A result that is directly linked to Theorem 4 is given as follows:

**Corollary 3.** *Assume that* $\alpha : [a,b]\times[c,d]\to\mathbb{C}$ *and* $Y : [a,b]\times[c,d]\to E_1\times E_2$ *are continuous and* $\frac{\partial Y}{\partial t}$, $\frac{\partial Y}{\partial w}$, $\frac{\partial^2 Y}{\partial t\partial w}$ *exist on* $(a,b)\times(c,d)$ *in the norm topology* $\|(x,y)\|_{X\times Y}=\|x\|_X+\|y\|_Y$ *of* $E_1\times E_2$, $(x,y)\in E_1\times E_2$, *then the following inequalities hold true for* $p$, $q>1$ *and* $\frac{1}{p}+\frac{1}{q}=1$*:*

$$\begin{aligned}
&\Bigg\|\left(\int_{\frac{a+b}{2}}^b\int_{\frac{c+d}{2}}^d\alpha(s,r)\,drds\right)Y(b,d)+\left(\int_a^{\frac{a+b}{2}}\int_{\frac{c+d}{2}}^d\alpha(s,r)\,drds\right)Y(a,d)\\
&\quad+\left(\int_{\frac{a+b}{2}}^b\int_c^{\frac{c+d}{2}}\alpha(s,r)\,drds\right)Y(b,c)+\left(\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}\alpha(s,r)\,drds\right)Y(a,c)\\
&\quad-\int_a^b Y(t,d)\left(\int_{\frac{c+d}{2}}^d\alpha(t,r)\,dr\right)dt-\int_a^b Y(t,c)\left(\int_c^{\frac{c+d}{2}}\alpha(t,r)\,dr\right)dt\\
&\quad-\int_c^d Y(b,w)\left(\int_{\frac{a+b}{2}}^b\alpha(s,w)\,ds\right)dw-\int_c^d Y(a,w)\left(\int_a^{\frac{a+b}{2}}\alpha(s,w)\,ds\right)dw\\
&\quad+\int_a^b\int_c^d\alpha(t,w)\,Y(t,w)\,dwdt\Bigg\|_{E_1\times E_2}\le C(\alpha,Y), \quad (2.27)
\end{aligned}$$

*where*

$$\begin{aligned}
C(\alpha,Y):=&\int_{\frac{a+b}{2}}^b\int_{\frac{c+d}{2}}^d\left(\int_{\frac{a+b}{2}}^t\int_{\frac{c+d}{2}}^w|\alpha(s,r)|\,drds\right)\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}dwdt\\
&+\int_a^{\frac{a+b}{2}}\int_{\frac{c+d}{2}}^d\left(\int_t^{\frac{a+b}{2}}\int_{\frac{c+d}{2}}^w|\alpha(s,r)|\,drds\right)\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}dwdt\\
&+\int_{\frac{a+b}{2}}^b\int_c^{\frac{c+d}{2}}\left(\int_{\frac{a+b}{2}}^t\int_w^{\frac{c+d}{2}}|\alpha(s,r)|\,drds\right)\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}dwdt\\
&+\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}\left(\int_t^{\frac{a+b}{2}}\int_w^{\frac{c+d}{2}}|\alpha(s,r)|\,drds\right)\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}dwdt. \quad (2.28)
\end{aligned}$$

