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Intuitionistic Fuzzy Structures on Sheffer Stroke UP-Algebras

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Abstract. The study defines an intuitionistic fuzzy SUP-subalgebra and a level set of an intuitionistic fuzzy UP-structure on Sheffer stroke UP-algebras. It appears that these concepts are integral to understanding the behavior of neutrosophic logic within the framework of Sheffer stroke UP-algebras. The study establishes a relationship between UP-subalgebras and level sets on Sheffer stroke UP-algebras. Specifically, it proves that the level set of intuitionistic fuzzy SUP-subalgebras on this algebra is its subalgebra, and vice versa. It is stated that the family of all intuitionistic fuzzy SUP-subalgebras of a Sheffer stroke UP-algebra forms a complete distributive lattice. Additionally, it is shown that every intuitionistic fuzzy SUP-ideal of a Sheffer stroke UP-algebra is also its intuitionistic fuzzy SUP-subalgebra, though the inverse is generally not true. This highlights the specific characteristics and behavior of intuitionistic fuzzy SUP-ideals within the given algebraic context.

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Key Words and Phrases: Sheffer stroke UP-algebra (SUP-algebra), subalgebra, ideal, intuitionistic fuzzy SUP-subalgebra, intuitionistic fuzzy SUP-ideal.

1. Introduction

The Sheffer operation, also known as the Sheffer stroke or NAND operator, was first introduced by Sheffer [12]. This operation holds significance because it can be used on its own, without any other logical operators, to construct a logical system. This means that any axiom of a logical system can be restated using only the Sheffer operation. Because of this property, it becomes easier to control specific properties of the newly constructed logical system. Additionally, it's worth noting that the axioms of Boolean algebra, which is the algebraic counterpart of classical propositional calculus, can be expressed solely

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using the Sheffer operation. This highlights the fundamental nature and versatility of the Sheffer operation in logical and algebraic systems.

Building on this foundation, Sheffer stroke UP-algebras introduce a unique algebraic structure that merges logical and algebraic principles. As detailed by Iampan [5] in 2017, UP-algebras represent a new branch of logical algebra, providing a versatile framework for studying algebraic systems with advanced logical constructs. This innovation facilitates a deeper understanding of logical operations and their algebraic counterparts, paving the way for applications in fields such as decision theory.

The concept of fuzzy sets [13] and their extensions has been pivotal in exploring algebraic structures under uncertainty. Platil and Vilela [11] introduced anti fuzzy substructures in KS-semigroups, which provide a foundation for examining fuzzy properties in specialized algebraic systems. Their approach offers valuable insights into how dual properties (such as anti-fuzziness) can coexist with standard fuzzy frameworks, laying a conceptual groundwork for integrating intuitionistic fuzzy sets into Sheffer stroke UP-algebras. Additionally, Platil and Petalcorin [9] extended fuzzy set theory to Γ-semimodules over Γ-semirings, showcasing the adaptability of fuzzy structures in modular algebraic frameworks. These studies underscore the importance of developing fuzzy-based generalizations in algebra, a direction closely aligned with the intuitionistic fuzzy structures in Sheffer stroke UP-algebras. By building on these foundational works, our study aims to further generalize and apply intuitionistic fuzzy concepts, offering a deeper understanding of their behavior and applications in novel algebraic contexts. Intuitionistic fuzzy sets (IFSs), introduced by Atanassov [3] in 1986, extend classical fuzzy sets by incorporating a membership function, a non-membership function, and a degree of hesitation, offering a robust framework for modeling uncertainty. Satisfying the condition $0 \leq \mu(x) + \nu(x) \leq 1$, IFSs have proven to be highly adaptable for analyzing imprecise information and complex systems. Their mathematical flexibility has spurred extensive research in various algebraic systems, such as groups, rings, semirings, lattices, and UP-algebras, where IFSs facilitate the study of generalized structures, ideals, and filters under uncertain conditions. For example, Platil and Tanaka [10] applied IFSs in multi-criteria evaluation based on set-relations, demonstrating their relevance in decision-making frameworks. Kesorn et al. [6] specifically investigated intuitionistic fuzzy sets in UP-algebras, offering a direct connection to the present study. Their analysis of intuitionistic fuzzy filters and ideals in UP-algebras highlights the potential for applying intuitionistic fuzzy logic to broader classes of algebraic operations. Adak et al. [1] explored the properties of generalized intuitionistic fuzzy nilpotent matrices over distributive lattices, while Ebrahimnejad et al. [4] investigated eigenvalues of intuitionistic fuzzy matrices, extending matrix theory within a fuzzy context. Additionally, Adak et al. [2] developed a ranking method for multi-criteria decision-making problems utilizing generalized intuitionistic fuzzy information, highlighting real-world applications of IFSs in areas such as engineering and operations research. Furthermore, the integration of IFSs into algebraic frameworks like Sheffer stroke UP-algebras has unveiled deeper connections between fuzzy logic principles and algebraic operations, underscoring their transformative potential. These advancements confirm the enduring significance of IFSs in both theoretical explorations and practical problem-solving, ensuring their continued relevance in modern mathematical and computational studies.

In this paper, we define an intuitionistic fuzzy SUP-subalgebra and a level set of an intuitionistic fuzzy UP-structure on Sheffer stroke UP-algebras. It appears that these concepts are integral to understanding the behavior of neutrosophic logic within the framework of Sheffer stroke UP-algebras. The study establishes a relationship between subalgebras and level sets on Sheffer stroke UP-algebras. Specifically, it proves that the level set of intuitionistic fuzzy SUP-subalgebras on this algebra is its subalgebra, and vice versa. This indicates a tight connection between two concepts within the given algebraic structure. It is stated that the family of all intuitionistic fuzzy SUP-subalgebras of a Sheffer stroke UP-algebra forms a complete distributive lattice. This suggests that there is a well-defined structure and order among these subalgebras, allowing for systematic analysis. The study describes an intuitionistic fuzzy SUP-ideal of a Sheffer stroke UP-algebra and provides some properties. Additionally, it is shown that every intuitionistic fuzzy SUP-ideal of a Sheffer stroke UP-algebra is also its intuitionistic fuzzy SUP-subalgebra, though the inverse is generally not true. This highlights the specific characteristics and behavior of intuitionistic fuzzy SUP-ideals within the given algebraic context.

2. Preliminaries

Sheffer Stroke UP-algebras represent a compelling intersection of algebra and logic, characterized by the Sheffer stroke operation, a key connective in propositional calculus. This algebraic structure not only enhances our understanding of logical operations but also has significant implications in fields like computer science and decision theory. This article will delve into the definitions and foundational aspects of Sheffer Stroke UP-algebras, highlighting their relevance within algebraic theory.

