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Characterizing 2-Distance Certified Hop Dominating Sets Using Co-Certified Pointwise Non-domination Concept

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Abstract. Let G be a graph. Then $C \subseteq V(G)$ is called a 2-distance certified hop dominating if $\forall\ x \in V(G) \backslash C$, there exists $y \in C$ such that $d_G(x, y) = 2$ and $\forall\ a \in C$, $|N_G^2(a) \backslash C| = 0$ or $|N_G^2(a) \backslash C| \geq 2$. The 2-distance certified hop domination number of G, denoted by $\gamma_{2ch}(G)$, is the minimum cardinality among all 2-distance certified hop dominating sets of G. In this study, the researchers give some properties of this new concept on some graphs, and present some its connections with others parameters. Also, the researchers introduce co-certified pointwise non-domination (co-certified pnd) to characterize the 2-distance certified hop dominating sets in the join of two graphs. They obtain some simplified formulas of the said parameter on this graph using this newly defined concept and some characterizations formulated.

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1. Introduction

Domination in graph theory is a fundamental concept that explores how subsets of vertices can control or influence the entire graph. A dominating set for a graph G is defined as a subset of vertices D such that every vertex in G is either included in D or is adjacent to at least one vertex in D. This concept is crucial in various applications such as in network design, resource allocation, social network analysis, and in infrastructure networks.

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The study of domination encompasses various parameters, including the domination number, which is the minimum size of a dominating set, and different types or variants of domination such as certified domination [5], Grundy hop domination variants [8, 9], convex hop domination [7], double domination [6], and many more. Each variation provides unique insights and tools for addressing specific problems within graphs. Some interesting studies related to domination, certified domination, and hop domination can be found in [1–4, 10, 11].

In this paper, new parameter called 2-distance certified hop domination in a graph is introduced and investigated. The researchers believe that his parameter and its results would give additional insights to researchers in the field and would lead to another interesting topics or network application in the future.

2. Terminology and Notation

Let $G = (V(G), E(G))$ be a simple and undirected graph. The *distance* $d_G(u, v)$ in G of two vertices u, v is the length of a shortest *u-v path* in G. The greatest distance between any two vertices in G, denoted by $diam(G)$, is called the *diameter* of G.

Two vertices x, y of G are *adjacent*, or *neighbors*, if xy is an edge of G. The *open neighborhood* of x in G is the set $N_G(x) = \{y \in V(G) : xy \in E(G)\}$. The *closed neighborhood* of x in G is the set $N_G[x] = N_G(x) \cup \{x\}$. If $X \subseteq V(G)$, the *open neighborhood* of X in G is the set $N_G(X) = \bigcup_{x \in X} N_G(x)$. The *closed neighborhood* of X in G is the set $N_G[X] = N_G(X) \cup X$. A subset S of $V(G)$ is called a *dominating* set of G if for every $a \in V(G) \backslash S$, there exists $b \in S$ such that $d_G(a, b) = 1$, that is, S is a dominating set of G if $N_G[S] = V(G)$. The minimum cardinality among all dominating sets of G, denoted by $\gamma(G)$, is called the *domination number* of G.

A dominating set $S \subseteq V(G)$ is called a *certified dominating* set of G if every $a \in S$, a has either zero or atleast two neighbors in $V(G) \setminus S$. The minimum cardinality among all certified dominating sets of G, denoted by $\gamma_{cer}(G)$, is called the *certified domination number* of G.

Let G be a graph. Then $S \subseteq V(G)$ is called a pointwise non-dominating (pnd) set of G if for each $v \in V(G) \backslash S$, there exists $w \in S$ such that $v \notin N_G(w)$.

The minimum cardinality of a pointwise non-dominating (pnd) set of G, is called the pointwise non-domination (pnd) number of G.

A vertex a in G is a *hop neighbor* of a vertex b in G if $d_G(a, b) = 2$. The set $N_G^2(a) = \{b \in V(G) : d_G(a, b) = 2\}$ is called the *open hop neighborhood* of a. The *closed hop neighborhood* of a in G is given by $N_G^2[a] = N_G^2(a) \cup \{a\}$. The *open hop neighborhood* of $S \subseteq V(G)$ is the set $N_G^2(S) = \bigcup_{a \in S} N_G^2(a)$. The *closed hop neighborhood* of S in G is the set $N_G^2[S] = N_G^2(S) \cup S$.