*We also have the following bounds for* $C(\alpha,Y)$*:*

$$C(\alpha,Y)$$

$$\leq \begin{cases} \int_{\frac{a+b}{2}}^{b}\int_{\frac{c+d}{2}}^{d}|\alpha(s,r)|\,drds\int_{\frac{a+b}{2}}^{b}\int_{\frac{c+d}{2}}^{d}\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}dwdt \\ +\int_{a}^{\frac{a+b}{2}}\int_{\frac{c+d}{2}}^{d}|\alpha(s,r)|\,drds\int_{a}^{\frac{a+b}{2}}\int_{\frac{c+d}{2}}^{d}\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}dwdt \\ +\int_{\frac{a+b}{2}}^{b}\int_{c}^{\frac{c+d}{2}}|\alpha(s,r)|\,drds\int_{\frac{a+b}{2}}^{b}\int_{c}^{\frac{c+d}{2}}\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}dwdt \\ +\int_{a}^{\frac{a+b}{2}}\int_{c}^{\frac{c+d}{2}}|\alpha(s,r)|\,drds\int_{a}^{\frac{a+b}{2}}\int_{c}^{\frac{c+d}{2}}\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}dwdt, \\[2ex] \left[\int_{\frac{a+b}{2}}^{b}\int_{\frac{c+d}{2}}^{d}\left(\int_{\frac{a+b}{2}}^{t}\int_{c}^{w}|\alpha(s,r)|\,drds\right)^p dwdt\right]^{\frac{1}{p}}\left[\int_{\frac{a+b}{2}}^{b}\int_{\frac{c+d}{2}}^{d}\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|^q_{E_1\times E_2}dwdt\right]^{\frac{1}{q}} \\ +\left[\int_{a}^{\frac{a+b}{2}}\int_{\frac{c+d}{2}}^{d}\left(\int_{t}^{\frac{a+b}{2}}\int_{\frac{c+d}{2}}^{w}|\alpha(s,r)|\,drds\right)^p dwdt\right]^{\frac{1}{p}}\left[\int_{a}^{\frac{a+b}{2}}\int_{\frac{c+d}{2}}^{d}\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|^q_{E_1\times E_2}dwdt\right]^{\frac{1}{q}} \\ +\left[\int_{\frac{a+b}{2}}^{b}\int_{c}^{\frac{c+d}{2}}\left(\int_{\frac{a+b}{2}}^{t}\int_{w}^{\frac{c+d}{2}}|\alpha(s,r)|\,drds\right)^p dwdt\right]^{\frac{1}{p}}\left[\int_{\frac{a+b}{2}}^{b}\int_{c}^{\frac{c+d}{2}}\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|^q_{E_1\times E_2}dwdt\right]^{\frac{1}{q}} \\ +\left[\int_{a}^{\frac{a+b}{2}}\int_{c}^{\frac{c+d}{2}}\left(\int_{t}^{\frac{a+b}{2}}\int_{w}^{\frac{c+d}{2}}|\alpha(s,r)|\,drds\right)^p dwdt\right]^{\frac{1}{p}}\left[\int_{a}^{\frac{a+b}{2}}\int_{c}^{\frac{c+d}{2}}\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|^q_{E_1\times E_2}dwdt\right]^{\frac{1}{q}}, \\[2ex] \int_{\frac{a+b}{2}}^{b}\int_{\frac{c+d}{2}}^{d}\left(\int_{\frac{a+b}{2}}^{t}\int_{\frac{c+d}{2}}^{w}|\alpha(s,r)|\,drds\right)dwdt\sup_{(t,w)\in\left[\frac{a+b}{2},b\right]\times\left[\frac{c+d}{2},d\right]}\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2} \\ +\int_{a}^{\frac{a+b}{2}}\int_{\frac{c+d}{2}}^{d}\left(\int_{t}^{\frac{a+b}{2}}\int_{\frac{c+d}{2}}^{w}|\alpha(s,r)|\,drds\right)dwdt\sup_{(t,w)\in\left[a,\frac{a+b}{2}\right]\times\left[\frac{c+d}{2},d\right]}\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2} \\ +\int_{\frac{a+b}{2}}^{b}\int_{c}^{\frac{c+d}{2}}\left(\int_{\frac{a+b}{2}}^{t}\int_{w}^{\frac{c+d}{2}}|\alpha(s,r)|\,drds\right)dwdt\sup_{(t,w)\in\left[\frac{a+b}{2},b\right]\times\left[c,\frac{c+d}{2}\right]}\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2} \\ +\int_{a}^{\frac{a+b}{2}}\int_{c}^{\frac{c+d}{2}}\left(\int_{t}^{\frac{a+b}{2}}\int_{w}^{\frac{c+d}{2}}|\alpha(s,r)|\,drds\right)dwdt\sup_{(t,w)\in\left[a,\frac{a+b}{2}\right]\times\left[c,\frac{c+d}{2}\right]}\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}. \end{cases} \tag{2.29}$$

With the assuptions of Corollary 3, one can obtain the following results:

**Corollary 4.** *With the assumptions of Corollary 3, we get*

$$\Bigg\|\left(\int_{\frac{a+b}{2}}^{b}\int_{\frac{c+d}{2}}^{d}\alpha(s,r)\,drds\right)Y(b,d)+\left(\int_{a}^{\frac{a+b}{2}}\int_{\frac{c+d}{2}}^{d}\alpha(s,r)\,drds\right)Y(a,d)$$
$$+\left(\int_{\frac{a+b}{2}}^{b}\int_{c}^{\frac{c+d}{2}}\alpha(s,r)\,drds\right)Y(b,c)+\left(\int_{a}^{\frac{a+b}{2}}\int_{c}^{\frac{c+d}{2}}\alpha(s,r)\,drds\right)Y(a,c)$$
$$-\int_{a}^{b}Y(t,d)\left(\int_{\frac{c+d}{2}}^{d}\alpha(t,r)\,dr\right)dt-\int_{a}^{b}Y(t,c)\left(\int_{c}^{\frac{c+d}{2}}\alpha(t,r)\,dr\right)dt$$
$$-\int_{c}^{d}Y(b,w)\left(\int_{\frac{a+b}{2}}^{b}\alpha(s,w)\,ds\right)dw-\int_{c}^{d}Y(a,w)\left(\int_{a}^{\frac{a+b}{2}}\alpha(s,w)\,ds\right)dw$$

$$+\int_a^b\int_c^d \alpha(t,w)\,Y(t,w)\,dwdt\Big\|_{E_1\times E_2}$$

$$\leq \int_{\frac{a+b}{2}}^b\int_{\frac{c+d}{2}}^d |\alpha(s,r)|\,drds\int_{\frac{a+b}{2}}^b\int_{\frac{c+d}{2}}^d\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}dwdt$$

