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Left and Right Regular Elements of Some Subsemigroups of the Linear Transformations Semigroups With Invariant Subspace

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Abstract. In this paper, we investigate the left regularity, right regularity, and complete regularity of elements in subsemigroups of the semigroups of linear transformations with invariant subspaces. We provide necessary and sufficient conditions for these subsemigroups to be left regular, right regular, and completely regular. Specifically, we examine semigroups of linear transformations with restricted range, invariant subspaces, and fixed subspaces. The results offer a comprehensive characterization of regular elements within these algebraic structures and extend existing work in this field. Our findings have potential applications in algebraic theory, particularly in the study of transformation semigroups and their subsemigroups.

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1. Introduction and Preliminaries

An element a of a semigroup S is called *left regular* if $a = xa^2$ for some $x \in S$, *right regular* if $a = a^2x$ for some $x \in S$, and *completely regular* if $a = axa$ and $ax = xa$ for some $x \in S$. For a semigroup S, let $LReg(S)$, $RReg(S)$ and $CReg(S)$ denote the set of all left regular elements, right regular elements, and completely regular elements of S, respectively. It is important to note that every completely regular element is also left and right regular. Additionally, Petrich and Reilly [10, Proposition 2.1.3] proved that an element a of a semigroup S is completely regular if and only if a is both a left and right regular element of S. A semigroup S is called left (right, completely) regular if all its elements are left (right, completely) regular, that is, $LReg(S) = S$ ($RReg(S) = S$, $CReg(S) = S$). The characterizations of left regularity, right regularity, and complete regularity for semigroups have been studied in detail, as seen in [2, 6–9, 12, 15].

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Let V be a vector space over a field, and let $L(V)$ be the semigroup (under composition) consisting of all linear operators on V. It is well known that $L(V)$ is a regular semigroup [4, page 63]. Moreover, Tantong [15] characterized left, right and completely regularity for elements of $L(V)$ and provided necessary and sufficient conditions for $L(V)$ to be left regular, right regular and completely regular in terms of the dimension of V. Let W be a fixed subspace of V. In 2008, Sullivan [14] defined a subsemigroup of the linear transformation semigroup as:

$$L(V, W) = \{\alpha \in L(V) : V\alpha \subseteq W\}.$$

This semigroup is called the *semigroup of linear transformations with restricted range.* The author examined Green's relations and ideals for the semigroup $L(V, W)$. Later, in 2019, Sangkhanan and Sanwong [11] proved certain isomorphism theorems and calculated the ranks of these semigroups for any proper subspace W of a finite dimensional vector space V over a finite field. Additionally, Sullivan [14] showed that:

$$Q = \{\alpha \in L(V, W) : V\alpha \subseteq W\alpha\}$$

is the largest regular subsemigroup of $L(V, W)$. In 2015, Sangkhanan and Sommanee [13] described all the maximal regular subsemigroups of Q when W is a finite dimensional subspace of V over a finite field. Furthermore, they computed the rank and idempotent rank of Q where W is an n-dimensional subspace of an m-dimensional vector space V over a finite field.

Let W be a subspace of a vector space V. The *semigroup of linear transformations with invariant subspace* are defined by:

$$S(V, W) = \{\alpha \in L(V) : W\alpha \subseteq W\}.$$

In 2012, Huisheng [5] described the relations $\mathcal{L}^*$ and $\mathcal{R}^*$ on $S(V, W)$. In the same year, Honyam and Sanwong [3] presented the relations of Green and ideals of the semigroup and proved that it is never isomorphic to $T(U)$ for any vector space U when W is a non-zero proper subspace of V. In 2019, Chaiya [1] characterized the natural partial order on $S(V, W)$ and determined the compatibility of their elements and found all maximal and minimal elements. Furthermore, she presented necessary and sufficient conditions for $S(V, W)$ to be factorizable, unit-regular, and directly finite.

For a fixed subspace W of a vector space V, let

$$Fix(V, W) = \{\alpha \in L(V) : w\alpha = w \text{ for all } w \in W\}.$$

Then, $Fix(V, W)$ is a subsemigroup of $S(V, W)$ and we call it the *semigroup of linear transformations with fixed subspaces.* In 2018, Chaiya et. al. [16] discussed the Green's relations, regularity, and ideals of $Fix(V, W)$, and characterized when $Fix(V, W)$ is factorisable, unit-regular, and directly finite.

The objective of this paper is to characterize the left, right, and complete regularity of elements within the semigroups $L(V, W)$, $S(V, W)$, Q, and $Fix(V, W)$. We also present a method to construct an element β in these semigroups such that it is left and right regular. Furthermore, we establish necessary and sufficient conditions for the semigroups $L(V, W)$, $S(V, W)$, Q, and $Fix(V, W)$ to be left regular, right regular, and completely regular.

2. Semigroups of Linear Transformations with Invariant Subspaces

In this section, we assume that V is a vector space over a field $\mathbb{F}$, and W is a subspace of V. Let S represent one of the semigroups $S(V,W)$, $L(V,W)$, and Q. We first characterize right regularity for elements of S.