A subset S of $V(G)$ is a *hop dominating* of G if for every $a \in V(G) \backslash S$, there exists $b \in S$ such that $d_G(a, b) = 2$, that is, S is a hop dominating set of G if $N_G^2[S] = V(G)$. The minimum cardinality among all hop dominating sets of G, denoted by $\gamma_h(G)$, is called the *hop domination number* of G.

Let G and H be any two graphs. The *join* of G and H, denoted by $G+H$ is the graph with vertex set $V(G+H) = V(G) \cup V(H)$ and edge set

$$E(G+H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}.$$

3. Results

4. Properties and Bounds of 2-Distance Certified Hop Domination in Some Graphs

Definition 1. Let G be a graph. Then a set $C \subseteq V(G)$ is called a 2-distance certified hop dominating if $\forall\ x \in V(G)\backslash C$, there exists $y \in C$ such that $d_G(x,y) = 2$ and $\forall\ a \in C$, $|N_G^2(a)\backslash C| = 0$ or $|N_G^2(a)\backslash C| \geq 2$. The 2-distance certified hop domination number of G, denoted by $\gamma_{2ch}(G)$, is the minimum cardinality among all 2-distace certified hop dominating sets of G.

Example 1. Below is an example of 2-distance certified hop domination.

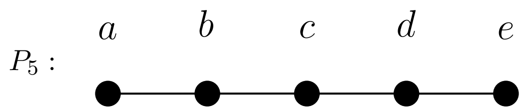


Figure 1: A path graph of P_5 with $\gamma_{2ch}(P_5) = 3$

Consider the Path graph given above. Let $C = \{b,c,d\}$. Then $N_G^2[b] = \{b,d\}, N_G^2[c] = \{a,c,e\}$ and $N_G^2[d] = \{b,d\}$. Thus, $N_G^2[C] = V(G)$, and so C is a hop dominating set of G. Observe that, vertices b and d have zero hop neighbor in $V(G)\backslash C$ and vertex c has two hop neighbors a,e in $V(G)\backslash C$. Therefore, C is a 2-distance certified hop dominating set of G. Moreover, it can be verified that $\gamma_{2ch}(G) = 3$.

Theorem 1. Let G be a graph. Then

(i) $\gamma_h(G) \leq \gamma_{2ch}(G)$

(ii) $1 \leq \gamma_{2ch}(G) \leq |V(G)|$

(iii) $\gamma_{2ch}(G) = 1$ if and only if $G = K_1$

Proof. (i) Let D be a minimum 2-distance certified hop dominating set of G. Then D is a hop dominating set of G and $\gamma_{2ch}(G) = |D|$. Thus, $\gamma_h(G) \leq |D| = \gamma_{2ch}(G)$ by definition.

(ii) Since $\gamma_h(G) \geq 1$ for any graph (G), $\gamma_{2ch}(G) \geq 1$ by (i). The upperbound is clear since any 2-distance certified hop dominating set is always a subset of $V(G)$.

(iii) Suppose that $\gamma_{2ch}(G) = 1$. Then $\gamma_h(G) = 1$ by (i). It follows that $G = K_1$. Conversely, Suppose that $G = K_1$. Then by (ii) $\gamma_{2ch}(G) = 1$

Theorem 2. Let n be a positive integer. Then $S \subseteq V(K_n)$ is a 2-distance certified hop dominating set if and only if $S = V(K_n)$.

Proof. Suppose that S is 2-distance certified hop dominating of K_n. Assume that $S \neq V(K_n)$. Then there exists at least one vertex $v \in V(K_n)$ such that $v \notin S$. Since the graph is complete, it implies that every vertex is adjacent to every other vertex. Then for S to be a 2-distance certified hop dominating, v must be included in S. Since $v \notin N_G^2[S]$, a contradiction.

For the converse, suppose that $S = V(K_n)$. Then S is a 2-distance certified hop dominating set of K_n.

Corollary 1. Let m be a positive integer. Then

$$\gamma_{2ch}(K_m) = m \ for \ all \ m \geq 1.$$

5. Co-Certified Pointwise Non-Domination in Graphs

The following definiton will be used to characterize the 2-distnace certified hop dominating sets in the join of two graphs as well as to solve its 2-distance certified hop domination numbers.

Definition 2. Let G be a graph. Then $S \subseteq V(G)$ is called a co-certified pointwise non-dominating (co-certified pnd) set of G if S satisfies the following two conditions:

(i) For each $u \in S$, there exist either zero or at least two vertices $x, y \in V(G)\backslash S$ such that $x, y \notin N_G(u)$.