$$+\int_a^{\frac{a+b}{2}}\int_{\frac{c+d}{2}}^d |\alpha(s,r)|\,drds\int_a^{\frac{a+b}{2}}\int_{\frac{c+d}{2}}^d\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}dwdt$$

$$+\int_{\frac{a+b}{2}}^b\int_c^{\frac{c+d}{2}} |\alpha(s,r)|\,drds\int_{\frac{a+b}{2}}^b\int_c^{\frac{c+d}{2}}\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}dwdt$$

$$+\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}} |\alpha(s,r)|\,drds\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}dwdt,$$

$$\leq\begin{cases}\max\Big\{\int_{\frac{a+b}{2}}^b\int_{\frac{c+d}{2}}^d|\alpha(s,r)|\,drds,\int_a^{\frac{a+b}{2}}\int_{\frac{c+d}{2}}^d|\alpha(s,r)|\,drds\\ \quad+\int_{\frac{a+b}{2}}^b\int_c^{\frac{c+d}{2}}|\alpha(s,r)|\,drds,\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}|\alpha(s,r)|\,drds\Big\}\\ \quad\times\int_a^b\int_c^d\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}dwdt,\\ \\ \max\Big\{\int_{\frac{a+b}{2}}^b\int_{\frac{c+d}{2}}^d\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}dwdt,\int_a^{\frac{a+b}{2}}\int_{\frac{c+d}{2}}^d\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}dwdt\\ \quad+\int_{\frac{a+b}{2}}^b\int_c^{\frac{c+d}{2}}\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}dwdt,\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}dwdt\Big\}\\ \quad\times\int_a^b\int_c^d|\alpha(s,r)|\,drds.\end{cases}$$

$$\leq \int_a^b\int_c^d|\alpha(s,r)|\,drds\int_a^b\int_c^d\frac{\partial^2}{\partial w\partial t}\|Y(t,w)\|_{E_1\times E_2}dwdt. \quad (2.30)$$

**Corollary 5.** *With the assumptions of Corollary 3, we get*

$$\Bigg\|\left(\int_{\frac{a+b}{2}}^b\int_{\frac{c+d}{2}}^d\alpha(s,r)\,drds\right)Y(b,d)+\left(\int_a^{\frac{a+b}{2}}\int_{\frac{c+d}{2}}^d\alpha(s,r)\,drds\right)Y(a,d)$$

$$+\left(\int_{\frac{a+b}{2}}^b\int_c^{\frac{c+d}{2}}\alpha(s,r)\,drds\right)Y(b,c)+\left(\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}\alpha(s,r)\,drds\right)Y(a,c)$$

$$-\int_a^b Y(t,d)\left(\int_{\frac{c+d}{2}}^d\alpha(t,r)\,dr\right)dt-\int_a^b Y(t,c)\left(\int_c^{\frac{c+d}{2}}\alpha(t,r)\,dr\right)dt$$

$$-\int_c^d Y(b,w)\left(\int_{\frac{a+b}{2}}^b\alpha(s,w)\,ds\right)dw-\int_c^d Y(a,w)\left(\int_a^{\frac{a+b}{2}}\alpha(s,w)\,ds\right)dw$$

$$+\int_a^b\int_c^d\alpha(t,w)\,Y(t,w)\,dwdt\Bigg\|_{E_1\times E_2}$$

$$\leq \sup_{(t,w)\in[a,b]\times[c,d]}\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}$$

$$\times\begin{cases}\frac{(b-a)(d-c)}{4}\int_a^b\int_c^d|\alpha(t,w)|dwdt,\\ \frac{[(b-a)(d-c)]^{\frac{1}{q}+1}}{4(q+1)^{\frac{2}{q}}}\left(\int_a^b\int_c^d|\alpha(t,w)|^p dwdt\right)^{\frac{1}{p}},\\ \frac{(b-a)(d-c)}{16}\sup\limits_{(t,w)\in[a,b]\times[c,d]}|\alpha(t,s)|,\end{cases}\tag{2.31}$$

*where* $p,q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$.

**Corollary 6.** *With the assumptions of Corollary 3, we get*

$$\begin{aligned}&\Bigg\|\left(\int_{\frac{a+b}{2}}^b\int_{\frac{c+d}{2}}^d\alpha(s,r)\,drds\right)Y(b,d)+\left(\int_a^{\frac{a+b}{2}}\int_{\frac{c+d}{2}}^d\alpha(s,r)\,drds\right)Y(a,d)\\&\quad+\left(\int_{\frac{a+b}{2}}^b\int_c^{\frac{c+d}{2}}\alpha(s,r)\,drds\right)Y(b,c)+\left(\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}\alpha(s,r)\,drds\right)Y(a,c)\\&\quad-\int_a^b Y(t,d)\left(\int_{\frac{c+d}{2}}^d\alpha(t,r)\,dr\right)dt-\int_a^b Y(t,c)\left(\int_c^{\frac{c+d}{2}}\alpha(t,r)\,dr\right)dt\\&\quad-\int_c^d Y(b,w)\left(\int_{\frac{a+b}{2}}^b\alpha(s,w)\,ds\right)dw-\int_c^d Y(a,w)\left(\int_a^{\frac{a+b}{2}}\alpha(s,w)\,ds\right)dw\\&\quad+\int_a^b\int_c^d\alpha(t,w)\,Y(t,w)\,dwdt\Bigg\|_{E_1\times E_2}\\&\le\left[\frac{(b-a)(d-c)}{4}\right]^{\frac{1}{p}}\Bigg[\left(\int_{\frac{a+b}{2}}^b\int_{\frac{c+d}{2}}^d|\alpha(s,r)|\,drds\right)^p+\left(\int_a^{\frac{a+b}{2}}\int_{\frac{c+d}{2}}^d|\alpha(s,r)|\,drds\right)^p\\&\quad+\left(\int_{\frac{a+b}{2}}^b\int_c^{\frac{c+d}{2}}|\alpha(s,r)|\,drds\right)^p+\left(\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}|\alpha(s,r)|\,drds\right)^p\Bigg]^{\frac{1}{p}}\\&\quad\times\left(\int_a^b\int_c^d\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}^q dwdt\right)^{\frac{1}{q}},\end{aligned}\tag{2.32}$$

