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s-(τ_1, τ_2)-continuity for Multifunctions

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Abstract. This paper deals with the concepts of upper s-(τ_1, τ_2)-continuous multifunctions and lower s-(τ_1, τ_2)-continuous multifunctions. Moreover, several characterizations and some properties concerning upper s-(τ_1, τ_2)-continuous multifunctions and lower s-(τ_1, τ_2)-continuous multifunctions are investigated.

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1. Introduction

It is well-known that the branch of mathematics called topology is concerned with all questions directly or indirectly related to continuity. Continuity is an important concept for the study and investigation in the theory of classical point set topology. Generalization of this concept by using stronger and weaker forms of open sets such as semi-open sets [43], preopen sets [45], α-open sets [46], β-open sets [34] and θ-open sets [61] is one of the main research topics of general topology. Viriyapong and Boonpok [63] investigated some characterizations of (Λ, sp)-continuous functions by utilizing the notions of (Λ, sp)-open sets and (Λ, sp)-closed sets due to Boonpok and Khampakdee [12]. Dungthaisong et al. [33] introduced and studied the concept of $g_{(m,n)}$-continuous functions. Duangphui et al. [32] introduced and investigated the notion of $(\mu, \mu')^{(m,n)}$-continuous functions. Furthermore, several characterizations and some properties concerning almost (Λ, p)-continuous functions, strongly $\theta(\Lambda, p)$-continuous functions, almost strongly $\theta(\Lambda, p)$-continuous functions, $\theta(\Lambda, p)$-continuous functions, weakly (Λ, b)-continuous functions, $\theta(\star)$-precontinuous functions, $(\Lambda, p(\star))$-continuous functions, $\star$-continuous functions, θ-$\mathscr{I}$-continuous functions, almost (g, m)-continuous functions,

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pairwise almost M-continuous functions, (τ_1, τ_2)-continuous functions, almost (τ_1, τ_2)-continuous functions, weakly (τ_1, τ_2)-continuous functions, faintly (τ_1, τ_2)-continuous functions, almost quasi (τ_1, τ_2)-continuous functions and weakly quasi (τ_1, τ_2)-continuous functions were presented in [56], [59], [16], [51], [25], [11], [8], [10], [4], [1], [2], [26], [23], [18], [57], [41] and [31], respectively. In 1965, Lee [42] studied the notion of semiconnected functions. Kohli [39] introduced the notion of s-continuous functions and investigated several characterizations of semilocally connected spaces in terms of s-continuous functions. The concept of s-continuity as a generalization of continuity and semiconnectedness. Moreover, Kohli [40] introduced the concepts of s-regular spaces and completely s-regular spaces and proved that s-regularity and complete s-regularity are preserved under certain s-continuous functions.