Theorem 1. *Let $\alpha \in S$. Then, $\alpha \in RReg(S)$ if and only if $\alpha|_{V\alpha}$ is a one-to-one transformations on $V\alpha$.*

Proof. Assume that α is a right regular element of S. Thus, there exists an element β of S such that $\alpha = \alpha^2\beta$. Let $v_1, v_2 \in V\alpha$, and suppose $v_1\alpha = v_2\alpha$. Then, there exist $v_1', v_2' \in V$ such that $v_1'\alpha = v_1$ and $v_2'\alpha = v_2$. Therefore,

$$v_1 = v_1'\alpha = v_1'\alpha^2\beta = v_1\alpha\beta = v_2\alpha\beta = v_2'\alpha^2\beta = v_2'\alpha = v_2.$$

Hence, $\alpha|_{V\alpha}$ is one-to-one.

Conversely, assume that $\alpha|_{V\alpha}$ is one-to-one. We will construct $\beta \in S$ such that $\alpha = \alpha^2\beta$. Let B be a basis for $V\alpha$, and let $B' = \{v\alpha : v \in B\}$. To show that B' is a linearly independent subset of V, let $a_1, a_2, \ldots, a_n \in \mathbb{F}$, and $v_1, v_2, \ldots, v_n \in B$ be such that

$$a_1(v_1\alpha) + a_2(v_2\alpha) + \ldots + a_n(v_n\alpha) = 0.$$

Thus, $(a_1v_1+a_2v_2+\ldots+a_nv_n)\alpha = 0$. Since $a_1v_1+a_2v_2+\ldots+a_nv_n \in V\alpha$ and $\alpha|_{V\alpha}$ is one-to-one, it follows that $a_1v_1+a_2v_2+\ldots+a_nv_n = 0$. Since $v_1, v_2, \ldots, v_n$ are linearly independent, we conclude that $a_i = 0$ for all $i = 1, 2, \ldots, n$. Hence, B' is linearly independent. We then construct a basis B'' for V such that $B' \subseteq B''$.

For each $u \in B'$, there exists a unique $u' \in B$ such that $u'\alpha = u$ by assumption. Define $\beta : B'' \to V$ by

$$v\beta = \begin{cases} v' & \text{if } v \in B', \\ 0 & \text{otherwise.} \end{cases}$$

Thus, β is well-defined and can be extended to a linear transformation on V. Let $v \in V$. Since $B' \subseteq B''$, there exist positive integers k and n such that $v_1, v_2, \ldots, v_k \in B'$ and $v_{k+1}, v_{k+2}, \ldots, v_n \in B'' \setminus B'$ and $a_1, a_2, \ldots, a_n \in \mathbb{F}$ with $v = a_1v_1 + a_2v_2 + \ldots + a_nv_n$. This implies that

$$\begin{aligned} v\beta &= (a_1v_1 + a_2v_2 + \ldots + a_kv_k + a_{k+1}v_{k+1} + a_{k+2}v_{k+2} + \ldots + a_nv_n)\beta \\ &= a_1(v_1\beta) + a_2(v_2\beta) + \ldots + a_k(v_k\beta) + a_{k+1}(v_{k+1}\beta) + a_{k+2}(v_{k+2}\beta) \\ &\quad + \ldots + a_n(v_n\beta) \\ &= a_1v_1' + a_2v_2' + \ldots + a_kv_k' + a_{k+1}(0) + a_{k+2}(0) + \ldots + a_n(0) \\ &= a_1v_1' + a_2v_2' + \ldots + a_kv_k'. \end{aligned}$$

Since $v_1', v_2', \ldots, v_k'$ are all elements in a basis B of $V\alpha$, we conclude that $a_1v_1' + a_2v_2' + \ldots + a_kv_k' \in V\alpha$. Therefore, $\beta \in L(V,W)$. This implies that $W\beta \subseteq V\beta \subseteq W$, and so

$\beta \in S(V, W)$. Moreover, we will now show that $\beta \in Q$. Let $w = a_1v_1 + a_2v_2 + \ldots + a_kv_k$. Then, $w \in W$, and

$$\begin{aligned} w\beta &= (a_1v_1 + a_2v_2 + \ldots + a_kv_k)\beta \\ &= a_1(v_1\beta) + a_2(v_2\beta) + \ldots + a_k(v_k\beta) \\ &= a_1v'_1 + a_2v'_2 + \ldots + a_kv'_k \\ &= v\beta. \end{aligned}$$

We can conclude that $\beta \in Q$. This shows that $\beta \in S$.

Finally, we show that $\alpha = \alpha^2\beta$. Let $v \in V$. Since $v\alpha \in V\alpha$, we can express $v\alpha = a_1u'_1 + a_2u'_2 + \ldots + a_nu'_n$ where $u'_1, u'_2, \ldots, u'_n \in B$ with $u'_i\alpha = u_i$ for all $i \in \{1, 2, \ldots, n\}$, and $a_1, a_2, \ldots, a_n \in \mathbb{F}$. Therefore,

$$\begin{aligned} v\alpha^2\beta &= (a_1u'_1 + a_2u'_2 + \ldots + a_nu'_n)\alpha\beta \\ &= (a_1u'_1\alpha + a_2u'_2\alpha + \ldots + a_nu'_n\alpha)\beta \\ &= (a_1u_1 + a_2u_2 + \ldots + a_nu_n)\beta \\ &= a_1u'_1 + a_2u'_2 + \ldots + a_nu'_n \\ &= v\alpha. \end{aligned}$$

Hence, α is right regular, as required.