Definition 1. *[12] Let $H = \langle H, |\rangle$ be a groupoid. The operation $|$ is said to be a Sheffer stroke operation if it satisfies the following conditions: for all $x, y, z \in H$,*

(*S-1*) $x|y = y|x$
(*S-2*) $(x|x)|(x|y) = x$
(*S-3*) $x|((y|z)|(y|z)) = ((x|y)|(x|y))|z$
(*S-4*) $(x|((x|x)|(y|y)))|(x|((x|x)|(y|y))) = x.$

Definition 2. *[7] A Sheffer stroke UP-algebra (briefly, SUP-algebra) is a structure $\langle H, |, 0\rangle$ of type $(2, 0)$ such that 0 is the fixed element in H and the following conditions are satisfied for all $x, y, z \in H$,*

(*SUP-1*) $(((z|(x|x))|(z|(x|x)))|(((y|(x|x))|(z|(y|y)))|((y|(x|x))|(z|(y|y)))))|(((z|(x|x))|(z|(x|x)))|(((y|(x|x))|(z|(y|y)))|((y|(x|x))|(z|(y|y))))) = 0$
(*SUP-2*) $x|x = x|(0|0)$
(*SUP-3*) $(x|(y|y))|(x|(y|y)) = 0$ *and* $(y|(x|x))|(y|(x|x)) = 0 \Rightarrow x = y.$

Proposition 1. *[7] Let $\langle H, |, 0\rangle$ be an SUP-algebra. Then the binary relation $x \leq y$ if and only if $(y|(x|x))|(y|(x|x)) = 0$ is a partial order on H.*

Lemma 1. *[7] Let $\langle H, |, 0\rangle$ be an SUP-algebra. Then for all $x, y, z \in H$, we have*

(1) $x \leq y \Rightarrow y|(z|z) \leq x|(z|z)$ *and* $z|(x|x) \leq z|(y|y)$

(2) $x \leq y \Leftrightarrow y|y \leq x|x$

(3) $y|(x|x) \leq x$

(4) $y \leq (y|(x|x))|(y|(x|x))$

(5) $x \leq y \Rightarrow x \leq (y|(z|z))|(y|(z|z))$

(6) $z|(y|y) \leq z|(y|(x|x))$

(7) $((z|(y|y))|(z|(y|y)))|(x|x) \leq z|(y|(x|x))$

(8) $x|((y|(z|z))|(y|(z|z))) \leq (x|(y|y))|((x|(z|z))|(x|(z|z)))$.

Definition 3. *[7] A nonempty subset G of an SUP-algebra H is called a subalgebra of H if $(x|(y|y))|(x|(y|y)) \in G$ for all $x, y \in G$.*

Definition 4. *[7, 8] A nonempty subset G of an SUP-algebra H is called an ideal of H if for all $x, y \in H$,*

(1) $y \in G \Rightarrow (y|(x|x))|(y|(x|x)) \in G$

(2) $(y|(x|x))|(y|(x|x)) \in G$ *and* $x \in G \Rightarrow y \in G$.

Definition 5. *[3] Let H be a nonempty set. The intuitionistic fuzzy set (IFS) $\mathcal{H} = (H, \mu, \gamma)$ is defined to be a structure*

$$\mathcal{H} := \{\langle x, \mu(x), \gamma(x)\rangle \mid x \in H\}, \tag{1}$$

where $\mu : H \to [0, 1]$ is the degree of membership of x to $\mathcal{H}$ and $\gamma : H \to [0, 1]$ is the degree of non-membership of x to $\mathcal{H}$ such that $0 \leq \mu(x) + \gamma(x) \leq 1$.

3. Intuitionistic fuzzy SUP-algebras

In this section, the study introduces the concepts of intuitionistic fuzzy SUP-subalgebras and intuitionistic fuzzy SUP-ideals within the context of SUP-algebras. It's worth noting that unless explicitly stated otherwise, H refers to an SUP-algebra.

Definition 6. *An IFS $\mathcal{H} = (H, \mu, \gamma)$ of H is called an intuitionistic fuzzy SUP-subalgebra of H if*

$$(\forall x, y \in H)\left(\begin{array}{l} \mu((x|(y|y))|(x|(y|y))) \geq \min\{\mu(x), \mu(y)\} \\ \gamma((x|(y|y))|(x|(y|y))) \leq \max\{\gamma(x), \gamma(y)\} \end{array}\right). \tag{2}$$

Theorem 1. *If $\mathcal{H} = (H, \mu, \gamma)$ is an intuitionistic fuzzy SUP-subalgebra of H, then $\mu(0) \geq \mu(x)$ and $\gamma(0) \leq \gamma(x)$ for all $x \in H$.*

Proof. For any $x \in H$,

$$\mu(0) = \mu((x|(x|x))|(x|(x|x))) \geq \min\{\mu(x), \mu(x)\} = \mu(x),$$

$$\gamma(0) = \gamma((x|(x|x))|(x|(x|x))) \leq \max\{\gamma(x), \gamma(x)\} = \gamma(x).$$

Definition 7. *Let μ be a fuzzy set on an SUP-algebra H and $\alpha \in [0, 1]$. We define the subsets $U(\mu, t) = \{x \in H : \mu(x) \geq t\}$ and $L(\mu, t) = \{x \in H : \mu(x) \leq t\}$ of H.*

Theorem 2. *An IFS $\mathcal{H} = (H, \alpha, \beta)$ in H is an intuitionistic fuzzy SUP-subalgebra of H if and only if the sets $L(\beta, s)$ and $U(\alpha, t)$ are subalgebras of H whenever they are nonempty for all $s, t \in [0, 1]$.*

Proof. Assume that $\mathcal{H} = (H, \alpha, \beta)$ is an intuitionistic fuzzy SUP-subalgebra of H and $L(\beta, s) \neq \emptyset \neq U(\alpha, t)$ for all $s, t \in [0, 1]$. Let $x, y, a, b \in H$ be such that $(x, a) \in L(\beta, s) \times U(\alpha, t)$ and $(y, b) \in L(\beta, s) \times U(\alpha, t)$. Then $\beta(x) \leq s$, $\beta(y) \leq s$, $\alpha(a) \geq t$ and $\alpha(b) \geq t$. Then $\beta((x|(y|y))|(x|(y|y))) \leq \max\{\beta(x), \beta(y)\} \leq s$ and $\alpha((a|(b|b))|(a|(b|b))) \geq \min\{\alpha(a), \alpha(b)\} \geq t$ and so $(x|(y|y))|(x|(y|y)), (a|(b|b))|(a|(b|b)) \in L(\beta, s) \times U(\alpha, t)$. Therefore, $L(\beta, s)$ and $U(\alpha, t)$ are SUP-subalgebras of H.

Conversely, let $\mathcal{H} = (H, \alpha, \beta)$ be an IFS in H for which $L(\beta, s)$ and $U(\alpha, t)$ are SUP-subalgebras of H whenever they are nonempty for all $s, t \in [0, 1]$. Suppose that $\beta((a|(b|b))|(a|(b|b))) > \max\{\beta(a), \beta(b)\}$ or $\alpha((x|(y|y))|(x|(y|y))) < \min\{\alpha(x), \alpha(y)\}$ for some $a, b, x, y \in H$. Then $a, b \in L(\beta, s)$ or $x, y \in U(\alpha, t)$ where $s = \max\{\beta(a), \beta(b)\}$ and $t = \min\{\alpha(x), \alpha(y)\}$. But $(a|(b|b))|(a|(b|b)) \notin L(\beta, s)$ or $(x|(y|y))|(x|(y|y)) \notin U(\alpha, t)$, a contradiction. Therefore,

$$\beta((a|(b|b))|(a|(b|b))) \leq \max\{\beta(a), \beta(b)\},$$

$$\alpha((x|(y|y))|(x|(y|y))) \geq \min\{\alpha(x), \alpha(y)\}$$

for all $a, b, x, y \in H$. Consequently, $\mathcal{H} = (H, \alpha, \beta)$ is an intuitionistic fuzzy SUP-subalgebra of H.

