EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 1, Article Number 5619
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Double Laplace-Sawi Transform

Monther Al-Momani[1], Ali Jaradat[1], Baha' Abughazaleh[2,*]

[1] *Department of Mathematics, Amman Arab University, Amman, Jordan*
[2] *Department of Mathematics, Isra University, Amman, Jordan*

Abstract. The primary objective of this study is to develop a new integral transform by combining the Laplace and Sawi transforms, and to investigate its key properties, existence, and the inversion theorem. Furthermore, we introduce new results related to partial differential equations in higher dimensions and extend the double convolution theorem to two dimensions. Using these new properties and theorems, we solve special type differential equations with some real applications in physics and related sciences.

2020 Mathematics Subject Classifications: 44A05, 44A10

Key Words and Phrases: Laplace transform, Sawi transform, Double integral transform, Laplace-Sawi transform

1. Introduction

Integral transforms are powerful mathematical tools that convert functions into new domains. After transforming the function can be returned to its original space by applying the inverse of the integral transform. By applying an integral transform, we generate a new function $G(\delta)$ through the integration of the product of $g(\eta)$ and $K(\eta, \delta)$ across the interval $[a, b]$ represented by:

$$\int_a^b g(\eta)K(\eta, \delta)d\eta$$

They are pivotal in engineering, economics, physics, and chemistry, serving as essential tools for understanding complex real-world phenomena. Thus, mathematicians relentlessly innovate and develop new techniques to tackle ever-broader classes of differential equations, and one of the most celebrated integral transforms is the Laplace transform, first introduced in 1780. Among the innovative integral transforms emerging in recent years is the Sawi transform introduced in 2021 by [1]. These transforms offer powerful new

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v18i1.5619

Email addresses: monther_ok@yahoo.com M. Al-Momani),
a.jaradat@aau.edu.jo (A. Jaradat), baha.abughazaleh@iu.edu.jo (B. Abughazaleh)

tools for tackling both ordinary and fractional differential equations, for more information about the Sawi transform, refer to [2], and for more details on other single transforms, one may refer to [3–5].

Additionally, some Double transforms exist to handle many-variable differential equations. In the wide range of double transforms, we notice fresh methods to help solve differential equations in more than one dimension. The double Laplace transform [6], the Double Laplace-Shehu transform [8], the Double Laplace ARA Transform [7], the Double Sawi transform [9] and Double Mellin-ARA Transform [10].

In the present work, we propose a double transform called the Double Laplace-Sawi Transform (DLSWT) aimed at globalizing differential equation analysis. We go down to its bedrock properties characterizing what is needed for it to exist and demonstrating their power in convolution theory and derivative operation. Applying this novel transform method, we identify new ways of dealing with partial differential equations and integral equations. The novelty of this work lies in the innovative combinations of the Laplace and Sawi transforms, creating a new approach that harnesses the strengths of both transforms. This combination enhances the simplicity and applicability in addressing complex mathematical problems.

2. Laplace and Sawi transforms

In this section, we provide an overview and highlight key properties of the single transforms, namely the Laplace and Sawi transforms.

2.1. Laplace transform

Definition 1. *The Laplace transform of a continuous function $p(\eta)$ on $(0,\infty)$ is defined as follows*

$$P(\delta) = L(p(\eta)) = \int_0^\infty e^{-\delta\eta} p(\eta) d\eta, \delta \in \mathbb{C}.$$

Some basic properties of the Laplace transform are now given.

Let $P(\delta) = L(p(\eta))$, then for nonzero constants u and v, we have

$$L(up_1(\eta) + vp_2(\eta)) = uL(p_1(\eta)) + vL(p_2(\eta)), \quad (1)$$

where $p_1(\eta)$ and $p_2(\eta)$ are continuous functions on $(0,\infty)$.

$$L(\eta^u) = \frac{\Gamma(u+1)}{\delta^{u+1}}, \quad (2)$$

$$L(e^{u\eta}) = \frac{1}{\delta - u}, \ u \in \mathbb{R}, \quad (3)$$

$$L(p'(\eta)) = \delta P(\delta) - p(0), \tag{4}$$

$$L(p''(\eta)) = \delta^2 P(\delta) - \delta p(0) - p'(0). \tag{5}$$

2.2. The Sawi transform

Definition 2. *The Sawi transform of a continuous function $q(\theta)$ on $(0,\infty)$ expressed as follows*

$$Q(\epsilon) = W(q(\theta)) = \frac{1}{\epsilon^2}\int_0^\infty e^{-\frac{\theta}{\epsilon}} q(\theta) d\theta.$$

Let us now explore the core properties that define the Sawi transform.

Suppose that $Q_1(\epsilon) = W(q_1(\theta))$ and $Q_2(\epsilon) = W(q_2(\theta))$,with u and v as nonzero real numbers, the following properties hold

$$W(uq_1(\theta) + vq_2(\theta)) = uW(q_1(\theta)) + vW(q_2(\theta)), \tag{6}$$

$$W(\theta^u) = \Gamma(u+1)\epsilon^{u-1}, \tag{7}$$

$$W(e^{v\theta}) = \frac{1}{\epsilon(1 - v\epsilon)}, \tag{8}$$

$$W(q'(\theta)) = \frac{1}{\epsilon}Q(\epsilon) - \frac{1}{\epsilon^2}q(0), \tag{9}$$

$$W(q''(\theta)) = \frac{1}{\epsilon^2}Q(\epsilon) - \frac{1}{\epsilon^3}q(0) - \frac{1}{\epsilon^2}q'(0). \tag{10}$$

3. Double Laplace-Sawi transform

This section announces the Double Laplace-Sawi Transformation (DLSWT). We start by stating the basic properties of the DLSWT, such as linearity and inversion. Then we state a new result regarding the partial derivatives and another new result regarding the convolution theorem. We also state how we use these results to compute the DLSWT of some basic functions. The definition of the DLSWT is:

$$G(\delta,\epsilon) = L_\eta W_\theta(g(\eta,\theta)) = \frac{1}{\epsilon^2}\int_0^\infty\int_0^\infty e^{-\delta\eta-\frac{\theta}{\epsilon}} g(\eta,\theta)\ d\eta d\theta, \tag{11}$$

where $g(\eta,\theta)$ is a continuous function on $(0,\infty)\times(0,\infty)$.

Clearly, $L_\eta W_\theta(g(\eta,\theta))$ is linear transformation. In fact, for nonzero constants u and v, we have

$$
\begin{aligned}
&L_\eta W_\theta(ug_1(\eta,\theta)+vg_2(\eta,\theta))\\
&= \frac{1}{\epsilon^2}\int_0^\infty\int_0^\infty e^{-\delta\eta-\frac{\theta}{\epsilon}}(ug_1(\eta,\theta)+vg_2(\eta,\theta))\,d\eta d\theta\\
&= u\frac{1}{\epsilon^2}\int_0^\infty\int_0^\infty e^{-\delta\eta-\frac{\theta}{\epsilon}}g_1(\eta,\theta)\,d\eta d\theta + v\frac{1}{\epsilon^2}\int_0^\infty\int_0^\infty e^{-\delta\eta-\frac{\theta}{\epsilon}}g_2(\eta,\theta)\,d\eta d\theta\\
&= uL_\eta W_\theta(g_1(\eta,\theta)) + vL_\eta W_\theta(g_2(\eta,\theta)).
\end{aligned}
$$