The following next theorem characterizes when S is a right regular semigroup.

Theorem 2. *The following statements are equivalent:*

(i) *S is a right regular semigroup.*

(ii) *$RReg(S)$ is a subsemigroup of S.*

(iii) $\dim(W) \leq 1$.

Proof. $(i) \Rightarrow (ii)$ This is clear by definition.

$(ii) \Rightarrow (iii)$ We will prove the contrapositive. Assume that $\dim(W) \geq 2$. Then, there exists a basis B_W of W such that $|B_W| \geq 2$. Now, let B be a basis for V such that $B_W \subseteq B$. Let a and b be distinct elements of B_W. Define two mappings α and β from B into V as follows:

$$x\alpha = \begin{cases} a & \text{if } x = b, \\ b & \text{if } x = a, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$x\beta = \begin{cases} b & \text{if } x = b, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to verify that α and β are well-defined and can be extended to linear transformations on V. From their definitions, $\alpha, \beta \in S$, and we will show that both are right regular elements of S, but their product $\alpha\beta$ is not right regular. To check that α is right regular, we use Theorem 1. Suppose $u \in V\alpha$ and $u\alpha = 0$. From the definition of α, we know that

$$V\alpha = \{k_1 a + k_2 b : k_1, k_2 \in \mathbb{F}\}.$$

Then, $u = k_1 a + k_2 b$ where $k_1, k_2 \in \mathbb{F}$. Thus $0 = u\alpha = k_1 b + k_2 a$. Since a, b are linearly independent, we must have $k_1 = k_2 = 0$, implying that $u = 0$. Thus $\ker(\alpha|_{V\alpha}) = \{0\}$, and hence, $\alpha|_{V\alpha}$ is one-to-one. By Theorem 1, α is right regular. Similarly, we can show that β is right regular by applying Theorem 1 to $\beta|_{V\beta}$.

Finally, we will show that $\alpha\beta$ is not right regular. Note that $0\alpha\beta = 0 = a\beta = b\alpha\beta$. Since $0, b \in V\alpha\beta$ and $0 \neq b$, it follows that $\alpha\beta|_{V\alpha\beta}$ is not one-to-one. From Theorem 1, $\alpha\beta$ is not right regular. Hence, $RReg(S)$ is not a subsemigroup of S.

$(iii) \Rightarrow (i)$ Suppose that $\dim(W) \leq 1$. Let $\alpha \in S$. If α is the zero transformation, then it is trivially right regular. Suppose that α is not the zero transformation. Then, $\dim(V\alpha) \geq 1$. Since $\dim(W) \leq 1$, by the Dimension Theorem, $\dim(\ker\alpha) = 0$, which implies that $\ker(\alpha|_{V\alpha}) = \{0\}$. Therefore, $\alpha|_{V\alpha}$ must be one-to-one, and by Theorem 1, α is right regular. Consequently, $S = RReg(S)$, meaning that S is a right regular semigroup.

Next, we give a characterization for left regular elements in $S \setminus Q$ and Q, respectively. Let S^* be either $L(V, W)$ or $S(V, W)$.

Theorem 3. *Let $\alpha \in S^*$. The following statements are equivalent:*

- (i) $\alpha \in LReg(S^*)$.
- (ii) $\alpha|_{W\alpha}$ *is an onto transformation on* $V\alpha$.
- (iii) $V\alpha \subseteq W\alpha^2$.
- (iv) *For every basis B for V, we have* $B\alpha \subseteq W\alpha^2$.
- (v) *There exists a basis B for V such that* $B\alpha \subseteq W\alpha^2$.

Proof. $(i) \Rightarrow (ii)$ Assume that α is left regular in S^*. Then, $\alpha = \beta\alpha^2$ for some $\beta \in S^*$. For any $y \in V\alpha$, then there exists $x \in V$ such that $y = x\alpha$. Thus, $x\beta \in W$, and $y = x\alpha = x\beta\alpha^2 = (x\beta\alpha)\alpha$, which proves that $\alpha|_{W\alpha}$ is onto.

$(ii) \Rightarrow (iii)$ If $\alpha|_{W\alpha} : W\alpha \to V\alpha$ is onto, then

$$W\alpha^2 = (W\alpha)\alpha = (W\alpha)\alpha|_{W\alpha} = V\alpha,$$

which proves that $V\alpha \subseteq W\alpha^2$.

$(iii) \Rightarrow (iv)$ and $(iv) \Rightarrow (v)$ are clear by definition.

$(v) \Rightarrow (i)$ Suppose there is a basis B for V such that $B\alpha \subseteq W\alpha^2$. For each $v \in B$, we choose and fix $v' \in W$ such that $v\alpha = v'\alpha^2$. Define $\beta : B \to V$ by

$$v\beta = v' \quad \text{for all } v \in B.$$

By the uniqueness condition, β is well-defined and can be extended to a linear transformation on V. We will now show that $\beta \in S^*$, and that $\alpha = \beta\alpha^2$, proving that α is left regular.

