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Existence of Solutions to a System of Fractional Differential Equations With Applications to Beam Deflections

Hasanen A. Hammad[1,2,*], Manal Elzain Mohamed Abdalla[3]

[1] *Department of Mathematics, College of Science, Qassim University, Buraydah 51452, Saudi Arabia*
[2] *Department of Mathematics, Faculty of Science, Sohag University, Sohag 82524, Egypt*
[3] *Department of Mathematics, College of Science and Arts, King Khalid University, Mahayil, Saudi Arabia*

Abstract. In this manuscript, we introduce a novel system of fractional differential equations incorporating both Caputo and conformable derivatives. We delve into the existence and uniqueness of solutions for this system, employing fixed-point techniques under appropriate conditions. To illustrate the practical applications of our theoretical findings, we investigate the traveling wave solutions of a tripled system of conformable fractional differential equations. Our analysis provides valuable insights into the dynamics and behavior of complex systems modeled by fractional differential equations.

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1. Background materials

Fractional differential equations (DEs) have gained significant attention due to their ability to model complex phenomena more accurately than classical integer-order DEs. Several analytical techniques have been developed to solve fractional DEs, including the Laplace transform, Fourier transform, Mellin transform, and operational calculus. These methods often involve transforming the fractional DE into an algebraic equation, solving the transformed equation, and then inverting the transform to obtain the solution. Additionally, various iterative methods like the Adomian decomposition method, variational iteration method, and homotopy perturbation method have been employed to find approximate solutions to nonlinear fractional DEs. These methods provide a powerful tool

*Corresponding author.
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Email addresses: `h.abdelwareth@qu.edu.sa` (H.A. Hammad), `mabdalla@kku.edu.sa` (M.E.M. Abdalla)

for investigating the behavior of systems described by fractional DEs, for more details; see[1, 3, 26, 30, 31].

The classical DEs can be extended to fractional DEs, where the order of differentiation might be non-integer or fractional. Since integer-order DEs and systems are unable to sufficiently represent complex systems with memory and hereditary qualities, these equations and systems have drawn a great deal of attention [2, 7, 10, 13, 14, 17, 28, 29, 37, 41, 46]. One of the formulations of fractional calculus (FC) that is frequently employed, the Caputo fractional derivative (CFD) [12, 28], is very helpful for initial value problems, which makes it appropriate for physical applications.

Fixed point (FP) theorems are a powerful tool in the study of fractional differential equations. By transforming fractional DEs into equivalent integral equations, FP techniques, such as the Banach contraction mapping principle or the Schauder FP theorem, can be applied to establish the existence and uniqueness of solutions. These techniques are particularly useful for nonlinear fractional DEs, where analytical solutions may be difficult to obtain. By employing FP methods, researchers can gain insights into the qualitative behavior of solutions, including their stability, boundedness, and asymptotic properties. See [4, 8, 9, 18–25, 32, 34, 38] for more information.

Beam deflection equations and systems are a significant area of study for fractional DEs. The ability of beams to sustain loads applied laterally to their axis makes them structural components. The bending and deflection of beams under varied stresses are described by the classical beam theory, which is frequently represented by fourth-order ordinary DEs [5, 6, 12, 29, 36, 43, 45]. Fractional derivatives (FDs), on the other hand, can give these models a more realistic depiction of materials with non-local behavior and viscoelastic characteristics, which are typical of sophisticated engineering materials and intricate structures.

Nevertheless, the inclusion of FDs in these models can yield a more precise portrayal of materials possessing viscoelastic characteristics and non-local behavior, which are prevalent in sophisticated engineering materials and intricate structures. These results in FD models that provide deeper insights and enhanced design skills in applied physics and engineering by more accurately predicting the deflection and dynamic response of beams.

Wang and Yang investigated in [43] whether positive solutions exist for the nonlinear fourth-order system that describes the deformation of an elastic beam as follows:

$$\begin{cases} \varphi^{(4)}(s) + \gamma_1 \varphi''(s) - \eta_1 \varphi(\zeta) = \ell_1\left(s, \varphi(s), \phi(s)\right), \ s \in V = [0,1], \\ \phi^{(4)}(s) + \gamma_2 \phi''(s) - \eta_2 \phi(\zeta) = \ell_2\left(s, \varphi(s), \phi(s)\right), \ s \in V, \end{cases}$$

via the stipulations

$$\begin{cases} \varphi(0) = \varphi(1) = \varphi''(0) = \varphi''(1) = 0, \\ \phi(0) = \phi(1) = \phi''(0) = \phi''(1) = 0, \end{cases}$$

where $\ell_i : V \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ are continuous functions, and $\gamma_i, \eta_i \in \mathbb{R}$ such that $\gamma_i \leq 2\pi^2$, $\frac{-\gamma_i^2}{4} \leq \eta_i$, $\frac{\eta_i}{\pi^4} + \frac{\gamma_i}{\pi^4} < 1$, $i = 1, 2$. The authors demonstrated the existence of positive solution results by providing a cone Q in $C(V) \times C(V)$. They built another successful solution outcome by building over a product cone.

The authors of [42] discussed the existence and uniqueness of solutions for the system that has Caputo sequential derivatives (CSDs) as the following:

$$\begin{cases} \left({}^cD^{\eta} + \vartheta\ {}^cD^{\eta-1}\right)\zeta(s) = \ell_1\left(s, \zeta(s), \omega(s), I_{0^+}^{r_1}\zeta(s), I_{0^+}^{r_2}\omega(s)\right), \ s \in V, \\ \left({}^cD^{\gamma} + \theta\ {}^cD^{\gamma-1}\right)\omega(s) = \ell_2\left(s, \zeta(s), \omega(s), I_{0^+}^{z_1}\zeta(s), I_{0^+}^{z_2}\omega(s)\right), \ s \in V, \end{cases}$$

under the constraints

$$\begin{cases} \zeta(0) = \zeta'(0) = 0, \ \zeta'(1) = 1, \ \zeta(1) = \int_0^1 \zeta(t)\, dR_1(t) + \int_0^1 \omega(t)\, dR_2(t), \\ \omega(0) = \omega'(0) = 0, \ \omega'(1) = 1, \ \omega(1) = \int_0^1 \zeta(t)\, dM_1(t) + \int_0^1 \omega(t)\, dM_2(t), \end{cases}$$

where $\eta, \gamma \in (3,4]$, $\vartheta, \theta > 0$, $r_1, r_2, z_1, z_2 > 0$, $\ell_1, \ell_2 : V \times \mathbb{R}^4 \to \mathbb{R}$ are continuous functions, $I_{0^+}^{r_1}, I_{0^+}^{r_2}, I_{0^+}^{z_1}, I_{0^+}^{z_2}$ are Riemann-Liouville (RL) integrals of orders r_1, r_2, z_1, z_2, and R_1, R_2, M_1, M_2 are Riemann-Stieltjes integrals. These systems have applications in biosciences (see to [7]). The authors discovered that the system's solutions exist and are unique.

The existence, uniqueness, and stability of the system in the Ulam-Hyers sense were examined by Bensassa et al. in [11]:

$$\begin{cases} D^{\eta_1}D^{\eta_2}\varphi(\zeta) = \ell_1(\zeta, \varphi(\zeta), \omega(\zeta)) + b_1\hbar_1(\zeta, \varphi(\zeta)) + c_1\varpi_1(\zeta, D^{\rho}\varphi(\zeta)), \\ D^{\gamma_1}D^{\gamma_2}\omega(\zeta) = \ell_2(\zeta, \varphi(\zeta), \omega(\zeta)) + b_2\hbar_2(\zeta, \omega(\zeta)) + c_2\varpi_2(\zeta, D^{\rho}\omega(\zeta)), \end{cases}$$

under the stipulations

$$\begin{cases} \varphi(0) = \varphi(1) = b, \quad \varphi'(0) = \varphi'(1) = 0, \\ \omega(0) = \omega(1) = c, \quad \omega'(0) = \omega'(1) = 0, \end{cases}$$

where $D^{\eta_i}, D^{\gamma_i}, D^{\rho}$ are CFDs, $\rho \in (0,1]$, $\ell_1, \ell_2 \in C([0,1] \times \mathbb{R} \times \mathbb{R}; \mathbb{R})$, $b_1, b_2, c_1, c_2 \in \mathbb{R}$, and $\hbar_1, \hbar_2, \varpi_1, \varpi_2 \in C([0,1] \times \mathbb{R}; \mathbb{R})$.

Drawing on the previously mentioned research and specifically from paper [43], the current study will address the following CSDs problem:

$$\begin{cases} D^{\eta_1}D^{\gamma_1}\varphi_1(\zeta) = \eth_1 Z_1\left(\zeta, \varphi_1(\zeta), \varphi_2(\zeta), D^{\rho-1}\varphi_1(\zeta), D^{\rho}\varphi_1(\zeta)\right) \\ D^{\eta_2}D^{\gamma_2}\varphi_2(\zeta) = \eth_2 Z_2\left(\zeta, \varphi_1(\zeta), \varphi_2(\zeta), D^{\rho-1}\varphi_2(\zeta), D^{\rho}\varphi_2(\zeta)\right) \end{cases} \tag{1}$$

with the conditions

$$\begin{cases} \varphi_j(0) = \varphi_j(1) = b_j \in \mathbb{R}, \\ D^{\gamma_j}\varphi_j(0) = D^{\gamma_j}\varphi_j(1) = 0, \ j = 1, 2, \end{cases} \tag{2}$$

where D^{η_1}, D^{γ_1}, and D^{ρ} are CFDs.

In order to ensure that there are no semi-group and commutativity properties on the CFD and to derive, as a specific case, the aforementioned Wang and Yang fourth-order system under various circumstances, we also assume that $\eta_j \in (2,3]$, $\gamma_j \in (0,1]$, and $\eth_j > 0$ $j = 1, 2$. It is also assumed that $\rho \in (1,2]$. As a limiting case of (1), we can achieve the aforesaid Wang and Yang [43] problem, where $Z_i : [0,1] \times \mathbb{R}^4 \to \mathbb{R}$ and $j = 1, 2$ are some continuous functions.