If $g(\eta,\theta)$ can be written as $g(\eta,\theta)=p(\eta)q(\theta)$ for some continuous functions p and q, then $L_\eta W_\theta(g(\eta,\theta)) = L(p(\eta))W(q(\theta))$. In fact

$$
\begin{aligned}
L_\eta W_\theta(g(\eta,\theta)) &= L_\eta W_\theta(p(\eta)q(\theta))\\
&= \frac{1}{\epsilon^2}\int_0^\infty\int_0^\infty e^{-\delta\eta-\frac{\theta}{\epsilon}}p(\eta)q(\theta)d\eta d\theta\\
&= \left(\int_0^\infty e^{-\delta\eta}p(\eta)d\eta\right)\left(\frac{1}{\epsilon^2}\int_0^\infty e^{-\frac{\theta}{\epsilon}}q(\theta)d\theta\right)\\
&= L(p(\eta))W(q(\theta)).
\end{aligned}
$$

3.1. Double Laplace-Sawi transform for some basic functions

(i)

$$L_\eta W_\theta(1) = \frac{1}{\epsilon^2}\int_0^\infty\int_0^\infty e^{-\delta\eta-\frac{\theta}{\epsilon}}d\eta d\theta$$

$$= \left(\int_0^\infty e^{-\delta\eta}d\eta\right)\left(\frac{1}{\epsilon^2}\int_0^\infty e^{-\frac{\theta}{\epsilon}}d\theta\right) = \frac{1}{\delta}\times\frac{1}{\epsilon} = \frac{1}{\delta\epsilon},\ \mathrm{Re}(\delta)>0.$$

(ii)

$$L_\eta W_\theta(\eta^u\theta^v) = \frac{1}{\epsilon^2}\int_0^\infty\int_0^\infty e^{-\delta\eta-\frac{\theta}{\epsilon}}\eta^u\theta^v d\eta d\theta$$

$$= \left(\int_0^\infty \eta^u e^{-\delta\eta}d\eta\right)\left(\frac{1}{\epsilon^2}\int_0^\infty \theta^v e^{-\frac{\theta}{\epsilon}}d\theta\right)$$

$$= \frac{\Gamma(u+1)}{\delta^{u+1}} \times \Gamma(v+1)\epsilon^{v-1}$$

$$= \frac{\epsilon^{v-1}}{\delta^{u+1}}\Gamma(u+1)\Gamma(v+1), \ \mathrm{Re}(\delta) > 0 \text{ and } \mathrm{Re}(u) > -1.$$

(iii)

$$L_\eta W_\theta(e^{u\eta+v\theta}) = \frac{1}{\epsilon^2}\int_0^\infty\int_0^\infty e^{-\delta\eta-\frac{\theta}{\epsilon}} e^{u\eta+v\theta} d\eta d\theta$$

$$= \left(\int_0^\infty e^{u\eta-\delta\eta} d\eta\right)\left(\frac{1}{\epsilon^2}\int_0^\infty e^{v\theta-\frac{\theta}{\epsilon}} d\theta\right) = \frac{1}{\delta-u} \times \frac{1}{\epsilon\,(1-v\epsilon)}$$

$$= \frac{1}{\epsilon\,(\delta-u)\,(1-v\epsilon)}, \ \mathrm{Re}(\delta) > \mathrm{Re}(u).$$

3.2. Existence condition for Double Laplace-Sawi transform

Definition 3. *A function $g(\eta,\theta)$ is said to be of exponential orders u and v on $0 \le \eta < \infty$ and $0 \le \theta < \infty$. If there exist $K, X, Y > 0$ such that $|g(\eta,\theta)| \le Ke^{u\eta+v\theta}$, for all $\eta > X$, $\theta > Y$.*

Theorem 1. *Let $g(\eta,\theta)$ be a continuous function on the region $[0,\infty) \times [0,\infty)$ of exponential orders u and v. Then $G(\delta,\epsilon)$ exists for δ, ϵ and γ whenever* $\mathrm{Re}\,(\delta) > u$ *and* $\mathrm{Re}\left(\frac{1}{\epsilon}\right) > v$.

Proof. We have

$$\begin{aligned}
|G(\delta,\epsilon)| &= \left|\frac{1}{\epsilon^2}\int_0^\infty\int_0^\infty e^{-\delta\eta-\frac{\theta}{\epsilon}} g(\eta,\theta)\ d\eta d\theta\right| \le \frac{1}{\epsilon^2}\int_0^\infty\int_0^\infty e^{-\delta\eta-\frac{\theta}{\epsilon}} \left|g(\eta,\theta)\right| d\eta d\theta \\
&\le K\frac{1}{\epsilon^2}\int_0^\infty\int_0^\infty e^{-\delta\eta-\frac{\theta}{\epsilon}} e^{u\eta+v\theta} d\eta d\theta = K\int_0^\infty\int_0^\infty \left(e^{-(\delta-u)\eta}\right)\left(\frac{1}{\epsilon^2}e^{-(\frac{1}{\epsilon}-v)\theta}\right) d\eta d\theta \\
&= K\left(\int_0^\infty e^{-(\delta-u)\eta} d\eta\right)\left(\frac{1}{\epsilon^2}\int_0^\infty e^{-(\frac{1}{\epsilon}-v)\theta} d\theta\right) \\
&= \frac{K}{\epsilon\,(\delta-u)\,(1-v\epsilon)},
\end{aligned}$$

where $\mathrm{Re}\,(\delta) > u$ and $\mathrm{Re}\left(\frac{1}{\epsilon}\right) > v$.

3.3. Derivatives properties

Now, we present some basic properties of the DLSWT

Let $G(\delta, \epsilon) = L_\eta W_\theta(g(\eta, \theta))$ where $g(\eta, \theta)$ is a continuous function on $(0, \infty) \times (0, \infty)$. Then

(i)

$$L_\eta W_\theta \left(\frac{\partial g(\eta, \theta)}{\partial \eta} \right) = \delta G(\delta, \epsilon) - W(g(0, \theta)), \tag{12}$$

(ii)

$$L_\eta W_\theta \left(\frac{\partial^2 g(\eta, \theta)}{\partial \eta^2} \right) = \delta^2 G(\delta, \epsilon) - \delta W(g(0, \theta)) - W(g_\eta(0, \theta)),$$

(iii)

$$L_\eta W_\theta \left(\frac{\partial g(\eta, \theta)}{\partial \theta} \right) = \frac{1}{\epsilon} G(\delta, \epsilon) - \frac{1}{\epsilon^2} L(g(\eta, 0)), \tag{13}$$

(iv)

$$L_\eta W_\theta \left(\frac{\partial^2 g(\eta, \theta)}{\partial \theta^2} \right) = \frac{1}{\epsilon^2} G(\delta, \epsilon) - \frac{1}{\epsilon^3} L(g(\eta, 0)) - \frac{1}{\epsilon^2} L(g_\theta(\eta, 0)), \tag{14}$$

(v)

$$L_\eta W_\theta \left(\frac{\partial^2 g(\eta, \theta)}{\partial \eta \partial \theta} \right) = \frac{\delta}{\epsilon} G(\delta, \epsilon) - \frac{\delta}{\epsilon^2} L(g(\eta, 0)) - \frac{1}{\epsilon} W(g(0, \theta)) + \frac{1}{\epsilon^2} g(0, 0). \tag{15}$$

Proof. (1) $L_\eta W_\theta \left(\frac{\partial g(\eta,\theta)}{\partial \eta} \right) = \frac{1}{\epsilon^2} \int_0^\infty \int_0^\infty e^{-\delta\eta - \frac{\theta}{\epsilon}} \frac{\partial g(\eta,\theta)}{\partial \eta} \, d\eta d\theta = \frac{1}{\epsilon^2} \int_0^\infty e^{-\frac{\theta}{\epsilon}} \int_0^\infty e^{-\delta\eta} \frac{\partial g(\eta,\theta)}{\partial \eta} d\eta d\theta.$
By integrating by parts, we get

