EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 1, Article Number 5617
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Employing the Limit Residual Function Method to Solve Systems of Fractional Differential Equations

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Abstract. This study aims to solve systems of fractional differential equations analytically using a simple new technique, the limit residual function method. This method relies on coupling the residual function with the limit to produce analytical and approximate solutions within rapidly converging series forms. This technique could be an alternative to the residual power series method which is an efficient and quick-to-solve system of fractional differential equations, both linear and nonlinear, that arise in numerous physical phenomena. To illustrate the methodology and confirm its effectiveness, the study explores three different applications. The proposed algorithm's reliability and accuracy can be easily described by contrasting the numerical results with the exact solutions.

2020 Mathematics Subject Classifications: 26A33, 32A05, 34A12

Key Words and Phrases: Fractional initial value problems, Fractional power series, Caputo's derivative operator, residual function

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1. Introduction

In applied sciences such as applied mathematics, mathematical biology, physics, and engineering, fractional calculus is currently an area of intense study. There are multiple definitions for the fractional derivative, making it a non-unique formula now. The most commonly used definition is the Riemann-Liouville (R-L) definition, while Caputo's definition (1967) of the fractional derivative is also widely used [6, 12, 13, 29, 31, 32, 37]. Differential equations of fractional order have been the subject of several investigations due to their frequent appearance in different scientific applications. The differential equations in several forms of fractional derivatives give different types of solutions. As a result, fractional differential equations cannot be solved using a standard approach. Therefore, a rapidly developing field of applied mathematics is the interpretation and solution of fractional differential equations. The Predictor-Corrector approach [16], the variational

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DOI: https://doi.org/10.29020/nybg.ejpam.v18i1.5617

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iteration approach [14, 23], the Homotopy perturbation approach [21], the Homotopy analysis method [22], the Adomian decomposition method [1, 26, 28], the operational matrix approach [34], the residual power series method [15, 20, 30], the Laplace residual power series method [3, 9, 10, 27, 36], the differential transformation technique [11], and the Taylor series method [25], are newly employed techniques for solving linear and nonlinear differential equations.

Recently, El-Ajou and Burgan introduced the limit residual function (LRF) method [18, 19] to find analytical series solutions for linear and nonlinear partial differential equations. The LRF approach is a semi-analytical technique that constructs an analytical solution using polynomials based on power series (PS) expansion, with the series coefficients determined by calculating the limit of the residual functions. Indeed, the LRF offers a power series solution for differential equations that is the same as the Taylor series solution. LRF is just a new, simple, and efficient technique for determining the coefficients of the Taylor series.

On the other hand, much literature has developed concerning the fractional system of differential equations and its applications [1, 2, 4, 5, 26, 35]. In this article, we present a new application of the LRF technique to yield approximate solutions for the system of fractional differential equations on the form

$$\begin{aligned} \mathcal{D}^{\alpha}u_1(x) &= H_1\left(x, u_1(x),\ u_2(x), \ldots, u_n(x)\right), \\ \mathcal{D}^{\alpha}u_2(x) &= H_2\left(x, u_1(x),\ u_2(x), \ldots, u_n(x)\right), \\ &\vdots \\ \mathcal{D}^{\alpha}u_n(x) &= H_1\left(x, u_1(x),\ u_2(x), \ldots, u_n(x)\right), \end{aligned} \tag{1}$$

where $\mathcal{D}^{\alpha},\ 0 < \alpha \leq 1$ is the derivative of order α in the sense of Caputo, subject to the initial conditions $u_1(0) = u_{1,0}, u_2(0) = u_{2,0},\ \ldots,\ u_n(0) = u_{n,0}$.

The structure of this paper is as follows: Section 2 revisits several important fundamental results of fractional calculus and fractional PS. Section 3 provides an overview of the suggested method for solving the system of fractional differential equations. In Section 4, three systems of fractional initial value problems are solved to explore the applicability and simplicity of the LRF technique. Section 5 provides the conclusion.

2. Essential Preliminaries and Notations

This section revisits several important fundamental results about the fractional PS, essential, to developing analytical solutions for systems of fractional differential equations using the LRF method. There are many definitions of fractional derivatives in the mathematical literature, such as the Riemann–Liouville fractional derivative, Caputo fractional derivative, Grünwald–Letnikov fractional derivative [12, 31], the two-scale fractal derivative [7], He's fractional derivative [24], conformable fractional derivatives, and Atangana-Baleanu's fractional derivative [8]. This article focuses on the Caputo derivative of order α defined for the function $u(x)$ as in the next definition. In the future, researchers may be able to adapt the LRF method to solve fractional differential equations using other types of fractional derivatives.

Definition 2.1. [12, 31] The Caputo derivative of order $\alpha \in (m-1, m]$, $m\in\mathbb{N}$ of the function $u(x)$ is defined as follows:

$$\mathcal{D}^{\alpha} u(x) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_{y_0}^{x} (x-y)^{\alpha-1}\, u^{(m)}(y)\, dy, & m-1 < \alpha < m\ ,\ x > y \geq y_0 \geq 0, \\ u^{(m)}(x) \quad , & \alpha = m. \end{cases} \tag{2}$$

Many properties of Caputo derivative can be found in the Refs. [12, 31].

In the following, the definitions and theorems relevant to the classical PS are expanded to include the fractional case in the Caputo sense.

Definition 2.2. [17] A PS representation of the form

$$\sum_{j=0}^{\infty} u_j\ (x-x_0)^{j\alpha} = u_0 + u_1 (x-x_0)^{\alpha} + u_2\ (x-x_0)^{2\alpha} + \ldots, \tag{3}$$

where $0 \leq m-1 < \alpha \leq m$, $x \geq x_0$ is called a fractional PS about x_0 where x is variable and u_j's are constants called the coefficients of the series.

Theorem 2.1. [33] The fractional PS $\sum_{j=0}^{\infty} u_j\ (x-x_0)^{j\alpha} = 0$ for all x in $|x-x_0| < s$ if and only if each coefficient u_j equals zero.

To determine the coefficients of the fractional PS solution, the primary principle for the LRFD approach has been presented and validated by the authors in references [18, 19], which is depicted in the following theorem.

Theorem 2.2. Suppose that $u(x)$ has the fractional PS expansion $u(x) = \sum_{j=0}^{\infty} u_j\ (x-x_0)^{j\alpha}$ and $u(x) = 0$ for all x in some interval I. Then

$$\lim_{x \to x_0} \frac{u_k(x)}{(x-x_0)^{(k-1)\alpha}} = 0, \quad k = 1, 2, \ldots, \quad x \neq x_0, \tag{4}$$

where

$$u_k(x) = \sum_{j=0}^{k} u_j\ (x-x_0)^{j\alpha}. \tag{5}$$

3. The methodology of the LRF method for solving systems of fractional differential equations

This section introduces the fundamental idea of the LRF technique for analytically solving specific systems of linear and non-linear fractional differential equations. The residual function and the limit at zero are the foundational concepts of this approach. To demonstrate the procedures of the LRF method, we will be studying the following class of fractional differential equation systems:

$$\mathcal{D}^{\alpha}u_i\left(x\right)=H_i\left(x,u_1\left(x\right),\ u_2\left(x\right),\ldots,u_n\left(x\right)\right),\quad x\geq 0,\ 0<\alpha\leq 1, \tag{6}$$

with the initial conditions

$$u_i\left(0\right)=u_{i,0}\ ,\ i=1,2,\ldots,n. \tag{7}$$

where α is the order of the Caputo fractional differential operator $\mathcal{D}^{\alpha}$, H_i are analytic functions, and $u_i\left(x\right)$ are unknown analytical smooth functions that will be identified.