In 1989, Lipski [44] extended the concept of s-continuous functions to the setting of multifunctions. Popa [47] introduced the concept of precontinuous multifunctions and showed that H-almost continuity and precontinuity are equivalent for multifunctions. Ewert and Lipski [35] introduced and investigated the concept of s-quasi-continuous multifunctions. Popa and Noiri [50] introduced a new class of multifunctions called s-precontinuous multifunctions is a generalization of s-continuous multifunctions and precontinuous multifunctions. Viriyapong and Boonpok [64] introduced and studied the concept of weakly quasi (Λ, sp)-continuous multifunctions. Moreover, several characterizations of $(\tau_1, \tau_2)\delta$-semicontinuous multifunctions, almost weakly (τ_1, τ_2)-continuous multifunctions, $\star$-continuous multifunctions, $\beta(\star)$-continuous multifunctions, α-$\star$-continuous multifunctions, almost α-$\star$-continuous multifunctions, almost quasi $\star$-continuous multifunctions, weakly α-$\star$-continuous multifunctions, $s\beta(\star)$-continuous multifunctions, weakly $s\beta(\star)$-continuous multifunctions, $\theta(\star)$-quasi continuous multifunctions, almost $\imath^\star$-continuous multifunctions, weakly (Λ, sp)-continuous multifunctions, $\alpha(\Lambda, sp)$-continuous multifunctions, almost $\alpha(\Lambda, sp)$-continuous multifunctions, weakly $\alpha(\Lambda, sp)$-continuous multifunctions, almost $\beta(\Lambda, sp)$-continuous multifunctions, slightly (Λ, sp)-continuous multifunctions, (τ_1, τ_2)-continuous multifunctions, almost (τ_1, τ_2)-continuous multifunctions, weakly (τ_1, τ_2)-continuous multifunctions, weakly quasi (τ_1, τ_2)-continuous multifunctions, c-(τ_1, τ_2)-continuous multifunctions, c-quasi (τ_1, τ_2)-continuous multifunctions and almost quasi (τ_1, τ_2)-continuous multifunctions were established in [5], [28], [3], [7], [17], [24], [6], [21], [20], [15], [9], [19], [22], [36], [13], [27], [58], [14], [54], [38], [60], [55], [37], [52] and [53], respectively. Popa and Noiri [49] introduced and studied the notion of s-β-continuous multifunctions. In particular, Popa and Noiri [48] introduced and investigated the concept of s-m-continuous multifunctions as multifunctions defined on a set satisfying some minimal conditions. Quite recently, Viriyapong et al. [66] introduced and studied the concept of s-$(\tau_1, \tau_2)p$-continuous multifunctions. In this paper, we introduce the concepts of upper s-(τ_1, τ_2)-continuous multifunctions and lower s-(τ_1, τ_2)-continuous multifunctions. We also investigate several characterizations of upper s-(τ_1, τ_2)-continuous multifunctions and lower s-(τ_1, τ_2)-continuous multifunctions.

2. Preliminaries

Throughout the present paper, spaces (X, τ_1, τ_2) and (Y, σ_1, σ_2) (or simply X and Y) always mean bitopological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a bitopological space (X, τ_1, τ_2). The closure of A and the interior of A with respect to τ_i are denoted by τ_i-Cl(A) and τ_i-Int(A), respectively, for $i = 1, 2$. A subset A of a bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$*-closed* [29] if $A = \tau_1$-Cl$(\tau_2$-Cl$(A))$. The complement of a $\tau_1\tau_2$-closed set is called $\tau_1\tau_2$*-open*. The intersection of all $\tau_1\tau_2$-closed sets of X containing A is called the $\tau_1\tau_2$*-closure* [29] of A and is denoted by $\tau_1\tau_2$-Cl(A). The union of all $\tau_1\tau_2$-open sets of X contained in A is called the $\tau_1\tau_2$*-interior* [29] of A and is denoted by $\tau_1\tau_2$-Int(A).

Lemma 1. [29] *Let A and B be subsets of a bitopological space (X, τ_1, τ_2). For the $\tau_1\tau_2$-closure, the following properties hold:*

(1) $A \subseteq \tau_1\tau_2\text{-}Cl(A)$ and $\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Cl(A)) = \tau_1\tau_2\text{-}Cl(A)$.

(2) If $A \subseteq B$, then $\tau_1\tau_2\text{-}Cl(A) \subseteq \tau_1\tau_2\text{-}Cl(B)$.

(3) $\tau_1\tau_2\text{-}Cl(A)$ is $\tau_1\tau_2$-closed.

(4) A is $\tau_1\tau_2$-closed if and only if $A = \tau_1\tau_2\text{-}Cl(A)$.

(5) $\tau_1\tau_2\text{-}Cl(X - A) = X - \tau_1\tau_2\text{-}Int(A)$.