$L_\eta W_\theta \left(\frac{\partial g(\eta,\theta)}{\partial \eta} \right) = \frac{1}{\epsilon^2} \int_0^\infty e^{-\frac{\theta}{\epsilon}} \left(-g(0, \theta) + \delta \int_0^\infty e^{-\delta\eta} g(\eta, \theta) \, d\eta \right) d\theta$
$= -\frac{1}{\epsilon^2} \int_0^\infty e^{-\frac{\theta}{\epsilon}} g(0, \theta) d\theta + \frac{\delta}{\epsilon^2} \int_0^\infty \int_0^\infty e^{-\delta\eta - \frac{\theta}{\epsilon}} g(\eta, \theta) \, d\eta d\theta$
$= \delta G(\delta, \epsilon) - W(g(0, \theta)).$

(2) $L_\eta W_\theta \left(\frac{\partial^2 g(\eta,\theta)}{\partial \eta^2} \right) = \frac{1}{\epsilon^2} \int_0^\infty \int_0^\infty e^{-\delta\eta - \frac{\theta}{\epsilon}} \frac{\partial^2 g(\eta,\theta)}{\partial \eta^2} \, d\eta d\theta = \frac{1}{\epsilon^2} \int_0^\infty e^{-\frac{\theta}{\epsilon}} \int_0^\infty e^{-\delta\eta} \frac{\partial^2 g(\eta,\theta)}{\partial \eta^2} d\eta d\theta.$

By integrating by parts, we get

$$L_\eta W_\theta\left(\frac{\partial^2 g(\eta,\theta)}{\partial \eta^2}\right) = \frac{1}{\epsilon^2}\int_0^\infty e^{-\frac{\theta}{\epsilon}}\left(-g_\eta(0,\theta) - \delta g(0,\theta) + \delta^2\int_0^\infty e^{-\delta\eta} g(\eta,\theta)d\eta\right)d\theta$$

$$= -\frac{1}{\epsilon^2}\int_0^\infty e^{-\frac{\theta}{\epsilon}} g_\eta(0,\theta)d\theta - \frac{\delta}{\epsilon^2}\int_0^\infty e^{-\frac{\theta}{\epsilon}} g(0,\theta)d\theta + \frac{\delta^2}{\epsilon^2}\int_0^\infty\int_0^\infty e^{-\delta\eta-\frac{\theta}{\epsilon}} g(\eta,\theta)d\eta d\theta$$

$$= \delta^2 G(\delta,\epsilon) - \delta W(g(0,\theta)) - W(g_\eta(0,\theta)).$$

(3) $L_\eta W_\theta\left(\frac{\partial g(\eta,\theta)}{\partial \theta}\right) = \frac{1}{\epsilon^2}\int_0^\infty\int_0^\infty e^{-\delta\eta-\frac{\theta}{\epsilon}}\frac{\partial g(\eta,\theta)}{\partial\theta}\ d\eta d\theta = \frac{1}{\epsilon^2}\int_0^\infty e^{-\delta\eta}\int_0^\infty e^{-\frac{\theta}{\epsilon}}\frac{\partial g(\eta,\theta)}{\partial\theta}d\theta d\eta.$

By integrating by parts, we get

$$L_\eta W_\theta\left(\frac{\partial g(\eta,\theta)}{\partial \theta}\right) = \frac{1}{\epsilon^2}\int_0^\infty e^{-\delta\eta}\left(-g(\eta,0) + \frac{1}{\epsilon}\int_0^\infty e^{-\frac{\theta}{\epsilon}} g(\eta,\theta)d\theta\right)d\eta$$

$$= -\frac{1}{\epsilon^2}\int_0^\infty e^{-\delta\eta} g(\eta,0)d\eta + \frac{1}{\epsilon^3}\int_0^\infty\int_0^\infty e^{-\delta\eta-\frac{\theta}{\epsilon}} g(\eta,\theta)\ d\theta d\eta$$

$$= \frac{1}{\epsilon}G(\delta,\epsilon) - \frac{1}{\epsilon^2}L(g(\eta,0)).$$

(4) $L_\eta W_\theta\left(\frac{\partial^2 g(\eta,\theta)}{\partial \theta^2}\right) = \frac{1}{\epsilon^2}\int_0^\infty\int_0^\infty e^{-\delta\eta-\frac{\theta}{\epsilon}}\frac{\partial^2 g(\eta,\theta)}{\partial\theta^2}\ d\eta d\theta = \frac{1}{\epsilon^2}\int_0^\infty e^{-\delta\eta}\int_0^\infty e^{-\frac{\theta}{\epsilon}}\frac{\partial^2 g(\eta,\theta)}{\partial\theta^2}d\theta d\eta.$

By integrating by parts, we get

$$L_\eta W_\theta\left(\frac{\partial^2 g(\eta,\theta)}{\partial \theta^2}\right) = \frac{1}{\epsilon^2}\int_0^\infty e^{-\delta\eta}\left(-g_\theta(\eta,0) - \frac{1}{\epsilon}g(\eta,0) + \frac{1}{\epsilon^2}\int_0^\infty e^{-\frac{\theta}{\epsilon}} g(\eta,\theta)d\theta\right)d\eta$$

$$= -\frac{1}{\epsilon^2}\int_0^\infty e^{-\delta\eta} g_\theta(\eta,0)d\eta - \frac{1}{\epsilon^3}\int_0^\infty e^{-\delta\eta} g(\eta,0)d\eta + \frac{1}{\epsilon^4}\int_0^\infty\int_0^\infty e^{-\delta\eta-\frac{\theta}{\epsilon}} g(\eta,\theta)d\theta d\eta$$

$$L_\eta W_\theta\left(\frac{\partial^2 g(\eta,\theta)}{\partial \theta^2}\right) = \frac{1}{\epsilon^2}G(\delta,\epsilon) - \frac{1}{\epsilon^3}L(g(\eta,0)) - \frac{1}{\epsilon^2}L(g_\theta(\eta,0)).$$

(5) $L_\eta W_\theta\left(\frac{\partial^2 g(\eta,\theta)}{\partial \eta\partial\theta}\right) = \frac{1}{\epsilon^2}\int_0^\infty\int_0^\infty e^{-\delta\eta-\frac{\theta}{\epsilon}}\frac{\partial^2 g(\eta,\theta)}{\partial\eta\partial\theta}d\eta d\theta = \frac{1}{\epsilon^2}\int_0^\infty e^{-\frac{\theta}{\epsilon}}\int_0^\infty e^{-\delta\eta}\frac{\partial^2 g(\eta,\theta)}{\partial\eta\partial\theta}d\eta d\theta$

By integrating by parts, we get

$$L_\eta W_\theta\left(\frac{\partial^2 g(\eta,\theta)}{\partial \eta\partial\theta}\right) = \frac{1}{\epsilon^2}\int_0^\infty e^{-\frac{\theta}{\epsilon}}\left(-g_\theta(0,\theta) + \delta\int_0^\infty e^{-\delta\eta} g_\theta(\eta,\theta)\ d\eta\right)d\theta$$

$$= -\frac{1}{\epsilon^2}\int_0^\infty e^{-\frac{\theta}{\epsilon}} g_\theta(0,\theta)d\theta + \frac{\delta}{\epsilon^2}\int_0^\infty\int_0^\infty e^{-\delta\eta-\frac{\theta}{\epsilon}} g_\theta(\eta,\theta)d\eta d\theta$$

$$= -W(g_\theta(0,\theta)) + \delta L_\eta W_\theta\left(g_\theta(\eta,\theta)\right)$$

Using Equations 9 and 13 we get

$$= \frac{\delta}{\epsilon}G(\delta,\epsilon) - \frac{\delta}{\epsilon^2}L(g(\eta,0)) - \frac{1}{\epsilon}W(g(0,\theta)) + \frac{1}{\epsilon^2}g(0,0).$$