The LRF method assumes writing the functions $u_i\left(x\right)$ using the following fractional PS expansions:

$$u_i\left(x\right)=\sum_{j=0}^{\infty}u_{i,j}\ x^{j\alpha},\quad i=1,2,\ldots,n,\quad x\geq 0. \tag{8}$$

Since $u_{i,0}=u_i\left(0\right)$, and by truncating the series (8), we obtain the kth approximate solution of the system (6)-(7) as follows:

$$u_{ik}\left(x\right)=u_{i,0}+\sum_{j=1}^{k}u_{i,j}\ x^{j\alpha},\quad i=1,2,\ldots,n,\quad x\geq 0, \tag{9}$$

in which the constants of the expansion series, $u_{i,n}$, can be obtained by identifying the residual functions as follows:

$$\mathcal{R}f\left(u_i\left(x\right)\right)=\mathcal{D}^{\alpha}u_i\left(x\right)-H_i\left(x,u_1\left(x\right),\ u_2\left(x\right),\ldots,u_n\left(x\right)\right),\quad i=1,2,\ldots,n,\quad x\geq 0. \tag{10}$$

So, the kth residual functions can be written as:

$$\mathcal{R}f\left(u_{ik}\left(x\right)\right)=\mathcal{D}^{\alpha}u_{ik}\left(x\right)-H_i\left(x,u_{1k}\left(x\right),\ u_{2k}\left(x\right),\ldots,u_{nk}\left(x\right)\right),\quad i=1,2,\ldots,n,\quad x\geq 0. \tag{11}$$

To find the kth approximate solution of the system (6)-(7), we need to determine the value of the coefficients, $u_{i,j}$, $i=1,2,\ldots,n$, and $j=1,2,\ldots,k$, in the series (9). The essential tool of the LRF method, which effectively identifies the unknown coefficients, as in [25], is

$$\lim_{x\to 0}\frac{\mathcal{R}f\left(u_{ik}\left(x\right)\right)}{x^{(k-1)\alpha}}\ =0,\quad i=1,2,\ ...,\ n. \tag{12}$$

To find the 1st approximate solution, we must determine the coefficients $u_{i,1}$ in the series (9). We can do this by substituting $u_{i1}\left(x\right)=u_{i,0}+u_{i,1}x^{\alpha}$ into $\mathcal{R}f\left(u_{i1}\left(x\right)\right),\ i=1,2,\ ...,\ n$, and then solving the equations $\lim_{x\to 0}\mathcal{R}f\left(u_{i1}\left(x\right)\right)\ =0$ for $u_{i,1}$, $i=1,2,\ldots,n$. This will lead us to our desire.

Similarly, we can get the actual 2nd approximate solution by substituting $u_{i2}\left(x\right)=u_{i,0}+u_{i,1}x^{\alpha}+u_{i,2}x^{2\alpha}$ into $\mathcal{R}f\left(u_{i2}\left(x\right)\right)$ and solving the equations $\lim_{x\to 0}\frac{\mathcal{R}f\left(u_{i2}\left(x\right)\right)}{x^{\alpha}}\ =0$ for $u_{i,2}$, $i=1,2,\ldots,n$.

Generally, the kth approximate solution of the system (6)-(7) is determined by substituting $u_{ik}\left(x\right)=u_{i(k-1)}\left(x\right)+u_{i,k}x^{k\alpha}$ into $\mathcal{R}f\left(u_{ik}\left(x\right)\right)$ and then solving the algebraic equations (12) for $u_{i,k}$.

4. Illustrative examples

This section tests the performance and implementation of the suggested method on four systems of linear and non-linear fractional differential equations. The accuracy of the technique will be evaluated by comparing the results obtained with the exact solutions.

Example 4.1. Consider the following system of linear fractional differential equations:

$$\mathcal{D}^{\alpha}u_1(x) = u_1(x) + u_2(x),$$
$$\mathcal{D}^{\alpha}u_2(x) = -u_1(x) + u_2(x), \tag{13}$$

where $0 < \alpha \leq 1, \quad 0 \leq x$ with the initial conditions

$$u_1(0) = 0, \quad u_2(0) = 1. \tag{14}$$

In the classical case ($\alpha = 1$), the exact solution may be obtained analytically and provided as follows:

$$u_1(x) = e^x \sin x\ , \quad u_2(x) = e^x \cos x\ . \tag{15}$$

Considering the LRF technique to create an analytical series solution for System (13)–(14), we start by assuming that the solution has the following fractional series expansions:

$$u_1(x) = \sum_{n=0}^{\infty} u_{1,n}\ x^{n\alpha}, \quad u_2(x) = \sum_{n=0}^{\infty} u_{2,n}\ x^{n\alpha}. \tag{16}$$

Based on the initial conditions specified in Equation (14), we have $u_{1,0} = 0,\ u_{2,0} = 1$. Thus, the kth approximation of $u_1(x)$ and $u_2(x)$ can be expressed as:

$$u_{1k}(x) = \sum_{n=1}^{k} u_{1,n}\ x^{n\alpha}, \quad u_{2k}(x) = 1 + \sum_{n=1}^{k} u_{2,n}\ x^{n\alpha}. \tag{17}$$

To identify the additional unknown coefficients of the series given in Equation (16), we proceed with the second step of the LRF approach by defining the residual functions of equations in (13) as follows:

$$\mathcal{R}f(u_1(x)) = \mathcal{D}^{\alpha}u_1(x) - u_1(x) - u_2(x),$$
$$\mathcal{R}f(u_2(x)) = \mathcal{D}^{\alpha}u_2(x) + u_1(x) - u_2(x), \tag{18}$$

and the kth residual functions are expressed as follows:

$$\mathcal{R}f(u_{1k}(x)) = \mathcal{D}^{\alpha}u_{1k}(x) - u_{1k}(x) - u_{2k}(x),$$
$$\mathcal{R}f(u_{2k}(x)) = \mathcal{D}^{\alpha}u_{2k}(x) + u_{1k}(x) - u_{2k}(x). \tag{19}$$

By substituting the first approximations $u_{11}(x) = u_{1,1}\ x^{\alpha}$, $u_{21}(x) = 1 + u_{2,1}x^{\alpha}$ into the first residual functions, $\mathcal{R}f(u_{11}(x))$, $\mathcal{R}f(u_{21}(x))$, we obtain

$$\begin{aligned} \mathcal{R}f(u_{11}(x)) &= u_{1,1}\Gamma(\alpha+1) - u_{1,1}x^{\alpha} - 1 - u_{2,1}x^{\alpha}, \\ \mathcal{R}f(u_{21}(x)) &= u_{2,1}\Gamma(\alpha+1) + u_{1,1}x^{\alpha} - 1 - u_{2,1}x^{\alpha}. \end{aligned} \tag{20}$$

Solving the equations $lim_{x\to 0}\mathcal{R}f(u_{11}(x)) = 0$, $lim_{x\to 0}\mathcal{R}f(u_{21}(x)) = 0$ for $u_{1,1}, u_{2,1}$, respectively, we have

$$u_{1,1} = \frac{1}{\Gamma(\alpha+1)}, \quad u_{2,1} = \frac{1}{\Gamma(\alpha+1)} \ . \tag{21}$$