*where* $p,q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$.

If $\alpha(s,r)=1$, $(u,v)\in[a,b]\times[c,d]$, then we can get the following very interesting inequalities:

**Theorem 5.** *Assume that* $Y:[a,b]\times[c,d]\to E_1\times E_2$ *is continuous and* $\frac{\partial Y}{\partial t}$, $\frac{\partial Y}{\partial w}$, $\frac{\partial^2 Y}{\partial t\partial w}$ *exist on* $(a,b)\times(c,d)$ *in the norm topology* $\|(x,y)\|_{X\times Y}=\|x\|_X+\|y\|_Y$ *of* $E_1\times E_2$,

*$(x,y) \in E_1 \times E_2$, then we get the following inequalities for all $(u,v) \in [a,b] \times [c,d]$ for $p$, $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$:*

$$\begin{aligned}
&\Bigg\| \int_u^b (b-u)(d-v) Y(b,d) + (u-a)(d-v) Y(a,d) \\
&\quad + (b-u)(v-c) Y(b,c) + (u-a)(v-c) Y(a,c) \\
&\quad - (d-v) \int_a^b Y(t,d)\, dt - (v-c) \int_a^b Y(t,c)\, dt \\
&\quad - (b-u) \int_c^d Y(b,w)\, dw - (u-a) \int_c^d Y(a,w)\, dw \\
&\quad + \int_a^b \int_c^d Y(t,w)\, dwdt \Bigg\|_{E_1 \times E_2} \leq C(Y,u,v), \qquad (2.33)
\end{aligned}$$

*where*

$$\begin{aligned}
C(Y,u,v) :=& \int_u^b \int_v^d (t-u)(w-v) \left\| \frac{\partial^2}{\partial w \partial t} Y(t,w) \right\|_{E_1 \times E_2} dwdt \\
&+ \int_a^u \int_v^d (u-t)(w-v) \left\| \frac{\partial^2}{\partial w \partial t} Y(t,w) \right\|_{E_1 \times E_2} dwdt \\
&+ \int_u^b \int_c^v (t-u)(v-w) \left\| \frac{\partial^2}{\partial w \partial t} Y(t,w) \right\|_{E_1 \times E_2} dwdt \\
&+ \int_a^u \int_c^v (u-t)(v-w) \left\| \frac{\partial^2}{\partial w \partial t} Y(t,w) \right\|_{E_1 \times E_2} dwdt. \qquad (2.34)
\end{aligned}$$

*We also have the following bounds for $C(Y,u,v)$:*

$$C(Y,u,v)$$

$$\leq \begin{cases} (b-u)(d-v)\int_u^b\int_v^d \left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2} dwdt \\ +(u-a)(d-v)\int_a^u\int_v^d \left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2} dwdt \\ +(b-u)(v-c)\int_u^b\int_c^v \left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2} dwdt \\ +(u-a)(v-c)\int_a^u\int_c^v \left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2} dwdt, \\ \\ \frac{[(b-u)(d-v)]^{1+\frac{1}{p}}}{(1+p)^{\frac{2}{p}}}\left[\int_u^b\int_v^d \left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|^q_{E_1\times E_2} dwdt\right]^{\frac{1}{q}} \\ +\frac{[(a-u)(d-v)]^{1+\frac{1}{p}}}{(1+p)^{\frac{2}{p}}}\left[\int_a^u\int_v^d \left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|^q_{E_1\times E_2} dwdt\right]^{\frac{1}{q}} \\ +\frac{[(b-u)(v-c)]^{1+\frac{1}{p}}}{(1+p)^{\frac{2}{p}}}\left[\int_u^b\int_c^v \left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|^q_{E_1\times E_2} dwdt\right]^{\frac{1}{q}} \\ +\frac{[(u-a)(v-c)]^{1+\frac{1}{p}}}{(1+p)^{\frac{2}{p}}}\left[\int_a^u\int_c^v \left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|^q_{E_1\times E_2} dwdt\right]^{\frac{1}{q}}, \\ \\ \frac{[(b-u)(d-v)]^2}{4}\sup_{(t,w)\in[u,b]\times[v,d]}\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2} \\ +\frac{[(a-u)(d-v)]^2}{4}\sup_{(t,w)\in[a,u]\times[v,d]}\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2} \\ +\frac{[(b-u)(v-c)]^2}{4}\sup_{(t,w)\in[u,b]\times[c,v]}\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2} \\ +\frac{[(u-a)(v-c)]^2}{4}\sup_{(t,w)\in[a,u]\times[c,v]}\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}. \end{cases} \quad (2.35)$$