3.4. Convolution Theorem of Double Laplace-Sawi transform

Let $H(\eta,\theta)$ represent the Heaviside unit step function, which is defined as follows:

$$H(\eta-u,\theta-v)=\begin{cases}1, & \eta>u \text{ and } \theta>v\\ 0, & \text{otherwise}\end{cases}$$

Then we have the following lemma

Lemma 1. *Let $g(\eta,\theta)$ be a continuous function on $(0,\infty)\times(0,\infty)$ and $H(\eta,\theta)$ be the Heaviside unit step function. Then $L_\eta W_\theta(g(\eta-u,\theta-v)H(\eta-u,\theta-v))=e^{-\delta u-\frac{v}{\epsilon}}L_\eta W_\theta(g(\eta,\theta)$.*

Proof. We have

$$L_\eta W_\theta(g(\eta-u,\theta-v)H(\eta-u,\theta-v)) \tag{16}$$

$$=\frac{1}{\epsilon^2}\int_0^\infty\int_0^\infty e^{-\delta\eta-\frac{\theta}{\epsilon}}g(\eta-u,\theta-v)H(\eta-u,\theta-v)d\eta d\theta$$

$$=\frac{1}{\epsilon^2}\int_u^\infty\int_v^\infty e^{-\delta\eta-\frac{\theta}{\epsilon}}g(\eta-u,\theta-v)d\eta d\theta.$$

Now, by making the substitution $z=\eta-u$ and $w=\theta-v$, equation 16 becomes:

$$\begin{aligned}L_\eta W_\theta(g(\eta-u,\theta-v)H(\eta-u,\theta-v)) &= \frac{1}{\epsilon^2}\int_0^\infty\int_0^\infty e^{-\delta(z+u)-\frac{(w+v)}{\epsilon}}g(z,w)dzdw\\ &= e^{-\delta u-\frac{v}{\epsilon}}L_\eta W_\theta(g(\eta,\theta)).\end{aligned}$$

Definition 4. *Let $g(\eta,\theta)$ and $k(\eta,\theta)$ be continuous functions. We define the convolution in the DLSWT as*

$$(g**k)(\eta,\theta)=\int_0^\eta\int_0^\theta g(\eta-u,\theta-v)k(u,v)dudv.$$

In the following theorem, we compute DLSWT of the convolution of two functions

Theorem 2. *Let $G(\delta,\epsilon)=L_\eta W_\theta(g(\eta,\theta))$ and $K(\delta,\epsilon)=L_\eta W_\theta(k(\eta,\theta))$. Then*

$$L_\eta W_\theta((g**k)(\eta,\theta))=\epsilon^2 G(\delta,\epsilon)K(\delta,\epsilon).$$

Proof.

$L_\eta W_\theta((g{*}{*}k)(\eta,\theta))$

$$\begin{aligned} &= \frac{1}{\epsilon^2}\int_0^\infty\int_0^\infty e^{-\delta\eta-\frac{\theta}{\epsilon}}(g**k)(\eta,\theta)d\eta d\theta \\ &= \frac{1}{\epsilon^2}\int_0^\infty\int_0^\infty e^{-\delta\eta-\frac{\theta}{\epsilon}}\left(\int_0^\eta\int_0^\theta g(\eta-u,\theta-v)k(u,v)dudv\right)d\eta d\theta. \end{aligned} \tag{17}$$

Using the Heaviside unit step function, We can write equation 17 as

$L_\eta W_\theta((g{*}{*}g)(\eta,\theta))$

$$\begin{aligned} &= \frac{1}{\epsilon^2}\int_0^\infty\int_0^\infty e^{-\delta\eta-\frac{\theta}{\epsilon}}\left(\int_0^\infty\int_0^\infty g(\eta-u,\theta-v)H(\eta-u,\theta-v)k(u,v))dudv\right)d\eta d\theta \\ &= \int_0^\infty\int_0^\infty k(u,v)\left(\frac{1}{\epsilon^2}\int_0^\infty\int_0^\infty e^{-\delta\eta-\frac{\theta}{\epsilon}}g(\eta-u,\theta-v)H(\eta-u,\theta-v)d\eta d\theta\right)dudv. \end{aligned}$$

So by Lemma 1, We have

$$\begin{aligned} L_\eta W_\theta((g**k)(\eta,\theta)) &= G(\delta,\epsilon)\int_0^\infty\int_0^\infty k(u,v)e^{-\delta u-\frac{v}{\epsilon}}dudv \\ &= \epsilon^2 G(\delta,\epsilon)K(\delta,\epsilon). \end{aligned}$$

In Table 1, we have the DAHT of some basic functions.

Table 1: Table of DAHT

$g(\eta,\theta)$	$L_\eta W_\theta(g(\eta,\theta))$
1	$\frac{1}{\delta\epsilon}$, $\mathrm{Re}(\delta)>0$
$\eta^u\theta^v$	$\frac{\epsilon^{v-1}}{\delta^{u+1}}\Gamma(u+1)\Gamma(v+1)$, $\mathrm{Re}(\delta)>0$ and $\mathrm{Re}(u)>-1$
$e^{u\eta+v\theta}$	$\frac{1}{\epsilon(\delta-u)(1-v\epsilon)}$, $\mathrm{Re}(\delta)>\mathrm{Re}(u)$
$e^{i(u\eta+v\theta)}$	$\frac{i}{\epsilon(\delta-iu)(i+v\epsilon)}$, $\mathrm{Im}(u)+\mathrm{Re}(\delta)>0$
$\sin(u\eta+v\theta)$	$\frac{u+\delta\epsilon v}{\epsilon(\delta^2+u^2)(1+v^2\epsilon^2)}$, $\lvert\mathrm{Im}(u)\rvert<\mathrm{Re}(\delta)$
$\cos(u\eta+v\theta)$	$\frac{\delta-\epsilon uv}{\epsilon(\delta^2+u^2)(1+v^2\epsilon^2)}$, $\lvert\mathrm{Im}(u)\rvert<\mathrm{Re}(\delta)$
$\sinh(u\eta+v\theta)$	$\frac{u+\delta\epsilon v}{\epsilon(\delta^2-u^2)(1-v^2\epsilon^2)}$, $\mathrm{Re}(\delta)>\mathrm{Re}(u)$ and $\mathrm{Re}(\delta)+\mathrm{Re}(u)>0$
$\cosh(u\eta+v\theta)$	$\frac{\delta+\epsilon uv}{\epsilon(\delta^2-u^2)(1-v^2\epsilon^2)}$, $\mathrm{Re}(\delta)>\mathrm{Re}(u)$ and $\mathrm{Re}(\delta)+\mathrm{Re}(u)>0$
$p(\eta)q(\theta)$	$L(p(\eta))W(q(\theta))$
$g(\eta-u,\theta-v)H(\eta-u,\theta-v)$	$e^{-\delta u-\frac{v}{\epsilon}}L_\eta W_\theta(g(\eta,\theta)$
$(g**k)(\eta,\theta)$	$\epsilon^2 L_\eta W_\theta(g(\eta,\theta))L_\eta W_\theta(k(\eta,\theta))$
$J_0\left(c\sqrt{\eta\theta}\right)$	$\frac{4}{\epsilon(4\delta+c^2\epsilon)}$, $\mathrm{Re}\left(\delta+\frac{c^2\epsilon}{4}\right)>0$

4. Applications

In this section, we use the DLSWT for solving PDEs and Integro PDEs

4.1. Double Laplace-Sawi transform for solving partial differential equations

Consider the PDE of the form

$$A_1 g_{\eta\eta} + A_2 g_{\eta\theta} + A_3 g_{\theta\theta} + A_4 g_\eta + A_5 g_\theta + A_6 g\left(\eta,\theta\right) = k\left(\eta,\theta\right), \tag{18}$$

With ICs

$g(\eta,0) = p_1\left(\eta\right)$, $g_\theta(\eta,0) = p_2\left(\eta\right)$,

and BCs

$g\left(0,\theta\right) = q_1\left(\theta\right)$, $g_\eta\left(0,\theta\right) = q_2\left(\theta\right)$,

and assuming $g\left(0,0\right) = \Phi$.