Theorem 3. *An IFS $\mathcal{H} = (H, \alpha, \beta)$ in H is an intuitionistic fuzzy SUP-subalgebra of H if and only if the fuzzy sets β_c and α are fuzzy SUP-subalgebras of H, where $\beta_c : L \to [0, 1],\ x \mapsto 1 - \beta(x)$.*

Proof. Assume that $\mathcal{H} = (H, \alpha, \beta)$ is an intuitionistic fuzzy SUP-subalgebra of H. It is clear that α is a fuzzy SUP-subalgebra of H. For every $x, y \in H$,

$$\begin{aligned} \beta_c((x|(y|y))|(x|(y|y))) &= 1 - \beta((x|(y|y))|(x|(y|y))) \\ &\geq 1 - \max\{\beta(x), \beta(y)\} \\ &= \min\{1 - \beta(x), 1 - \beta(y)\} \\ &= \min\{\beta_c(x), \beta_c(y)\}. \end{aligned}$$

Hence, β_c is a fuzzy SUP-subalgebra of H.

Conversely, let $\mathcal{H} = (H, \alpha, \beta)$ be an IFS of H for which β_c and α are fuzzy SUP-subalgebras of H. Let $x, y \in H$. Then

$$\begin{array}{rcl} 1 - \beta((x|(y|y))|(x|(y|y))) & = & \beta_c((x|(y|y))|(x|(y|y))) \\ & \geq & \min\{\beta_c(x), \beta_c(y)\} \\ & = & \min\{1-\beta(x), 1-\beta(y)\} \\ & = & 1 - \max\{\beta(x), \beta(y)\}, \\ \beta((x|(y|y))|(x|(y|y))) & \leq & \max\{\beta(x), \beta(y)\}. \end{array}$$

Hence, $\mathcal{H} = (H, \alpha, \beta)$ is an intuitionistic fuzzy SUP-subalgebra of H.

Theorem 4. *Given a nonempty subset F of H, let $\mathcal{H}_F = (H, \alpha_F, \beta_F)$ be an IFS in H defined as follows:* $\alpha(x) = \begin{cases} \alpha_0 & \text{if } x \in F \\ \alpha_1 & \text{otherwise,} \end{cases}$ $\beta(x) = \begin{cases} \beta_0 & \text{if } x \in F \\ \beta_1 & \text{otherwise} \end{cases}$ *for all $x \in H$ and $\alpha_i, \beta_i \in [0,1]$ such that $\alpha_0 > \alpha_1$, $\beta_0 < \beta_1$, and $\alpha_i + \beta_i \leq 1$ for $i = 0, 1$. Then $\mathcal{H}_F = (H, \alpha_F, \beta_F)$ be an intuitionistic fuzzy SUP-subalgebra of H if and only if F is an SUP-subalgebra of H.*

Proof. Assume that $\mathcal{H}_F = (H, \alpha_F, \beta_F)$ is an intuitionistic fuzzy SUP-subalgebra of H. Let $x, y \in H$ be such that $x, y \in F$. Then

$$\alpha((x|(y|y))|(x|(y|y))) \geq \min\{\alpha(x), \alpha(y)\} = \alpha_0,$$

$$\beta((x|(y|y))|(x|(y|y))) \leq \max\{\beta(x), \beta(y)\} = \beta_0,$$

and so $\alpha((x|(y|y))|(x|(y|y))) = \alpha_0$ and $\beta((x|(y|y))|(x|(y|y))) = \beta_0$. This shows that $(x|(y|y))|(x|(y|y)) \in F$. Therefore, F is an SUP-subalgebra of H.

Conversely, let F be an SUP-subalgebra of H. For every $x, y \in H$, if $x, y \in F$, then $x|(y|y))|(x|(y|y) \in F$ which implies that

$$\alpha((x|(y|y))|(x|(y|y))) = \alpha_0 = \min\{\beta(x), \beta(y)\},$$

$$\beta((x|(y|y))|(x|(y|y))) = \beta_0 = \max\{\beta(x), \beta(y)\}.$$

If $x \notin F$ of $y \notin F$, then

$$\alpha((x|(y|y))|(x|(y|y))) \geq \alpha_1 = \min\{\alpha(x), \alpha(y)\},$$

$$\beta((x|(y|y))|(x|(y|y))) \leq \beta_1 = \max\{\beta(x), \beta(y)\}.$$

Therefore, $\mathcal{H}_F = (H, \alpha_F, \beta_F)$ is an intuitionistic fuzzy SUP-subalgebra of H.

Definition 8. *An IFS $\mathcal{H} = (H, \mu, \gamma)$ of H is called an intuitionistic fuzzy SUP-ideal of H if*

$$(\forall x, y \in H) \left(\begin{array}{l} \mu((y|(x|x))|(y|(x|x))) \geq \mu(y) \geq \min\{\mu((y|(x|x))|(y|(x|x))), \mu(x)\} \\ \gamma((y|(x|x))|(y|(x|x))) \leq \gamma(y) \leq \max\{\gamma((y|(x|x))|(y|(x|x))), \gamma(x)\} \end{array} \right). \quad (3)$$

Theorem 5. *Every intuitionistic fuzzy SUP-ideal of H is an intuitionistic fuzzy SUP-subalgebra of H.*

Lemma 2. *If $\mathcal{H} = (H, \mu, \gamma)$ is an intuitionistic fuzzy SUP-ideal of H, then*

$$(\forall\, x, y \in H) \left(\quad x \leq y \ \Rightarrow \begin{cases} \mu(y) \geq \mu(x) \\ \gamma(y) \leq \gamma(x) \end{cases} \right). \tag{4}$$

Proof. Let $\mathcal{H} = (H, \mu, \gamma)$ be an intuitionistic fuzzy SUP-ideal of H and $x \leq y$. Then $\mu(y) \geq \min\{\mu(x), \mu(0)\} = \mu(0)$, $\gamma(y) \leq \max\{\gamma(x), \gamma(0)\} = \gamma(x)$ for all $x, y \in H$.