In order to determine the second coefficients $u_{1,2}$ and $u_{2,2}$, substitute the second approximations $u_{12}(x) = \frac{x^{\alpha}}{\Gamma(\alpha+1)} + u_{1,2}x^{2\alpha}$, $u_{22}(x) = 1 + \frac{x^{\alpha}}{\Gamma(\alpha+1)} + u_{2,2}x^{2\alpha}$ into $\mathcal{R}f(u_{12}(x))$, $\mathcal{R}f(u_{22}(x))$ to have

$$\begin{aligned} \mathcal{R}f(u_{12}(x)) &= 1 + u_{1,2}\ \frac{\Gamma(2\alpha+1)\ x^{\alpha}}{\Gamma(\alpha+1)} - \left(\frac{x^{\alpha}}{\Gamma(\alpha+1)} + u_{1,2}\ x^{2\alpha}\right) \\ &\quad - \left(1 + \frac{x^{\alpha}}{\Gamma(\alpha+1)} + u_{2,2}\ x^{2\alpha}\right), \\ \mathcal{R}f(u_{22}(x)) &= 1 + u_{2,2}\ \frac{\Gamma(2\alpha+1)\ x^{\alpha}}{\Gamma(\alpha+1)} + \left(\frac{x^{\alpha}}{\Gamma(\alpha+1)} + u_{1,2}\ x^{2\alpha}\right) \\ &\quad - \left(1 + \frac{x^{\alpha}}{\Gamma(\alpha+1)} + u_{2,2}\ x^{2\alpha}\right). \end{aligned} \tag{22}$$

Making simple calculations, the limits $lim_{x\to 0}\frac{\mathcal{R}f(u_{12}(x))}{x^{\alpha}} = 0$, $lim_{x\to 0}\frac{\mathcal{R}f(u_{22}(x))}{x^{\alpha}} = 0$, yield

$$u_{1,2} = \frac{2}{\Gamma(2\alpha+1)}, \quad u_{2,2} = 0. \tag{23}$$

The values of the coefficients $u_{1,3}$ and $u_{2,3}$ are also obtained by substituting $u_{13}(x) = \frac{x^{\alpha}}{\Gamma(\alpha+1)} + \frac{2}{\Gamma(2\alpha+1)}x^{2\alpha} + \ u_{1,3}\ x^{3\alpha}$ and $u_{23}(x) = 1 + \frac{x^{\alpha}}{\Gamma(\alpha+1)} + \ u_{2,3}\ x^{3\alpha}$ into $\mathcal{R}f(u_{13}(x))$, $\mathcal{R}f(u_{23}(x))$, and then we solve the following equations:

$$\begin{aligned} \lim_{x\to 0}\frac{\mathcal{R}f(u_{13}(x))}{x^{2\alpha}} &= u_{1,3}\ \frac{\Gamma(3\alpha+1)}{\Gamma(2\alpha+1)} - \frac{2}{\Gamma(2\alpha+1)} = 0, \\ \lim_{x\to 0}\frac{\mathcal{R}f(u_{23}(x))}{x^{2\alpha}} &= u_{2,3}\ \frac{\Gamma(3\alpha+1)}{\Gamma(2\alpha+1)} + \frac{2}{\Gamma(2\alpha+1)} = 0. \end{aligned} \tag{24}$$

Then we have

$$u_{1,3} = \frac{2}{\Gamma(3\alpha+1)}, \quad u_{2,3} = -\frac{2}{\Gamma(3\alpha+1)}. \tag{25}$$

Proceeding in the same manner, we get

$$u_{1,4} = 0, \quad u_{2,4} = -\frac{4}{\Gamma(4\alpha+1)}. \tag{26}$$

Table 1: The coefficients of the 7th approximation of $u_1(x)$, $u_2(x)$ for the System (13)-(14).

k	$u_{1,k}$	$u_{2,k}$
0	0	1
1	$\frac{1}{\Gamma(\alpha+1)}$	$\frac{1}{\Gamma(\alpha+1)}$
2	$\frac{2}{\Gamma(2\alpha+1)}$	0
3	$\frac{2}{\Gamma(3\alpha+1)}$	$-\frac{2}{\Gamma(3\alpha+1)}$
4	0	$-\frac{4}{\Gamma(4\alpha+1)}$
5	$-\frac{4}{\Gamma(5\alpha+1)}$	$-\frac{4}{\Gamma(5\alpha+1)}$
6	$-\frac{8}{\Gamma(6\alpha+1)}$	0
7	$-\frac{8}{\Gamma(7\alpha+1)}$	$\frac{8}{\Gamma(7\alpha+1)}$

For $k = 5, 6,\ 7$, the algebraic equations $lim_{x\to 0} \frac{\mathcal{R}f(u_{1k}(x))}{x^{(k-1)\alpha}} = 0,\ \ lim_{x\to 0} \frac{\mathcal{R}f(u_{2k}(x))}{x^{(k-1)\alpha}} = 0$ can be solved repeatedly to give the coefficients of the 7th approximation. The needed coefficients are summarized in Table 1.

So, the LRF solution of the System (13)-(14) has the following series expansions:

$$u_1(x) = \frac{x^{\alpha}}{\Gamma(\alpha+1)} + \frac{2x^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{2x^{3\alpha}}{\Gamma(3\alpha+1)} - \frac{4\ x^{5\alpha}}{\Gamma(5\alpha+1)} - \frac{8\ x^{6\alpha}}{\Gamma(6\alpha+1)} - \frac{8\ x^{7\alpha}}{\Gamma(7\alpha+1)} + \ldots,$$

$$u_2(x) = 1 + \frac{x^{\alpha}}{\Gamma(\alpha+1)} - \frac{2x^{3\alpha}}{\Gamma(3\alpha+1)} - \frac{4\ x^{4\alpha}}{\Gamma(4\alpha+1)} - \frac{4\ x^{5\alpha}}{\Gamma(5\alpha+1)} + \frac{8\ x^{7\alpha}}{\Gamma(7\alpha+1)} + \ldots. \quad (27)$$

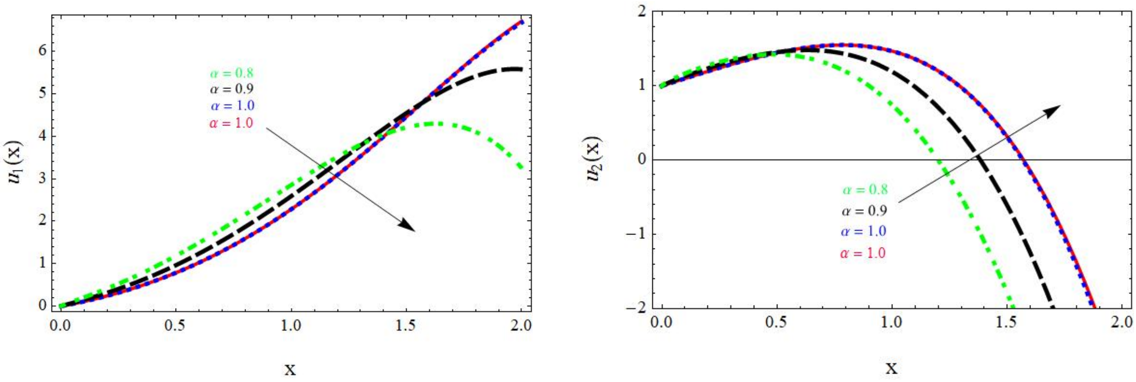
When $\alpha = 1$, the series solution (27) becomes as follows:

$$u_1(x) = x + x^2 + \frac{x^3}{3} - \frac{x^5}{30} - \frac{x^6}{90} - \frac{x^7}{630} + \ldots,$$

$$u_2(x) = 1 + x - \frac{x^3}{3} - \frac{x^4}{6} - \frac{x^5}{30} + \frac{x^7}{630} + \ldots, \quad (28)$$

which is the expansion of the exact solution given in (15).