It is noteworthy that our problem is more substantial than the studies cited above, and that most of them are not the same because we include CFD in the beginning circumstances. This inclusion strengthens our problem's relevance and applicability in physical environments by adding a layer of complexity and realism. These constraints are compatible with observable initial circumstances and effective numerical techniques, enabling improved capturing of memory effects and non-local behaviors of the system.

In addition, our system has three generic functions, Z_1, Z_2 and Z_3, which broaden the system's applicability beyond issues that have already been researched and offer a more thorough framework for comprehending the dynamics of beam deflection. In engineering and applied physics, this more comprehensive approach enables the modeling of more complicated situations, leading to deeper insights and enhanced forecasting skills.

Furthermore, the problem (1) fills the gap that exists between contemporary viscoelastic models and traditional beam deflection theories. In particular, our problem (1) may be reduced to the above fourth-order ODEs that describe the deformation of an elastic beam [43] if we consider the specific situation:

$$Z_j\left(\zeta, \varphi_1(\zeta), \varphi_2(\zeta), D^{\rho-1}\varphi_1(\zeta), D^{\rho}\varphi_1(\zeta)\right) = \ell_j\left(\zeta, \varphi_1(\zeta), \varphi_2(\zeta)\right) - b_j\varphi_j'' + a_j\varphi_j',$$

where $\rho = 2$, $\eta_j = 3$, $\gamma_j = 1$, for $j = 1, 2$. Furthermore, (1) offers a powerful tool for more in-depth and accurate research on a variety of physical phenomena. Because of this, our work is conceptually stimulating as well as practically important.

The Tanh approach [15, 16] will be applied in the second section of our study to get new traveling wave solutions for the tripled beam issue with generalized conformable FDs [27] as follows:

$$\begin{cases} \Im_s^{2\eta}\varphi + \Im_\zeta^{4\gamma}\varphi + \eth_1 K\left(\varphi, \omega, \cdots\right) = 0, \\ \Im_s^{2\eta}\omega + \Im_\zeta^{4\gamma}\omega + \eth_2 N\left(\varphi, \omega, \cdots\right) = 0. \end{cases}$$

The relationship and utility of FDs-both Caputo and conformable-in tackling and resolving complex mathematical issues are illustrated via the application of fractional calculus to expand and improve classical models. A range of complex systems can be studied and solved with the help of different FDs, as demonstrated by this cohesive method.

The Tanh method was chosen because it is generally less computationally demanding than other approaches, and it is also very simple to apply. Because of this, it's a useful tool for problem solving. It also offers explicit analytical solutions, which are helpful for verifying numerical simulations and comprehending the qualitative behavior of the solutions. The Tanh technique aids in understanding the physical processes, such as wave propagation, solitons, and other localized structures, represented by the equations by providing correct traveling wave solutions. The Tanh technique is a useful tool for studying traveling waves in nonlinear equations because of these benefits. To obtain additional information about this approach and other significant related methods, see articles [33, 35, 44].

The results of this study have significant implications for engineering and applied physics, particularly in areas where beam deflection models with fractional derivatives could offer a more accurate and comprehensive representation of real-world phenomena. By considering the memory and hereditary effects inherent in materials, these models can provide valuable insights into the long-term behavior of structures and systems. This could lead to improved design and analysis techniques, enabling engineers to develop more reliable and efficient structures. Additionally, the application of fractional calculus to beam deflection problems can contribute to advancements in fields such as vibration analysis, control systems, and materials science, ultimately leading to innovative solutions and technological breakthroughs.

2. Basic facts

This section is devoted to presenting some definitions and lemmas concerning the FC. We can find these results in [12, 28].

Definition 1. *For any function $\varpi \in C(V)$, $V = [0,1]$, the fractional integral of order η is described as*

$$I^{\eta}\varpi(s) = \frac{1}{\Gamma(\eta)} \int_0^s (s-r)^{\eta-1} \varpi(r)\, dr,$$

where $\Gamma(.)$ is the Euler gamma function.

Definition 2. *For the function $\varpi \in C^m(V)$, the CFD of order η is given by*

$$D^{\eta}\varpi(s) = \frac{1}{\Gamma(m-\eta)} \int_0^s (s-r)^{m-\eta-1} \varpi^{(m)}(r)\, dr, \ m = [\eta] + 1,$$

where $[\eta]$ represents the integer part of η.

Lemma 1. *[28] Assume that $\eta > 0$, the solution of the equation*

$$D^{\eta}\varpi(s) = 0, \ s \in V$$

is written as

$$\varpi(s) = k_0 + k_1 s + k_2 s^2 + \cdots + k_{m-1} s^{m-1},$$

where $k_v \in \mathbb{R}$, $v = 0, 1, 2, \cdots, m-1$, $m = [\eta] + 1$.

Lemma 2. *[28] For any $\eta > 0$, we have*

$$I_{0^+}^{\eta} D^{\eta}\varpi(s) = \varpi(s) + k_0 + k_1 s + k_2 s^2 + \cdots + k_{m-1} s^{m-1},$$

where $k_v \in \mathbb{R}$, $v = 0, 1, 2, \cdots, m-1$, $m = [\eta] + 1$.

Remark 1. *For the function $\varpi \in C(V)$, we have the following results:*

(i) For each $\eta, \gamma > 0$ and for all $s \in V$, $I^{\eta} I^{\gamma}\varpi(s) = I^{\eta+\gamma}\varpi(s)$;

(ii) According to Definition 2 and the property (i), if $0 < \eta \leq \gamma$, we get $D^{\eta} I^{\gamma}\varpi(s) = I^{\eta-\gamma}\varpi(s)$;

(iii) If $\eta = \gamma$ in (ii), we have $D^{\eta} I^{\gamma} \varpi(s) = \varpi(s)$.

It is very important to present the equivalent integral eqution to the problem (1).

Lemma 3. *Assume that $R_1, R_2 \in C(V, \mathbb{R})$, $\eta_j \in (2,3]$, $\gamma_j \in (0,1]$, and $\eth_j > 0$, $j = 1.2$. The problem*

$$\begin{cases} D^{\eta_1} D^{\gamma_1} \varphi_1(\zeta) = \eth_1 R_1(\zeta), \\ D^{\eta_2} D^{\gamma_2} \varphi_2(\zeta) = \eth_2 R_2(\zeta), \end{cases} \tag{3}$$

via the conditions

$$\begin{cases} \varphi_j(0) = \varphi_j(1) = b_j \in \mathbb{R}, \\ D^{\lambda_j} \varphi_j(0) = D^{\lambda_j} \varphi_j(1) = 0, \end{cases}$$

is equivalent to

$$\begin{cases} \varphi_1(\zeta) = b_1 + \eth_1 I^{\eta_1+\gamma_1} R_1(\zeta) \\ + \left(\frac{2\eth_1}{\gamma_1 \Gamma(\eta_1)} \int_0^1 (1-r)^{\eta_1-1} R_1(r)\, dr - \frac{\eth_1 \Gamma(\gamma_1+3)}{\gamma_1 \Gamma(\eta_1+\gamma_1)} \int_0^1 (1-r)^{\eta_1+\gamma_1-1} R_1(r)\, dr \right) \zeta^{\gamma_1+1} \\ + \left(\frac{\eth_1 \Gamma(\gamma_1+3)}{\gamma_1 \Gamma(\eta_1+\gamma_1)} \int_0^1 (1-r)^{\eta_1+\gamma_1-1} R_1(r)\, dr - \frac{\eth_1 (\gamma_1+2)}{\gamma_1 \Gamma(\eta_1)} \int_0^1 (1-r)^{\eta_1-1} R_1(r)\, dr \right) \zeta^{\gamma_1+2}, \\ \varphi_2(\zeta) = b_2 + \eth_2 I^{\eta_2+\gamma_2} R_2(\zeta) \\ + \left(\frac{2\eth_2}{\gamma_2 \Gamma(\eta_2)} \int_0^1 (1-r)^{\eta_2-1} R_2(r)\, dr - \frac{\eth_2 \Gamma(\gamma_2+3)}{\gamma_2 \Gamma(\eta_2+\gamma_2)} \int_0^1 (1-r)^{\eta_2+\gamma_2-1} R_2(r)\, dr \right) \zeta^{\gamma_2+1} \\ + \left(\frac{\eth_2 \Gamma(\gamma_2+3)}{\gamma_2 \Gamma(\eta_2+\gamma_2)} \int_0^1 (1-r)^{\eta_2+\gamma_2-1} R_2(r)\, dr - \frac{\eth_2 (\gamma_2+2)}{\gamma_2 \Gamma(\eta_2)} \int_0^1 (1-r)^{\eta_2-1} R_2(r)\, dr \right) \zeta^{\gamma_2+2}. \end{cases} \tag{4}$$

Proof. Applying $I^{\eta_j+\gamma_j}$, $(j = 1,2)$ on (3) and using Lemma 2, we have

$$\begin{cases} \varphi_1(\zeta) = \eth_1 I^{\eta_1+\gamma_1} R_1(\zeta) + \frac{k_0 \zeta^{\gamma_1}}{\Gamma(\gamma_1+1)} + \frac{k_1 \zeta^{\gamma_1+1}}{\Gamma(\gamma_1+2)} + \frac{k_2 \zeta^{\gamma_1+2}}{\Gamma(\gamma_1+3)} + k_3, \\ \varphi_2(\zeta) = \eth_2 I^{\eta_2+\gamma_2} R_2(\zeta) + \frac{c_0 \zeta^{\gamma_2}}{\Gamma(\gamma_2+1)} + \frac{c_1 \zeta^{\gamma_2+1}}{\Gamma(\gamma_2+2)} + \frac{c_2 \zeta^{\gamma_2+2}}{\Gamma(\gamma_2+3)} + c_3. \end{cases} \tag{5}$$

Because $\varphi_j(0) = b_j$ $(j = 1,2)$, we conclude that $k_3 = b_1$, and $c_3 = b_2$. Applying D^{γ_j} $(j = 1,2)$ on (5), we get

$$\begin{cases} D^{\gamma_1} \varphi_1(\zeta) = \frac{\eth_1}{\Gamma(\eta_1)} \int_0^{\zeta} (\zeta-r)^{\eta_1-1} R_1(r)\, dr + k_0 + k_1 \zeta + k_2 \zeta^2, \\ D^{\gamma_2} \varphi_2(\zeta) = \frac{\eth_2}{\Gamma(\eta_2)} \int_0^{\zeta} (\zeta-r)^{\eta_2-1} R_2(r)\, dr + c_0 + c_1 \zeta + c_2 \zeta^2. \end{cases} \tag{6}$$