A subset A of a bitopological space (X, τ_1, τ_2) is said to be $\tau_1\tau_2$*-clopen* [29] if A is both $\tau_1\tau_2$-open and $\tau_1\tau_2$-closed. A subset A of a bitopological space (X, τ_1, τ_2) is called $(\tau_1, \tau_2)r$*-open* [62] (resp. $(\tau_1, \tau_2)s$*-open* [5], $(\tau_1, \tau_2)p$*-open* [5], $(\tau_1, \tau_2)\beta$*-open* [5], $\alpha(\tau_1, \tau_2)$*-open*) [65]) if $A = \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))$ (resp. $A \subseteq \tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(A))$, $A \subseteq \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))$, $A \subseteq \tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A)))$, $A \subseteq \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(A)))$). The complement of a $(\tau_1, \tau_2)r$-open (resp. $(\tau_1, \tau_2)s$-open, $(\tau_1, \tau_2)p$-open, $(\tau_1, \tau_2)\beta$-open, $\alpha(\tau_1, \tau_2)$-open) set is called $(\tau_1, \tau_2)r$*-closed* (resp. $(\tau_1, \tau_2)s$*-closed*, $(\tau_1, \tau_2)p$*-closed*, $(\tau_1, \tau_2)\beta$*-closed*, $\alpha(\tau_1, \tau_2)$*-closed*).

By a multifunction $F : X \to Y$, we mean a point-to-set correspondence from X into Y, and we always assume that $F(x) \neq \emptyset$ for all $x \in X$. For a multifunction $F : X \to Y$, we shall denote the upper and lower inverse of a set B of Y by $F^+(B)$ and $F^-(B)$, respectively, that is, $F^+(B) = \{x \in X \mid F(x) \subseteq B\}$ and $F^-(B) = \{x \in X \mid F(x) \cap B \neq \emptyset\}$. In particular, $F^-(y) = \{x \in X \mid y \in F(x)\}$ for each point $y \in Y$. For each $A \subseteq X$, $F(A) = \cup_{x\in A} F(x)$.

3. Upper and lower s-(τ_1, τ_2)-continuous multifunctions

In this section, we introduce the notions of upper s-(τ_1, τ_2)-continuous multifunctions and lower s-(τ_1, τ_2)-continuous multifunctions. Moreover, some characterizations of upper s-(τ_1, τ_2)-continuous multifunctions and lower s-(τ_1, τ_2)-continuous multifunctions are discussed.

Definition 1. *A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be upper s-(τ_1, τ_2)-continuous at $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y containing $F(x)$ and having $\sigma_1\sigma_2$-connected complement, there exists a $\tau_1\tau_2$-open set U of X containing x such that $F(U) \subseteq V$. A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be upper s-(τ_1, τ_2)-continuous if F is upper s-(τ_1, τ_2)-continuous at each point x of X.*

Theorem 1. *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is upper s-(τ_1, τ_2)-continuous;

(2) $F^+(V)$ is $\tau_1\tau_2$-open in X for every $\sigma_1\sigma_2$-open set V of Y having $\sigma_1\sigma_2$-connected complement;

(3) $F^-(K)$ is $\tau_1\tau_2$-closed in X for every $\sigma_1\sigma_2$-connected $\sigma_1\sigma_2$-closed set K of Y;

(4) $\tau_1\tau_2$-Cl$(F^-(B)) \subseteq F^-(\sigma_1\sigma_2$-Cl$(B))$ for every subset B of Y having the $\sigma_1\sigma_2$-connected $\sigma_1\sigma_2$-closure;

(5) $F^+(\sigma_1\sigma_2$-Int$(B)) \subseteq \tau_1\tau_2$-Int$(F^+(B))$ for every subset B of Y such that $Y - \sigma_1\sigma_2$-Int(B) is $\sigma_1\sigma_2$-connected.

Proof. (1) $\Rightarrow$ (2): Let V be any $\sigma_1\sigma_2$-open set of Y having $\sigma_1\sigma_2$-connected complement and $x \in F^+(V)$. Then, there exists a $\tau_1\tau_2$-open set U of X containing x such that $F(U) \subseteq V$. Therefore, we have $x \in U \subseteq F^+(V)$ and hence $x \in \tau_1\tau_2$-Int$(F^+(V))$. Thus, $F^+(V) \subseteq \tau_1\tau_2$-Int$(F^+(V))$ and so $F^+(V)$ is $\tau_1\tau_2$-open in X.

(2) $\Rightarrow$ (3): The proof follows immediately from the fact that $F^+(Y - B) = X - F^-(B)$ for every subset B of Y.