Theorem 6. *An IFS $\mathcal{H} = (H, \alpha, \beta)$ in H is an intuitionistic fuzzy SUP-ideal of H if and only if the sets $L(\beta, s)$ and $U(\alpha, t)$ are SUP-ideals of H whenever they are nonempty for all $s, t \in [0, 1]$.*

Proof. Assume that $\mathcal{H} = (H, \alpha, \beta)$ is an intuitionistic fuzzy SUP-ideal of H and $L(\beta, s) \neq \emptyset \neq U(\alpha, t)$ for all $s, t \in [0, 1]$. Let $x, y, a, b \in H$ be such that $(y, b) \in L(\beta, s) \times U(\alpha, t)$. Then $\beta(y) \leq s$ and $\alpha(b) \geq t$. Then $\beta((y|(x|x))|(y|(x|x))) \leq \beta(y) \leq s$ and $\alpha((a|(b|b))|(a|(b|b))) \geq \alpha(b) \geq t$ and so $(y|(x|x))|(y|(x|x))$, $(a|(b|b))|(a|(b|b)) \in L(\beta, s) \times U(\alpha, t)$. Let $x, y, a, b \in H$ be such that $((y|(x|x))|(y|(x|x)), (a|(b|b))|(a|(b|b))) \in L(\beta, s) \times U(\alpha, t)$ and $(x, a) \in L(\beta, s) \times U(\alpha, t)$. Then $\beta((y|(x|x))|(y|(x|x))) \leq s$, $\beta(x) \leq s$, $\alpha((a|(b|b))|(a|(b|b)))) \geq t$ and $\alpha(a) \geq t$. Then $\beta(y) \leq \max\{\beta(((y|(x|x))|(y|(x|x))), \beta(x)\} \leq s$ and $\alpha(b) \geq \min\{\alpha((a|(b|b))|(a|(b|b))), \alpha(a)\} \geq t$ and so $(y, b) \in L(\beta, s) \times U(\alpha, t)$. Therefore, $L(\beta, s)$ and $U(\alpha, t)$ are SUP-ideals of H.

Conversely, let $\mathcal{H} = (H, \alpha, \beta)$ be an IFS in H for which its negative s-cut and positive t-cut are SUP-ideals of H whenever they are nonempty for all $s, t \in [0, 1]$. Suppose that $\beta((a|(b|b))|(a|(b|b))) > \beta(b)$ for some $a, b \in H$. Then $b \in L(\beta, \beta(b))$ but $(a|(b|b))|(a|(b|b)) \notin L(\beta, \beta(b))$, a contradiction. Hence, $\beta((y|(x|x))|(y|(x|x))) \leq \beta(y)$ for all $x, y \in H$. Suppose that $\alpha((x|(y|y))|(x|(y|y))) < \alpha(y)$ for some $x, y \in H$. Then $y \in U(\alpha, \alpha(y))$ but $(x|(y|y))|(x|(y|y)) \notin U(\alpha, \alpha(y))$, a contradiction. Hence, $\beta((a|(b|b))|(a|(b|b))) \geq \beta(b)$ for all $a, b \in H$. Suppose that

$$\beta(b) > \max\{\beta((a|(b|b))|(a|(b|b))), \beta(a)\}$$

or

$$\alpha(y) < \min\{\alpha((x|(y|y))|(x|(y|y))), \alpha(x)\}$$

for some $a, b, x, y \in H$. Then $(a|(b|b))|(a|(b|b)), a \in L(\beta, s)$ or $(x|(y|y))|(x|(y|y)), x \in U(\alpha, t)$ where $s = \max\{\beta((a|(b|b))|(a|(b|b))), \beta(a)\}$ and $t = \min\{\alpha((x|(y|y))|(x|(y|y))), \alpha(x)\}$. But $b \notin L(\beta, s)$ or $y \notin U(\alpha, t)$, a contradiction. Therefore,

$$\beta(y) \leq \max\{\beta((x|(y|y))|(x|(y|y))), \beta(x)\},$$

$$\alpha((x|(y|y))|(x|(y|y))) \geq \min\{\alpha(x), \alpha(y)\}$$

for all $x, y \in H$. Consequently, $\mathcal{H} = (H, \alpha, \beta)$ is an intuitionistic fuzzy SUP-ideal of H.

Theorem 7. *An IFS $\mathcal{H} = (H, \alpha, \beta)$ in H is an intuitionistic fuzzy SUP-ideal of H if and only if the fuzzy sets β_c and α are fuzzy SUP-ideals of H, where $\beta_c : L \to [0,1]$, $x \mapsto 1 - \beta(x)$.*

Proof. Assume that $\mathcal{H} = (H, \alpha, \beta)$ is an intuitionistic fuzzy SUP-ideal of H. It is clear that α is a fuzzy SUP-ideal of H. For every $x, y \in H$,

$$\begin{aligned} \beta_c((x|(y|y))|(x|(y|y))) &= 1 - \beta((x|(y|y))|(x|(y|y))) \\ &\geq 1 - \beta(y) \\ &= 1 - \beta(y) \\ &= \beta_c(y), \end{aligned}$$

$$\begin{aligned} \beta_c(y) &= 1 - \beta(y) \\ &\geq 1 - \max\{\beta(x), \beta(y)\} \\ &= \min\{1 - \beta((x|(y|y))|(x|(y|y))), 1 - \beta(x)\} \\ &= \min\{\beta_c((x|(y|y))|(x|(y|y))), \beta_c(x)\}. \end{aligned}$$

Hence, β_c is a fuzzy SUP-ideal of H.

Conversely, let $\mathcal{H} = (H, \alpha, \beta)$ be an IFS of H for which β_c and α are fuzzy SUP-ideals of H. Let $x, y \in H$. Then

$$\begin{aligned} 1 - \beta((x|(y|y))|(x|(y|y))) &= \beta_c((x|(y|y))|(x|(y|y))) \\ &\geq \beta_c(y) \\ &= 1 - \beta(y), \\ \beta((x|(y|y))|(x|(y|y))) &\leq \beta(y), \end{aligned}$$

$$\begin{aligned} 1 - \beta(y) &= \beta_c(y) \\ &\geq \min\{\beta_c(x), \beta_c(y)\} \\ &= \min\{1 - \beta((x|(y|y))|(x|(y|y))), 1 - \beta(x)\} \\ &= 1 - \max\{\beta((x|(y|y))|(x|(y|y))), \beta(x)\}, \\ \beta(y) &\leq \max\{\beta((x|(y|y))|(x|(y|y))), \beta(x)\}. \end{aligned}$$

Hence, $\mathcal{H} = (H, \alpha, \beta)$ is an intuitionistic fuzzy SUP-ideal of H.

Theorem 8. *Given a nonempty subset F of H, let $\mathcal{H}_F = (H, \alpha_F, \beta_F)$ be an IFS in H defined as follows:*

$$\alpha_F : L \to [0,1], x \mapsto \begin{cases} \alpha_0 & if\ x \in F, \\ \alpha_1 & otherwise \end{cases} \quad \beta_F : L \to [0,1], a \mapsto \begin{cases} \beta_0 & if\ a \in F, \\ \beta_1 & otherwise, \end{cases}$$

where $\beta_0 < \beta_1$ in $[0,1]$ and $\alpha_0 > \alpha_1$ in $[0,1]$. Then $\mathcal{H}_F = (H, \alpha_F, \beta_F)$ is an intuitionistic fuzzy SUP-ideal of H if and only if F is an SUP-ideal of H.