Letting $\zeta = 0$ in (5), we have $k_0 = c_0 = 0$. Taking $\zeta = 1$ in (5) and (6), we obtain that

$$\begin{cases} \frac{\eth_1}{\Gamma(\eta_1+\gamma_1)} \int_0^1 (1-r)^{\eta_1+\gamma_1-1} R_1(r)\, dr + \frac{k_1}{\Gamma(\gamma_1+2)} + \frac{2k_2}{\Gamma(\gamma_1+3)} = 0, \\ \frac{\eth_1}{\Gamma(\eta_1)} \int_0^1 (1-r)^{\eta_1-1} R_1(r)\, dr + k_1 + k_2 = 0, \\ \frac{\eth_2}{\Gamma(\eta_2+\gamma_2)} \int_0^1 (1-r)^{\eta_2+\gamma_2-1} R_2(r)\, dr + \frac{c_1}{\Gamma(\gamma_2+2)} + \frac{2c_2}{\Gamma(\gamma_2+3)} = 0, \\ \frac{\eth_2}{\Gamma(\eta_2)} \int_0^1 (1-r)^{\eta_2-1} R_2(r)\, dr + c_1 + c_2 = 0. \end{cases}$$

After solving the above equations, we have

$$\begin{cases} k_1 = -\frac{\eth_1 \Gamma(\gamma_1+3)}{\gamma_1 \Gamma(\eta_1+\gamma_1)} \int_0^1 (1-r)^{\eta_1+\gamma_1-1} R_1(r)\, dr + \frac{2\eth_1}{\gamma_1 \Gamma(\eta_1)} \int_0^1 (1-r)^{\eta_1-1} R_1(r)\, dr, \\ k_2 = \frac{\eth_1 \Gamma(\gamma_1+3)}{\gamma_1 \Gamma(\eta_1+\gamma_1)} \int_0^1 (1-r)^{\eta_1+\gamma_1-1} R_1(r)\, dr - \frac{\eth_1 (\gamma_1+2)}{\gamma_1 \Gamma(\eta_1)} \int_0^1 (1-r)^{\eta_1-1} R_1(r)\, dr, \\ c_1 = -\frac{\eth_2 \Gamma(\gamma_2+3)}{\gamma_2 \Gamma(\eta_2+\gamma_2)} \int_0^1 (1-r)^{\eta_2+\gamma_2-1} R_2(r)\, dr + \frac{2\eth_2}{\gamma_2 \Gamma(\eta_2)} \int_0^1 (1-r)^{\eta_2-1} R_2(r)\, dr, \\ c_2 = \frac{\eth_2 \Gamma(\gamma_2+3)}{\gamma_2 \Gamma(\eta_2+\gamma_2)} \int_0^1 (1-r)^{\eta_2+\gamma_2-1} R_2(r)\, dr - \frac{\eth_2 (\gamma_2+2)}{\gamma_2 \Gamma(\eta_2)} \int_0^1 (1-r)^{\eta_2-1} R_2(r)\, dr. \end{cases}$$

Substituting from the values of k_j and c_j $(j = 1, 2)$ in (5), we get (4).

3. Fixed point approaches

In this section, we illustrate that the mechanism of the FP for discussing the existence of the solution of our problem.

We define the Banach space (BS)

$$\Xi = \left\{\varphi \in C(V, \mathbb{R}),\ D^{\rho-1}\varphi \in C(V, \mathbb{R}),\ \text{and } D^{\rho}\varphi \in C(V, \mathbb{R})\right\}$$

under the norm

$$\|\varphi\|_{\Xi} = \|\varphi\|_{\infty} + \left\|D^{\rho-1}\varphi\right\|_{\infty} + \|D^{\rho}\varphi\|_{\infty},$$

where

$$\|\varphi\|_{\infty} = \sup_{\zeta\in V}|\varphi(\zeta)|,\ \ \left\|D^{\rho-1}\varphi\right\|_{\infty} = \sup_{\zeta\in V}\left|D^{\rho-1}\varphi(\zeta)\right|,\ \text{and}\ \|D^{\rho}\varphi\|_{\infty} = \sup_{\zeta\in V}|D^{\rho}\varphi(\zeta)|,\ V = [0,1].$$

Consider the product space $\Xi \times \Xi$ defined on the norm

$$\|(\varphi, \omega)\|_{\Xi\times\Xi} = \|\varphi\|_{\Xi} + \|\omega\|_{\Xi}.$$

Clearly, the pair $(\Xi \times \Xi, \|.\|)$ is a BS.

Now, in order to apply the technique of the FP, we consider the operator $\mho = (\mho_1, \mho_2)$ such that $\mho : \Xi \times \Xi \to \Xi \times \Xi$. This operator can be written as

$$\mho(\varphi(\zeta), \omega(\zeta)) = \mho_1(\varphi(\zeta), \omega(\zeta)) + \mho_2(\varphi(\zeta), \omega(\zeta)),$$

where

$$\begin{aligned}
&\mho_j(\varphi(\zeta), \omega(\zeta))\\
=\ & b_j + \eth_j I^{\eta_j+\gamma_j} Z_j\left(\zeta, \varphi(\zeta), \omega(\zeta), D^{\rho-1}\varphi(\zeta), D^{\rho}\omega(\zeta)\right)\\
&+\left(\frac{2\eth_j}{\gamma_j\Gamma(\eta_j)}\int_0^1 (1-r)^{\eta_j-1} Z_j\left(r, \varphi(r), \omega(r), D^{\rho-1}\varphi(r), D^{\rho}\omega(r)\right)dr\right.\\
&\left.-\frac{\eth_j\Gamma(\gamma_j+3)}{\gamma_j\Gamma(\eta_j+\gamma_j)}\int_0^1 (1-r)^{\eta_j+\gamma_j-1} Z_j\left(r, \varphi(r), \omega(r), D^{\rho-1}\varphi(r), D^{\rho}\omega(r)\right)dr\right)\zeta^{\gamma_j+1}\\
&+\left(\frac{\eth_j\Gamma(\gamma_j+3)}{\gamma_j\Gamma(\eta_j+\gamma_j)}\int_0^1 (1-r)^{\eta_j+\gamma_j-1} Z_j\left(r, \varphi(r), \omega(r), D^{\rho-1}\varphi(r), D^{\rho}\omega(r)\right)dr\right.\\
&\left.-\frac{\eth_j(\gamma_j+2)}{\gamma_j\Gamma(\eta_j)}\int_0^1 (1-r)^{\eta_j-1} Z_j\left(r, \varphi(r), \omega(r), D^{\rho-1}\varphi(r), D^{\rho}\omega(r)\right)dr\right)\zeta^{\gamma_j+2}
\end{aligned}$$

Therefore, the FP of the operator $\mho$ is equivalent to the solution of the system (1).

4. Existence and uniqueness results

We begin this part with the following assertion:

(A$_1$) For all $\zeta \in V$ and $(\varphi, \omega, \delta, \varrho), \left(\widetilde{\varphi}, \widetilde{\omega}, \widetilde{\delta}, \widetilde{\varrho}\right) \in \mathbb{R}^4$, there exists a matrix of positive functions M_{lj}, $l = 1, 2, 3, 4$, $j = 1, 2$ such that

$$\begin{aligned} \left|Z_j(\zeta, \varphi, \omega, \delta, \varrho) - Z_j\left(\zeta, \widetilde{\varphi}, \widetilde{\omega}, \widetilde{\delta}, \widetilde{\varrho}\right)\right| &\leq M_{1j}(\zeta)\left|\varphi - \widetilde{\varphi}\right| + M_{2j}(\zeta)\left|\omega - \widetilde{\omega}\right| \\ &\quad + M_{3j}(\zeta)\left|\delta - \widetilde{\delta}\right| + M_{1j}(\zeta)\left|\varrho - \widetilde{\varrho}\right|, \end{aligned}$$

with $m_{lj} = \sup_{\zeta \in V}\{M_{lj}\}$.

The first main result of this part is as follows:

Theorem 1. *Assume that the assertion (A_1) is true. The system (1) under conditions (3) admits a unique solution, provided that $\Omega_1 + \Omega_2 < 1$, where*

$$\Omega_j = B_j \max\{(m_{1j} + m_{3j}), (m_{2j} + m_{4j})\}, \ j = 1, 2,$$

and

$$\begin{aligned} B_j &= \eth_j \left\{ \left(\frac{\gamma_j + 2\Gamma(\gamma_j + 3)}{\gamma_j \Gamma(\eta_j + \gamma_j)} + \frac{\gamma_j + 4}{\gamma_j \Gamma(\eta_j)} \right) + \frac{1}{\Gamma(\eta_j + \gamma_j - \rho + 2)} \right. \\ &\quad + \frac{\Gamma(\gamma_j + 2)}{\gamma_j \Gamma(\eta_j + 1)\Gamma(\gamma_j - \rho + 4)} \left[2(\gamma_j - \rho + 3) + (\gamma_j + 2)^2 \right] \\ &\quad + \frac{\Gamma(\gamma_j + 2)\Gamma(\gamma_j + 3)}{\gamma_j \Gamma(\gamma_j + \eta_j + 1)\Gamma(\gamma_j - \rho + 4)} [2\gamma_j - \rho + 5] + \frac{1}{\Gamma(\eta_j + \gamma_j - \rho + 1)} \\ &\quad + \frac{\Gamma(\gamma_j + 2)}{\gamma_j \Gamma(\eta_j + 1)\Gamma(\gamma_j - \rho + 3)} \left[2(\gamma_j - \rho + 2) + (\gamma_j + 2)^2 \right] \\ &\quad \left. + \frac{\Gamma(\gamma_j + 2)\Gamma(\gamma_j + 3)}{\gamma_j \Gamma(\gamma_j + \eta_j + 1)\Gamma(\gamma_j - \rho + 3)} [2\gamma_j - \rho + 4] \right\} \end{aligned}$$

for $j = 1, 2$.