For any $v \in V$, there exist $v_1, v_2, \ldots, v_n \in B$ and $a_1, a_2, \ldots, a_n \in \mathbb{F}$ such that $v = a_1v_1 + a_2v_2 + \ldots + a_nv_n$. Thus,

$$\begin{aligned} v\beta &= (a_1v_1 + a_2v_2 + \ldots + a_nv_n)\beta \\ &= a_1(v_1\beta) + a_2(v_2\beta) + \ldots + a_n(v_n\beta) \\ &= a_1v'_1 + a_2v'_2 + \ldots + a_nv'_n \in W. \end{aligned}$$

Hence $\beta \in L(V, W)$ or $\beta \in S(V, W)$, depending on the semigroup. Finally, for each $v \in B$, we have

$$v\beta\alpha^2 = v\beta\alpha\alpha = v'\alpha\alpha = v\alpha,$$

showing that $\beta\alpha^2 = \alpha$, as required.

The following result follows from Theorems 1 and 3.

Corollary 1. *Let $\alpha \in S^*$. The following statements are equivalent:*

(i) $\alpha \in CReg(S^*)$.

(ii) $\alpha|_{V\alpha} : V\alpha \to V\alpha$ *is one-to-one, and* $\alpha|_{W\alpha} : W\alpha \to V\alpha$ *is onto.*

(iii) *For every $v \in V$, there exists a unique $v' \in W\alpha$ such that $v\alpha = v'\alpha$.*

(iv) *For every basis B of V, and for every $v \in B$, there exists a unique $v' \in W\alpha$ such that $v\alpha = v'\alpha$.*

(v) *There exists a basis B of V, and for every $v \in B$, there exists a unique $v' \in W\alpha$ such that $v\alpha = v'\alpha$.*

Next, we give a necessary and sufficient condition when the semigroup S^* to be left regular.

Theorem 4. *The following statements are equivalent:*

(i) *S^* is a left regular semigroup.*

(ii) *$LReg(S^*)$ is a subsemigroup of S^*.*

(iii) $\dim(W) \leq 1$.

Proof. (i) $\Rightarrow$ (ii) This is clear by definition.

(ii) $\Rightarrow$ (iii) We will prove by contrapositive. Assume that $\dim(W) \geq 2$. Then, there exists a basis B_W of W such that $|B_W| \geq 2$. Let B be a basis for V such that $B_W \subseteq B$.

Let a and b be distinct elements of B_W. Define two transformations α and β on B as follows:

$$x\alpha = \begin{cases} a & \text{if } x = b, \\ b & \text{if } x = a, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$x\beta = \begin{cases} b & \text{if } x = b, \\ 0 & \text{otherwise.} \end{cases}$$

Both α and β are well-defined and can be extended to the linear transformations on V. Since $a, b, 0 \in W$, it follows that $\alpha, \beta \in S^*$. From the previous theorem (Theorem 3), we know that α and β are left regular because $B\alpha \subseteq W\alpha^2$ and $B\beta \subseteq W\beta^2$. However, their product $\alpha\beta$ is not left regular.

To see this, note that:

$$a\alpha = b, \quad b\alpha = a, \quad a\beta = 0, \quad b\beta = b.$$

Thus, $a\alpha\beta = b$ and $b\alpha\beta = 0 = 0\alpha\beta$, showing that $\alpha\beta$ is not one-to-one on its image. Therefore, $\alpha\beta$ is not left regular. This implies that $LReg(S^*)$ is not a subsemigroup of S^*.

$(iii) \Rightarrow (i)$ Suppose that $\dim(W) \leq 1$. Let $\alpha \in S^*$. If α is the zero transformation, it is trivially left regular. Now, assume that α is not the zero transformation. Then, $\dim(V\alpha) \neq 0$. Since $\dim(W) \leq 1$, $V\alpha \subseteq W$, and thus α is surjective on W. By assumption, we have $\dim(V\alpha) = 1$. Hence, $LReg(S^*) = S^*$, meaning that S^* is a left regular semigroup.

Corollary 2. *The following statements are equivalent:*

(i) *S^* is a left regular semigroup.*

(ii) *$CReg(S^*)$ is a subsemigroup of S^*.*

(iii) $\dim(W) \leq 1$.

The following theorem gives a necessary and sufficient condition for an element of Q to be left regular.

Theorem 5. *Let $\alpha \in Q$. The following statements are equivalent:*

(i) *$\alpha \in LReg(Q)$.*

(ii) *$\alpha|_{W\alpha}$ is an onto transformation on $V\alpha$.*

(iii) *$V\alpha \subseteq W\alpha^2$.*

Proof. $(i) \Rightarrow (ii)$ and $(ii) \Rightarrow (iii)$ These follow directly from Theorem 3.