Proof. Assume that $\mathcal{H}_F = (H, \alpha_F, \beta_F)$ is an intuitionistic fuzzy SUP-ideal of H. Let $x, y \in H$ be such that $x, y \in F$. Then

$$\alpha((x|(y|y))|(x|(y|y))) \geq \alpha(y) = \alpha_0,$$

$$\beta((x|(y|y))|(x|(y|y))) \leq \beta(y) = \beta_0,$$

and so $\alpha((x|(y|y))|(x|(y|y))) = \alpha_0$ and $\beta((x|(y|y))|(x|(y|y))) = \beta_0$. This shows that $(x|(y|y))|(x|(y|y)) \in F$. Then

$$\alpha(y) \geq \min\{\alpha((x|(y|y))|(x|(y|y))), \alpha(x)\} = \alpha_0,$$

$$\beta(y) \leq \max\{\beta((x|(y|y))|(x|(y|y))), \beta(x)\} = \beta_0,$$

and so $\alpha(y) = \alpha_0$ and $\beta(y) = \beta_0$. This shows that $y \in F$. Therefore, F is an SUP-ideal of H.

Conversely, let F be an SUP-ideal of H. For every $x, y \in H$, if $(x|(y|y))|(x|(y|y)) \in F$, then $y \in F$ which implies that

$$\alpha(y) = \alpha_0 = \beta((x|(y|y))|(x|(y|y))),$$

$$\beta(y) = \beta_0 = \beta((x|(y|y))|(x|(y|y))).$$

If $(x|(y|y))|(x|(y|y)) \notin F$, then

$$\alpha((x|(y|y))|(x|(y|y))) = \alpha_1 < \beta(y),$$

$$\beta((x|(y|y))|(x|(y|y))) = \beta_1 > \beta(y).$$

For every $x, y \in H$, if $x, y \in F$, then $x|(y|y))|(x|(y|y) \in F$ which implies that

$$\alpha((x|(y|y))|(x|(y|y))) = \alpha_0 = \min\{\alpha(x), \alpha(y)\},$$

$$\beta((x|(y|y))|(x|(y|y))) = \beta_0 = \max\{\beta(x), \beta(y)\}.$$

If $x \notin F$ of $y \notin F$, then

$$\alpha((x|(y|y))|(x|(y|y))) \geq \alpha_1 = \min\{\alpha(x), \alpha(y)\},$$

$$\beta((x|(y|y))|(x|(y|y))) \leq \beta_1 = \max\{\beta(x), \beta(y)\}.$$

Therefore, $\mathcal{H}_F = (H, \alpha_F, \beta_F)$ is an intuitionistic fuzzy SUP-ideal of H.

Proposition 2. *If $\mathcal{H}_i = \{(H, \alpha_i, \beta_i) : i \in \Delta\}$ is a family of intuitionistic fuzzy SUP-ideals of H, then $\bigwedge_{i\in\Delta} \mathcal{X}_i$ is an intuitionistic fuzzy SUP-ideal of H.*

Proof. Let $\mathcal{H}_i = \{(H, \alpha_i, \beta_i) : i \in \Delta\}$ be a family of intuitionistic fuzzy SUP-ideals of H. Let $x, y \in H$. Then

$$(\bigwedge_{i\in\Delta} \alpha_i)((x|(y|y))|(x|(y|y))) = \inf_{i\in\Delta}\{\alpha_i((x|(y|y))|(x|(y|y)))\} \geq \inf_{i\in\Delta}\{\alpha_i(y)\} = (\bigwedge_{i\in\Delta} \alpha_i)(y),$$

$$(\bigwedge_{i\in\Delta} \beta_i)((x|(y|y))|(x|(y|y))) = \sup_{i\in\Delta}\{\beta_i((x|(y|y))|(x|(y|y)))\} \leq \sup_{i\in\Delta}\{\beta_i(y)\} = (\bigwedge_{i\in\Delta} \beta_i)(y).$$

Let $x, y \in H$. Then

$$\begin{array}{rcl} (\bigwedge_{i\in\Delta} \alpha_i)(y) & = & \inf_{i\in\Delta}\{\alpha_i(y)\} \\ & \geq & \inf_{i\in\Delta}\{\min\{\alpha_i((x|(y|y))|(x|(y|y))), \alpha_i(x)\}\} \\ & = & \min\{\inf_{i\in\Delta} \alpha_i((x|(y|y))|(x|(y|y))), \inf_{i\in\Delta} \alpha_i(x)\} \\ & = & \min\{(\bigwedge_{i\in\Delta} \alpha_i)((x|(y|y))|(x|(y|y))), (\bigwedge_{i\in\Delta} \alpha_i)(x)\}, \end{array}$$

$$\begin{array}{rcl} (\bigwedge_{i\in\Delta} \beta_i)(y) & = & \sup_{i\in\Delta}\{\beta_i(y)\} \\ & \leq & \sup_{i\in\Delta}\{\max\{\beta_i((x|(y|y))|(x|(y|y))), \beta_i(x)\}\} \\ & = & \max\{\sup_{i\in\Delta} \beta_i((x|(y|y))|(x|(y|y))), \sup_{i\in\Delta} \beta_i(x)\} \\ & = & \max\{(\bigwedge_{i\in\Delta} \beta_i)((x|(y|y))|(x|(y|y))), (\bigwedge_{i\in\Delta} \beta_i)(x)\}. \end{array}$$

Hence, $\bigwedge_{i\in\Delta} \mathcal{X}_i$ is an intuitionistic fuzzy SUP-ideal of an SUP-algebra H.

Theorem 9. *[7] Let $\langle A, |_A, 0_A\rangle$ and $\langle B, |_B, 0_B\rangle$ be SUP-algebras. Then $\langle A\times B, |_{A\times B}, 0_{A\times B}\rangle$ is an SUP-algebra where the set $A \times B$ is the Cartesian product of A and B and the operation $|_{A\times B}$ on this set is defined by $(a_1, b_1)|_{A\times B}(a_2, b_2) = (a_1|_A a_2, b_1|_B b_2)$, and the fixed element is $0_{A\times B} = (0_A, 0_B)$.*

Theorem 10. *Let $\langle A, |_A, 0_A\rangle$ and $\langle B, |_B, 0_B\rangle$ be SUP-algebras. If $\mathcal{H}_A = (A, \mu_A, \gamma_A)$ and $\mathcal{H}_B = (B, \mu_B, \gamma_B)$ are intuitionistic fuzzy SUP-subalgebras of $\langle A, |_A, 0_A\rangle$ and $\langle B, |_B, 0_B\rangle$, respectively, then $\mathcal{H}_{A\times B} = (A \times B, \mu_{A\times B}, \gamma_{A\times B})$ is an intuitionistic fuzzy SUP-subalgebra of $\langle A \times B, |_{A\times B}, 0_{A\times B}\rangle$.*