In Figure 1, the graphs show the 7th approximate solution of the system (13)-(14) at different values of α, as well as the exact solution at $\alpha = 1$. The graph demonstrates strong agreement between the 7th approximate solution and the exact solution at $\alpha = 1$. Furthermore, it illustrates the impact of the derivative order on the solution behavior, revealing that the solution curve decreases as the order of the fractional derivative decreases.



(a) (b)

Figure 1: The curves of the 7th approximate and exact (solid curve) solutions of System (13)-(14) at different values of α.

Example 4.2. Consider the following system of non-linear fractional differential equations

$$\mathcal{D}^{\alpha}u_1(x) = u_1(x),$$
$$\mathcal{D}^{\alpha}u_2(x) = 2u_1^2(x),$$
$$\mathcal{D}^{\alpha}u_3(x) = 3u_1(x)\ u_2(x), \tag{29}$$

where $0 < \alpha \leq 1, \quad 0 \leq x$ with the initial conditions

$$u_1(0) = 1, \quad u_2(0) = 1, \quad u_3(0) = 1. \tag{30}$$

In the classical case ($\alpha = 1$), the exact solution may be obtained analytically and provided as follows:

$$u_1(x) = e^{x}, \quad u_2(x) = e^{2x}, \quad u_3(x) = e^{3x}\ . \tag{31}$$

Assume the solution of the System (29)-(30) has the following fractional series expansions:

$$u_1(x) = \sum_{n=0}^{\infty} u_{1,n}\ \ x^{n\alpha}, \quad u_2(x) = \sum_{n=0}^{\infty} u_{2,n}\ \ x^{n\alpha}, \quad u_3(x) = \sum_{n=0}^{\infty} u_{3,n}\ \ x^{n\alpha}. \tag{32}$$

Employing the initial conditions (30), then the kth approximate solution becomes as follows:

$$u_{1k}(x) = 1+\sum_{n=1}^{k} u_{1,n}\ \ x^{n\alpha}, \quad u_{2k}(x) = 1+\sum_{n=1}^{k} u_{2,n}\ \ x^{n\alpha}, \quad u_{3k}(x) = 1+\sum_{n=1}^{k} u_{3,n}\ \ x^{n\alpha}. \tag{33}$$

To get the kth approximate solution of System (29)-(30), we define the kth residual functions of equations (29) as follows:

$$\begin{aligned}\mathcal{R}f\left(u_{1k}\left(x\right)\right) &= \mathcal{D}^{\alpha}u_{1k}\left(x\right)-u_{1k}\left(x\right),\\ \mathcal{R}f\left(u_{2k}\left(x\right)\right) &= \mathcal{D}^{\alpha}u_{2k}\left(x\right)-2u_{1_k}^{2}(x),\\ \mathcal{R}f\left(u_{3k}\left(x\right)\right) &= \mathcal{D}^{\alpha}u_{3k}\left(x\right)-3u_{1k}\left(x\right)u_{2k}\left(x\right).\end{aligned} \tag{34}$$

After substituting the kth approximations (33) into Equation (34), then the kth residual functions becomes as:

$$\begin{aligned}\mathcal{R}f\left(u_{1k}\left(x\right)\right) &= \sum_{n=1}^{k}u_{1,n}\ \frac{\Gamma\left(n\alpha+1\right)}{\Gamma\left(\left(n-1\right)\alpha+1\right)}\ x^{(n-1)\alpha}-\sum_{n=1}^{k}u_{1,n}\ \ x^{n\alpha}-1,\\ \mathcal{R}f\left(u_{2k}\left(x\right)\right) &= \sum_{n=1}^{k}u_{2,n}\ \frac{\Gamma\left(n\alpha+1\right)}{\Gamma\left(\left(n-1\right)\alpha+1\right)}\ x^{(n-1)\alpha}-2\left(\sum_{n=1}^{k}u_{1,n}\ \ x^{n\alpha}+1\right)^{2},\\ \mathcal{R}f\left(u_{3k}\left(x\right)\right) &= \sum_{n=1}^{k}u_{3,n}\ \frac{\Gamma\left(n\alpha+1\right)}{\Gamma\left(\left(n-1\right)\alpha+1\right)}\ x^{(n-1)\alpha}-3\left(1+\sum_{n=1}^{k}u_{1,n}\ \ x^{n\alpha}\right)\\ &\quad\times\left(1+\sum_{n=1}^{k}u_{2,n}\ \ x^{n\alpha}\right).\end{aligned} \tag{35}$$

According to the formulation used in the previous section, the kth approximations can be achieved by obtaining the coefficients $u_{1,j},\ u_{2,j},\ u_{3,j}$ for $j=1,\ldots,\ k$ via solving the following algebraic equations iteratively:

$$\lim_{x\to 0}\frac{\mathcal{R}f\left(u_{1j}\left(x\right)\right)}{x^{(j-1)\alpha}}=0\ \ ,\ \ \lim_{x\to 0}\frac{\mathcal{R}f\left(u_{2j}\left(x\right)\right)}{x^{(j-1)\alpha}}=0,\ \ \lim_{x\to 0}\frac{\mathcal{R}f\left(u_{3j}\left(x\right)\right)}{x^{(j-1)\alpha}}=0\quad ,j=1,\ldots,\ k. \tag{36}$$

These coefficients are summarized in Table 2.

So, the LRF solution of the System (29)-(30) has the following series expansions:

$$\begin{aligned}u_1\left(x\right) &= 1+\frac{x^{\alpha}}{\Gamma\left(\alpha+1\right)}+\frac{x^{2\alpha}}{\Gamma\left(2\alpha+1\right)}+\frac{x^{3\alpha}}{\Gamma\left(3\alpha+1\right)}+\frac{x^{4\alpha}}{\Gamma\left(4\alpha+1\right)}+\ldots,\\ u_2\left(x\right) &= 1+\frac{2x^{\alpha}}{\Gamma\left(\alpha+1\right)}+\frac{4x^{2\alpha}}{\Gamma\left(2\alpha+1\right)}+\left(\frac{4}{\Gamma\left(3\alpha+1\right)}+\ \frac{2\ \Gamma\left(2\alpha+1\right)}{\Gamma^{2}\left(\alpha+1\right)\Gamma\left(3\alpha+1\right)}\right)x^{3\alpha},\\ &\quad+\left(\frac{4}{\Gamma\left(4\alpha+1\right)}+\ \frac{4\ \Gamma\left(3\alpha+1\right)}{\Gamma\left(\alpha+1\right)\Gamma\left(2\alpha+1\right)\Gamma\left(4\alpha+1\right)}\right)x^{4\alpha}+\ldots,\\ u_3\left(x\right) &= 1+\frac{3x^{\alpha}}{\Gamma\left(\alpha+1\right)}+\frac{9x^{2\alpha}}{\Gamma\left(2\alpha+1\right)}+\left(\frac{15}{\Gamma\left(3\alpha+1\right)}+\ \frac{6\ \Gamma\left(2\alpha+1\right)}{\Gamma^{2}\left(\alpha+1\right)\Gamma\left(3\alpha+1\right)}\right)x^{3\alpha}\end{aligned}$$