Given that $g\left(\eta,\theta\right)$ is the unknown function, $k\left(\eta,\theta\right)$ is the source term, and $A_1, A_2, ..., A_6$ and Φ are constants, we aim to apply the DLSWT to Equation 18.

To achieve this, we first apply the single Laplace transform to the ICs and the single Sawi transform to the BCs.

$L\left(p_1\left(\eta\right)\right) = P_1(\eta)$, $L\left(p_2\left(\eta\right)\right) = P_2(\eta)$, $W\left(q_1\left(\theta\right)\right) = Q_1(\theta)$ and $W\left(q_2\left(\theta\right)\right) = Q_2(\theta)$.

By applying the DLSWT to Equation (18), we have

$$\begin{aligned} A_1 L_\eta W_\theta\left(g_{\eta\eta}\right) + A_2 L_\eta W_\theta\left(g_{\eta\theta}\right) + A_3 L_\eta W_\theta\left(g_{\theta\theta}\right) + A_4 L_\eta W_\theta\left(g_\eta\right) \\ + A_5 L_\eta W_\theta\left(g_\theta\right) + A_6 L_\eta W_\theta\left(g\left(\eta,\theta\right)\right) = L_\eta W_\theta\left(k\left(\eta,\theta\right)\right). \end{aligned} \tag{19}$$

By the properties of the derivatives in Equations $(12) - (15)$, we get

$$A_1\left(\delta^2 G(\delta,\epsilon) - \delta^2 Q_1(\theta) - \delta Q_2(\theta)\right) \tag{20}$$

$$+A_2\left(\frac{\delta\epsilon}{\gamma} G(\delta,\epsilon) - \delta P_1(\eta) - \frac{\delta\epsilon}{\gamma} Q_1(\theta) + \delta\Phi\right)$$

$$+A_3\left(\frac{1}{\epsilon}G(\delta,\epsilon)-\frac{1}{\epsilon}P_1(\eta)-P_2(\eta)\right)+A_4\left(\delta G(\delta,\epsilon)-\delta Q_1(\theta)\right)$$

$$+A_5\left(\frac{1}{\epsilon}G(\delta,\epsilon)G(\delta,\epsilon)-P_1(\eta)\right)+A_6G(\delta,\epsilon)=K(\delta,\epsilon).$$

Simplify Equation 20 as follows

$$G(\delta,\epsilon)=$$

$$\frac{\left(A_1\delta^2+A_2\frac{\delta\epsilon}{\gamma}+A_4\delta\right)Q_1+A_1\delta Q_2+\left(A_2\delta+A_3\frac{1}{\epsilon}+A_5\right)P_1+A_3P_2-A_2\delta\Phi+K}{A_1\delta^2+A_2\frac{\delta\epsilon}{\gamma}+A_3\frac{1}{\epsilon}+A_4\delta+A_5\frac{1}{\epsilon}+A_6}. \quad (21)$$

Example 1. *Consider the wave equation*

$$g_{\eta\eta}-g_{\theta\theta}=0,\ \textit{where}\ \eta,\theta\geq 0,$$

With ICs
$g(\eta,0)=5\eta,\ g_\theta(\eta,0)=\cos\eta,$
and BCs
$g\left(0,\theta\right)=\sin\theta,\ g_\eta\left(0,\theta\right)=5.$

Solution 1. *By applying the single Laplace transform to the ICs and the single Sawi transform to the BCs, I get*
$P_1=\frac{5}{\delta^2},\ P_2=\frac{\delta}{1+\delta^2},\ Q_1=\frac{1}{1+\epsilon^2},\ Q_2=\frac{5}{\epsilon}$
Substitute in Equation (21) $A_1=1,\ A_3=-1,\ A_2=A_4=A_5=A_6=0$ *and the values of* $P_1,\ P_2,\ Q_1$ *and* Q_2*, we get*

$$\begin{aligned}G(\delta,\epsilon) &= \frac{\frac{\delta}{1+\epsilon^2}+\frac{5}{\epsilon}-\frac{5}{\delta^2\epsilon^3}-\frac{\delta}{\epsilon^2(1+\delta^2)}}{\delta^2-\frac{1}{\epsilon^2}} \quad (22)\\ &= \frac{\frac{5\left(\delta^2\epsilon^2-1\right)}{\delta^2\epsilon}+\frac{\delta\left(\delta^2\epsilon^2-1\right)}{(1+\delta^2)(1+\epsilon^2)}}{\delta^2\epsilon^2-1}\\ &= \frac{5}{\delta^2\epsilon}+\frac{\delta}{\left(1+\delta^2\right)\left(1+\epsilon^2\right)}.\end{aligned}$$

So,

$$g(\eta,\theta)=L_\eta^{-1}W_\theta^{-1}\left(\frac{5}{\delta^2\epsilon}+\frac{\delta}{\left(1+\delta^2\right)\left(1+\epsilon^2\right)}\right)=5\eta+\cos\eta\sin\theta.$$

Its graph is

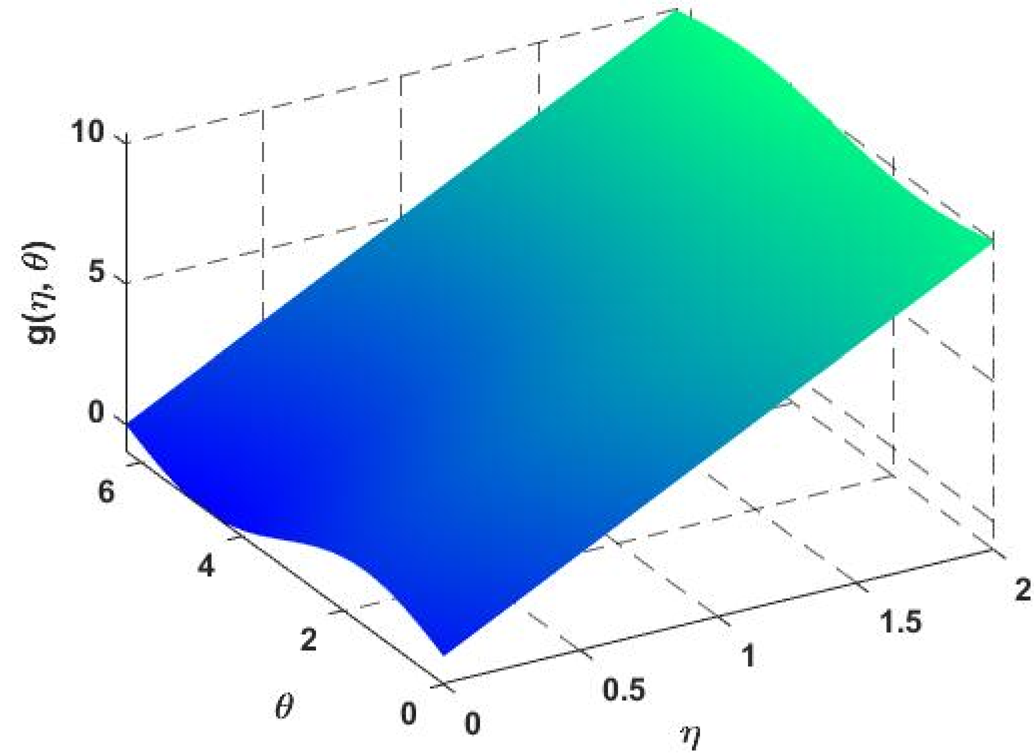

Figure 1: The solution of Example 1

Example 2. *Consider the Advection-Diffusion equation*

$$g_\delta + 2g_{\epsilon\epsilon} = 2g_\epsilon, \text{ where } \eta, \theta \geq 0,$$