Proof. Let $\langle A, |_A, 0_A\rangle$ and $\langle B, |_B, 0_B\rangle$ be SUP-algebras. If $\mathcal{H}_A = (A, \mu_A, \gamma_A)$ and $\mathcal{H}_B = (B, \mu_B, \gamma_B)$ are intuitionistic fuzzy SUP-subalgebras of $\langle A, |_A, 0_A\rangle$ and $\langle B, |_B, 0_B\rangle$, respectively. Let $(a_1, b_1), (a_2, b_2) \in A \times B$. Then

$$\begin{array}{rl} & \mu_{A\times B}((((a_1, b_1)|_{A\times B}((a_2, b_2)|_{A\times B}(a_2, b_2)))|_{A\times B}((a_1, b_1)|_{A\times B}((a_2, b_2)|_{A\times B}(a_2, b_2)))) \\ = & \mu_{A\times B}((a_1|_A(a_2|_A a_2))|_A(a_1|_A(a_2|_A a_2)), (b_1|_B(b_2|_B b_2))|_B(b_1|_B(b_2|_B b_2))) \\ = & \max\{\mu_A((a_1|_A(a_2|_A a_2))|_A(a_1|_A(a_2|_A a_2)), T_B(b_1|_B(b_2|_B b_2))|_B(b_1|_B(b_2|_B b_2)))\} \\ \geq & \max\{\min\{\mu_A(a_1), \mu_A(a_2)\}, \min\{\mu_B(b_1), \mu_B(b_2)\}\} \\ = & \max\{\min\{\mu_A(a_1), \mu_B(b_1)\}, \min\{\mu A(a_2), i_B(b_2)\}\} \\ = & \max\{\mu_{A\times B}(a_1, b_1), i_{A\times B}(a_2, b_2)\}, \end{array}$$

$$\begin{array}{rl} & \gamma_{A\times B}((((a_1, b_1)|_{A\times B}((a_2, b_2)|_{A\times B}(a_2, b_2)))|_{A\times B}((a_1, b_1)|_{A\times B}((a_2, b_2)|_{A\times B}(a_2, b_2)))) \\ = & \gamma_{A\times B}((a_1|_A(a_2|_A a_2))|_A(a_1|_A(a_2|_A a_2)), (b_1|_B(b_2|_B b_2))|_B(b_1|_B(b_2|_B b_2))) \\ = & \max\{\gamma_A((a_1|_A(a_2|_A a_2))|_A(a_1|_A(a_2|_A a_2)), \gamma_B(b_1|_B(b_2|_B b_2))|_B(b_1|_B(b_2|_B b_2)))\} \\ \leq & \max\{\max\{\gamma_A(a_1), \gamma_A(a_2)\}, \max\{\gamma_B(b_1), \gamma_B(b_2)\}\} \\ = & \max\{\max\{\gamma_A(a_1), \gamma_B(b_1)\}, \max\{\gamma_A(a_2), \gamma_B(b_2)\}\} \\ = & \max\{\gamma_{A\times B}(a_1, b_1), \gamma_{A\times B}(a_2, b_2)\}. \end{array}$$

Hence, $\mathcal{H}_{A\times B} = (A \times B, \mu_{A\times B}, \gamma_{A\times B})$ is an intuitionistic fuzzy SUP-subalgebra of $\langle A \times B, |_{A\times B}, 0_{A\times B}\rangle$.

Theorem 11. *Let $\langle A, |_A, 0_A\rangle$ and $\langle B, |_B, 0_B\rangle$ be SUP-algebras. If $\mathcal{H}_A = (A, \mu_A, \gamma_A)$ and $\mathcal{H}_B = (B, \mu_B, \gamma_B)$ are intuitionistic fuzzy SUP-ideal of $\langle A, |_A, 0_A\rangle$ and $\langle B, |_B, 0_B\rangle$, respectively, then $\mathcal{H}_{A\times B} = (A \times B, \mu_{A\times B}, \gamma_{A\times B})$ is an intuitionistic fuzzy SUP-ideal of $\langle A \times B, |_{A\times B}, 0_{A\times B}\rangle$.*

Proof. Let $\langle A, |_A, 0_A\rangle$ and $\langle B, |_B, 0_B\rangle$ be SUP-algebras and $\mathcal{H}_A = (A, \mu_A, \gamma_A)$ and $\mathcal{H}_B = (B, \mu_B, \gamma_B)$ be intuitionistic fuzzy SUP-ideal of $\langle A, |_A, 0_A\rangle$ and $\langle B, |_B, 0_B\rangle$, respectively. Let $(a_1, b_1), (a_2, b_2) \in A \times B$. Then

$$\begin{aligned}
& \mu_{A\times B}(((a_2,b_2)|_{A\times B}((a_1,b_1)|_{A\times B}(a_1,b_1)))|_{A\times B}((a_2,b_2)|_{A\times B}((a_1,b_1)|_{A\times B}(a_1,b_1)))) \\
= \;& \max\{\mu_A((a_2|_A(a_1|_Aa_1))|_A(a_2|_A(a_1|_Aa_1))), \mu_B((b_2|_B(b_1|_Bb_1))|_B(b_2|_B(b_1|_Bb_1)))\} \\
\geq \;& \max\{\mu_A(a_2), \mu_B(b_2)\} \\
= \;& \mu_{A\times B}(a_2, b_2),
\end{aligned}$$

$$\begin{aligned}
& \gamma_{A\times B}(((a_2,b_2)|_{A\times B}((a_1,b_1)|_{A\times B}(a_1,b_1)))|_{A\times B}((a_2,b_2)|_{A\times B}((a_1,b_1)|_{A\times B}(a_1,b_1)))) \\
= \;& \max\{\gamma_A((a_2|_A(a_1|_Aa_1))|_A(a_2|_A(a_1|_Aa_1))), F_B((b_2|_B(b_1|_Bb_1))|_B(b_2|_B(b_1|_Bb_1)))\} \\
\leq \;& \max\{\gamma_A(a_2), \gamma_B(b_2)\} \\
= \;& \gamma_{A\times B}(a_2, b_2),
\end{aligned}$$

$$\begin{aligned}
& \mu_{A\times B}(a_2, b_2) \\
= \;& \max\{\mu_A(a_2), \mu_B(b_2)\} \\
\geq \;& \max\{\min\{\mu_A(a_1), \mu_A((a_2|_A(a_1|_Aa_1))|_A(a_2|_A(a_1|_Aa_1)))\}, \\
& \min\{\mu_B(b_1), \mu_B((b_2|_B(b_1|_Bb_1))|_B(b_2|_B(b_1|_Bb_1)))\}\} \\
= \;& \max\{\min\{\mu_A(a_1), \mu_B(b_1)\}, \min\{\mu_A((a_2|_A(a_1|_Aa_1))|_A(a_2|_A(a_1|_Aa_1))), \\
& \mu_B((b_2|_B(b_1|_Bb_1))|_B(b_2|_B(b_1|_Bb_1)))\}\} \\
= \;& \max\{\mu_{A\times B}(a_1, b_1), \mu_{A\times B}(((a_2,b_2)|_{A\times B} \\
& ((a_1,b_1)|_{A\times B}(a_1,b_1)))|_{A\times B}((a_2,b_2)|_{A\times B}((a_1,b_1)|_{A\times B}(a_1,b_1))))\},
\end{aligned}$$