Table 2: The coefficients of the 4th approximations of $u_1(x)$, $u_2(x)$, $u_3(x)$ for the System (29)-(30).

k	$u_{1,k}$	$u_{2,k}$	$u_{3,k}$
0	1	1	1
1	$\frac{1}{\Gamma(\alpha+1)}$	$\frac{2}{\Gamma(\alpha+1)}$	$\frac{3}{\Gamma(\alpha+1)}$
2	$\frac{1}{\Gamma(2\alpha+1)}$	$\frac{4}{\Gamma(2\alpha+1)}$	$\frac{9}{\Gamma(2\alpha+1)}$
3	$\frac{1}{\Gamma(3\alpha+1)}$	$\frac{4}{\Gamma(3\alpha+1)}+\frac{2\ \Gamma(2+1)}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)}$	$\frac{15}{\Gamma(3\alpha+1)}+\frac{6\ \Gamma(2+1)}{\Gamma^2(+1)\Gamma(3\alpha+1)}$
4	$\frac{1}{\Gamma(4\alpha+1)}$	$\frac{4}{\Gamma(4\alpha+1)}+\frac{4\ \Gamma(3+1)}{\Gamma(\alpha+1)\Gamma(2\alpha+1)\Gamma(4\alpha+1)}$	$\frac{15}{\Gamma(4\alpha+1)}+\frac{6\ \Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(4\alpha+1)}$ $+\frac{18\ \Gamma(3\alpha+1)}{\Gamma(\alpha+1)\Gamma(2\alpha+1)\Gamma(4\alpha+1)}$

$$+\left(\frac{15}{\Gamma\left(4\alpha+1\right)}+\frac{6\ \Gamma\left(2\alpha+1\right)}{\Gamma^2\left(\alpha+1\right)\Gamma\left(4\alpha+1\right)}+\frac{18\ \Gamma\left(3\alpha+1\right)}{\Gamma\left(\alpha+1\right)\Gamma\left(2\alpha+1\right)\Gamma\left(4\alpha+1\right)}\right)x^{4\alpha}+\dots. \tag{37}$$

For $\alpha=1$, the expansions in (37) become as follows:

$$u_1(x)=1+x+\frac{x^2}{2}+\frac{x^3}{3!}+\frac{x^4}{4!}+\dots,$$

$$u_2(x)=1+2x+2x^2+\frac{4x^3}{3}+\frac{2x^4}{3}+\dots,$$

$$u_3(x)=1+3x+\frac{9x^2}{2}+\frac{9x^3}{2}+\frac{27x^4}{8}+\dots, \tag{38}$$

which are the expansions of the exact solution indicated in (31).

Figure 2 shows the graphs of the 4th approximate solution of the system (29)-(30) at different values of α, as well as the exact solution at $\alpha=1$. The graph shows agreement between the approximate and actual solutions at $\alpha=1$. It also shows the effect of the derivative's order on the solution's behavior. It is noticeable that the convergence intervals of the solution decrease due to the action and effects of nonlinearity in the equation defining the dependent variables.

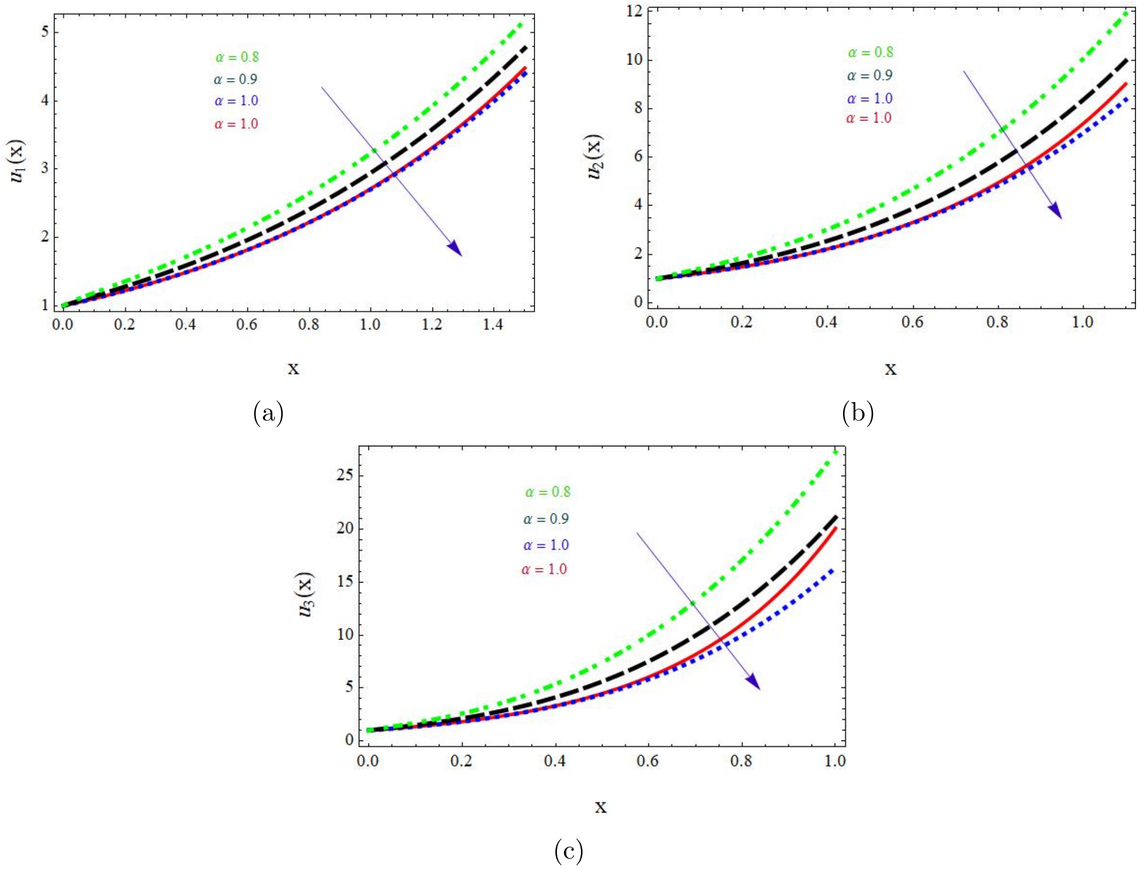


Figure 2: The curves of the 4th approximate and exact (solid curve) solutions of System (29)-(30) at different values of α.