With IC
$g(\eta, 0) = 2\delta - 1, g_\epsilon(\delta, 0) = 0,$
and BCs
$g(0, \theta) = \epsilon - e^\epsilon.$

Solution 2. *By applying the single Laplace transform to the ICs and the single Sawi transform to the BCs, we get*
$P_1 = \frac{1}{\delta - 1}$, $P_2 = 0$, $Q_1 = \frac{1}{\epsilon(1+2\epsilon)}$
Substitute in Equation (21) $A_3 = 2$, $A_4 = 1$, $A_5 = -2$, $A_1 = A_2 = A_6 = 0$ *and the values of* P_1, P_2, Q_1 *and* Q_2, *we get*

$$G(\delta, \epsilon) = \frac{1 - \frac{1}{\epsilon(1-\epsilon)} + \left(\frac{2}{\epsilon^3} - \frac{2}{\epsilon^2}\right) \times \left(\frac{2}{\delta^2} - \frac{1}{\delta}\right)}{\frac{2}{\epsilon^2} + \delta - \frac{2}{\epsilon}}.$$

By simplifying, we get

$$G(\delta, \epsilon) = \frac{2}{\delta^2 \epsilon} - \frac{1}{\delta\epsilon(1-\epsilon)} + \frac{1}{\delta}.$$

So,

$$g(\eta,\theta) = L_\eta^{-1} W_\theta^{-1}\left(\frac{2}{\delta^2\epsilon} - \frac{1}{\delta\epsilon\,(1-\epsilon)} + \frac{1}{\delta}\right) = 2\eta - e^\theta + \theta.$$

Its graph is

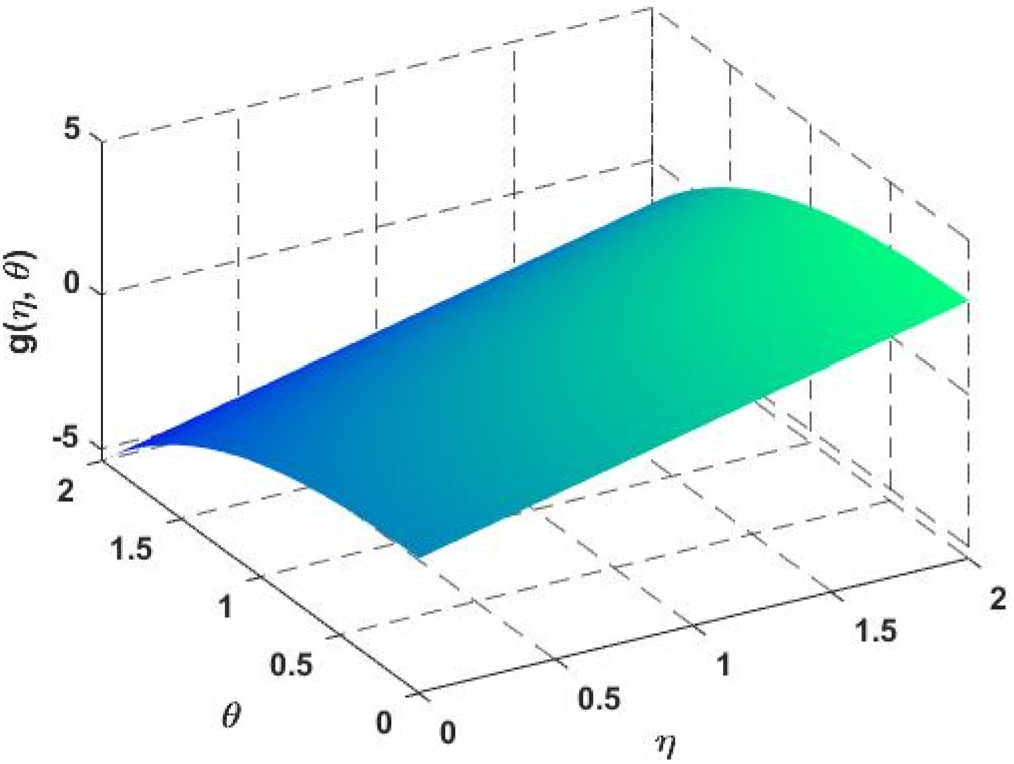

Figure 2: The solution of Example 2

Example 3. *Consider the telegraph equation*

$$2g_{\eta\eta} + g_{\theta\theta} - g_\eta = 5g(\eta,\theta),\ \textit{where}\ \eta,\theta \geq 0,$$

With ICs
$g(\eta,0) = e^\eta$, $g_\theta(\eta,0) = -2e^\eta$,
and BCs
$g\,(0,\theta) = e^{-2\theta}$, $g_\eta\,(0,\theta) = e^{-2\theta}$.

Solution 3. *By applying the single Laplace transform to the ICs and the single Sawi transform to the BCs, we get*
$P_1 = \frac{1}{\delta-1}$, $P_2 = \frac{-2}{\delta-1}$, $Q_1 = \frac{1}{\epsilon(1+2\epsilon)}$, $Q_2 = \frac{1}{\epsilon(1+2\epsilon)}$.
Substitute in Equation (21) $A_1 = 2$, $A_3 = 1$, $A_4 = -1$, $A_6 = -5$, $A_2 = A_5 = 0$ *and the*

values of P_1, P_2, Q_1 *and* Q_2, *we get*

$$\begin{aligned} G(\delta,\epsilon) &= \frac{\frac{2\delta-1}{\epsilon(1+2\epsilon)}+\frac{2}{\epsilon(1+2\epsilon)}+\frac{1}{\epsilon^3(\delta-1)}-\frac{2}{\epsilon^2(\delta-1)}}{2\delta^2-\frac{1}{\epsilon^2}-\delta-5} \qquad (23)\\ &= \frac{\frac{\epsilon^2(\delta-1)(2\delta+1)+(1+2\epsilon)-2\epsilon(1+2\epsilon)}{\epsilon^3(\delta-1)(1+2\epsilon)}}{\frac{2\delta^2\epsilon^2-\delta\epsilon^2-5\epsilon^2+1}{\epsilon^2}}. \end{aligned}$$

By simplify,

$$G(\delta,\epsilon)=\frac{1}{\epsilon\left(\delta-1\right)\left(1+2\epsilon\right)}.$$

So,

$$g(\eta,\theta)=L_\eta^{-1}W_\theta^{-1}\left(\frac{1}{\epsilon\left(\delta-1\right)\left(1+2\epsilon\right)}\right)=e^{\eta-2\theta}.$$