Proof. It is enough to prove that the mapping $\mho$ is a Banach contraction principle to finish the proof.

Assume that (φ_1, ω_1) and (φ_2, ω_2) be arbitrary elements in $\Xi \times \Xi$. Then for each $\zeta \in V$ and for $j = 1, 2$, we have

$$\begin{aligned} &\left|\mho_j(\varphi_1(\zeta), \omega_1(\zeta)) - \mho_j(\varphi_2(\zeta), \omega_2(\zeta))\right| \\ \leq\ & \frac{\eth_j}{\Gamma(\eta_j + \gamma_j)} \left| \int_0^\zeta (\zeta - r)^{\eta_j + \gamma_j - 1} \left[Z_j\left(r, \varphi_1(r), \omega_1(r), D^{\rho-1}\varphi_1(r), D^\rho \omega_1(r)\right) \right. \right. \\ & \left. \left. - \ Z_j\left(r, \varphi_2(r), \omega_2(r), D^{\rho-1}\varphi_2(r), D^\rho \omega_2(r)\right) \right] dr \right| \end{aligned}$$

$$
\begin{aligned}
&+\frac{2\eth_j}{\gamma_j\Gamma(\eta_j)}\left|\int_0^1 (1-r)^{\eta_j-1}\left[Z_j\left(r,\varphi_1(r),\omega_1(r),D^{\rho-1}\varphi_1(r),D^{\rho}\omega_1(r)\right)\right.\right.\\
&\left.\left.-\ Z_j\left(r,\varphi_2(r),\omega_2(r),D^{\rho-1}\varphi_2(r),D^{\rho}\omega_2(r)\right)\right]dr\right|\\
&+\frac{\eth_j\Gamma(\gamma_j+3)}{\gamma_j\Gamma(\eta_j+\gamma_j)}\left|\int_0^1 (1-r)^{\eta_j+\gamma_j-1}\left[Z_j\left(r,\varphi_1(r),\omega_1(r),D^{\rho-1}\varphi_1(r),D^{\rho}\omega_1(r)\right)\right.\right.\\
&\left.\left.-\ Z_j\left(r,\varphi_2(r),\omega_2(r),D^{\rho-1}\varphi_2(r),D^{\rho}\omega_2(r)\right)\right]dr\right|\\
&+\frac{\eth_j\Gamma(\gamma_j+3)}{\gamma_j\Gamma(\eta_j+\gamma_j)}\left|\int_0^1 (1-r)^{\eta_j+\gamma_j-1}\left[Z_j\left(r,\varphi_1(r),\omega_1(r),D^{\rho-1}\varphi_1(r),D^{\rho}\omega_1(r)\right)\right.\right.\\
&\left.\left.-\ Z_j\left(r,\varphi_2(r),\omega_2(r),D^{\rho-1}\varphi_2(r),D^{\rho}\omega_2(r)\right)\right]dr\right|\\
&+\frac{\eth_j(\gamma_j+2)}{\gamma_j\Gamma(\eta_j)}\left|\int_0^1 (1-r)^{\eta_j-1}\left[Z_j\left(r,\varphi_1(r),\omega_1(r),D^{\rho-1}\varphi_1(r),D^{\rho}\omega_1(r)\right)\right.\right.\\
&\left.\left.-\ Z_j\left(r,\varphi_2(r),\omega_2(r),D^{\rho-1}\varphi_2(r),D^{\rho}\omega_2(r)\right)\right]dr\right|\\
=\ &\frac{\eth_j\gamma_j+2\eth_j\Gamma(\gamma_j+3)}{\gamma_j\Gamma(\eta_j+\gamma_j)}\left|\int_0^\zeta (\zeta-r)^{\eta_j+\gamma_j-1}\left[Z_j\left(r,\varphi_1(r),\omega_1(r),D^{\rho-1}\varphi_1(r),D^{\rho}\omega_1(r)\right)\right.\right.\\
&\left.\left.-\ Z_j\left(r,\varphi_2(r),\omega_2(r),D^{\rho-1}\varphi_2(r),D^{\rho}\omega_2(r)\right)\right]dr\right|\\
&+\frac{\eth_j(\gamma_j+4)}{\gamma_j\Gamma(\eta_j)}\left|\int_0^1 (1-r)^{\eta_j-1}\left[Z_j\left(r,\varphi_1(r),\omega_1(r),D^{\rho-1}\varphi_1(r),D^{\rho}\omega_1(r)\right)\right.\right.\\
&\left.\left.-\ Z_j\left(r,\varphi_2(r),\omega_2(r),D^{\rho-1}\varphi_2(r),D^{\rho}\omega_2(r)\right)\right]dr\right|.
\end{aligned}
$$

Applying the hypothesis (A_1), we have

$$
\begin{aligned}
&\left\|\mho_j(\varphi_1,\omega_1)-\mho_j(\varphi_2,\omega_2)\right\|_\infty\\
\leq\ &\left(\frac{\eth_j\gamma_j+2\eth_j\Gamma(\gamma_j+3)}{\gamma_j\Gamma(\eta_j+\gamma_j)}+\frac{\eth_j(\gamma_j+4)}{\gamma_j\Gamma(\eta_j)}\right)\left(M_{1j}\left\|\varphi_1-\varphi_2\right\|_\infty+M_{2j}\left\|\omega_1-\omega_2\right\|_\infty\right.\\
&\left.+M_{3j}\left\|D^{\rho-1}\varphi_1-D^{\rho-1}\varphi_2\right\|_\infty+M_{4j}\left\|D^{\rho}\omega_1-D^{\rho}\omega_2\right\|_\infty\right). \qquad (7)
\end{aligned}
$$

Using the CFD, we get

$$
\begin{aligned}
&D^{\rho-1}\left(\mho_j(\varphi(\zeta),\omega(\zeta))\right)\\
=\ &\eth_j I^{\eta_j+\gamma_j-\rho+1}Z_j\left(\zeta,\varphi(\zeta),\omega(\zeta),D^{\rho-1}\varphi(\zeta),D^{\rho}\omega(\zeta)\right)\\
&+\frac{\eth_j(\gamma_j+2)}{\Gamma(\gamma_j-\rho+3)}\zeta^{\gamma_j-\rho+2}\left(\frac{2}{\gamma_j\Gamma(\eta_j)}\int_0^1 (1-r)^{\eta_j-1}Z_j\left(r,\varphi(r),\omega(r),D^{\rho-1}\varphi(r),D^{\rho}\omega(r)\right)dr\right.\\
&\left.-\ \frac{\Gamma(\gamma_j+3)}{\gamma_j\Gamma(\eta_j+\gamma_j)}\int_0^1 (1-r)^{\eta_j+\gamma_j-1}Z_j\left(r,\varphi(r),\omega(r),D^{\rho-1}\varphi(r),D^{\rho}\omega(r)\right)dr\right)\\
&+\frac{\eth_j(\gamma_j+3)}{\Gamma(\gamma_j-\rho+4)}\zeta^{\gamma_j-\rho+3}\left(\frac{\Gamma(\gamma_j+3)}{\gamma_j\Gamma(\eta_j+\gamma_j)}\int_0^1 (1-r)^{\eta_j+\gamma_j-1}Z_j\left(r,\varphi(r),\omega(r),D^{\rho-1}\varphi(r),D^{\rho}\omega(r)\right)dr\right.\\
&\left.-\ \frac{\gamma_j+2}{\gamma_j\Gamma(\eta_j)}\int_0^1 (1-r)^{\eta_j-1}Z_j\left(r,\varphi(r),\omega(r),D^{\rho-1}\varphi(r),D^{\rho}\omega(r)\right)dr\right),
\end{aligned}
$$

and

$$\begin{aligned}
& D^{\rho}\left(\mho_j\left(\varphi(\zeta),\omega(\zeta)\right)\right) \\
= \; & \eth_j I^{\eta_j+\gamma_j-\rho} Z_j\left(\zeta,\varphi(\zeta),\omega(\zeta),D^{\rho-1}\varphi(\zeta),D^{\rho}\omega(\zeta)\right) \\
& +\frac{\eth_j\left(\gamma_j+2\right)}{\Gamma\left(\gamma_j-\rho+2\right)}\zeta^{\gamma_j-\rho+1}\left(\frac{2}{\gamma_j\Gamma\left(\eta_j\right)}\int_0^1(1-r)^{\eta_j-1}Z_j\left(r,\varphi(r),\omega(r),D^{\rho-1}\varphi(r),D^{\rho}\omega(r)\right)dr\right. \\
& \left.-\frac{\Gamma\left(\gamma_j+3\right)}{\gamma_j\Gamma\left(\eta_j+\gamma_j\right)}\int_0^1(1-r)^{\eta_j+\gamma_j-1}Z_j\left(r,\varphi(r),\omega(r),D^{\rho-1}\varphi(r),D^{\rho}\omega(r)\right)dr\right) \\
& +\frac{\eth_j\left(\gamma_j+3\right)}{\Gamma\left(\gamma_j-\rho+3\right)}\zeta^{\gamma_j-\rho+2}\left(\frac{\Gamma\left(\gamma_j+3\right)}{\gamma_j\Gamma\left(\eta_j+\gamma_j\right)}\int_0^1(1-r)^{\eta_j+\gamma_j-1}Z_j\left(r,\varphi(r),\omega(r),D^{\rho-1}\varphi(r),D^{\rho}\omega(r)\right)dr\right. \\
& \left.-\frac{\gamma_j+2}{\gamma_j\Gamma\left(\eta_j\right)}\int_0^1(1-r)^{\eta_j-1}Z_j\left(r,\varphi(r),\omega(r),D^{\rho-1}\varphi(r),D^{\rho}\omega(r)\right)dr\right).
\end{aligned}$$