$$\begin{aligned}
& \gamma_{A\times B}(a_2, b_2) \\
= \;& \max\{\gamma_A(a_2), \gamma_B(b_2)\} \\
\leq \;& \max\{\max\{\gamma_A(a_1), \gamma_A((a_2|_A(a_1|_Aa_1))|_A(a_2|_A(a_1|_Aa_1)))\}, \\
& \max\{\gamma_B(b_1), \gamma_B((b_2|_B(b_1|_Bb_1))|_B(b_2|_B(b_1|_Bb_1)))\}\} \\
= \;& \max\{\max\{\gamma_A(a_1), \gamma_B(b_1)\}, \max\{\gamma_A((a_2|_A(a_1|_Aa_1))|_A(a_2|_A(a_1|_Aa_1))), \\
& \gamma_B((b_2|_B(b_1|_Bb_1))|_B(b_2|_B(b_1|_Bb_1)))\}\} \\
= \;& \max\{\gamma_{A\times B}(a_1, b_1), \gamma_{A\times B}(((a_2,b_2)|_{A\times B} \\
& ((a_1,b_1)|_{A\times B}(a_1,b_1)))|_{A\times B}((a_2,b_2)|_{A\times B}((a_1,b_1)|_{A\times B}(a_1,b_1))))\}.
\end{aligned}$$

Hence, $\mathcal{H}_{A\times B} = (A\times B, \mu_{A\times B}, \gamma_{A\times B})$ is an intuitionistic fuzzy SUP-ideal of $\langle A\times B, |_{A\times B}, 0_{A\times B}\rangle$.

Definition 9. *[7] Let $\langle A, |_A, 0_A\rangle$ and $\langle B, |_B, 0_B\rangle$ be SUP-algebras. Then a mapping $f : A \to B$ is called a homomorphism if $f(x|_Ay) = f(x)|_Bf(y)$ for all $x, y \in H$ and $f(0_A) = 0_B$.*

Theorem 12. *Let $\langle A, |_A, 0_A\rangle$ and $\langle B, |_B, 0_B\rangle$ be SUP-algebras, $f : A \to B$ be a surjective homomorphism, and $\mathcal{B} = (B, \mu, \gamma)$ be an intuitionistic fuzzy UP-structure on B. Then $\mathcal{B} = (B, \mu, \gamma)$ is an intuitionistic fuzzy SUP-ideal of B if and only if $\mathcal{B}^f = (B, \mu^f, \gamma^f)$ is an intuitionistic fuzzy SUP-ideal of A.*

Proof. Let $\langle A, |_A, 0_A\rangle$ and $\langle B, |_B, 0_B\rangle$ be SUP-algebras, $f : A \to B$ be a surjective homomorphism, and $\mathcal{B} = (B, \mu, \gamma)$ be an intuitionistic fuzzy SUP-ideal of B. Let $x_1, x_2 \in A$. Then

$$\begin{aligned} & \mu^f((x_2|_A(x_1|_Ax_1))|_A(x_2|_A(x_1|_Ax_1))) \\ = \ & \mu(f((x_2|_A(x_1|_Ax_1))|_A(x_2|_A(x_1|_Ax_1)))) \\ = \ & \mu((f(x_2)|_B(f(x_1)|_Bf(x_1)))|_B(f(x_2)|_B(f(x_1)|_Bf(x_1)))) \\ \geq \ & \mu(f(x_2)) \\ = \ & \mu^f(x_2), \end{aligned}$$

$$\begin{aligned} & \gamma^f((x_2|_A(x_1|_Ax_1))|_A(x_2|_A(x_1|_Ax_1))) \\ = \ & \gamma(f((x_2|_A(x_1|_Ax_1))|_A(x_2|_A(x_1|_Ax_1)))) \\ = \ & \gamma((f(x_2)|_B(f(x_1)|_Bf(x_1)))|_B(f(x_2)|_B(f(x_1)|_Bf(x_1)))) \\ \leq \ & \gamma(f(x_2)) \\ = \ & \gamma^f(x_2), \end{aligned}$$

$$\begin{aligned} \mu^f(x_2) &= \mu(f(x_2)) \\ &\geq \min\{\mu(f(x_1)), \mu((f(x_2)|_B(f(x_1)|_Bf(x_1)))|_B(f(x_2)|_B(f(x_1)|_Bf(x_1))))\} \\ &= \min\{\mu(f(x_1)), \mu(f((x_2|_A(x_1|_Ax_1))|_A(x_2|_A(x_1|_Ax_1))))\} \\ &= \min\{\mu^f(x_1), \mu^f((x_2|_A(x_1|_Ax_1))|_A(x_2|_A(x_1|_Ax_1)))\}, \end{aligned}$$

$$\begin{aligned} \gamma^f(x_2) &= \gamma(f(x_2)) \\ &\leq \max\{\gamma(f(x_1)), \gamma((f(x_2)|_B(f(x_1)|_Bf(x_1)))|_B(f(x_2)|_B(f(x_1)|_Bf(x_1))))\} \\ &= \max\{\gamma(f(x_1)), \gamma(f((x_2|_A(x_1|_Ax_1))|_A(x_2|_A(x_1|_Ax_1))))\} \\ &= \max\{\gamma^f(x_1), \gamma^f((x_2|_A(x_1|_Ax_1))|_A(x_2|_A(x_1|_Ax_1)))\}. \end{aligned}$$

Hence, $\mathcal{B}^f = (B, \mu^f, \gamma^f)$ is an intuitionistic fuzzy SUP-ideal of A.

Conversely, let $\mathcal{B}^f = (B, \mu^f, \gamma^f)$ be an intuitionistic fuzzy SUP-ideal of A. Let $y_1, y_2 \in B$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$ for $x_1, x_2 \in A$. Then

$$\begin{aligned} & \mu((y_2|_B(y_1|_By_1))|_B(y_2|_B(y_1|_By_1))) \\ = \ & \mu((f(x_2)|_B(f(x_1)|_Bf(x_1)))|_B(f(x_2)|_B(f(x_1)|_Bf(x_1)))) \\ = \ & \mu^f((x_2|_A(x_1|_Ax_1))|_A(x_2|_A(x_1|_Ax_1))) \\ \geq \ & \mu^f(x_2) \\ = \ & \mu(f(x_2)) \\ = \ & \mu(y_2), \end{aligned}$$

$$\begin{aligned} & \gamma((y_2|_B(y_1|_By_1))|_B(y_2|_B(y_1|_By_1))) \\ = \ & \gamma((f(x_2)|_B(f(x_1)|_Bf(x_1)))|_B(f(x_2)|_B(f(x_1)|_Bf(x_1)))) \\ = \ & \gamma^f((x_2|_A(x_1|_Ax_1))|_A(x_2|_A(x_1|_Ax_1))) \\ \leq \ & \gamma^f(x_2) \\ = \ & \gamma(f(x_2)) \\ = \ & \gamma(y_2), \end{aligned}$$