(3) $\Rightarrow$ (4): Let B be any subset of Y having the $\sigma_1\sigma_2$-connected $\sigma_1\sigma_2$-closure. Thus by (3), $\tau_1\tau_2$-Cl$(F^-(B)) \subseteq \tau_1\tau_2$-Cl$(F^-(\sigma_1\sigma_2$-Cl$(B))) = F^-(\sigma_1\sigma_2$-Cl$(B))$.

(4) $\Rightarrow$ (5): Let B be any subset of Y such that $Y - \sigma_1\sigma_2$-Int(B) is $\sigma_1\sigma_2$-connected. By (4), we have

$$\begin{aligned}
X - \tau_1\tau_2\text{-Int}(F^+(B)) &= \tau_1\tau_2\text{-Cl}(X - F^+(B)) \\
&= \tau_1\tau_2\text{-Cl}(F^-(Y - B)) \\
&\subseteq F^-(\sigma_1\sigma_2\text{-Cl}(Y - B)) \\
&= F^-(Y - \sigma_1\sigma_2\text{-Int}(B)) \\
&= X - F^+(\sigma_1\sigma_2\text{-Int}(B))
\end{aligned}$$

and hence $F^+(\sigma_1\sigma_2$-Int$(B)) \subseteq \tau_1\tau_2$-Int$(F^+(B))$.

(5) $\Rightarrow$ (1): Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y containing $F(x)$ and having $\sigma_1\sigma_2$-connected complement. By (5), $F^+(V) = F^+(\sigma_1\sigma_2$-Int$(V)) \subseteq \tau_1\tau_2$-Int$(F^+(V))$. Then, there exists a $\tau_1\tau_2$-open set U of X containing x such that $U \subseteq F^+(V)$. Thus, $F(U) \subseteq V$. This shows that F is upper s-(τ_1, τ_2)-continuous.

Definition 2. *A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be lower s-(τ_1, τ_2)-continuous at $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y such that $F(x) \cap V \neq \emptyset$ and having $\sigma_1\sigma_2$-connected complement, there exists a $\tau_1\tau_2$-open set U of X containing x such that $F(z) \cap V \neq \emptyset$ for each $z \in U$. A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be lower s-(τ_1, τ_2)-continuous if F is lower s-(τ_1, τ_2)-continuous at each point x of X.*

Theorem 2. *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is lower s-(τ_1, τ_2)-continuous;

(2) $F^-(V)$ is $\tau_1\tau_2$-open in X for every $\sigma_1\sigma_2$-open set V of Y having $\sigma_1\sigma_2$-connected complement;

(3) $F^+(K)$ is $\tau_1\tau_2$-closed in X for every $\sigma_1\sigma_2$-connected $\sigma_1\sigma_2$-closed set K of Y;

(4) $\tau_1\tau_2$-$Cl(F^+(B)) \subseteq F^+(\sigma_1\sigma_2$-$Cl(B))$ for every subset B of Y having the $\sigma_1\sigma_2$-connected $\sigma_1\sigma_2$-closure;

(5) $F^-(\sigma_1\sigma_2$-$Int(B)) \subseteq \tau_1\tau_2$-$Int(F^-(B))$ for every subset B of Y such that $Y - \sigma_1\sigma_2$-$Int(B)$ is $\sigma_1\sigma_2$-connected.

Proof. The proof is similar to that of Theorem 1.

Corollary 1. *A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is upper s-(τ_1, τ_2)-continuous if $F^-(B)$ is $\tau_1\tau_2$-closed in X for every $\sigma_1\sigma_2$-connected set B of Y.*

Proof. Let V be any $\sigma_1\sigma_2$-open set of Y having $\sigma_1\sigma_2$-connected complement. Then, $Y - V$ is $\sigma_1\sigma_2$-connected and $\sigma_1\sigma_2$-closed. By the hypothesis, $F^-(Y - V)$ is $\tau_1\tau_2$-closed in X. Thus, $F^+(V)$ is $\tau_1\tau_2$-open in X and by Theorem 1, F is upper s-(τ_1, τ_2)-continuous.

Corollary 2. *A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is lower s-(τ_1, τ_2)-continuous if $F^+(B)$ is $\tau_1\tau_2$-closed in X for every $\sigma_1\sigma_2$-connected set B of Y.*

Proof. The proof is similar to that of Corollary 1.