By using the hypothesis (A_1), and similar to (7), we get

$$\begin{aligned}
& \left\|D^{\rho-1}\mho_j\left(\varphi_1,\omega_1\right)-D^{\rho-1}\mho_j\left(\varphi_2,\omega_2\right)\right\|_{\infty} \\
\leq \; & \left(\frac{\eth_j}{\Gamma\left(\eta_j+\gamma_j-\rho+2\right)}+\frac{\eth_j\Gamma\left(\gamma_j+2\right)}{\gamma_j\Gamma\left(\eta_j+1\right)\Gamma\left(\gamma_j-\rho+4\right)}\left[2\left(\gamma_j-\rho+3\right)+\left(\gamma_j+2\right)^2\right]\right. \\
& \left.+\frac{\eth_j\Gamma\left(\gamma_j+2\right)\Gamma\left(\gamma_j+3\right)}{\gamma_j\Gamma\left(\gamma_j+\eta_j+1\right)\Gamma\left(\gamma_j-\rho+4\right)}\left[2\gamma_j-\rho+5\right]\right)\left[M_{1j}\left\|\varphi_1-\varphi_2\right\|_{\infty}+M_{2j}\left\|\omega_1-\omega_2\right\|_{\infty}\right. \\
& \left.+M_{3j}\left\|D^{\rho-1}\varphi_1-D^{\rho-1}\varphi_2\right\|_{\infty}+M_{4j}\left\|D^{\rho}\omega_1-D^{\rho}\omega_2\right\|_{\infty}\right] \qquad (8)
\end{aligned}$$

Replacing $\rho-1$ with ρ in (8) except in the term within the large parentheses, one has

$$\begin{aligned}
& \left\|D^{\rho}\mho_j\left(\varphi_1,\omega_1\right)-D^{\rho}\mho_j\left(\varphi_2,\omega_2\right)\right\|_{\infty} \\
\leq \; & \left(\frac{\eth_j}{\Gamma\left(\eta_j+\gamma_j-\rho+1\right)}+\frac{\eth_j\Gamma\left(\gamma_j+2\right)}{\gamma_j\Gamma\left(\eta_j+1\right)\Gamma\left(\gamma_j-\rho+3\right)}\left[2\left(\gamma_j-\rho+2\right)+\left(\gamma_j+2\right)^2\right]\right. \\
& \left.+\frac{\eth_j\Gamma\left(\gamma_j+2\right)\Gamma\left(\gamma_j+3\right)}{\gamma_j\Gamma\left(\gamma_j+\eta_j+1\right)\Gamma\left(\gamma_j-\rho+3\right)}\left[2\gamma_j-\rho+4\right]\right)\left[M_{1j}\left\|\varphi_1-\varphi_2\right\|_{\infty}+M_{2j}\left\|\omega_1-\omega_2\right\|_{\infty}\right. \\
& \left.+M_{1j}\left\|D^{\rho-1}\varphi_1-D^{\rho-1}\varphi_2\right\|_{\infty}+M_{2j}\left\|D^{\rho}\omega_1-D^{\rho}\omega_2\right\|_{\infty}\right] \qquad (9)
\end{aligned}$$

It follows from (7)-(9) and the definition of the norm on Ξ that

$$\begin{aligned}
& \left\|\mho_j\left(\varphi_1,\omega_1\right)-\mho_j\left(\varphi_2,\omega_2\right)\right\|_{\Xi} \\
\leq \; & \eth_j\left\{\left(\frac{\gamma_j+2\Gamma\left(\gamma_j+3\right)}{\gamma_j\Gamma\left(\eta_j+\gamma_j\right)}+\frac{\gamma_j+4}{\gamma_j\Gamma\left(\eta_j\right)}\right)+\frac{1}{\Gamma\left(\eta_j+\gamma_j-\rho+2\right)}\right. \\
& +\frac{\Gamma\left(\gamma_j+2\right)}{\gamma_j\Gamma\left(\eta_j+1\right)\Gamma\left(\gamma_j-\rho+4\right)}\left[2\left(\gamma_j-\rho+3\right)+\left(\gamma_j+2\right)^2\right] \\
& +\frac{\Gamma\left(\gamma_j+2\right)\Gamma\left(\gamma_j+3\right)}{\gamma_j\Gamma\left(\gamma_j+\eta_j+1\right)\Gamma\left(\gamma_j-\rho+4\right)}\left[2\gamma_j-\rho+5\right]+\frac{1}{\Gamma\left(\eta_j+\gamma_j-\rho+1\right)} \\
& +\frac{\Gamma\left(\gamma_j+2\right)}{\gamma_j\Gamma\left(\eta_j+1\right)\Gamma\left(\gamma_j-\rho+3\right)}\left[2\left(\gamma_j-\rho+2\right)+\left(\gamma_j+2\right)^2\right]
\end{aligned}$$

$$
\begin{aligned}
& + \frac{\Gamma(\gamma_j+2)\Gamma(\gamma_j+3)}{\gamma_j \Gamma(\gamma_j+\eta_j+1)\Gamma(\gamma_j-\rho+3)}[2\gamma_j-\rho+4] \Big\} \\
& \times \left[(M_{1j}+M_{3j})\|\varphi_1-\varphi_2\|_\infty + (M_{2j}+M_{4j})\|\omega_1-\omega_2\|_\infty\right] \\
\leq \; & B_j \left[(M_{1j}+M_{3j})\|\varphi_1-\varphi_2\|_\infty + (M_{2j}+M_{4j})\|\omega_1-\omega_2\|_\infty\right].
\end{aligned}
$$

By the definition of the norm on $\Xi \times \Xi$ and Ω_1, Ω_2, we have

$$
\|\mho(\varphi_1,\omega_1)-\mho(\varphi_2,\omega_2)\|_{\Xi\times\Xi} \leq (\Omega_1+\Omega_2)\|(\varphi_1,\omega_1)-(\varphi_2,\omega_2)\|.
$$

Since $\Omega_1+\Omega_2<1$, we conclude that $\mho$ is a contraction mapping. This completes the proof.

The example below support the above theorem.

Example 1. *Consider the following problem:*

$$
\begin{cases}
D^{\eta_1}D^{\gamma_1}\varphi_1(\zeta) = \eth_1 Z_1\left(\zeta,\varphi_1(\zeta),\varphi_2(\zeta),D^{\rho-1}\varphi_1(\zeta),D^{\rho}\varphi_1(\zeta)\right) \\
D^{\eta_2}D^{\gamma_2}\varphi_2(\zeta) = \eth_2 Z_2\left(\zeta,\varphi_1(\zeta),\varphi_2(\zeta),D^{\rho-1}\varphi_2(\zeta),D^{\rho}\varphi_2(\zeta)\right) \\
\varphi_1(0)=\varphi_1(1)=\varphi_2(0)=\varphi_2(1)=0, \\
D^{\gamma_1}\varphi_1(0)=D^{\gamma_1}\varphi_1(1)=D^{\gamma_2}\varphi_1(0)=D^{\gamma_2}\varphi_1(1)=0.
\end{cases}
$$

We consider that

$$
\eta_1=\frac{7}{3},\ \gamma_1=\frac{2}{3},\ \eta_2=\frac{5}{2},\ \gamma_2=\frac{1}{2},\ \eth_1=\eth_2=1,\ \text{and } \rho=\frac{3}{2}.
$$

Moreover,

$$
\begin{aligned}
Z_1\left(\zeta,\varphi_1(\zeta),\varphi_2(\zeta),D^{\rho-1}\varphi_1(\zeta),D^{\rho}\varphi_1(\zeta)\right) &= \frac{1}{205e^{\zeta}}\varphi_1(\zeta)+\left(\frac{\cos(2+\zeta^2)}{40(5+\zeta^2)}\right)\varphi_2(\zeta) \\
&\quad +\left(\frac{\sin(\zeta)}{25(\zeta^2+10)}\right)D^{\frac{1}{2}}\varphi_1(\zeta)+\frac{1}{450e^{\zeta}}D^{\frac{3}{2}}\varphi_1(\zeta), \\
Z_2\left(\zeta,\varphi_1(\zeta),\varphi_2(\zeta),D^{\rho-1}\varphi_1(\zeta),D^{\rho}\varphi_1(\zeta)\right) &= \frac{1}{30}\left(\frac{1}{10\pi}\ln(1+\zeta)\right)\varphi_1(\zeta)+\left(\frac{\sin(1+\zeta)}{250e^{\zeta}}\right)\varphi_2(\zeta) \\
&\quad +\left(\frac{\cos(\zeta)}{16(e^{\zeta}+10)}\right)D^{\frac{1}{2}}\varphi_1(\zeta)+\frac{1}{180\pi e^{4\zeta}}D^{\frac{3}{2}}\varphi_1(\zeta),
\end{aligned}
$$

Clearly, for all $\zeta\in[0,1]$ and $\varphi,\omega,\delta,\varrho,\widetilde{\varphi},\widetilde{\omega},\widetilde{\delta},\widetilde{\varrho}\in\mathbb{R}$, we get

$$
\begin{aligned}
\left|Z_1(\zeta,\varphi,\omega,\delta,\varrho)-Z_1\left(\zeta,\widetilde{\varphi},\widetilde{\omega},\widetilde{\delta},\widetilde{\varrho}\right)\right| &\leq M_{11}(\zeta)|\varphi-\widetilde{\varphi}|+M_{21}(\zeta)|\omega-\widetilde{\omega}| \\
&\quad +M_{31}(\zeta)\left|\delta-\widetilde{\delta}\right|+M_{41}(\zeta)|\varrho-\widetilde{\varrho}|,
\end{aligned}
$$

and

$$
\left|Z_2(\zeta,\varphi,\omega,\delta,\varrho)-Z_2\left(\zeta,\widetilde{\varphi},\widetilde{\omega},\widetilde{\delta},\widetilde{\varrho}\right)\right| \leq M_{12}(\zeta)|\varphi-\widetilde{\varphi}|+M_{22}(\zeta)|\omega-\widetilde{\omega}|
$$

$$+M_{32}(\zeta)\left|\delta-\widetilde{\delta}\right|+M_{42}(\zeta)\left|\varrho-\widetilde{\varrho}\right|,$$

with

$$\begin{cases} M_{11}(\zeta)=\frac{1}{205e^{\zeta}},\ M_{21}(\zeta)=\left(\frac{\cos\left(2+\zeta^2\right)}{40(5+\zeta^2)}\right),\ M_{31}(\zeta)=\left(\frac{\sin(\zeta)}{25(\zeta^2+10)}\right),\ M_{41}(\zeta)=\frac{1}{450e^{\zeta}}, \\ M_{12}(\zeta)=\frac{1}{30}\left(\frac{1}{10\pi}\ln\left(1+\zeta\right)\right),\ M_{22}(\zeta)=\left(\frac{\sin(1+\zeta)}{250e^{\zeta}}\right),\ M_{32}(\zeta)=\left(\frac{\cos(\zeta)}{16\left(e^{\zeta}+10\right)}\right),\ M_{42}(\zeta)=\frac{1}{180\pi e^{4\zeta}}. \end{cases}$$