The following twoo results are consequences of Theorem 5:

**Corollary 7.** *With the assumptions of Theorem 5, we get the following inequalities:*

$$\begin{aligned} \Bigg\|\int_u^b & (b-u)(d-v)Y(b,d)+(u-a)(d-v)Y(a,d) \\ & +(b-u)(v-c)Y(b,c)+(u-a)(v-c)Y(a,c) \\ & -(d-v)\int_a^b Y(t,d)\,dt-(v-c)\int_a^b Y(t,c)\,dt \\ & -(b-u)\int_c^d Y(b,w)\,dw-(u-a)\int_c^d Y(a,w)\,dw \\ & +\int_a^b\int_c^d Y(t,w)\,dwdt\Bigg\|_{E_1\times E_2} \\ \leq & \sup_{(t,w)\in[a,b]\times[c,d]}\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}\left\{\frac{1}{2}(b-a)+\left|u-\frac{a+b}{2}\right|\right\} \\ & \times\left\{\frac{1}{2}(d-c)+\left|v-\frac{c+d}{2}\right|\right\}\int_a^b\int_c^d|\alpha(t,w)|dwdt, \quad (2.36) \end{aligned}$$

*for all* $(u,v) \in [a,b] \times [c,d]$.

**Corollary 8.** *With the assumptions of Theorem 5, we get the following inequalities:*

$$\begin{aligned}
&\Bigg\| \int_u^b (b-u)(d-v)Y(b,d) + (u-a)(d-v)Y(a,d) \\
&\qquad + (b-u)(v-c)Y(b,c) + (u-a)(v-c)Y(a,c) \\
&\qquad - (d-v)\int_a^b Y(t,d)\,dt - (v-c)\int_a^b Y(t,c)\,dt \\
&\qquad - (b-u)\int_c^d Y(b,w)\,dw - (u-a)\int_c^d Y(a,w)\,dw \\
&\qquad + \int_a^b \int_c^d Y(t,w)\,dwdt \Bigg\|_{E_1\times E_2} \\
&\leq \sup_{(t,w)\in[a,b]\times[c,d]} \left\| \frac{\partial^2}{\partial w \partial t} Y(t,w) \right\|_{E_1\times E_2} \Big\{ [(b-u)(d-v)]^{1+p} + [(u-a)(d-v)]^{1+p} \\
&\qquad + [(v-c)(d-v)]^{1+p} + [(u-a)(v-c)]^{1+p} \Big] \int_a^b \int_c^d |\alpha(t,w)| dwdt, \quad (2.37)
\end{aligned}$$

*for all* $(u,v) \in [a,b] \times [c,d]$.

**Theorem 6.** *Assume that* $\alpha : [a,b] \times [c,d] \to \mathbb{C}$, $Y : [a,b] \times [c,d] \to E_1 \times E_2$ *are continuous. If* $\frac{\partial \alpha}{\partial t}$, $\frac{\partial \alpha}{\partial t}$, $\frac{\partial^2 \alpha}{\partial t \partial w}$ *exist on* $(a,b) \times (c,d)$ *and are continuous, then we get the following inequalities for all* $(u,v) \in [a,b] \times [c,d]$ *for* $p,\, q > 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$*:*

$$\begin{aligned}
&\Bigg\| \left( \int_u^b \int_v^d \alpha(s,r)\,drds \right) Y(b,d) + \left( \int_a^u \int_v^d \alpha(s,r)\,drds \right) Y(a,d) \\
&\qquad + \left( \int_u^b \int_c^v \alpha(s,r)\,drds \right) Y(b,c) + \left( \int_a^u \int_c^v \alpha(s,r)\,drds \right) Y(a,c) \\
&\qquad - \int_a^b \alpha(t,d) \left( \int_v^d Y(t,r)\,dr \right) dt - \int_a^b \alpha(t,c) \left( \int_c^v Y(t,r)\,dr \right) dt \\
&\qquad - \int_c^d \alpha(b,w) \left( \int_u^b Y(s,w)\,ds \right) dw - \int_c^d \alpha(a,w) \left( \int_a^u Y(s,w)\,ds \right) dw \\
&\qquad + \int_a^b \int_c^d \alpha(t,w) Y(t,w)\,dwdt \Bigg\|_{E_1\times E_2} \leq \tilde{C}(\alpha, Y, u, v), \quad (2.38)
\end{aligned}$$