Definition 3. *A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be s-(τ_1, τ_2)-continuous if for $x \in X$ and each $\sigma_1\sigma_2$-open set V of Y containing $f(x)$ and having $\sigma_1\sigma_2$-connected complement, there exists a $\tau_1\tau_2$-open set U of X containing x such that $f(U) \subseteq V$.*

Corollary 3. *For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) f is s-(τ_1, τ_2)-continuous;

(2) $f^{-1}(V)$ is $\tau_1\tau_2$-open in X for every $\sigma_1\sigma_2$-open set V of Y having $\sigma_1\sigma_2$-connected complement;

(3) $f^{-1}(K)$ *is* $\tau_1\tau_2$*-closed in* X *for every* $\sigma_1\sigma_2$*-connected* $\sigma_1\sigma_2$*-closed set* K *of* Y*;*

(4) $\tau_1\tau_2$*-Cl*$(f^{-1}(B)) \subseteq f^{-1}(\sigma_1\sigma_2$*-Cl*$(B))$ *for every subset* B *of* Y *having the* $\sigma_1\sigma_2$*-connected* $\sigma_1\sigma_2$*-closure;*

(5) $f^{-1}(\sigma_1\sigma_2$*-Int*$(B)) \subseteq \tau_1\tau_2$*-Int*$(f^{-1}(B))$ *for every subset* B *of* Y *such that* $Y-\sigma_1\sigma_2$*-Int*(B) *is* $\sigma_1\sigma_2$*-connected.*

For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, a multifunction

$$\mathrm{Cl}F_{\circledast} : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$$

is defined in [29] as follows: $\mathrm{Cl}F_{\circledast}(x) = \sigma_1\sigma_2\text{-Cl}(F(x))$ for each $x \in X$.

Definition 4. [29] *A subset* A *of a bitopological space* (X, τ_1, τ_2) *is said to be:*

(1) $\tau_1\tau_2$*-paracompact if every cover of* A *by* $\tau_1\tau_2$*-open sets of* X *is refined by a cover of* A *which consists of* $\tau_1\tau_2$*-open sets of* X *and is* $\tau_1\tau_2$*-locally finite in* X*;*

(2) $\tau_1\tau_2$*-regular if for each* $x \in A$ *and each* $\tau_1\tau_2$*-open set* U *of* X *containing* x*, there exists a* $\tau_1\tau_2$*-open set* V *of* X *such that* $x \in V \subseteq \tau_1\tau_2$*-Cl*$(V) \subseteq U$*.*

Lemma 2. [29] *If* A *is a* $\tau_1\tau_2$*-regular* $\tau_1\tau_2$*-paracompact set of a bitopological space* (X, τ_1, τ_2) *and* U *is a* $\tau_1\tau_2$*-open neighbourhood of* A*, then there exists a* $\tau_1\tau_2$*-open set* V *of* X *such that* $A \subseteq V \subseteq \tau_1\tau_2$*-Cl*$(V) \subseteq U$*.*

Lemma 3. [29] *If* $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *is a multifunction such that* $F(x)$ *is* $\tau_1\tau_2$*-regular and* $\tau_1\tau_2$*-paracompact for each* $x \in X$*, then* $\mathit{Cl}F^{+}_{\circledast}(V) = F^{+}(V)$ *for each* $\sigma_1\sigma_2$*-open set* V *of* Y*.*

Theorem 3. *Let* $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *be a multifunction such that* $F(x)$ *is* $\sigma_1\sigma_2$*-paracompact and* $\sigma_1\sigma_2$*-regular for each* $x \in X$*. Then,* F *is upper* s*-*(τ_1, τ_2)*-continuous if and only if* $\mathit{Cl}F_{\circledast} : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *is upper* s*-*(τ_1, τ_2)*-continuous.*