Its graph is

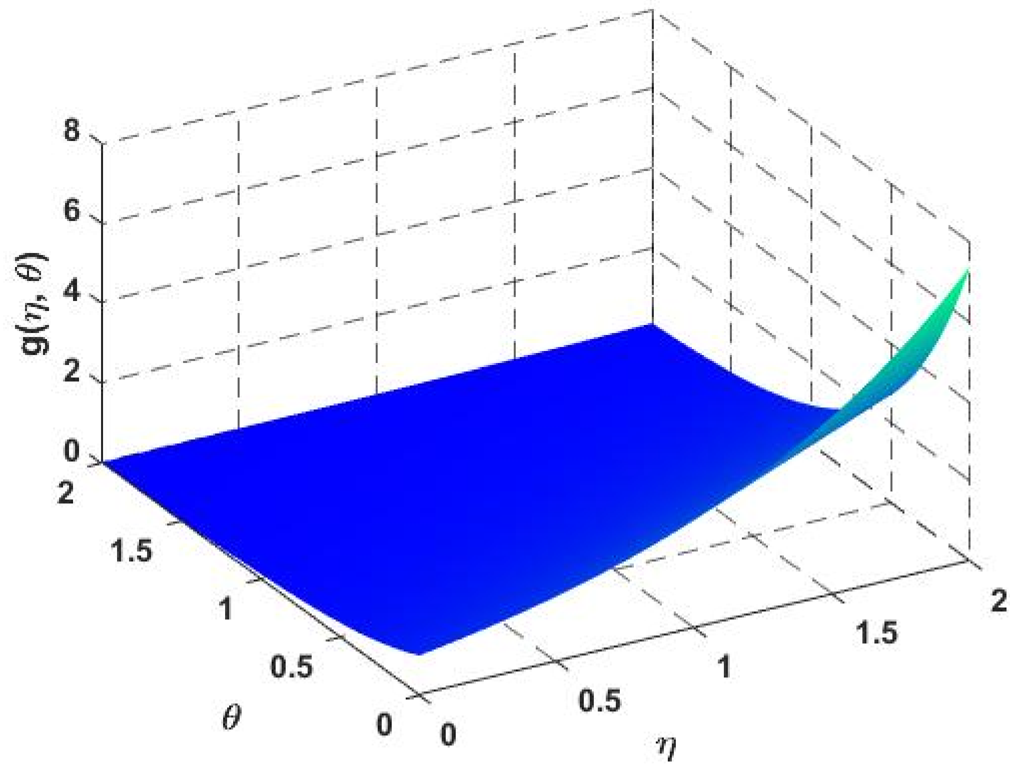

Figure 3: The solution of Example 3

4.2. Double Laplace-Sawi transform for solving Integro partial differential equations

Example 4. *Consider the equation of Volterra Integro PDE.*

$$g_\eta + g_\theta - e^{2\eta} - e^\theta - 2e^{2\eta+\theta} + 1 = 2\int_0^\eta \int_0^\theta g(u,v))dudv, \text{ where } \eta, \theta \geq 0, \tag{24}$$

With ICs
$g(\eta, 0) = e^{2\eta}$, $g(0, \theta) = e^\theta$.

Solution 4. *By applying the single Laplace transform and the single Sawi transform to the ICs, we get*
$P_1 = \frac{1}{\delta - 2}$, $Q_1 = \frac{1}{\epsilon(1-\epsilon)}$.
By Definition 4 and Theorem 2, we have

$$\int_0^\eta \int_0^\theta g(u,v))dudv = (1 * *g)(\eta, \theta). \tag{25}$$

Apply the DLSWT to Equation 25*, we get*

$$\begin{aligned} & \delta G(\delta,\epsilon) - \frac{1}{\epsilon(1-\epsilon)} + \frac{1}{\epsilon}G(\delta,\epsilon) - \frac{1}{\epsilon^2(\delta-2)} - \frac{1}{\epsilon(\delta-2)} \\ -\frac{1}{\delta\epsilon(1-\epsilon)} - \frac{2}{\epsilon(\delta-2)(1-\epsilon)} + \frac{1}{\delta\epsilon} \;=\; & \frac{2\epsilon}{\delta}G(\delta,\epsilon). \end{aligned}$$

So,

$$\begin{aligned} \frac{\delta^2\epsilon + \delta - 2\epsilon^2}{\delta\epsilon} \times G(\delta,\epsilon) \;=\; & \\ & \frac{\delta\epsilon(\delta-2) + \delta(1-\epsilon) + \delta\epsilon(1-\epsilon) + \epsilon(\delta-2) + 2\delta\epsilon - \epsilon(\delta-2)(1-\epsilon)}{\delta\epsilon^2(\delta-2)(1-\epsilon)}. \end{aligned}$$

Thus,

$$\begin{aligned} G(\delta,\epsilon) &= \frac{\delta^2\epsilon + \delta - 2\epsilon^2}{\epsilon(\delta-2)(1-\epsilon)(\delta^2\epsilon + \delta - 2\epsilon^2)} \\ &= \frac{1}{\epsilon(\delta-2)(1-\epsilon)}. \end{aligned}$$

Therefore,

$$g(\eta,\theta) = L_\eta^{-1} W_\theta^{-1}\left(\frac{1}{\epsilon(\delta-2)(1-\epsilon)}\right) = e^{2\eta+\theta}.$$

Its graph is

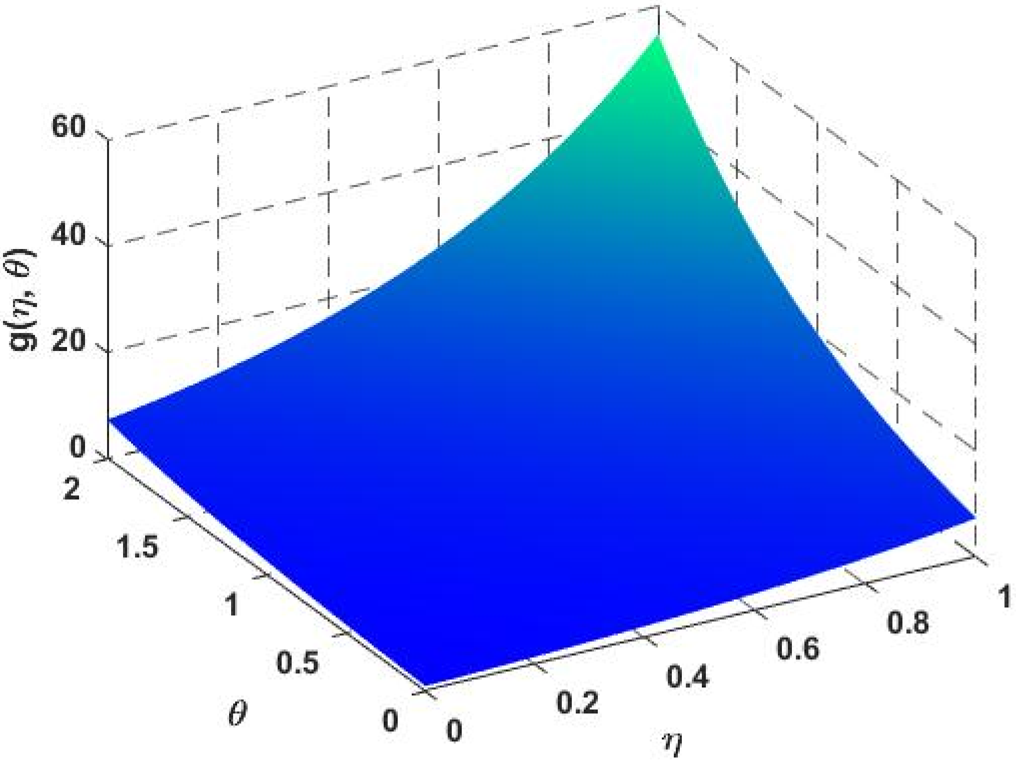

Figure 4: The solution of Example 4

Example 5. *Consider the equation of Integro PDE.*

$$g_{\eta\theta} + g_{\eta} - 2e^{\theta} + \eta^2 e^{\theta} - \eta^2 = 2\int_0^{\eta}\int_0^{\theta} g(u,v))dudv, \text{ where } \eta, \theta \geq 0, \tag{26}$$

With ICs
$g(\eta, 0) = \eta$, $g(0, \theta) = 0$.