Since $m_{lj}=\sup_{\zeta\in V}\{M_{lj}\}$, *we have*

$$\begin{cases} m_{11}=\frac{1}{205},\ m_{21}=\frac{1}{240},\ m_{31}=\frac{1}{250},\ m_{41}=\frac{1}{450}, \\ m_{12}=\frac{1}{300\pi},\ m_{22}(\zeta)=\frac{1}{250},\ m_{32}(\zeta)=\frac{1}{160},\ m_{42}(\zeta)=\frac{1}{180\pi}. \end{cases}$$

By simple calculations, and the definition of B_j $(j=1,2)$, *we can write* $B_1=25.0783$ *and* $B_2=29.5855$. *Further,*

$$\begin{aligned} \Omega_1 &= 25.0783\max\left\{\left(\frac{1}{205}+\frac{1}{250}\right),\left(\frac{1}{240}+\frac{1}{450}\right)\right\}\approx 0.22265, \\ \Omega_2 &= 29.5855\max\left\{\left(\frac{1}{300\pi}+\frac{1}{160}\right),\left(\frac{1}{250}+\frac{1}{180\pi}\right)\right\}\approx 0.634706, \end{aligned}$$

Hence, $\Omega_1+\Omega_2<1$. *Therefore, all requirements of Theorem 1 are fulfilled. So, the problem has at least one solution.*

5. Applications to beam systems

The technique of Tanh [15, 33, 39] will be applied in this section to discuss the traveling wave solutions for a tripled system of conformable FDs in the form of

$$\begin{cases} \aleph_s^{2\eta}\varphi\left(s,\zeta\right)+\aleph_\zeta^{4\gamma}\varphi\left(s,\zeta\right)+\eth_1 K\left(\varphi,\omega,\aleph_\zeta^{2\gamma}\left(\varphi,\omega\right)\right)\left(s,\zeta\right)=0, \\ \aleph_s^{2\eta}\omega\left(s,\zeta\right)+\aleph_\zeta^{4\gamma}\omega\left(s,\zeta\right)+\eth_2 N\left(\varphi,\omega,\aleph_\zeta^{2\gamma}\varphi\left(\varphi,\omega\right)\right)\left(s,\zeta\right)=0, \end{cases} \tag{10}$$

where $\eta,\gamma\in(0,1]$, K,N are given function, and $\aleph_s^{\eta}\varphi\left(s,\zeta\right)$ refers to the conformable FD. $\aleph_s^{\eta}\varphi\left(s,\zeta\right)$ is defined by Khalil as follows:

$$\Im_s^{\eta}\varphi\left(s,\zeta\right)=\frac{\partial^\eta\varphi\left(s,\zeta\right)}{\partial s^\eta}=\lim_{\varepsilon\to 0}\left(\frac{\varphi\left(s+\varepsilon s^{1-\eta},\zeta\right)-\varphi\left(s,\zeta\right)}{\varepsilon}\right),\ \eta\in(0,1], \tag{11}$$

where $\varphi\left(s,\zeta\right)$ is unknown function. Also, by using (11), we can present $\aleph_\zeta^{\gamma}$.

Clearly, the classical coupled beam equations [40, 47] can be obtained if we put $\eta=\gamma=1$ and $\eth_1=\eth_2=1$ in (10). It takes the shape

$$\begin{cases} \varphi_{ss}+\varphi_{\zeta\zeta\zeta\zeta}+\eth_1 K\left(\varphi,\omega,\left(\varphi,\omega\right)_{\zeta\zeta}\right)=0, \\ \omega_{ss}+\omega_{\zeta\zeta\zeta\zeta}+\eth_1 K\left(\varphi,\omega,\left(\varphi,\omega\right)_{\zeta\zeta}\right)=0. \end{cases}$$

5.1. Steps of the Tanh method

In this part, we summarize the important steps of the Tanh technique involving conformable FDs as follows:

(i) We begin with the following coupled system:

$$\begin{cases} \Phi_1\left(\varphi,\omega,\aleph_s^{\eta}\varphi,\aleph_\zeta^{\gamma}\varphi,\aleph_s^{\eta}\omega,\aleph_\zeta^{\gamma}\omega,\aleph_s^{2\eta}\varphi,\aleph_s^{\eta}\left(\aleph_\zeta^{\gamma}\varphi\right),\aleph_\zeta^{2\gamma}\varphi,\aleph_s^{2\eta}\omega,\aleph_s^{\eta}\left(\aleph_\zeta^{\gamma}\omega\right),\aleph_\zeta^{2\gamma}\omega,\cdots\right)=0,\\ \Phi_2\left(\varphi,\omega,\aleph_s^{\eta}\varphi,\aleph_\zeta^{\gamma}\varphi,\aleph_s^{\eta}\omega,\aleph_\zeta^{\gamma}\omega,\aleph_s^{2\eta}\varphi,\aleph_s^{\eta}\left(\aleph_\zeta^{\gamma}\varphi\right),\aleph_\zeta^{2\gamma}\varphi,\aleph_s^{2\eta}\omega,\aleph_s^{\eta}\left(\aleph_\zeta^{\gamma}\omega\right),\aleph_\zeta^{2\gamma}\omega,\cdots\right)=0.\end{cases} \tag{12}$$

(ii) We use the relation

$$\phi=\frac{\widetilde{k}}{\eta}s^{\eta}+\frac{\nu}{\gamma}s^{\gamma}, \tag{13}$$

for simplicity, the model (12) can be written as

$$\begin{cases}\Psi_1\left(W,E,W',E',W'',E'',W''',E''',\cdots\right)=0,\\ \Psi_2\left(W,E,W',E',W'',E'',W''',E''',\cdots\right)=0.\end{cases}$$

(iii) We apply the transformation

$$\Re=\tanh\left(\phi\right),$$

$$\begin{aligned}
\frac{d}{d\phi} &= \left(1-\Re^2\right)\frac{d}{d\Re},\\
\frac{d^2}{d\phi^2} &= -2\Re\left(1-\Re^2\right)\frac{d}{d\Re}+\left(1-\Re^2\right)^2\frac{d^2}{d\Re^2},\\
\frac{d^3}{d\phi^3} &= -2\left(1-\Re^2\right)\left(3\Re^2-1\right)\frac{d}{d\Re}-6\Re\left(1-\Re^2\right)^2\frac{d^2}{d\Re^2}+\left(1-\Re^2\right)^3\frac{d^3}{d\Re^3},\\
\frac{d^4}{d\phi^4} &= -8\Re\left(1-\Re^2\right)\left(3\Re^2-2\right)\frac{d}{d\Re}+4\left(1-\Re^2\right)^2\left(9\Re^2-2\right)\frac{d^2}{d\varrho^2}\\
&\quad -12\Re\left(1-\Re^2\right)^3\frac{d^3}{d\Re^3}+\left(1-\Re^2\right)^4\frac{d^4}{d\Re^4}.
\end{aligned} \tag{14}$$

(iv) We consider that

$$\begin{cases}\varphi\left(\zeta,s\right)=W\left(\phi\right)=Q\left(\Re\right)=\sum_{j=0}^{m}b_j\Re^j,\\ \omega\left(\zeta,s\right)=E\left(\phi\right)=\mho\left(\Re\right)=\sum_{j=0}^{n}c_j\Re^j.\end{cases} \tag{15}$$

(v) Lastly, we obtain the required solutions for the constants b_j and c_j by using Wazwaz term-balancing [44].

5.2. Traveling wave solutions

In this part, as a practical application, we suggest identifying traveling wave solutions for the coupled problem

$$\begin{cases} \aleph_{\zeta}^{4\gamma}(\varphi)+\aleph_{s}^{2\eta}(\varphi)+\aleph_{\zeta}^{2\gamma}(\varphi)+2\eth_1 c\aleph_{\zeta}^{\gamma}\left(\left(\aleph_{\zeta}^{\gamma}\omega\right)\omega\right)=0,\\ \aleph_{\zeta}^{4\gamma}(\omega)+\aleph_{s}^{2\eta}(\omega)+\eth_2 h\aleph_{\zeta}^{2\gamma}(\varphi)(\varphi\omega+e\omega)=0, \end{cases} \tag{16}$$

where $\eth_1, \eth_2$ are positive real constants and $h, c \in \mathbb{R}$.