Example 4.3. Consider the following system of non-linear fractional differential equations

$$\mathcal{D}^{\alpha} u_1(x) = -1002\ u_1(x) + 1000\ u_2^2(x),$$
$$\mathcal{D}^{\alpha} u_2(x) = u_1(x) - u_2(x) - u_2^2(x), \tag{39}$$

where $0 < \alpha \leq 1,\ \ 0 \leq x$ with the initial conditions

$$u_1(0) = 1, \quad u_2(0) = 1. \tag{40}$$

In the classical case ($\alpha = 1$), the exact solution is given as follows:

$$u_1(x) = e^{-2x}, \quad u_2(x) = e^{-x}. \tag{41}$$

According to the methodology of the LRF method, we assume the solution of the System (39)-(40) has fractional series expansions as:

$$u_1(x) = \sum_{n=0}^{\infty} u_{1,n}\ \ x^{n\alpha}, \quad u_2(x) = \sum_{n=0}^{\infty} u_{2,n}\ \ x^{n\alpha}. \tag{42}$$

Based on the initial conditions in Equation (40), the kth approximate solution can be written as

$$u_{1k}(x) = 1 + \sum_{n=1}^{k} u_{1,n} \ x^{n\alpha}, \quad u_{2k}(x) = 1 + \sum_{n=1}^{k} u_{2,n} \ x^{n\alpha}. \tag{43}$$

The residual and kth residual functions of Equation (39) can be given respectively as follows:

$$\begin{aligned} \mathcal{R}f(u_1(x)) &= \mathcal{D}^{\alpha} u_1(x) + 1002 \ u_1(x) - 1000 \ u_2^2(x), \\ \mathcal{R}f(u_2(x)) &= \mathcal{D}^{\alpha} u_2(x) - u_1(x) + u_2(x) + u_2^2(x). \end{aligned} \tag{44}$$

$$\begin{aligned} \mathcal{R}f(u_{1k}(x)) &= \mathcal{D}^{\alpha} u_{1k}(x) + 1002 \ u_{1k}(x) - 1000 \ u_{2_k}^2(x), \\ \mathcal{R}f(u_{2k}(x)) &= \mathcal{D}^{\alpha} u_{2k}(x) - u_{1k}(x) + u_{2k}(x) + u_{2_k}^2(x). \end{aligned} \tag{45}$$

Substituting the kth approximation (43) into the kth residual function (45) gives the series form of the kth residual function as follows:

$$\begin{aligned} \mathcal{R}f(u_{1k}(x)) &= \sum_{n=1}^{k} u_{1,n} \frac{\Gamma(n\alpha+1)}{\Gamma((n-1)\alpha+1)} x^{(n-1)\alpha} + 1002\left(1 + \sum_{n=1}^{k} u_{1,n} \ x^{n\alpha}\right) \\ &\quad -1000\left(1 + \sum_{n=1}^{k} u_{2,n} \ x^{n\alpha}\right)^2, \\ \mathcal{R}f(u_{2k}(x)) &= \sum_{n=1}^{k} u_{2,n} \frac{\Gamma(n\alpha+1)}{\Gamma((n-1)\alpha+1)} x^{(n-1)\alpha} - \sum_{n=1}^{k} u_{1,n} \ x^{n\alpha} + \sum_{n=1}^{k} u_{2,n} \ x^{n\alpha} \\ &\quad + \left(1 + \sum_{n=1}^{k} u_{2,n} \ x^{n\alpha}\right)^2. \end{aligned} \tag{46}$$

According to the formulation used in the previous section, the kth approximations can be achieved by obtaining the coefficients $u_{1,j}, u_{2,j}$ for $j = 1, \ldots, k$ via solving the following algebraic equations iteratively:

$$\lim_{x \to 0} \frac{\mathcal{R}f\left(u_{1j}(x)\right)}{x^{(j-1)\alpha}} = 0, \quad \lim_{x \to 0} \frac{\mathcal{R}f\left(u_{2j}(x)\right)}{x^{(j-1)\alpha}} = 0. \tag{47}$$

These coefficients are summarized in Table 3.

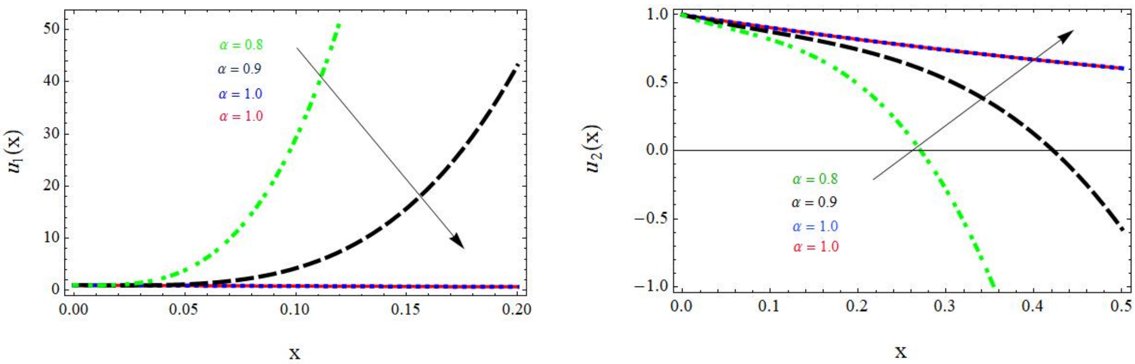
So, the LRF solution of the System (39)-(40) has the following series expansions:

$$\begin{aligned} u_1(x) &= 1 - \frac{2x^{\alpha}}{\Gamma(\alpha+1)} + \frac{4x^{2\alpha}}{\Gamma(2\alpha+1)} + \left(-\frac{2008}{\Gamma(3\alpha+1)} + \frac{1000\ \Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)}\right) x^{3\alpha} \\ &+ \left(\frac{2014016}{\Gamma(4\alpha+1)} - \frac{1004000\ \Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(4\alpha+1)} - \frac{2000\ \Gamma(3\alpha+1)}{\Gamma(\alpha+1)\Gamma(2\alpha+1)\Gamma(4\alpha+1)}\right) x^{4\alpha} + \ldots, \end{aligned}$$

Table 3: The coefficients of the 4th approximations of $u_1(x)$, $u_2(x)$ for the System (39)-(40).

k	$u_{1,k}$	$u_{2,k}$
0	1	1
1	$-\frac{2}{\Gamma(\alpha+1)}$	$-\frac{1}{\Gamma(\alpha+1)}$
2	$\frac{4}{\Gamma(2\alpha+1)}$	$\frac{1}{\Gamma(2\alpha+1)}$
3	$-\frac{2008}{\Gamma(3\alpha+1)}+\frac{1000\ \Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)}$	$\frac{1}{\Gamma(3\alpha+1)}-\frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)}$
4	$\frac{2014016}{\Gamma(4\alpha+1)}-\frac{1004000\ \Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(4\alpha+1)}-\frac{2000\ \Gamma(3\alpha+1)}{\Gamma(\alpha+1)\Gamma(2\alpha+1)\Gamma(4\alpha+1)}$	$\frac{-2011}{\Gamma(4\alpha+1)}+\frac{1003\ \Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(4\alpha+1)}+\frac{2\ \Gamma(3\alpha+1)}{\Gamma(\alpha+1)\Gamma(2\alpha+1)\Gamma(4\alpha+1)}$

$$u_2(x)=1-\frac{x^{\alpha}}{\Gamma(\alpha+1)}+\frac{x^{2\alpha}}{\Gamma(2\alpha+1)}+\left(\frac{1}{\Gamma(3\alpha+1)}-\frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)}\right)x^{3\alpha}$$
$$+\left(\frac{-2011}{\Gamma(4\alpha+1)}+\frac{1003\ \Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(4\alpha+1)}+\frac{2\ \Gamma(3\alpha+1)}{\Gamma(\alpha+1)\Gamma(2\alpha+1)\Gamma(4\alpha+1)}\right)x^{4\alpha}+\ldots. \tag{48}$$



(a) (b)

Figure 3: The curves of the 4th approximate and exact (solid curve) solutions of System (39)-(40) at different values of α.