$$\begin{aligned}\mu(y_2) &= \mu(f(x_2))\\ &= \mu^f(x_2)\\ &\geq \min\{\mu^f(x_1), \mu^f((x_2|_A(x_1|_Ax_1))|_A(x_2|_A(x_1|_Ax_1)))\}\\ &= \min\{\mu(f(x_1)), \mu(f((x_2|_A(x_1|_Ax_1))|_A(x_2|_A(x_1|_Ax_1))))\}\\ &= \min\{\mu(f(x_1)), \mu((f(x_2)|_B(f(x_1)|_Bf(x_1)))|_B(f(x_2)|_B(f(x_1)|_Bf(x_1))))\}\\ &= \min\{\mu(y_1), \mu((y_2|_B(y_1|_By_1))|_B(y_2|_B(y_1|_By_1)))\},\end{aligned}$$

$$\begin{aligned}\gamma(y_2) &= \gamma(f(x_2))\\ &= \gamma^f(x_2)\\ &\leq \max\{\gamma^f(x_1), \gamma^f((x_2|_A(x_1|_Ax_1))|_A(x_2|_A(x_1|_Ax_1)))\}\\ &= \max\{\gamma(f(x_1)), \gamma(f((x_2|_A(x_1|_Ax_1))|_A(x_2|_A(x_1|_Ax_1))))\}\\ &= \max\{\gamma(f(x_1)), \gamma((f(x_2)|_B(f(x_1)|_Bf(x_1)))|_B(f(x_2)|_B(f(x_1)|_Bf(x_1))))\}\\ &= \max\{\gamma(y_1), \gamma((y_2|_B(y_1|_By_1))|_B(y_2|_B(y_1|_By_1)))\}.\end{aligned}$$

Hence, $\mathcal{B} = (B, \mu, \gamma)$ is an intuitionistic fuzzy SUP-ideal of B.

Theorem 13. *Let $\langle A, |_A, 0_A\rangle$ and $\langle B, |_B, 0_B\rangle$ be SUP-algebras, $f : A \to B$ be a surjective homomorphism, and $\mathcal{B} = (B, \mu, \gamma)$ be an intuitionistic fuzzy UP-structure on B. Then $\mathcal{B} = (B, \mu, \gamma)$ is an intuitionistic fuzzy SUP-subalgebra of B if and only if $\mathcal{B}^f = (B, \mu^f, \gamma^f)$ is an intuitionistic fuzzy SUP-subalgebra of A.*

Proof. Let $\langle A, |_A, 0_A\rangle$ and $\langle B, |_B, 0_B\rangle$ be SUP-algebras, $f : A \to B$ be a surjective homomorphism, and $\mathcal{B} = (B, \mu, \gamma)$ be an intuitionistic fuzzy SUP-subalgebra of B. Let $x_1, x_2 \in A$. Then

$$\begin{aligned}&\mu^f((x_1|_A(x_2|_Ax_2))|_A(x_1|_A(x_2|_Ax_2)))\\ =\ &\mu(f((x_1|_A(x_2|_Ax_2))|_A(x_1|_A(x_2|_Ax_2))))\\ =\ &\mu(f(x_1)|_B(f(x_2)|_Bf(x_2))|_B(f(x_1)|_B(f(x_2)|_Bf(x_2))))\\ \geq\ &\min\{\mu(f(x_1)), \mu(f(x_2))\}\\ =\ &\min\{\mu^f(x_1), \mu^f(x_2)\},\end{aligned}$$

$$\begin{aligned}&\gamma^f((x_1|_A(x_2|_Ax_2))|_A(x_1|_A(x_2|_Ax_2)))\\ =\ &\gamma(f((x_1|_A(x_2|_Ax_2))|_A(x_1|_A(x_2|_Ax_2))))\\ =\ &\gamma(f(x_1)|_B(f(x_2)|_Bf(x_2))|_B(f(x_1)|_B(f(x_2)|_Bf(x_2))))\\ \leq\ &\max\{\gamma(f(x_1)), \gamma(f(x_2))\}\\ =\ &\max\{\gamma^f(x_1), \gamma^f(x_2)\}.\end{aligned}$$

Hence, $B^f = (A, \mu^f, \gamma^f)$ is an intuitionistic fuzzy SUP-subalgebra of A.

Conversely, let $B^f = (A, \mu^f, \gamma^f)$ be an intuitionistic fuzzy SUP-subalgebra of A. Let $y_1, y_2 \in B$ be such that $f(x_1) = y_1$ and $f(x_2) = y_2$ for $x_1, x_2 \in A$. Then

$$\begin{aligned}&\mu((y_1|_B(y_2|_By_2))|_B(y_1|_B(y_2|_By_2)))\\ =\ &\mu((f(x_1)|_B(f(x_2)|_Bf(x_2)))|_B(f(x_1)|_B(f(x_2)|_Bf(x_2))))\\ =\ &\mu(f((x_1|_A(x_2|_Ax_2))|_A(x_1|_A(x_2|_Ax_2))))\\ =\ &\mu^f((x_1|_A(x_2|_Ax_2))|_A(x_1|_A(x_2|_Ax_2)))\\ \geq\ &\min\{\mu^f(x_1), \mu^f(x_2)\}\\ =\ &\min\{\mu(f(x_1)), \mu(f(x_2))\}\\ =\ &\min\{\mu(y_1), \mu(y_2)\},\end{aligned}$$

$$
\begin{aligned}
& \gamma((y_1|_B(y_2|_By_2))|_B(y_1|_B(y_2|_By_2))) \\
= \; & \gamma((f(x_1)|_B(f(x_2)|_Bf(x_2)))|_B(f(x_1)|_B(f(x_2)|_Bf(x_2)))) \\
= \; & \gamma(f((x_1|_A(x_2|_Ax_2))|_A(x_1|_A(x_2|_Ax_2)))) \\
= \; & \gamma^f((x_1|_A(x_2|_Ax_2))|_A(x_1|_A(x_2|_Ax_2))) \\
\leq \; & \max\{\gamma^f(x_1), \gamma^f(x_2)\} \\
= \; & \max\{\gamma(f(x_1)), \gamma(f(x_2))\} \\
= \; & \max\{\gamma(y_1), \gamma(y_2)\}.
\end{aligned}
$$

Hence, $\mathcal{B} = (B, \mu, \gamma)$ is an intuitionistic fuzzy SUP-subalgebra of B.

4. Conclusion

This study introduces the concepts of intuitionistic fuzzy SUP-subalgebras and level sets within SUP-algebras, emphasizing their significance in understanding neutrosophic logic in this context. We establish a crucial relationship between subalgebras and level sets, demonstrating that the level set of an intuitionistic fuzzy SUP-subalgebra is also a subalgebra, and vice versa. Furthermore, we show that the collection of all intuitionistic fuzzy SUP-subalgebras in an SUP-algebra forms a complete distributive lattice. Additionally, we highlight that while every intuitionistic fuzzy SUP-ideal is an intuitionistic fuzzy SUP-subalgebra, the reverse does not hold true, thereby illustrating the unique characteristics of intuitionistic fuzzy SUP-ideals within this algebraic structure.

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