$(iii) \Rightarrow (i)$ Assume that $V\alpha \subseteq W\alpha^2$. Let B_W be a basis for W, and let B be a basis for V such that $B_W \subseteq B$. For each $v \in B \setminus B_W$, since $V\alpha \subseteq W\alpha$, we choose and fix

$v' \in W$ such that $v\alpha = v'\alpha$. For each $w \in W$, we let $w' = w$. For each $w' \in W$ and by assumption, we choose and fix $w'' \in W$ such that $w'\alpha = w''\alpha^2$. Define $\beta : B \to V$ by

$$v\beta = v'' \quad \text{for all } v \in B$$

It is easy to verify that β is well-defined, and it can be extended to a linear transformation on V. By the method of constructing β, it is easy to see that $\beta \in L(V, W)$. We now show that $V\beta \subseteq W\beta$. Let $v \in V$. Then, $v = a_1v_1 + a_2v_2 + \ldots + a_nv_n$ where $v_1, v_2, \ldots, v_k \in B$, and $a_1, a_2, \ldots, a_k \in \mathbb{F}$. This implies that

$$\begin{aligned} v\beta &= (a_1v_1 + a_2v_2 + \ldots + a_kv_k)\beta \\ &= a_1(v_1\beta) + a_2(v_2\beta) + \ldots + a_k(v_k\beta) \\ &= a_1v_1'' + a_2v_2'' + \ldots + a_kv_k'' \\ &= a_1v_1'\beta + a_2v_2'\beta + \ldots + a_kv_k'\beta \\ &= (a_1v_1' + a_2v_2' + \ldots + a_kv_k')\beta. \end{aligned}$$

Since W is a subspace of V and $v_1', v_2', \ldots, v_k' \in W$, we have that $a_1v_1' + a_2v_2' + \ldots + a_kv_k' \in W$. This implies that $\beta \in Q$. Moreover, we will now show that $\alpha = \beta\alpha^2$. For each $v \in B$, we have

$$v\beta\alpha^2 = v''\alpha\alpha = v'\alpha = v\alpha.$$

Hence, $\beta\alpha^2 = \alpha$, proving that α is left regular.

The following corollary follows directly from Theorems 1 and 5.

Corollary 3. *Let $\alpha \in Q$. The following statements are equivalent:*

(i) $\alpha \in CReg(Q)$.

(ii) $\alpha|_{V\alpha} : V\alpha \to V\alpha$ *is one-to-one, and* $\alpha|_{W\alpha} : W\alpha \to V\alpha$ *is onto.*

(iii) *For every $w \in W$, there exists a unique $w' \in W\alpha$ such that $w\alpha = w'\alpha$.*

Lastly, we characterize the left regular semigroup structure of the semigroup Q.

Theorem 6. *The following statements are equivalent:*

(i) *Q is a left regular semigroup.*

(ii) *$CReg(Q)$ is a subsemigroup of S^*.*

(iii) $\dim(W) \leq 1$.

Proof. The proof can be established in the same way as Theorem 4.

The following corollary is an immediate consequence of Theorems 2 and 6.

Corollary 4. *The following statements are equivalent:*

(i) *Q is a left regular semigroup.*

(ii) *$CReg(Q)$ is a subsemigroup of S^*.*

(iii) $\dim(W) \leq 1$.

3. Semigroups of Linear Transformations with Fixed Subspaces

In this section, we now explore the necessary and sufficient conditions for an element of $Fix(V, W)$ to be right regular.

Theorem 7. *Let $\alpha \in Fix(V, W)$. Then, $\alpha \in RReg(Fix(V, W))$ if and only if $\alpha|_{V\alpha}$ is a one-to-one transformation on $V\alpha$.*

Proof. The necessity follows directly from Theorem 1. To prove the sufficiency, we suppose that $\alpha|_{V\alpha}$ is one-to-one. Since $\alpha \in Fix(V, W)$, we get that $W \subseteq V\alpha$. Let B_W be a basis for W and B be a basis for $V\alpha$ such that $B_W \subseteq B$. Define $B' = \{u\alpha \mid u \in B\}$. Then, $B_W \subseteq B'$. By the same reasoning used in Theorem 1, B' is linearly independent, and there exists a basis B'' for V such that $B' \subseteq B''$.

For each $u \in B'$, there exists a unique $u' \in B$ such that $u'\alpha = u$ by assumption. Define $\beta : B'' \to V$ by

$$v\beta = \begin{cases} v' & \text{if } v \in B', \\ 0 & \text{otherwise.} \end{cases}$$

By the uniqueness, β is well-defined. Hence, β can be extended to a linear transformation on V. Let $w \in W$. Then, there are $w_1, w_2, \ldots, w_k \in B_W$, and $a_1, a_2, \ldots, a_k \in \mathbb{F}$ such that $w = a_1w_1 + a_2w_2 + \ldots + a_kw_k$. Since $B_W \subseteq B$, and by the definition of α, we observe that $w_i'\alpha = w_i = w_i\alpha$. It follows from assumption that $w_i = w_i'$. This implies that

$$\begin{aligned} w\beta &= (a_1w_1 + a_2w_2 + \ldots + a_kw_k)\beta \\ &= a_1(w_1\beta) + a_2(w_2\beta) + \ldots + a_k(w_k\beta) \\ &= a_1w_1 + a_2w_2 + \ldots + a_kw_k \\ &= w. \end{aligned}$$

It implies that β belong to $Fix(V, W)$. We now demonstrate that $\alpha = \alpha^2\beta$. If $v \in V$, then we have $v\alpha \in V\alpha$, and we can write $v\alpha = a_1u_1' + a_2u_2' + \ldots + a_nu_n'$ where $u_1', u_2', \ldots, u_n' \in B$ with $u_i'\alpha = u_i$ for all $i \in \{1, 2, \ldots, n\}$ and $a_1, a_2, \ldots, a_n \in \mathbb{F}$. Therefore,

$$\begin{aligned} v\alpha^2\beta &= (a_1u_1' + a_2u_2' + \ldots + a_nu_n')\alpha\beta \\ &= (a_1u_1'\alpha + a_2u_2'\alpha + \ldots + a_nu_n'\alpha)\beta \\ &= (a_1u_1 + a_2u_2 + \ldots + a_nu_n)\beta \\ &= a_1u_1' + a_2u_2' + \ldots + a_nu_n' \\ &= v\alpha. \end{aligned}$$

Hence, α is right regular. This completes the proof of theorem.