For $\alpha=1$, the expansions in (48) become as follows:

$$u_1(x)=1-2x+2x^2-\frac{4x^3}{3}+\frac{2x^4}{3}+\ldots,$$
$$u_2(x)=1-x+\frac{x^2}{2}-\frac{x^3}{6}+\frac{x^4}{24}+\ldots. \tag{49}$$

which are the expansions of the exact solution indicated in (41).

Figure 3 shows graphs of the 4th approximate solution of the system (39)-(40) for different values of α. The graphs show the intervals of convergence and the effect of the order of the derivative on the behavior of the solution.

Example 4.4. Consider the following system of non-linear fractional differential equations

$$\mathcal{D}^{\alpha} u_1(x) = u_1^2(x) + u_2(x),$$
$$\mathcal{D}^{\alpha} u_2(x) = u_2(x)\ \cos(u_1(x)), \tag{50}$$

where $0 < \alpha \leq 1$, $x \geq 0$, with the initial conditions

$$u_1(0) = 0, \quad u_2(0) = 1. \tag{51}$$

According to the methodology of the LRF method, we assume the solution of the System (50)-(51) has fractional PS expansions as:

$$u_1(x) = \sum_{n=0}^{\infty} u_{1,n}\ \ x^{n\alpha}, \quad u_2(x) = \sum_{n=0}^{\infty} u_{2,n}\ \ x^{n\alpha}. \tag{52}$$

Based on the initial conditions in Equation (51), the kth approximate solution can be written as

$$u_{1k}(x) = \sum_{n=1}^{k} u_{1,n}\ \ x^{n\alpha}, \quad u_{2k}(x) = 1 + \sum_{n=1}^{k} u_{2,n}\ \ x^{n\alpha}. \tag{53}$$

The residual and kth residual functions of Equation (50) can be given respectively as follows:

$$\mathcal{R}f(u_1(x)) = \mathcal{D}^{\alpha} u_1(x) - u_1^2(x) - u_2(x),$$
$$\mathcal{R}f(u_2(x)) = \mathcal{D}^{\alpha} u_2(x) - u_2(x)\ \cos(u_1(x))\ . \tag{54}$$
$$\mathcal{R}f(u_{1k}(x)) = \mathcal{D}^{\alpha} u_{1k}(x) - u_{1_k}^2(x) - u_{2k}(x),$$
$$\mathcal{R}f(u_{2k}(x)) = \mathcal{D}^{\alpha} u_{2k}(x) - u_{2k}(x)\cos(u_{1k}(x))\ . \tag{55}$$

By substituting the first approximations $u_{11}(x) = u_{1,1}\ x^{\alpha}$, $u_{21}(x) = 1 + u_{2,1}x^{\alpha}$ into the first residual functions, $\mathcal{R}f(u_{11}(x))$, $\mathcal{R}f(u_{21}(x))$, we obtain

$$\mathcal{R}f(u_{11}(x)) = u_{1,1}\Gamma(\alpha+1) - (u_{1,1}x^{\alpha})^2 - 1 - u_{2,1}x^{\alpha},$$
$$\mathcal{R}f(u_{21}(x)) = u_{2,1}\Gamma(\alpha+1) - (1 + u_{2,1}x^{\alpha})\cos(u_{1,1}\ x^{\alpha})\ . \tag{56}$$

Solving the equations $lim_{x\to 0}\mathcal{R}f(u_{11}(x)) = 0$, $lim_{x\to 0}\mathcal{R}f(u_{21}(x)) = 0$ for $u_{1,1}, u_{2,1}$, respectively, we have

$$u_{1,1} = \frac{1}{\Gamma(\alpha+1)}, \quad u_{2,1} = \frac{1}{\Gamma(\alpha+1)}\ . \tag{57}$$

In order to determine the second coefficients $u_{1,2}$ and $u_{2,2}$, substitute the second approximations $u_{12}(x) = \frac{x^{\alpha}}{\Gamma(\alpha+1)} + u_{1,2}x^{2\alpha}$, $u_{22}(x) = 1 + \frac{x^{\alpha}}{\Gamma(\alpha+1)} + u_{2,2}x^{2\alpha}$ into $\mathcal{R}f(u_{12}(x))$, $\mathcal{R}f(u_{22}(x))$ to have

$$\mathcal{R}f(u_{12}(x)) = 1 + u_{1,2}\ \frac{\Gamma(2\alpha+1)\ x^{\alpha}}{\Gamma(\alpha+1)} - \left(\frac{x^{\alpha}}{\Gamma(\alpha+1)} + u_{1,2}\ x^{2\alpha}\right)^2 - \left(1 + \frac{x^{\alpha}}{\Gamma(\alpha+1)} + u_{2,2}\ x^{2\alpha}\right),$$

$$\mathcal{R}f\left(u_{22}\left(x\right)\right)=1+u_{2,2}\ \frac{\Gamma\left(2\alpha+1\right)\ x^{\alpha}}{\Gamma\left(\alpha+1\right)}-\left(1+\frac{x^{\alpha}}{\Gamma\left(\alpha+1\right)}+u_{2,2}x^{2\alpha}\right)\cos\left(\frac{x^{\alpha}}{\Gamma\left(\alpha+1\right)}+u_{1,2}x^{2\alpha}\right). \tag{58}$$

Making simple calculations, the limits $lim_{x\to 0}\frac{\mathcal{R}f(u_{12}(x))}{x^{\alpha}}=0,\ lim_{x\to 0}\frac{\mathcal{R}f(u_{22}(x))}{x^{\alpha}}=0,$ yield

$$u_{1,2}=\frac{1}{\Gamma\left(2\alpha+1\right)},\quad u_{2,2}=\frac{1}{\Gamma\left(2\alpha+1\right)}. \tag{59}$$

The values of the coefficients $u_{1,3}$ and $u_{2,3}$ are also obtained by substituting $u_{13}\left(x\right)=\frac{x^{\alpha}}{\Gamma(\alpha+1)}+\frac{2}{\Gamma(2\alpha+1)}x^{2\alpha}+\ u_{1,3}\ x^{3\alpha}$ and $u_{23}\left(x\right)=1+\frac{x^{\alpha}}{\Gamma(\alpha+1)}+\ u_{2,3}\ x^{3\alpha}$ into $\mathcal{R}f\left(u_{13}\left(x\right)\right),\ \mathcal{R}f\left(u_{23}\left(x\right)\right)$, and then we solve the following equations:

$$\lim_{x\to 0}\frac{\mathcal{R}f\left(u_{13}\left(x\right)\right)}{x^{2\alpha}}=0,\quad \lim_{x\to 0}\frac{\mathcal{R}f\left(u_{23}\left(x\right)\right)}{x^{2\alpha}}=0. \tag{60}$$