*where*

$$\begin{aligned}
&\Bigg\| \alpha(b,d) \left( \int_u^b \int_v^d Y(s,r)\,drds \right) + \alpha(a,d) \left( \int_a^u \int_v^d Y(s,r)\,drds \right) \\
&\qquad + \alpha(b,c) \left( \int_u^b \int_c^v Y(s,r)\,drds \right) + \alpha(a,c) \left( \int_a^u \int_c^v Y(s,r)\,drds \right)
\end{aligned}$$

$$
\begin{aligned}
&-\int_a^b \alpha(t,d)\left(\int_v^d Y(t,r)\,dr\right)dt-\int_a^b \alpha(t,c)\left(\int_c^v Y(t,r)\,dr\right)dt\\
&-\int_c^d \alpha(b,w)\left(\int_u^b Y(s,w)\,ds\right)dw-\int_c^d \alpha(a,w)\left(\int_a^u Y(s,w)\,ds\right)dw\\
&+\int_a^b\int_c^d \alpha(t,w)\,Y(t,w)\,dwdt\Big\|_{E_1\times E_2}\\
&\le \int_u^b\int_v^d \left|\frac{\partial^2}{\partial w\partial t}\alpha(t,w)\right|\left(\int_u^t\int_v^w \|Y(s,r)\|_{E_1\times E_2}\,drds\right)dwdt\\
&+\int_a^u\int_v^d \left|\frac{\partial^2}{\partial w\partial t}\alpha(t,w)\right|\left(\int_t^u\int_v^w \|Y(s,r)\|_{E_1\times E_2}\,drds\right)dwdt\\
&+\int_u^b\int_c^v \left|\frac{\partial^2}{\partial w\partial t}\alpha(t,w)\right|\left(\int_u^t\int_w^v \|Y(s,r)\|_{E_1\times E_2}\,drds\right)dwdt\\
&+\int_a^u\int_c^v \left|\frac{\partial^2}{\partial w\partial t}\alpha(t,w)\right|\left(\int_t^u\int_w^v \|Y(s,r)\|_{E_1\times E_2}\,drds\right)dwdt\\
&:=\tilde{C}(\alpha,Y,u,v)\,. \qquad (2.39)
\end{aligned}
$$

*We also have the following bounds for* $\tilde{C}(\alpha,Y,u,v)$*:*

$\tilde{C}(\alpha,Y,u,v)$

$$\leq \begin{cases} \int_u^b \int_v^d \left| \frac{\partial^2}{\partial w \partial t} \alpha(s,r) \right| drds \int_u^b \int_v^d \|Y(t,w)\|_{E_1 \times E_2} dwdt \\ + \int_a^u \int_v^d \left| \frac{\partial^2}{\partial w \partial t} \alpha(s,r) \right| drds \int_a^u \int_v^d \|Y(t,w)\|_{E_1 \times E_2} dwdt \\ + \int_u^b \int_c^v \left| \frac{\partial^2}{\partial w \partial t} \alpha(s,r) \right| drds \int_u^b \int_c^v \|Y(t,w)\|_{E_1 \times E_2} dwdt \\ + \int_a^u \int_c^v \left| \frac{\partial^2}{\partial w \partial t} \alpha(s,r) \right| drds \int_a^u \int_c^v \|Y(t,w)\|_{E_1 \times E_2} dwdt, \\ \\ \left[ \int_u^b \int_v^d \left( \frac{\partial^2}{\partial w \partial t} |\alpha(t,w)| drds \right)^p dwdt \right]^{\frac{1}{p}} \left[ \int_u^b \int_v^d \left( \int_u^t \int_c^w \|Y(s,r)\|_{E_1 \times E_2} \right)^q drdst \right]^{\frac{1}{q}} \\ + \left[ \int_a^u \int_v^d \left( \frac{\partial^2}{\partial w \partial t} |\alpha(t,w)| \right)^p dwdt \right]^{\frac{1}{p}} \left[ \int_a^u \int_v^d \left( \int_t^u \int_v^w \|Y(s,r)\|_{E_1 \times E_2} drds \right)^q dwdt \right]^{\frac{1}{q}} \\ + \left[ \int_u^b \int_c^v \left( \frac{\partial^2}{\partial w \partial t} |\alpha(t,w)| \right)^p dwdt \right]^{\frac{1}{p}} \left[ \int_u^b \int_u^t \left( \int_w^v \|Y(s,r)\|_{E_1 \times E_2} drds \right)^q dwdt \right]^{\frac{1}{q}} \\ + \left[ \int_a^u \int_c^v \left( \frac{\partial^2}{\partial w \partial t} |\alpha(t,w)| \right)^p dwdt \right]^{\frac{1}{p}} \left[ \int_a^u \int_c^v \left( \int_t^u \int_w^v \|Y(s,r)\|_{E_1 \times E_2} drds \right)^q dwdt \right]^{\frac{1}{q}}, \\ \\ \int_u^b \int_v^d \left( \int_u^t \int_v^w \|Y(s,r)\|_{E_1 \times E_2} drds \right) dwdt \sup_{(t,w) \in [u,b] \times [v,d]} \left| \frac{\partial^2}{\partial w \partial t} Y(t,w) \right| \\ + \int_a^u \int_v^d \left( \int_t^u \int_v^w \|Y(s,r)\|_{E_1 \times E_2} drds \right) dwdt \sup_{(t,w) \in [a,u] \times [v,d]} \left| \frac{\partial^2}{\partial w \partial t} Y(t,w) \right| \\ + \int_u^b \int_c^v \left( \int_u^t \int_w^v \|Y(s,r)\|_{E_1 \times E_2} drds \right) dwdt \sup_{(t,w) \in [u,b] \times [c,v]} \left| \frac{\partial^2}{\partial w \partial t} Y(t,w) \right| \\ + \int_a^u \int_c^v \left( \int_t^u \int_w^v \|Y(s,r)\|_{E_1 \times E_2} drds \right) dwdt \sup_{(t,w) \in [a,u] \times [c,v]} \left| \frac{\partial^2}{\partial w \partial t} Y(t,w) \right|. \end{cases} \tag{2.40}$$