We apply the relation (13) to convert the problem (16) to the following integral equation:

$$\begin{cases} \nu^4 W_{\phi\phi\phi\phi}+\widetilde{k}^2 W_{\phi\phi}+\nu^2 W_{\phi\phi}+2c\eth_1\nu^2\left(E_{\phi}E\right)_{\phi}=0,\\ \nu^4 E_{\phi\phi\phi\phi}+\widetilde{k}^2 V_{\phi\phi}+\eth_2 h\nu^2(WE)_{\phi\phi}+e\nu^2 V_{\phi\phi}=0. \end{cases} \tag{17}$$

Integrating (17), one has

$$\begin{cases} \nu^4 W_{\phi\phi}+\widetilde{k}^2 W+\nu^2 W+c\eth_1\nu^2 E^2=0,\\ \nu^4 E_{\phi\phi}+\widetilde{k}^2 V+\eth_2 h\nu^2(WE)+e\nu^2 V=0. \end{cases} \tag{18}$$

The first equation of (18) becomes the following equation when (14) and (15) are substituted into (18):

$$\nu^4\left[-2\Re\left(1-\Re^2\right)\frac{dQ}{d\Re}+\left(1-\Re^2\right)^2\frac{d^2Q}{d\Re^2}\right]+\widetilde{k}^2 Q+c\eth_1\nu^2\mho^2+e\nu^2 Q=0. \tag{19}$$

It is possible to convert the second equation of (18) into

$$\nu^4\left[-2\Re\left(1-\Re^2\right)\frac{d\mho}{d\Re}+\left(1-\Re^2\right)^2\frac{d^2\mho}{d\Re^2}\right]+\widetilde{k}^2\mho+h\eth_2\nu^2(Q\mho)+e\nu^2\mho=0. \tag{20}$$

We balance $\Re^4\frac{d^2Q}{d\Re^2}$ with $\mho^2$ in (19) to $2+m=2n$. Moreover, with (20), we can apply the same method to get $2+n=n+m$. Consequently, we are able to write

$$\begin{cases} Q(\Re)=b_0+b_1\Re+b_2\Re^2,\\ \mho(\Re)=c_0+c_1\Re+c_2\Re^2. \end{cases} \tag{21}$$

We note that when we substitute (21) for (19),

$$\begin{aligned} & -2\nu^4\Re\left(1-\Re^2\right)(b_1+2b_2\Re)+2b_2\nu^4\left(1-\Re^2\right)^2+\widetilde{k}^2\left(b_0+b_1\Re+b_2\Re^2\right)\\ & +c\eth_1\nu^2\left(c_0+c_1\Re+c_2\Re^2\right)^2+e\nu^2\left(b_0+b_1\Re+b_2\Re^2\right)\\ = & \ 0. \end{aligned}$$

Moreover, when we change (21) into (20), we obtain

$$\begin{aligned} & -2\nu^4\Re\left(1-\Re^2\right)(c_1+2c_2\Re)+2c_2\nu^4\left(1-\Re^2\right)^2+\widetilde{k}^2\left(c_0+c_1\Re+c_2\Re^2\right)\\ & +h\eth_2\nu^2\left(c_0+c_1\Re+c_2\Re^2\right)\left(b_0+b_1\Re+b_2\Re^2\right)+e\nu^2\left(c_0+c_1\Re+c_2\Re^2\right)\\ = & \ 0. \end{aligned}$$

As a result, we have the following two sets:

Set 1:

$$\begin{cases} \Re^0 : c\eth_1\nu^2c_0^2 - 2\nu^2b_1 + 2\nu^4b_2 + \widetilde{k}^2b_0 + e\nu^2b_0 = 0, \\ \Re^1 : 2c\nu^2c_0c_1 - 4\nu^2b_2 + \widetilde{k}^2b_1 + \nu^2b_1 = 0, \\ \Re^2 : 2c\nu^2c_0c_2 + c\eth_1\nu^2c_1^2 + 2\nu^2b_1 - 4\nu^2b_2 + \widetilde{k}^2b_2 + \nu^2b_2 = 0, \\ \Re^3 : 2c\nu^2c_1c_2 + 4\nu^2b_2 = 0, \\ \Re^4 : c\eth_1\nu^2c_2^2 + 2\nu^2b_2 = 0, \end{cases} \tag{22}$$

Set 2:

$$\begin{cases} \Re^0 : h\nu^2\eth_2c_0b_0 - 2\nu^4c_1 + 2\nu^4c_2 + e\nu^2c_0 + \widetilde{k}^2c_0 = 0, \\ \Re^1 : h\nu^2\eth_2c_0b_1 + h\nu^2\eth_2c_1b_0 - 4\nu^2c_2 + e\nu^2b_1 + \widetilde{k}^2c_1 = 0, \\ \Re^2 : h\nu^2\eth_2c_0b_2 + h\nu^2\eth_2c_1b_1 + h\nu^2\eth_2c_2b_0 + 2\nu^2c_1 - 4\nu^4c_2 + e\nu^2c_2 + \widetilde{k}^2c_2 = 0, \\ \Re^3 : h\nu^2\eth_2c_1b_2 + h\nu^2\eth_2c_2b_1 + 4\nu^4c_2 = 0, \\ \Re^4 : h\nu^2\eth_2c_2b_2 + 2\nu^4b_2 = 0. \end{cases} \tag{23}$$

With the aid of Maple, for $\eth_1 = \eth_1 = 1$, the constants of (22) and (23) can be obtained as follows:

Case 1: For the set 1, we have

$$\nu = e,\ \widetilde{k} = \pm e\sqrt{4e^2 - 1},\ b_0 = c_0 = 0,\ b_1 = b_2 = -\frac{2e^2}{h},\ \text{and } c_1 = c_2 = \pm\sqrt{\frac{1}{ch}}. \tag{24}$$

From (24) in (21), the solution of (16) can be written as

$$\begin{cases} \varphi(\zeta, s) = -\frac{2e^2}{h}\tanh(\phi) - \frac{2e^2}{h}\tanh^2(\phi), \\ \omega(\zeta, s) = \pm\sqrt{\frac{1}{ch}}\tanh(\phi) \pm \sqrt{\frac{1}{ch}}\tanh^2(\phi). \end{cases} \tag{25}$$

Under particular conditions, we now trace the two parts of this traveling wave solution in Figure 1.

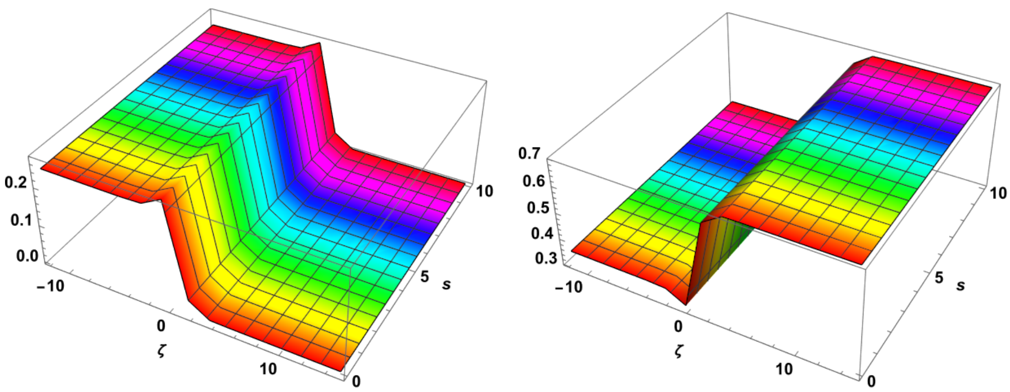

Figure 1: 3D shapes for the solutions of (25) with $-10 \le \zeta \le 10$, $0 \le s \le 10$, $\nu = e = 2$, $h = 1$, $\eta = \frac{2}{5}$, and $\gamma = \frac{1}{5}$.

Case 2: For the set 2, we have

$$\nu = e,\ \widetilde{k} = \widetilde{k},\ b_0 = \frac{4e^2}{h},\ b_1 = b_2 = -\frac{2e^2}{h},\ c_0 = \pm 4e^2\sqrt{\frac{1}{ch}} \text{ and } c_1 = c_2 = \pm 2e^2\sqrt{\frac{1}{ch}}. \tag{26}$$

From (26) in (21), the solution of (16) takes the form

$$\begin{cases} \varphi(\zeta, s) = \frac{4e^2}{h} - \frac{2e^2}{h}\tanh(\phi) - \frac{2e^2}{h}\tanh^2(\phi), \\ \omega(\zeta, s) = \pm 4e^2\sqrt{\frac{1}{ch}} \pm 2e^2\sqrt{\frac{1}{ch}}\tanh(\phi) \pm 2e^2\sqrt{\frac{1}{ch}}\tanh^2(\phi). \end{cases} \tag{27}$$

As previously mentioned, Figure 2 shows the two parts of this traveling wave solution for a given set of parameters.

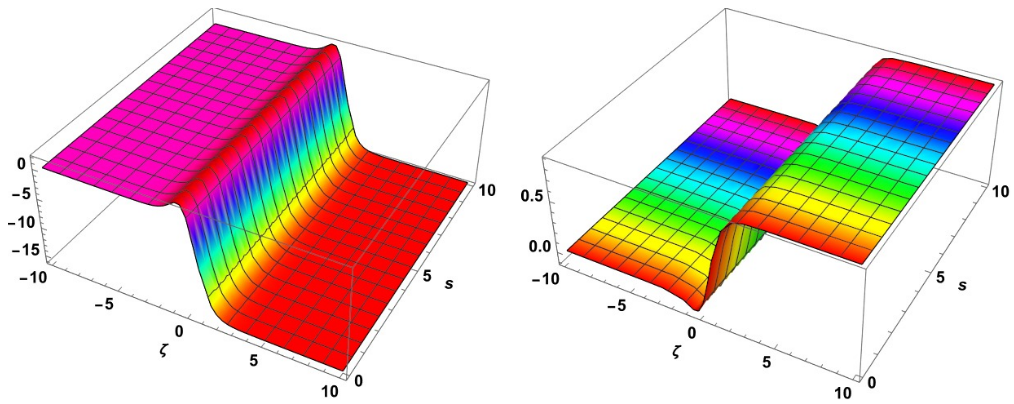

Figure 2: 3D shapes for the solutions of (27) with $-10 \leq \zeta \leq 10$, $0 \leq s \leq 10$, $\nu = e = \frac{3}{7}$, $h = 3$, $c = \frac{3}{2}$, $\eta = \frac{7}{9}$, and $\gamma = \frac{4}{15}$.

6. Abbreviations

DE→differential equation
FC→fractional calculus
CFD→Caputo fractional derivative
FD→Fractional derivative
CSD→Caputo sequential derivative
RL→Riemann-Liouville
FP→fixed point
BS→Banach space

7. Conclusion

In this work, we explored two interconnected mathematical problems. Firstly, we analyzed a fractional system involving Caputo derivatives, which generalizes a beam detection-type system. Our primary objective was to establish the existence and uniqueness of solutions for this system. Secondly, we investigated a system composed of two tripled evolution equations utilizing conformable FDs as defined by Khalil. We obtained an ordinary differential system with moving waves by changing this conformable fractional system. The application of FC to expand and improve classical models is what connects these two sections, showcasing the adaptability and strength of FDs both Caputo and conformable in tackling and resolving complex mathematical problems. This integrated method enhances our knowledge and proficiency in mathematical modeling and analysis by demonstrating how various FDs can be used to investigate and resolve a range of complicated problems.