Then we have

$$u_{1,3}=\frac{\Gamma^{2}\left(\alpha+1\right)+\Gamma\left(2\alpha+1\right)}{\Gamma^{2}\left(\alpha+1\right)\Gamma\left(3\alpha+1\right)},\quad u_{2,3}=\frac{2\Gamma^{2}\left(\alpha+1\right)-\Gamma\left(2\alpha+1\right)}{2\Gamma^{2}\left(\alpha+1\right)\Gamma\left(3\alpha+1\right)}. \tag{61}$$

Similarly, we can find the following coefficients, which are as follows:

$$u_{1,4}=\frac{2\Gamma^{2}\left(\alpha+1\right)\Gamma\left(2\alpha+1\right)-\Gamma^{2}\left(2\alpha+1\right)+4\Gamma\left(\alpha+1\right)\Gamma\left(3\alpha+1\right)}{2\Gamma^{2}\left(\alpha+1\right)\Gamma\left(2\alpha+1\right)\Gamma\left(4\alpha+1\right)},$$

$$u_{2,4}=\frac{\Gamma\left(\alpha+1\right)\left(2\Gamma^{2}\left(\alpha+1\right)-\Gamma\left(2\alpha+1\right)\right)\Gamma\left(2\alpha+1\right)-\left(2\Gamma^{2}\left(\alpha+1\right)+\Gamma\left(2\alpha+1\right)\right)\Gamma\left(3\alpha+1\right)}{2\Gamma^{3}\left(\alpha+1\right)\Gamma\left(2\alpha+1\right)\Gamma\left(4\alpha+1\right)}. \tag{62}$$

So, the LRF solution of the System (50)-(51) has the following series expansions:

$$\begin{aligned}u_{1}\left(x\right)=&\frac{x^{\alpha}}{\Gamma\left(\alpha+1\right)}+\frac{x^{2\alpha}}{\Gamma\left(2\alpha+1\right)}+\frac{\Gamma^{2}\left(\alpha+1\right)+\Gamma\left(2\alpha+1\right)}{\Gamma^{2}\left(\alpha+1\right)\Gamma\left(3\alpha+1\right)}x^{3\alpha}\\&+\frac{2\Gamma^{2}\left(\alpha+1\right)\Gamma\left(2\alpha+1\right)-\Gamma^{2}\left(2\alpha+1\right)+4\Gamma\left(\alpha+1\right)\Gamma\left(3\alpha+1\right)}{2\Gamma^{2}\left(\alpha+1\right)\Gamma\left(2\alpha+1\right)\Gamma\left(4\alpha+1\right)}x^{4\alpha}+\ldots,\\u_{2}\left(x\right)=&1+\frac{x^{\alpha}}{\Gamma\left(\alpha+1\right)}+\frac{x^{2\alpha}}{\Gamma\left(2\alpha+1\right)}+\frac{2\Gamma^{2}\left(\alpha+1\right)-\Gamma\left(2\alpha+1\right)}{2\Gamma^{2}\left(\alpha+1\right)\Gamma\left(3\alpha+1\right)}x^{3\alpha}\\&+\frac{\Gamma\left(\alpha+1\right)\left(2\Gamma^{2}\left(\alpha+1\right)-\Gamma\left(2\alpha+1\right)\right)\Gamma\left(2\alpha+1\right)-\left(2\Gamma^{2}\left(\alpha+1\right)+\Gamma\left(2\alpha+1\right)\right)\Gamma\left(3\alpha+1\right)}{2\Gamma^{3}\left(\alpha+1\right)\Gamma\left(2\alpha+1\right)\Gamma\left(4\alpha+1\right)}x^{4\alpha}+\ldots.\end{aligned} \tag{63}$$

For $\alpha=1$, the solution (63) becomes as follows:

$$\begin{aligned}u_{1}\left(x\right)&=x+\frac{x^{2}}{2}+\frac{x^{3}}{2}+\frac{x^{4}}{4}+\ldots,\\u_{2}\left(x\right)&=1+x+\frac{x^{2}}{2}-\frac{x^{4}}{4}+\ldots.\end{aligned} \tag{64}$$

This solution coincides with the solution attained by using the Adomian decomposition method [38].

Tables 4 and 5 present numerical values for the 4th approximation of the solution to the IVP (50)-(51) and the residual error of the approximate solution at various α values. Since there is no exact solution available for this problem, we focus solely on the residual error, which is defined for the problem as follows:

$$\mathcal{R}f.\ Err.\,(x) = \left|\mathcal{D}^{\alpha}u_{1k}\,(x) - u_{1_k}^{2}\,(x) - u_{2k}\,(x)\right|,$$
$$\mathcal{R}f.\ Err.\,(x) = \left|\mathcal{D}^{\alpha}u_{2k}\,(x) - u_{2k}\,(x)\cos\left(u_{1k}\,(x)\right)\right|. \tag{65}$$

Table 4: The 4th approximation of $u_1\,(x)$, the IVP (33)-(34) solution, and the residual error for $\alpha = 1$ and $\alpha = 0.8$.

	$\alpha = 1$		$\alpha = 0.8$	
t	$u_{14}\,(x)$	$\mathcal{R}f.\ Err.$	$u_{14}\,(x)$	$\mathcal{R}f.\ Err.$
0.0	0	0	0	0
0.1	0.105525	1.10526×10^{-4}	0.191650	1.47602×10^{-3}
0.2	0.224400	1.95536×10^{-3}	0.371681	1.58855×10^{-2}
0.3	0.360525	1.09533×10^{-2}	0.573817	6.71247×10^{-2}
0.4	0.518400	3.83386×10^{-2}	0.807786	1.92998×10^{-1}
0.5	0.703125	1.0376×10^{-1}	1.080985	4.48851×10^{-1}

The residual error serves as an indicator of the solution's accuracy, although it does not precisely measure the error in the same way that absolute error does. Even though the residual error remains within a range of 1 to 3 and is concentrated in a small interval, the solution period can be extended by employing alternative techniques, such as multi-step methods.

Table 5: The 4th approximation of $u_2\,(x)$, the IVP (33)-(34) solution, and the residual error for $\alpha = 1$ and $\alpha = 0.8$.

	$\alpha = 1$		$\alpha = 0.8$	
t	$u_{24}\,(x)$	$\mathcal{R}f.\ Err.$	$u_{24}\,(x)$	$\mathcal{R}f.\ Err.$
0.0	1	0	1	0
0.1	1.104975	1.71532×10^{-4}	1.187653	2.13071×10^{-3}
0.2	1.219600	2.97806×10^{-3}	1.347857	2.22820×10^{-2}
0.3	1.342975	1.63625×10^{-2}	1.504286	9.14918×10^{-2}
0.4	1.473600	5.60118×10^{-2}	1.657010	2.53724×10^{-1}
0.5	1.609375	1.47328×10^{-1}	1.803767	5.60315×10^{-1}

5. Conclusion

In this article, we introduced a new analytical iterative technique called the LRF method, which we used to solve linear and nonlinear systems of Caputo-fractional differential equations. This approach simplifies discovering exact solution patterns while reducing the need for complex computational calculations. The main advantage of this method is its straightforwardness in computing the coefficients of the series solution by evaluating the limit of a form that includes the residual function for the given equations. This is a departure from other well-known analytic techniques that rely on differential and integral operators, which can be challenging in the fractional case. We plan to modify our approach to handle more complex real-world applications in future work.

Acknowledgements

The authors express their gratitude to the dear referees, who wish to remain anonymous, and the editor for their helpful suggestions that improved the final version of this paper. This research is funded by Zarqa University-Jordan.

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