(ii) For each $v \in V(G)\backslash S$, there exists $w \in S$ such that $v \notin N_G(w)$.

The minimum cardinality of a co-certified pointwise non-dominating (co-certified pnd) set of G, is called the co-certified pointwise non-domination (co-certified pnd) number of G.

Remark 1. Let G be a graph. Then

(i) pnd(G) $\leq$ ccpnd(G); and

(ii) $1 \leq$ ccpnd(G) $\leq |V(G)|$.

Theorem 3. Let G be a graph. Then $ccpnd(G) \neq |V(G)|$ if and only if

$$ccpnd(G) \leq |V(G)| - 2.$$

Proof. Suppose that $ccpnd(G) = |V(G)| - 1$, say S is the minimum co-certified pnd set. Then there exists $x \in V(G)$ such that $x \notin S$. Let $y \in S$. If x and y are non-adjacent.

Then $x \notin N_G(y)$. That is, y has only one non-neighbor x in $V(G)\backslash S$, a contradiction. Therefore, the assertion follows.

The converse is clear.

Proposition 1. Let k be a positive integer. Then,

$$(i) \quad ccpnd(P_k) = \begin{cases} k, & \text{if } k = 1, 2, 3, 4 \\ 2, & \text{if } k \geq 5; \end{cases}$$

$$(ii) \quad ccpnd(C_k) = \begin{cases} k, & \text{if } k = 3, 4 \\ 2, & \text{if } k \geq 5; \end{cases}$$

$$(iii) \quad ccpnd(K_k) = k \text{ for all k} \geq 1; \text{ and}$$

$$(iv) \quad ccpnd(\bar{K_k}) = \begin{cases} 1, & \text{if } for\ all\ k \geq 3 \\ k, & \text{if } k = 1, 2. \end{cases}$$

Proof. (i) Since $pnd(P_k) = k$ for $k = 1, 2$, it follows that $ccpnd(P_k) = k$ for $k = 1, 2$ by Remark 1. For $k = 3$, let $V(P_3) = \{a_1, a_2, a_3\}$. Consider $R = \{a_1, a_2\}$. Then R is a minimum pnd set of P_3, and so by Remark 1, $ccpnd(P_3) \geq 2$. By Theorem 3, $ccpnd(P_3) = 3$. For $k = 4$, let $V(P_4) = \{a_1, a_2, a_3, a_4\}$. Consider $Q = \{a_1, a_2\}$. Then Q is a minimum pnd set of P_4. Thus, $ccpnd(P_4) \geq 2$ by Remark 1. Suppose that $ccpnd(G) = 2$, say M is a minimum co-certified pnd set of P_4. Then M is either of the following sets:

$$\{a_1, a_2\}, \{a_2, a_3\}, \{a_3, a_4\} \ or \ \{a_1, a_4\}.$$

However, either of these cases contradicts our assumption of being a co-certified pnd set of P_4. Therefore, by Theorem 3, $ccpnd(P_4) = 4$.

Next, for $k \geq 5$, let $V(P_k) = \{v_1, v_2, ..., v_k\}$. Consider $N = \{v_1, v_2\}$. Then N is a minimum pnd set of P_k. Since $k \geq 5$, both v_1 and v_2 has at least two non-neighbors in $V(P_k)\backslash N$, respectively. Therefore, N is a minimum co-certified pnd set of P_k, and so $ccpnd(P_k) = 2$ for all $k \geq 5$.

(ii) Since $pnd(C_3) = 3$, it follows that $ccpnd(C_3) = 3$ by Remark 1. For $n = 4$, let $V(C_4) = \{u_1, u_2, u_3, u_4\}$. Consider $Q = \{u_1, u_2\}$. Then Q is a minimum pnd set of C_4. By Remark 1, $ccpnd(C_4) \geq 2$. Suppose that $ccpnd(C_4) = 2$, say R is a minimum co-certified pnd set of C_4. Then R is either of the following sets:

$$\{u_1, u_2\}, \{u_2, u_3\}, \{u_3, u_4\} \ or \ \{u_4, u_1\}.$$

However, either of these cases violates the properties of a co-certified pnd set. By Theorem 3, $ccpnd(C_4) = 4$.

Now, suppose that $k \geq 5$. Let $V(C_k) = \{u_1, u_2, ..., u_k\}$. Consider $P = \{u_1, u_2\}$. Then P is a minimum pnd set of C_k. Since $k \geq 5$, u_1 and u_2 has at least two neighbors in $V(C_k)\backslash P$. Therefore, P is a minimum co-certified pnd set of C_k, and so $ccpnd(C_k) = 2$ for all $k \geq 5$.