*Proof.* By integration by parts, we can observe that for all $(u,v) \in [a,b] \times [c,d]$, we have the following equality:

$$\int_u^b \int_v^d \frac{\partial^2}{\partial w \partial t} \alpha(t,w) \left( \int_u^t \int_v^w Y(s,r)\, drds \right) dwdt$$

$$\int_u^b \left[ \int_v^d \frac{\partial^2}{\partial w \partial t} \alpha(t,w) \left( \int_u^t \int_v^w Y(s,r)\, drds \right) dw \right] dt$$

$$= \int_u^b \left[ \frac{\partial}{\partial t} \alpha(t,w) \left( \int_u^t \int_v^w Y(s,r)\, drds \right) \Big|_v^d - \int_v^d \left( \frac{\partial}{\partial t} \alpha(t,w) \int_u^t Y(s,w)\, ds \right) dw \right] dt$$

$$= \int_u^b \frac{\partial}{\partial t} \alpha(t,d) \left( \int_u^t \int_v^d Y(s,r)\, drds \right) dt - \int_v^d \left[ \int_u^b \left( \frac{\partial}{\partial t} \alpha(t,w) \int_u^t Y(s,w)\, ds \right) dt \right] dw$$

$$= \alpha(t,d) \left( \int_u^t \int_v^d Y(s,r)\, drds \right) \Big|_u^b - \int_u^b \alpha(t,d) \left( \int_v^d Y(t,r)\, dr \right) dt$$

$$- \int_v^d \alpha(t,w) \left( \int_u^t Y(s,w)\, ds \right) dw \Big|_u^b + \int_u^b \int_v^d \alpha(t,w) Y(t,w)\, dwdt$$

$$= \alpha(b,d) \left( \int_u^b \int_v^d Y(s,r)\, drds \right) - \int_u^b \alpha(t,d) \left( \int_v^d Y(t,r)\, dr \right) dt$$

$$-\int_v^d \alpha(b,w)\left(\int_u^b Y(s,w)\,ds\right)dw+\int_u^b\int_v^d \alpha(t,w)\,Y(t,w)\,dwdt \quad (2.41)$$

In a similar way, we can get the following equalities:

$$\begin{aligned}
&\int_a^u\int_v^d \frac{\partial^2}{\partial w\partial t}\alpha(t,w)\left(\int_t^u\int_v^w Y(s,r)\,drds\right)dwdt\\
&\quad=-\alpha(a,d)\left(\int_a^u\int_v^d Y(s,r)\,drds\right)+\int_a^u \alpha(t,d)\left(\int_v^d Y(t,r)\,dr\right)dt\\
&\qquad+\int_v^d \alpha(a,w)\left(\int_a^u Y(s,w)\,ds\right)dw-\int_a^u\int_v^d \alpha(t,w)\,Y(t,w)\,dwdt, \quad (2.42)
\end{aligned}$$

$$\begin{aligned}
&\int_u^b\int_c^v \frac{\partial^2}{\partial w\partial t}\alpha(t,w)\left(\int_u^t\int_w^v Y(s,r)\,drds\right)dwdt\\
&\quad=-\alpha(b,c)\left(\int_u^b\int_c^v Y(s,r)\,drds\right)+\int_u^b \alpha(t,c)\left(\int_c^v Y(t,r)\,dr\right)dt\\
&\qquad+\int_c^v \alpha(b,w)\left(\int_u^b Y(s,w)\,ds\right)dw-\int_u^b\int_c^v \alpha(t,w)\,Y(t,w)\,dwdt \quad (2.43)
\end{aligned}$$

and

$$\begin{aligned}
&\int_a^u\int_c^v \frac{\partial^2}{\partial w\partial t}\alpha(t,w)\left(\int_t^u\int_w^v Y(s,r)\,drds\right)dwdt\\
&\quad=\alpha(a,c)\left(\int_a^u\int_c^v Y(s,r)\,drds\right)-\int_a^u \alpha(t,c)\left(\int_c^v Y(t,r)\,dr\right)dt\\
&\qquad-\int_c^v \alpha(a,w)\left(\int_a^u Y(s,w)\,ds\right)dw+\int_a^u\int_c^v \alpha(t,w)\,Y(t,w)\,dwdt. \quad (2.44)
\end{aligned}$$