Proof. We put $G = \mathrm{Cl}F_{\circledast}$. Suppose that F is upper s-(τ_1, τ_2)-continuous. Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y containing $G(x)$ and having $\sigma_1\sigma_2$-connected complement. By Lemma 3, we have $x \in G^{+}(V) = F^{+}(V)$ and hence there exists a $\tau_1\tau_2$-open set U of X containing x such that $F(U) \subseteq V$. Since $F(z)$ is $\sigma_1\sigma_2$-paracompact and $\sigma_1\sigma_2$-regular for each $z \in U$, by Lemma 2 there exists a $\tau_1\tau_2$-open set W of Y such that $F(z) \subseteq W \subseteq \sigma_1\sigma_2\text{-Cl}(W) \subseteq V$; hence $G(z) \subseteq \sigma_1\sigma_2\text{-Cl}(W) \subseteq V$ for each $z \in U$. Thus, $G(U) \subseteq V$. This shows that G is upper s-(τ_1, τ_2)-continuous.

Conversely, suppose that G is upper s-(τ_1, τ_2)-continuous. Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y containing $F(x)$ and having $\sigma_1\sigma_2$-connected complement. By Lemma 3, we have $x \in F^{+}(V) = G^{+}(V)$ and hence $G(x) \subseteq V$. There exists a $\tau_1\tau_2$-open set U of X containing x such that $G(U) \subseteq V$. Thus, $U \subseteq G^{+}(V) = F^{+}(V)$ and so $F(U) \subseteq V$. This shows that F is upper s-(τ_1, τ_2)-continuous.

Lemma 4. [29] *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, $ClF^-_{\circledast}(V) = F^-(V)$ for each $\sigma_1\sigma_2$-open set V of Y.*

Theorem 4. *A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is lower s-(τ_1, τ_2)-continuous if and only if $ClF_{\circledast} : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is lower s-(τ_1, τ_2)-continuous.*

Proof. By using Lemma 4 this is shown similarly as in Theorem 3.

The $\tau_1\tau_2$-*frontier* [26] of a subset A of a bitopological space (X, τ_1, τ_2), denoted by $\tau_1\tau_2$-fr(A), is defined by

$$\tau_1\tau_2\text{-fr}(A) = \tau_1\tau_2\text{-Cl}(A) \cap \tau_1\tau_2\text{-Cl}(X - A) = \tau_1\tau_2\text{-Cl}(A) - \tau_1\tau_2\text{-Int}(A).$$

Theorem 5. *The set of all points x of X at which a multifunction*

$$F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$$

is not upper s-(τ_1, τ_2)-continuous is identical with the union of the $\tau_1\tau_2$-frontier of the upper inverse images of $\sigma_1\sigma_2$-open sets containing $F(x)$ and having $\sigma_1\sigma_2$-connected complement.

Proof. Let x be a point of X at which F is not upper s-(τ_1, τ_2)-continuous. Then, there exists a $\sigma_1\sigma_2$-open set V of Y containing $F(x)$ and having $\sigma_1\sigma_2$-connected complement such that $U \cap (X - F^+(V)) \neq \emptyset$ for every $\tau_1\tau_2$-open set U of X containing x. Therefore, we have $x \in \tau_1\tau_2\text{-Cl}(X - F^+(V))$ and hence $x \in \tau_1\tau_2\text{-fr}(F^+(V))$.

Conversely, suppose that V is a $\sigma_1\sigma_2$-open set of Y containing $F(x)$ and having $\sigma_1\sigma_2$-connected complement such that $x \in \tau_1\tau_2\text{-fr}(F^+(V))$. If F is upper s-(τ_1, τ_2)-continuous at $x \in X$, then there exists a $\tau_1\tau_2$-open set U of X containing x such that $U \subseteq F^+(V)$; hence $x \in \tau_1\tau_2\text{-Int}(F^+(V))$. This is a contradiction and so F is not upper s-(τ_1, τ_2)-continuous at x.

Theorem 6. *The set of all points x of X at which a multifunction*

$$F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$$

is not lower s-(τ_1, τ_2)-continuous is identical with the union of the $\tau_1\tau_2$-frontier of the lower inverse images of $\sigma_1\sigma_2$-open sets meeting $F(x)$ and having $\sigma_1\sigma_2$-connected complement.

Proof. The proof is similar to that of Theorem 5.