(iii) Let $S = V(K_k) = \{a_1, a_2, ..., a_k\}$. Then S is a co-certified pnd set of K_k. Suppose that S is not a minimum co-certified pnd set of K_k. Then there exists $x \in V(K_k)$ such that $x \notin S$. However, x is adjacent to every other vertex in $V(K_k)\backslash\{x\}$, a contradiction to the fact that S is a pnd set of K_k. Therefore, $S = V(K_k)$ is a minimum co-certified pnd set of K_k, and so $ccpnd(K_k) = k \; \forall \; k \geq 1$

(iv) Clearly, $ccpnd(\bar{K_k}) = 1$. For $k = 2$, let $V(\bar{K_k}) = \{v_1, v_2\}$. Consider $R = \{v_1\}$. Then R is a minimum pnd set of $(\bar{K_2})$, and so $pnd(\bar{K_2}) = 1$. Thus, $ccpnd(\bar{K_2}) \geq 1$. Assume that $ccpnd(\bar{K_2}) = 1$. Then either $\{v_1\}$ or $\{v_2\}$ is a minimum co-certified pnd set of $\bar{K_2}$. Suppose that $M = \{v_2\}$ is a minimum co-certifed pnd set of $(\bar{K_2})$. However, v_2 has only one non-neighbor $v_1 \in V(\bar{K_2})\backslash M$, a contradiction. Similarly the assertion follows when $M = \{v_1\}$. Thus, $ccpnd(\bar{K_2}) = 2$.

Next, suppose that $k \geq 3$. Let $V(\bar{K_k}) = \{a_1, a_2, ..., a_k\}$. Consider $B = \{a_1\}$. Then B is a minimum pnd set of $(\bar{K_k})$. Since $k \geq 3$, a_1 has at least two non-neighbors in $V(\bar{K}) \setminus B$. Therefore, $ccpnd(\bar{K_k}) = 1 \; for \; all \; k \geq 3$.

6. 2-Distance Certified Hop Domination in the Join of Two Graphs

Theorem 4. Let G and H be graphs. Then $O \subseteq (G + H)$ is a 2-distance certified hop dominating set of $G + H$ if and only if $O = O_G \cup O_H$, where O_G and O_H are co-certified pnd sets in G and H, respectively.

Proof. Suppose that $O \subseteq V(G + H)$ is a 2-distance certified hop dominating set of $G + H$. Then O is a hop dominating set of $G + H$. If $O \subseteq V(G)$, then $N_G^2[O] \subseteq V(G)$, a contradiction. Hence, $O \nsubseteq V(G)$. Similarly, $O \nsubseteq V(H)$. Thus, $O = O_G \cup O_H$, where $O_G \subseteq V(G)$ and $O_H \subseteq V(H)$. Let $x \in V(G + H)\backslash O$. Assume that $x \in V(G)\backslash O_G$. Since O is a hop dominating, there exists $y \in O$ such that $d_{G+H}(x, y) = 2$. It follows that $x \notin N_G(y)$, and so O_G is a pnd set of G. Since O is a 2-distance certified set, for every $w \in O_G$, there exist either zero or at least two vertices $u, v \in V(G)\backslash O_G$ such that $d_{G+H}(w, u) = (w, v) = 2$. Hence, $u, v \notin N_G(w)$. Consequently, O_G is a co-certified pnd set of G. Similarly, O_H is a co-certified pnd set of H.

Conversely, suppose that $O = O_G \cup O_H$, where O_G and O_H are co-certified pnd sets in G and H, respectively. Let $a \in V(G + H)\backslash O$. Assume that $a \in V(G)\backslash O_G$. Since O_G is a co-certified pnd set of G, there exists $b \in O_G$ such that $a \notin N_G(b)$ and for each $q \in O_G$, there exist either zero or at least two neighbor $r, t \in V(G)\backslash O_G$ such that $r, t \notin N_G(q)$. This means that $d_{G+H}(a, b) = 2$ and q has either zero or at least two hop neighbors $r, t \in G + H$. Thus, O is a 2-distance certified hop dominating set of $G + H$. Similarly, when $a \in V(H)\backslash O_H$, then O is a 2-distance certified hop dominating set of $G + H$.