From (2.41)-(2.44), we have the following equality:

$$\begin{aligned}
&\int_u^b\int_v^d \frac{\partial^2}{\partial w\partial t}\alpha(t,w)\left(\int_u^t\int_v^w Y(s,r)\,drds\right)dwdt\\
&\qquad-\int_a^u\int_v^d \frac{\partial^2}{\partial w\partial t}\alpha(t,w)\left(\int_t^u\int_v^w Y(s,r)\,drds\right)dwdt\\
&\qquad-\int_u^b\int_c^v \frac{\partial^2}{\partial w\partial t}\alpha(t,w)\left(\int_u^t\int_w^v Y(s,r)\,drds\right)dwdt\\
&\qquad+\int_a^u\int_c^v \frac{\partial^2}{\partial w\partial t}\alpha(t,w)\left(\int_t^u\int_w^v Y(s,r)\,drds\right)dwdt\\
&\quad=\alpha(b,d)\left(\int_u^b\int_v^d Y(s,r)\,drds\right)+\alpha(a,d)\left(\int_a^u\int_v^d Y(s,r)\,drds\right)
\end{aligned}$$

$$+\alpha(b,c)\left(\int_u^b\int_c^v Y(s,r)\,drds\right)+\alpha(a,c)\left(\int_a^u\int_c^v Y(s,r)\,drds\right)$$
$$-\int_a^b \alpha(t,d)\left(\int_v^d Y(t,r)\,dr\right)dt-\int_a^b \alpha(t,c)\left(\int_c^v Y(t,r)\,dr\right)dt$$
$$-\int_c^d \alpha(b,w)\left(\int_u^b Y(s,w)\,ds\right)dw-\int_c^d \alpha(a,w)\left(\int_a^u Y(s,w)\,ds\right)dw$$
$$+\int_a^b\int_c^d \alpha(t,w)\,Y(t,w)\,dwdt \quad (2.45)$$

By taking the norm $\|\cdot\|_{E_1\times E_2}$ on both sides (2.6), we get

$$\Bigg\|\alpha(b,d)\left(\int_u^b\int_v^d Y(s,r)\,drds\right)+\alpha(a,d)\left(\int_a^u\int_v^d Y(s,r)\,drds\right)$$
$$+\alpha(b,c)\left(\int_u^b\int_c^v Y(s,r)\,drds\right)+\alpha(a,c)\left(\int_a^u\int_c^v Y(s,r)\,drds\right)$$
$$-\int_a^b \alpha(t,d)\left(\int_v^d Y(t,r)\,dr\right)dt-\int_a^b \alpha(t,c)\left(\int_c^v Y(t,r)\,dr\right)dt$$
$$-\int_c^d \alpha(b,w)\left(\int_u^b Y(s,w)\,ds\right)dw-\int_c^d \alpha(a,w)\left(\int_a^u Y(s,w)\,ds\right)dw$$
$$+\int_a^b\int_c^d \alpha(t,w)\,Y(t,w)\,dwdt\Bigg\|_{E_1\times E_2}$$
$$\leq\int_u^b\int_v^d\left|\frac{\partial^2}{\partial w\partial t}\alpha(t,w)\right|\left(\int_u^t\int_v^w\|Y(s,r)\|_{E_1\times E_2}\,drds\right)dwdt$$
$$+\int_a^u\int_v^d\left|\frac{\partial^2}{\partial w\partial t}\alpha(t,w)\right|\left(\int_t^u\int_v^w\|Y(s,r)\|_{E_1\times E_2}\,drds\right)dwdt$$
$$+\int_u^b\int_c^v\left|\frac{\partial^2}{\partial w\partial t}\alpha(t,w)\right|\left(\int_u^t\int_w^v\|Y(s,r)\|_{E_1\times E_2}\,drds\right)dwdt$$
$$+\int_a^u\int_c^v\left|\frac{\partial^2}{\partial w\partial t}\alpha(t,w)\right|\left(\int_t^u\int_w^v\|Y(s,r)\|_{E_1\times E_2}\,drds\right)dwdt$$
$$:=\tilde{C}(\alpha,Y,u,v)\,. \quad (2.46)$$

Now by following the similar steps as in the proof of Theorem 4, we can get the inequalities (2.38), where the bounds for $\tilde{C}(\alpha,Y,u,v)$ are given in (2.40).

**Remark 2.** *A number of results can be obtained from Theorem 6, however, we leave their proofs to be investigated by the interested readers.*

## 3. Conclusion

This paper provides new weighted Trapezoidal-type inequalities for the product of two functions of two variables, one of which takes values in Banach spaces and the other in the complex plane. This is performed using the integration by parts method for Bochner integrals, as well as analytical techniques for the product of functions with values in Banach spaces. Our findings extend and generalize several previously reported results for functions of a single variable to functions of two variables. Indeed, our inequalities generalize those results of Dragomir from [9] to the product of two functions of two variables of which one function has values in the direct product of two Banach spaces and the other function contains values in the product of the complex plane. The findings of this study can be applied to a variety of different fields of study that focus on Banach spaced-valued functions, leading to new theoretical and practical discoveries. Our unique method also allows us to determine error bounds for numerical integration.

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