For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the graph $G(F) = \{(x, F(x)) \mid x \in X\}$ is said to be *strongly* (τ_1, τ_2)-*closed* if for each $(x, y) \in (X \times Y) - G(F)$, there exists a $\tau_1\tau_2$-open set U of X containing x and a $\sigma_1\sigma_2$-open set V of Y containing y such that $[U \times \sigma_1\sigma_2\text{-Cl}(V)] \cap G(F) = \emptyset$.

Lemma 5. *A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ has a strongly (τ_1, τ_2)-closed graph if and only if for each $(x, y) \in (X \times Y) - G(F)$, there exists a $\tau_1\tau_2$-open set U of X containing x and a $\sigma_1\sigma_2$-open set V of Y containing y such that $F(U) \cap \sigma_1\sigma_2$-$Cl(V) = \emptyset$.*

Proof. This proof is obvious.

Recall that a bitopological space (X, τ_1, τ_2) is said to be (τ_1, τ_2)-*regular* [30] if for each $\tau_1\tau_2$-closed set F and each point $x \in X - F$, there exist disjoint $\tau_1\tau_2$-open sets U and V such that $x \in U$ and $F \subseteq V$.

Definition 5. *A bitopological space (X, τ_1, τ_2) is said to be locally $\tau_1\tau_2$-connected if for each $x \in X$ and each $\tau_1\tau_2$-open set G of X containing x, there exists a $\tau_1\tau_2$-open $\tau_1\tau_2$-connected set V such that $x \in V \subseteq G$.*

Theorem 7. *Let (Y, σ_1, σ_2) be a (σ_1, σ_2)-regular and locally $\sigma_1\sigma_2$-connected space. If $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is an upper s-(τ_1, τ_2)-continuous multifunction such that $F(x)$ is $\sigma_1\sigma_2$-closed for each $x \in X$, then $G(F)$ is strongly (τ_1, τ_2)-closed.*

Proof. Let $(x, y) \in (X \times Y) - G(F)$. Then, $y \in Y - F(x)$. Since (Y, σ_1, σ_2) is (σ_1, σ_2)-regular, there exist disjoint $\sigma_1\sigma_2$-open sets V and V' of Y such that $F(x) \subseteq V$ and $y \in V'$. Moreover, since (Y, σ_1, σ_2) is locally $\sigma_1\sigma_2$-connected, there exists a $\sigma_1\sigma_2$-open $\sigma_1\sigma_2$-connected set W of Y such that $y \in W \subseteq \sigma_1\sigma_2\text{-Cl}(W) \subseteq V'$. Since F is upper s-(τ_1, τ_2)-continuous and $Y - \sigma_1\sigma_2\text{-Cl}(W)$ is a $\sigma_1\sigma_2$-open set having $\sigma_1\sigma_2$-connected complement, there exists a $\tau_1\tau_2$-open set U of X containing x such that $F(U) \subseteq Y - \sigma_1\sigma_2\text{-Cl}(W)$. Thus, $F(U) \cap \sigma_1\sigma_2\text{-Cl}(W) = \emptyset$ and by Lemma 5, $G(F)$ is strongly (τ_1, τ_2)-closed.

Definition 6. [54] *A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be lower (τ_1, τ_2)-continuous if for each $x \in X$ and each $\sigma_1\sigma_2$-open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists a $\tau_1\tau_2$-open set U of X containing x such that $F(z) \cap V \neq \emptyset$ for each $z \in U$.*

Lemma 6. [54] *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is lower (τ_1, τ_2)-continuous;

(2) $F^-(V)$ is $\tau_1\tau_2$-open in X for every $\sigma_1\sigma_2$-open set V of Y;

(3) $F^+(K)$ is $\tau_1\tau_2$-closed in X for every $\sigma_1\sigma_2$-closed set K of Y;

(4) $\tau_1\tau_2\text{-Cl}(F^+(B)) \subseteq F^+(\sigma_1\sigma_2\text{-Cl}(B))$ for every subset B of Y;

(5) $F(\tau_1\tau_2\text{-Cl}(A)) \subseteq \sigma_1\sigma_2\text{-Cl}(F(A))$ for every subset A of X;

(6) $F^-(\sigma_1\sigma_2\text{-Int}(B)) \subseteq \tau_1\tau_2\text{-Int}(F^-(B))$ for every subset B of Y.