Solution 5. *By applying the single Laplace transform and the single Sawi transform to the ICs, we get*
$P_1 = \frac{1}{\delta^2}$, $Q_1 = 0$.
Apply the DLSWT to Equation 26*, we get*

$$\frac{\delta}{\epsilon}G(\delta,\epsilon) - \frac{1}{\delta\epsilon^2} + \delta G(\delta,\epsilon) - \frac{2}{\delta\epsilon\,(1-\epsilon)}$$

$$+\frac{2}{\delta^3\epsilon\left(1-\epsilon\right)}-\frac{2}{\delta^3\epsilon}-\frac{1}{\epsilon\left(\delta-2\right)} = \frac{2\epsilon}{\delta}G(\delta,\epsilon).$$

So,

$$\frac{\delta^2\epsilon+\delta^2-2\epsilon^2}{\delta\epsilon}\times G(\delta,\epsilon) = \frac{\delta^2\left(1-\epsilon\right)+2\delta^2\epsilon-2\epsilon+2\epsilon\left(1-\epsilon\right)}{\delta^3\epsilon\left(1-\epsilon\right)}.$$

Thus,

$$\begin{aligned} G(\delta,\epsilon) &= \frac{\delta^2\epsilon+\delta^2-2\epsilon^2}{\delta^2\epsilon\left(1-\epsilon\right)\left(\delta^2\epsilon+\delta^2-2\epsilon^2\right)} \\ &= \frac{1}{\delta^2\epsilon\left(1-\epsilon\right)}. \end{aligned}$$

Therefore,

$$g(\eta,\theta) = L_\eta^{-1}W_\theta^{-1}\left(\frac{1}{\delta^2\epsilon\left(1-\epsilon\right)}\right) = \eta e^\theta.$$

Its graph is

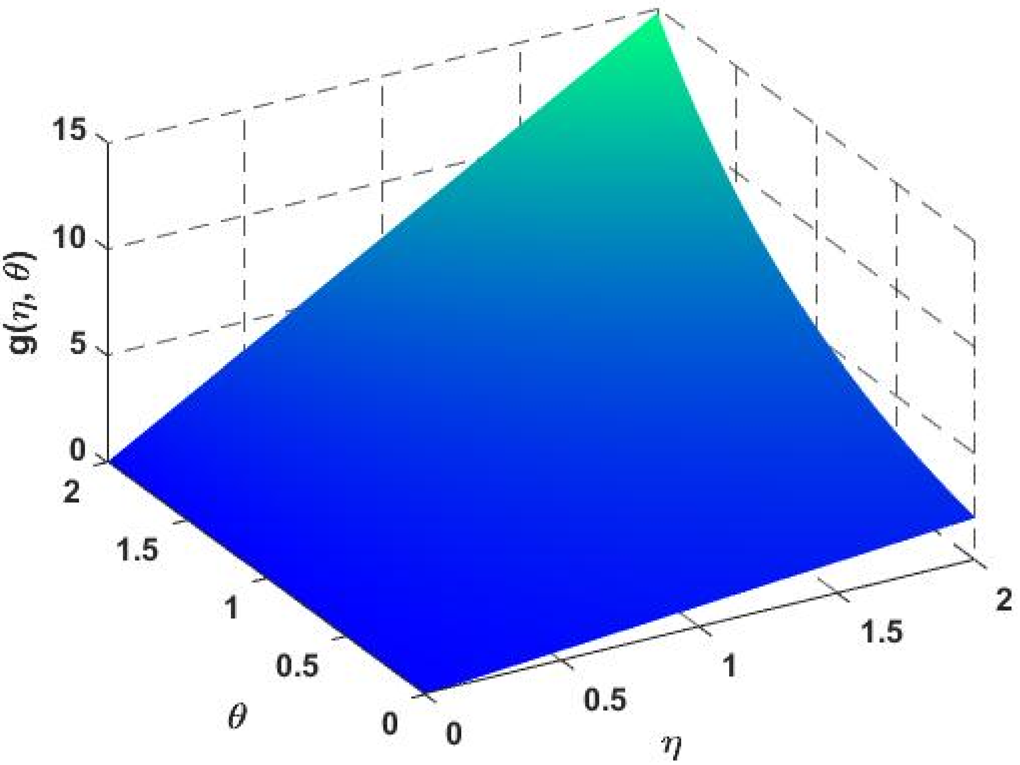

Figure 5: The solution of Example 5

5. Conclusion

In this paper, we introduce the Double Laplace-Sawi Transform (DLSWT) and we have delved deeply into the foundational properties of the proposed hybrid double transform, rigorously characterizing the necessary conditions for its existence. Through this exploration, we have demonstrated the transformative power of these properties in the realms of convolution theory and derivative operations. By establishing the theoretical framework and validating its applicability. Our discussion is realistic in that where appropriate we specify earlier numerical procedures that benefitted from our previous research while highlighting the key advantages of the DLSWT in problem solving. We believe that the future of the DLSWT is profound in the area of fractional and conformable PDEs and Integro PDEs with coefficients that vary. More related results on fractional and conformable PDEs and Integro PDEs can be found in [11–14].

Author contribution statement

The authors listed have significantly contributed to the development and the writing of this article.

Data availability statement
No data was used for the research described in the article.

Conflict of interest
The authors declare that they have no conflict of interest.

References

[1] M Mahgoub and M Mohand. The new integral transform "Sawi Transform". Advances in Theoretical and Applied Mathematics, 14(1): 81-87, 2019.

[2] M Higazy and S Aggarwal. Sawi transformation for system of ordinary differential equations with application. Ain Shams Engineering Journal, 12: 3173-3182, 2021.

[3] GK Watugala. Sumudu transform: a new integral transform to solve differential equations and control engineering problems. International Journal of Mathematical Education in Science and Technology, 24(1): 35-43, 1993.

[4] R Saadeh A Qazza and A Burqan. A new integral transform: ARA transform and its properties and applications. Symmetry, 12: 925, 2020.

[5] K Aboodh I Abdullahi and R Nuruddeen. On the Aboodh transform connections with some famous integral transforms. Int. J.Eng. Inform. Syst., 1: 143-151, 2017.

[6] A Aghili and B Parsa Moghaddam. Certain theorems on two dimensional Laplace transform and non-homogeneous parabolic partial differential equations. Surv. Math. Its Appl., 6: 165-174, 2011.

[7] AK Sedeeg ZI Mahamoud and R Saadeh. Using Double Integral Transform (Laplace-ARA Transform) in Solving Partial Differential Equations. Symmetry, 14(11): 2418, 2022.

[8] M Hunaiber and A Al-Aati. On Double Laplace-Shehu Transform and its Properties with Applications. Turkish Journal of Mathematics and Computer Science, 15(2): 218-226, 2023.

[9] S Khan A Ullah M De la Sen and S Ahmad. Double Sawi Transform: Theory and Applications to Boundary Values Problems. Symmetry, 15(4): 921, 2023.

[10] B Abughazaleh MA Amleh A Al-Natoor and R Saadeh. Double Mellin-ARA Transform. Springer Proceedings in Mathematics and Statistics, 466: 383-394, 2024.

[11] HK Jassim and H Kadhim. Fractional Sumudu decomposition method for solving PDEs of fractional order. J. Appl. Comput. Mech., 7, 302-311, 2021.

[12] H Jafari HK Jassim and C & Ãœnlãœ. Laplace decomposition method for solving the two-dimensional diffusion problem in fractal heat transfer. FRACTALS (fractals), 32(4): 1-6, 2024.

[13] MA Amleh B Abughazaleh and A Al-Natoor. Conformable fractional Lomax probability distribution. J. Math. Comput. Sci., 12, Article-ID 130, 2022.

[14] A Qazza A Burqan R Saadeh and R Khalil. Applications on double ARA–Sumudu transform in solving fractional partial differential equations. Symmetry, 14(9): 1817, 2022.