Next, we give a necessary and sufficient condition when the semigroup $Fix(V, W)$ to be left regular. Tantong [15], prove that $L(V)$ is a right regular semigroup if and only if $\dim(V) \leq 1$. We will use this result in the proof of the next theorem.

Theorem 8. *The following statements are equivalent:*

(*i*) *$Fix(V,W)$ is a right regular semigroup.*

(*ii*) *$RReg(Fix(V,W))$ is a subsemigroup of $Fix(V,W)$.*

(*iii*) $V = W$ *or* $\dim(V) \leq 1$.

Proof. $(i) \Rightarrow (ii)$ This is clear by definition.

$(ii) \Rightarrow (iii)$ We will prove the contrapositive. Assume that $W \neq V$ and $\dim(V) > 1$. If $\dim(W) = 0$, then $Fix(V,W) = L(V)$ and hence $RReg(Fix(V,W))$ is not a subsemigroup of $Fix(V,W)$. Suppose that $\dim(W) > 0$. Let a be a non-zero element of W. Then, there is a basis B_W for W such that $a \in B_W$. Let B be a basis for V with $B_W \subseteq B$. By assumption, let $b \in B \setminus B_W$. Define transformations α and β as follows:

$$x\alpha = \begin{cases} x & \text{if } x \in B_W, \\ b & \text{if } x = b, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$x\beta = \begin{cases} x & \text{if } x \in B_W, \\ a & \text{if } x = b, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to verify that α and β are well-defined and can be extended to the linear transformations on V. By the definitions of α and β, we see that both $\alpha|_W$ and $\beta|_W$ are the identity transformation on W, so that $\alpha, \beta \in Fix(V,W)$. Next, we check that α and β are right regular elements of $Fix(V,W)$ by using Theorem 7. Let $u \in V\alpha$ be such that $u\alpha = 0$. By the definition of α, we know that

$$V\alpha = \{w + kb : w \in W \text{ and } k \in \mathbb{F}\}.$$

Then, $u = w + kb$ where $k \in \mathbb{F}$ and $w \in W$. Thus

$$\begin{aligned} 0 &= u\alpha \\ &= (w + kb)\alpha \\ &= w\alpha + b\alpha \\ &= w + kb \\ &= u \end{aligned}$$

Hence, $\ker(\alpha|_{V\alpha}) = \{0\}$ and so $\alpha|_{V\alpha}$ is one-to-one. It follows from Theorem 7 that α is right regular of $Fix(V,W)$. Similarly, we can show that $\beta|_{V\beta}$ is one-to-one. It follows from Theorem 7 that β is right regular.

Finally, we will show that $\alpha\beta$ is not right regular. Note that $b\alpha\beta = b\beta = a = a\beta = a\alpha\beta$. Since $a, b \in V\alpha\beta$ and $a \neq b$, it follows that $\alpha\beta|_{V\alpha\beta}$ is not one-to-one. From Theorem 7, $\alpha\beta$ is not right regular. Hence, $RReg(Fix(V,W))$ is not a subsemigroup of $Fix(V,W)$.

$(iii) \Rightarrow (i)$ Assume that $W = V$ or $\dim(V) \leq 1$. If $W = V$, then $Fix(V,W)$ consists only of the identity transformation, and hence, $Fix(V,W)$ is trivially a right regular semigroup. If $\dim(V) \leq 1$, then $Fix(V,W) = L(V)$ and hence, it is a right regular semigroup.

Secondly, we investigate the condition under which an element of $Fix(V,W)$ is left regular.

Theorem 9. *Let $\alpha \in Fix(V,W)$. The following statements are equivalent:*

(i) *$\alpha \in LReg(Fix(V,W))$.*

(ii) *$\alpha|_{V\alpha}$ is an onto transformation on $V\alpha$.*

(iii) *$V\alpha \subseteq V\alpha^2$.*

(iv) *For every basis B for V, we have $B\alpha \subseteq V\alpha^2$.*

(v) *There exist two basses B for V and B_W for W such that $B_W \subseteq B$ and $B\alpha \subseteq V\alpha^2$.*

Proof. $(i) \Rightarrow (ii)$ Assume that α is left regular of $Fix(V,W)$. Then, $\alpha = \beta\alpha^2$ for some $\beta \in Fix(V,W)$. If $y \in V\alpha$, then $y = x\alpha$ for some $x \in V$. Thus, $y = x\alpha = x\beta\alpha^2 = (x\beta\alpha)\alpha$. This prove that $\alpha|_{V\alpha} : V\alpha \to V\alpha$ is onto.