Theorem 8. *If $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is lower s-(τ_1, τ_2)-continuous and $F(A)$ is $\sigma_1\sigma_2$-connected for every subset A of X, then F is lower (τ_1, τ_2)-continuous.*

Proof. Let A be any subset of X. Since $\sigma_1\sigma_2\text{-Cl}(F(A))$ is $\sigma_1\sigma_2$-closed and $\sigma_1\sigma_2$-connected, by Theorem 2 we have $F^+(\sigma_1\sigma_2\text{-Cl}(F(A))) = \tau_1\tau_2\text{-Cl}(F^+(\sigma_1\sigma_2\text{-Cl}(F(A))))$ and $A \subseteq F^+(F(A)) \subseteq F^+(\sigma_1\sigma_2\text{-Cl}(F(A)))$. Thus, $F(\sigma_1\sigma_2\text{-Cl}(A)) \subseteq \sigma_1\sigma_2\text{-Cl}(F(A))$. It follows from Lemma 6 that F is lower (τ_1, τ_2)-continuous.

Recall that a bitopological space (X, τ_1, τ_2) is said to be $\tau_1\tau_2$-*connected* [29] if X cannot be written as the union of two disjoint nonempty $\tau_1\tau_2$-open sets.

Definition 7. *A bitopological space (X, τ_1, τ_2) is said to be s-$\tau_1\tau_2$-connected if X cannot be written as the union of two disjoint nonempty $\tau_1\tau_2$-open sets having $\tau_1\tau_2$-connected complement.*

Theorem 9. *If $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is an upper or lower s-(τ_1, τ_2)-continuous surjective multifunction such that $F(x)$ is $\sigma_1\sigma_2$-connected for each $x \in X$ and (X, τ_1, τ_2) is $\tau_1\tau_2$-connected, then (Y, σ_1, σ_2) is s-$\sigma_1\sigma_2$-connected.*

Proof. Suppose that (Y, σ_1, σ_2) is not s-$\sigma_1\sigma_2$-connected. There exist nonempty $\sigma_1\sigma_2$-open sets U and V of Y having $\sigma_1\sigma_2$-connected complement such that $U \cap V = \emptyset$ and $U \cup V = Y$. Since $F(x)$ is $\sigma_1\sigma_2$-connected for each $x \in X$, either $F(x) \subseteq U$ or $F(x) \subseteq V$. If $x \in F^+(U \cup V)$, then $F(x) \subseteq U \cup V$ and hence $x \in F^+(U) \cup F^+(V)$. Moreover, since F is surjective, there exist x and y in X such that $F(x) \subseteq U$ and $F(y) \subseteq V$; hence $x \in F^+(U)$ and $y \in F^+(V)$. Therefore, we obtain the following:

(1) $F^+(U) \cup F^+(V) = F^+(U \cup V) = X$;

(2) $F^+(U) \cap F^+(V) = F^+(U \cap V) = \emptyset$;

(3) $F^+(U) \neq \emptyset$ and $F^+(V) \neq \emptyset$.

Next, we shall show that $F^+(U)$ and $F^+(V)$ are $\tau_1\tau_2$-open in X. (i) Let F be upper s-(τ_1, τ_2)-continuous. By Theorem 1, $F^+(U)$ and $F^+(V)$ are $\tau_1\tau_2$-open in X. (ii) Let F be lower s-(τ_1, τ_2)-continuous. Since V is a $\sigma_1\sigma_2$-clopen set with $\sigma_1\sigma_2$-connected complement, by Theorem 2, $F^+(V)$ is $\tau_1\tau_2$-closed in X. Therefore, $F^+(U)$ is $\tau_1\tau_2$-open in X. Similarly, we obtain $F^+(V)$ is $\tau_1\tau_2$-open in X. Thus, (X, τ_1, τ_2) is not $\tau_1\tau_2$-connected.

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