Theorem 5. Let G and H be a graphs. Then

$$\gamma_{2ch}(G + H) = ccpnd(G) + ccpnd(H).$$

Proof. Suppose that $O = O_G \cup O_H$ is a minimum 2-distance certified hop dominating set of $G + H$. Then by Theorem 4, O_G and O_H are co-certified pnd sets of G and H, respectively. Thus,

$$\gamma_{2ch}(G+H) = |O| = |O_G| + |O_H| \geq ccpnd(G) + ccpnd(H) \qquad (i)$$

On the other hand, suppose that $O = O_G \cup O_H$, where O_G and O_H are both minimum co-certified pnd sets of G and H, respectively. Then by Theorem 4, O is a 2-distance certified hop domknating set of $G+H$. Therefore,

$$ccpnd(G) + ccpnd(H) = |O_G| + |O_H| = |O| \geq \gamma_{2ch}(G+H) \qquad (ii)$$

Combining (i) and (ii), we have $\gamma_{2ch}(G+H) = ccpnd(G) + ccpnd(H)$.

The following result follows from Proposition 1 and Theorem 5.

Corollary 2. Let n be a positive integer. Then

(i) $\gamma_{2ch}(P_n + P_n) = \begin{cases} 2n, \text{ if } n = 1,2,3,4 \\ 4, \text{ if } n \geq 5; \end{cases}$

(ii) $\gamma_{2ch}(C_n + C_n) = \begin{cases} 2n, \text{ if } n = 3,4 \\ 4, \text{ if } n \geq 5; \end{cases}$

(iii) $\gamma_{2ch}(K_n + K_n) = 2n$ for all n ≥ 1;

(iv) $\gamma_{2ch}(S_n) = \begin{cases} n, \text{ if } n = 1,2 \\ 2, \text{ if } n \geq 3; \end{cases}$

(v) $\gamma_{2ch}(K_{m,n}) = \begin{cases} 4, \text{ if } m,n = 2 \\ 3, \text{ if } m = 2 \text{ and } n \geq 3 \text{ or } m \geq 3 \text{ and } n = 2 \\ 2, \text{ if } m,n \geq 3 \text{ or } m,n = 1; \end{cases}$

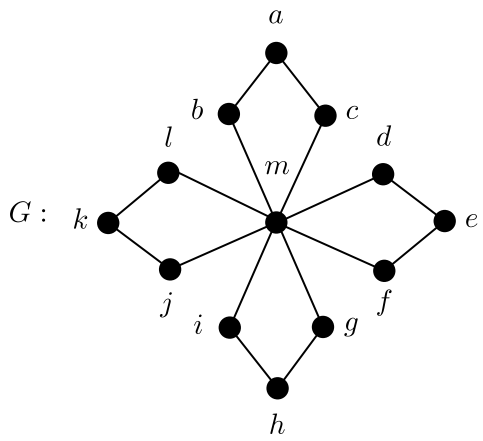
(vi) $\gamma_{2ch}(F_{1,n}) = \begin{cases} n+1, \text{ if } n = 1,2,3,4 \\ 3, \text{ if } n \geq 5; \text{ and} \end{cases}$

(vii) $\gamma_{2ch}(W_n) = \begin{cases} n+1, \text{ if } n = 3,4 \\ 3, \text{ if } n \geq 5. \end{cases}$

7. Incomparability of 2-distance certified hop domination with certified domination

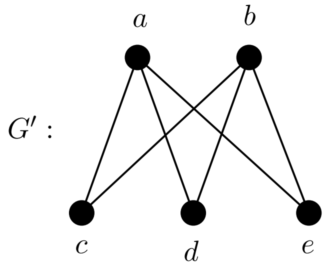
Remark 2. Let H be any graph. Then certified domination and 2-distance certified hop domination parameters are incomparable.

Consider the graph G below.



Let $S_1 = \{a, e, h, k, m\}$ and $S_2 = \{c, m\}$. Then S_1 is minimum certified dominating set of (G). Thus $\gamma_{cer}(G) = 5$. Moreover, S_2 is minimum 2-distance certified hop dominating set of G. Hence, $\gamma_{2ch}(G) = 2$. Therefore, $\gamma_{cer}(G) > \gamma_{2ch}(G)$.

Next, consider the graph G' below.



Let $U_1 = \{a, b\}$ and $U_2 = \{a, b, d\}$. Then U_1 is minimum certified dominating set of G'. Thus, $\gamma_{cer} = 2$. Additionally, U_2 is minimum 2-distance certified hop dominating set of G'. Hence, $\gamma_{2ch} = 3$. Therefore, $\gamma_{cer} < \gamma_{2ch}$.

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