$(ii) \Rightarrow (iii)$ Assume that $\alpha|_{V\alpha} : V\alpha \to V\alpha$ is onto. Then,

$$V\alpha^2 = (V\alpha)\alpha = (V\alpha)\alpha|_{V\alpha} = V\alpha,$$

which proves that $V\alpha \subseteq V\alpha^2$.

$(iii) \Rightarrow (iv)$ This implication is clear by definition. If $V\alpha \subseteq V\alpha^2$, then for every basis B of V, we will have $B\alpha \subseteq V\alpha^2$

$(iv) \Rightarrow (v)$ Suppose that the condition (iv) holds. Let B_W be a basis for W. Then, there exists a basis B for V such that $B_W \subseteq B$. By the assumption, we have $B\alpha \subseteq V\alpha^2$, which implies that condition (v) holds.

$(v) \Rightarrow (i)$ Suppose that there is a basis B for V and a basis B_W for W such that

$$B_W \subseteq B \text{ and } B\alpha \subseteq V\alpha^2.$$

For each $u \in B \setminus B_W$, we choose and fix $u' \in V$ such that $u\alpha = u'\alpha^2$. For each $u \in B_W$, we set $u' = u$. Then, $u\alpha\alpha = u\alpha = u'\alpha$. Define $\beta : B \to V$ by

$$v\beta = v' \quad \text{for all } v \in B.$$

It is easy to verify that β is well-defined and can be extended to a linear transformation on V. We will now show that $\beta \in Fix(V,W)$, and that $\alpha = \beta\alpha^2$. For each $w \in W$, we have $w = a_1w_1 + a_2w_2 + \ldots + a_nw_n$ for some $w_1, w_2, \ldots, w_n \in B_W$, and scalars $a_1, a_2, \ldots, a_n \in \mathbb{F}$. Therefore,

$$\begin{aligned} w\beta &= (a_1w_1 + a_2w_2 + \ldots + a_nw_n)\beta \\ &= a_1(w_1\beta) + a_2(w_2\beta) + \ldots + a_n(w_n\beta) \end{aligned}$$

$$\begin{aligned} &= a_1w_1' + a_2w_2' + \ldots + a_nw_n' \\ &= a_1w_1 + a_2w_2 + \ldots + a_nw_n \\ &= w. \end{aligned}$$

Thus, $\alpha \in Fix(V, W)$. Finally, for every $v \in B$, we have

$$v\beta\alpha^2 = v\beta\alpha\alpha = v'\alpha\alpha = v\alpha.$$

Therefore $\beta\alpha^2 = \alpha$, which shows that α is left regular, as required.

Next corollary is result from Theorems 7 and 9.

Corollary 5. *Let $\alpha \in Fix(V, W)$. The following statements are equivalent:*

(i) $\alpha \in CReg(Fix(V, W))$.

(ii) $\alpha|_{V\alpha} : V\alpha \to V\alpha$ *is a bijective tranformation.*

(iii) *For every $v \in V$, there exists a unique $v' \in V\alpha$ such that $v\alpha = v'\alpha$.*

(iv) *For every basis B for V, and for every $v \in B$, there exists a unique $v' \in V\alpha$ such that $v\alpha = v'\alpha$.*

(v) *There exist two basses B for V, and B_W for W such that $B_W \subseteq B$, and for every $v \in B$, there exists a unique $v' \in V\alpha$ such that $v\alpha = v'\alpha$.*

Finally, we show that for the semigroup $Fix(V, W)$ to be left regular whenever V is a finite dimensional vector space.

Theorem 10. *Let V be a finite dimensional vector space. The following statements are equivalent:*

(i) $Fix(V, W)$ *is a left regular semigroup.*

(ii) $LReg(Fix(V, W))$ *is a subsemigroup of* $Fix(V, W)$.

(iii) $V = W$ *or* $\dim(V) = \dim(W) + 1$.

Proof. (i) $\Rightarrow$ (ii) This is clear by definition.

(ii) $\Rightarrow$ (iii) Suppose that $W \neq V$ and $\dim(V) \neq \dim(W) + 1$. Then, $\dim(V) - \dim(W) > 1$. Let B_W be a basis for W and extend it to a basis B for V. Let a and b be distinct elements of $B \setminus B_W$. Define two mappings α and β from B into V as follows:

$$x\alpha = \begin{cases} a & \text{if } x = b, \\ b & \text{if } x = a, \\ x & \text{otherwise,} \end{cases}$$

and

$$x\beta = \begin{cases} x & \text{if } x \in B_W \cup \{a\}, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, both α and β are well-defined and can be extended to the linear transformation on V. Let $w \in W$. Then,

$$w = k_1w_1 + k_2w_2 + \ldots + k_nw_n$$

for some $k_1, k_2, \ldots, k_n \in \mathbb{F}$ and $w_1, w_2, \ldots, w_n \in B$. Therefore,

$$\begin{aligned} w\alpha &= (k_1w_1 + k_2w_2 + \ldots + k_nw_n)\alpha \\ &= (k_1w_1)\alpha + (k_2w_2) + \ldots + (k_nw_n) \\ &= k_1w_1\alpha + k_2w_2\alpha + \ldots + k_nw_n\alpha \\ &= k_1w_1 + k_2w_2 + \ldots + k_nw_n \\ &= w. \end{aligned}$$

Hence, $\alpha \in Fix(V, W)$. By the symmetry, we can show that $\beta \in Fix(V, W)$. It follows from $B \subseteq V\alpha$ and Theorem 9 that α is left regular. Let $v \in B$. If $v \in B_W \cup \{a\}$, then $v\beta = v = v\beta^2$. Otherwise, $v\beta = 0 = (v\beta)\beta$. This implies that β is a left regular element of $Fix(V, W)$ by Theorem 9.