While this study offers a solid theoretical framework for understanding fractional differential equations and their applications, it's essential to acknowledge certain limitations and assumptions. One significant limitation lies in the analytical nature of the solutions, which might not always be practical for complex systems. Additionally, the choice of fractional derivative, such as the Caputo or conformable derivative, can significantly influence the system's behavior. Future research could concentrate on numerical implementations of the proposed fractional systems, enabling the exploration of a wider range of parameter values and initial conditions. Furthermore, empirical validation of the traveling wave solutions through experiments or simulations would allow for a direct comparison between theoretical predictions and real-world observations. This could lead to refinements in the model and its parameters. Another potential area for future research is the investigation of fractional systems with more intricate boundary conditions and external forcing terms. These extensions could have significant implications for diverse physical and engineering applications. Additionally, exploring the stability analysis of fractional systems would provide valuable insights into their long-term behavior.

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Data Availability

No data is associated with this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors Contributions

All authors contributed equally and significantly in writing this article.

References

[1] S Abbasbandy. A new application of he's variational iteration method for quadratic riccati differential equation by using adomian's polynomials. *J. Comput. Appl. Math.*, 207(1):59–63, 2007.

[2] T Abdeljawad. On conformable fractional calculus. *J. Comput. Appl. Math.*, 279:57–66, 2015.

[3] M Adel, M M Khader, and T A Assiri. Comparative analysis of modified admoian decomposition method and homotopy perturbation mohand transform method for solving burger's equations. *European J. Pure Appl. Math.*, 17(4):2812–2827, 2024.

[4] H Afshari, H Hosseinpour, and H R Marasi. Application of some new contractions for existenceand uniqueness of differential equations involving caputofabrizio derivative. *Adv. Differ. Equ.*, 2021:321, 2021.

[5] AR Aftabizadeh. Existence and uniqueness theorems for fourth-order boundary value problems. *J. Math. Anal. Appl.*, 116:415–426, 1986.

[6] R Agarwal. On fourth-order boundary value problems arising in beam analysis. *Differ. Integr. Equ.*, 2:91–110, 1989.

[7] B Ahmad and S K Ntouyas. Existence results for a coupled system of caputo type sequential fractional differential equations with nonlocal integral boundary conditions. *Appl. Math. Comput.*, 266:615–622, 2015.

[8] S A Al-Mezel an J Ahmad. Smigroups without convexity. *Proc. Amer. Math.*, 122:1175 – 1179, 1994.

[9] H A Hammad an R A Rashwan, A Nafea, M E Samei, and De la Sen. Stability and existence of solutions for a tripled problem of fractional hybrid delay differential equations. *Symmetry.*, 14:2579, 2022.

[10] I M Batiha, S A Njadat, R M Batyha, A Zraiqat, A Dababneh, and S Momani. Design frac-tionalorder pid controllers for single-joint robot arm model. *Int. J. Adv. Soft Comput. Appl.*, 14:96–114, 2022.

[11] K Bensaassa, Z Dahmani, M Rakah, and M Z Sarikaya. Beam deflection coupled systems of fractional differential equations: Existence of solutions, ulam-hyers stability and travelling waves. *Anal. Math. Phys.*, 14:29, 2024.

[12] A Carpinteri and F Mainardi. *Fractional calculus in continuum mechanics, 2 Eds.* New York: Academic Press., 1997.

[13] Y Chatibi, El-H El Kinani, and A Ouhadan. Lie symmetry analysis and conservation laws for the time-fractional black-scholes equation. *Int. J. Geometric Methods Modern Phy.*, 17(01):2050010, 2020.

[14] Y Chatibi, EL-H El Kinani, and A Ouhadan. Continuous and discrete symmetry methods for fractional differential equations, fractional-order model. *Dynamic Sys. Appl. Opti., Signal Proc.*, 2:1– 35, 2022.

[15] Z Dahmani, A Anber an Y Gouari, M Kaid, and I Jebril. Extension of a method for solving nonlinear evolution equations via conformable fractional approach. *Int. Conf. Infor. Tech.*, pages 38–42, 2021.

[16] Z Dahmani, A Anber, and I Jebril. Solving conformable evolution equations by an extended numerical method. *Jordan J. Math. Stat.*, 15:363–380, 2022.

[17] Y Gouari, Z Dahmani, and I Jebril. Application of fractional calculus on a new differential problem of duffing type. *Adv. Math. Sci. J.*, 9:10989–11002, 2020.

[18] H A Hammad, H Aydi, H Isik, and M De la Sen. Stability and controllability study for mixed integral fractional delay dynamic systems endowed with impulsive effects on time scales. *AIMS Math.*, 8(3):6913–6941, 2023.

[19] H A Hammad, H Aydi, and M Zayed. On the qualitative evaluation of the variable order coupled boundary value problems with a fractional delay. *J. Inequal. Appl.*, 2023:105, 2023.

[20] H A Hammad and M De la Sen. Analytical solution of urysohn integral equations by fixed point technique in complex valued metric spaces. *Mathematics.*, 7:852, 2019.

[21] H A Hammad and M De la Sen. Solutions of fractional differential type equations by fixed point techniques for multivalued contractions. *Complexity.*, 2021:5730853, 2021.

[22] H A Hammad and M De la Sen. Stability and controllability study for mixed integral fractional delay dynamic systems endowed with impulsive effects on time scales. *Fractal Fract.*, 7:92, 2023.

[23] H A Hammad, MDe la Sen, and H Aydi. Generalized dynamic process for an extended multivaluedf-contraction in metric-like spaces with applications. *Alex. Eng. J.*, 59(5):5730853, 2020.

[24] H A Hammad, R A Rashwan, A Nafea, M E Samei, and S Noeiaghdam. Stability analysis for a tripled system of fractional pantograph differential equations with nonlocal conditions. *J. Vib. Control.*, 30(3-4):632–647, 2024.

[25] M Q Iqbal and A Hussain. Existence criteria via $\alpha - \psi$- contractive mappings of $\phi-$-fractional differential nonlocal boundary value problems. *AIMS Math.*, 2021:350, 2021.

[26] L Kexue and P Jigen. Laplace transform and fractional differential equations. *Appl. Math. Lett.*, 24(12):2019–2023, 2011.

[27] R Khalil, M Al Horani, A Yousef, and M Sababheh. A new definition of fractional derivative. *J. Comput. Appl. Math.*, 264:65–70, 2014.

[28] A A Kilbas, H M Srivastava, and J J Trujillo. *Theory and applications of fractional differential equations.* Amsterdam: Elsevier B.V, 2006.

[29] P Li, Y Lu, C Xu, and J Ren. Dynamic exploration and control of bifurcation in a fractional-order lengyel-epstein model owing time delay. *MATCH Commun. Math. Comput. Chem.*, 92:472–482, 2024.

[30] A Kılıçman and M Omran. Note on fractional mellin transform and applications. *Springerplus.*, 5:100, 2016.

[31] Y F Luchko, H Matrínez, and J J Trujillo. Fractional fourier transform and some of its applications. *Fract. Calc. Appl. Anal.*, 11(4):1–15, 2008.

[32] Z Ma, H Zahed, and J Ahmad. Fixed point results with applications to fractional

differential equations of anomalous diffusionential equations. *Fractal Fract.*, 8:318, 2024.

[33] W Maliet and W Hereman. The tanh method: I. exact solutions of nonlinear evolution and wave equations. *Phys. Scripta.*, 54:563–568, 1996.

[34] M Manigandan, K Manikandan, H A Hammad, and M De la Sen. Applying fixed point techniques to solve fractional differential inclusions under new boundary conditions. *AIMS Math.*, 9(6):15505–15542, 2024.

[35] M Marin, A Hobiny, and I Abbas. Finite element analysis of nonlinear bioheat model in skintissue due to external thermal sources. *Mathematics.*, page 1459, 2021.

[36] M Marin, A Öchsner, and M M Bhatti. Some results in moore-gibson-thompson thermoelasticity of dipolar bodies. *ZAMM J. Appl. Math. Mech.*, 100:202000090, 2020.

[37] M A Noor, K I Noor, S Iftikhar, and M U Awan. Fractal integral inequalities for harmonic convex functions.. *Appl. Math. Inf. Sci.*, 12(4):831 – 839, 2018.

[38] S K Panda, T Abdeljawad, and F Jarad. Chaotic attractors and fixed point methods in piece wise fractional derivatives and multi-term fractional delay differential equations. *Results Phy.*, 46:106313, 2023.

[39] M Rakah, Z Dahmani, and A Senouci. New uniqueness results for fractional differential equations with a caputo and khalil derivatives. *Appl. Math. Inf. Sci.*, 16:943–952, 2022.

[40] U Sadiya, M Inc, M A Aren, and M H Uddin. Consistent travelling waves solutions to the non-linear time fractional klein-gordon and sine-gordon equations through extended tanh function approach. *J. Taibah Univ. Sci.*, 16:594–607, 2022.

[41] A B Shaalan, N F Habubi, S S Chiad, and Z A Toma. New design of hairpin-koch fractal filter for suppression of spurious band. *Int. J. Thin Film Sci. Tech.*, 2(3):217 – 221, 2013.

[42] A Tudorache and R Luca. On a system of sequential caputo fractional differential equations with nonlocal boundary conditions. *Frac. Fract.*, 7:1–23, 2023.

[43] Q Wang and L Yang. Positive solution for a nonlinear system of fourth-order ordinary differential equations. *Electr. J. Differ. Equ.*, 220:1–15, 2020.

[44] A M Wazwaz. The tanh method for compact and non compact solutions for variants of the kdv-burger and the$k(n,n)-$burger equations. *Phys. Nonlinear Phen.*, 213:147–157, 2006.

[45] M Xia, X Zhang, D Kang, and C Liu. Existence and concentration of nontrivial solutions for an elastic beam equation with local nonlinearity. *AIMS Math.*, 7:579–605, 2021.

[46] X J Yang, M Abdel-Aty, and C Cattani. A new general fractional-order derivative with rabotnov fractional-exponential kernel applied to model the anomalous heat transfer. *Thermal Sci.*, 23(3):1677–1681, 2019.

[47] M Younis. Soliton solutions of fractional order kdv-burger's equation. *J. Adv. Phys.*, 3:325–328, 2014.