Finally, we will show that $\beta\alpha$ is not left regular. Note that $a\beta\alpha = a\alpha = b$. Thus, $b \in V\beta\alpha$. Claim that $b \neq v\beta\alpha$ for all $v \in V\beta\alpha$. Suppose that $v\beta\alpha = b$ for some $v \in V\beta\alpha$. Then, $v = v'\beta\alpha$ for some $v' \in V$. Thus, $v' = a_1v_1+a_2v_2+\ldots+a_nv_n$ where $v_1, v_2, \ldots, v_n \in B$ and $a_1, a_2, \ldots, a_n \in \mathbb{F}$. If $v' \in W$, then $v = v'\beta\alpha = v'$, and so $b = v\beta\alpha = v \in W$, which is a contradiction. Hence, $v' \notin W$. Since $b \neq 0$ and $v\beta\alpha = b$, we get that $v \neq 0$, so that $0 \neq v'\beta\alpha = a_1v_1\beta\alpha + a_2v_2\beta\alpha + \ldots + a_nv_n\beta\alpha$. By the definition of β, there exists $k \in \{1, 2, \ldots, n\}$ such that $v_k\beta = a$ and so $v = a_kv_k\beta\alpha = a_ka\alpha = a_kb$. Therefore, $b = v\beta\alpha = a_kb\beta\alpha = a_k(0\alpha) = 0$, which is a contradiction. So we have the claim. Hence, $\beta\alpha|_{V\beta\alpha}$ is not onto. From Theorem 9, $\beta\alpha$ is not left regular. Hence, $LReg(Fix(V, W))$ is not a subsemigroup of $Fix(V, W)$.

(iii) $\Rightarrow$ *(i)* Assume that $W = V$ or $\dim(W) + 1 = \dim(V)$. If $W = V$, then $Fix(V, W)$ contains only the identity transformation and is trivially left regular. Suppose that $\dim(W) + 1 = \dim(V)$. Let $\alpha \in Fix(V, W)$. Then, by Theorem 9, we get that $\alpha|_{V\alpha} : V\alpha \to V\alpha$ is onto. By the Dimension Theorem and assumptions, we have

$$\dim(W) + 1 = \dim(\ker \alpha) + \dim(V\alpha).$$

Since $V\alpha$ contains the subspace W, clarifying this will help in explaining the two cases.

Case 1. $\dim(W) = \dim(V\alpha)$ and $\dim(\ker \alpha) = 1$. Then, $V\alpha = W$. Since $\alpha|_W$ is the identity transformation on W, by Theorem 9, we have $\alpha \in LReg(Fix(V, W))$.

Case 2. $\dim(W) + 1 = \dim(V\alpha)$ and $\dim(\ker \alpha) = 0$. Then, $V\alpha = V$. Therefore, α is onto. Hence, by Theorem 9, we have $\alpha \in LReg(Fix(V, W))$.
Hence, $Fix(V, W)$ is a left regular semigroup, as required.

The next corollary is a result from Theorems 8 and 10.

Corollary 6. *The following statements are equivalent:*

(*i*) *$Fix(V, W)$ is a completely regular semigroup.*

(*ii*) *$CReg(Fix(V, W))$ is a subsemigroup of $Fix(V, W)$.*

(*iii*) *$V = W$ or $\dim(V) \leq 1$.*

4. Conclusions

In this paper, we have investigated the left, right, and complete regularity of elements in the semigroups of linear transformations with invariant subspaces. Specifically, we focused on the subsemigroups $L(V, W)$, $S(V, W)$, Q, and $Fix(V, W)$. We provided necessary and sufficient conditions for these semigroups to exhibit left, right, and complete regularity. Our results contribute to the characterization of regular elements within these mathematical structures.

One of the key findings is the relationship between the subspaces V and W in determining the regularity of the transformations. For instance, we have shown that an element of $L(V, W)$ is right regular if and only if its restriction to its image is a one-to-one transformation. Similarly, left regularity is characterized by onto mappings in certain conditions. These results align with and extend previous work, offering a broader understanding of regular semigroups and their elements.

The significance of this study lies in its application to algebraic structures such as semigroups, transformation semigroups, and subspaces. Specifically, Cayley's Theorem states that every semigroup can be embedded in a full transformation semigroup. Our characterization of subsemigroups of transformation semigroups provides deeper insights into their structure, allowing for further exploration of their properties and potential applications.

In future research, this framework can be expanded to explore other algebraic structures, including transformation semigroups that preserve different types of equivalence relations. Additionally, studying the interaction between these regular elements and more complex algebraic operations could lead to new theoretical developments and applications in fields such as linear algebra, automata theory, and representation theory.

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