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Spectral Properties of Structured Matrices in Transportation Problems

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Abstract. The Hitchcock-Koopmans transportation problem is a well-known and fundamental optimization problem which focuses on minimization of the objective function which is basically the transportation cost from multiple sources to multiple destinations. In this article, we present some novel results on the spectral properties of structured matrices appearing in Hitchcock-Koopmans transportation problems. The results on the computation of singular values are presented with usage of tools from linear algebra and matrix analysis. The new results are derived on interconnection between structured singular values of pseudo-inverse and D-stable matrices of Hitchcock-Koopmans transportation models. The numerical experimentation shows the behavior of singular values. The Matlab EigTool is used for the computation of pseudo-spectrum of pseudo-inverse matrix corresponding to the transportation model.

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1. Introduction

The transportation theory is a name given to study of optimal transportation and the allocation of resources. In optimization, the transportation problem was earlier formalized by a French mathematician Gaspard Monge, and then A.N. Talstoi first studied transport problem mathematically in 1920 and published an article with titled: Methods of finding minimal kilometrage in the Cargo-Transportation in space.

The transportation problem for m numbers of sources $x_1, x_2, \cdots, x_m$ for a given commodity having $a(x_i)$ number of units of supply at x_i points and n number of sinks $y_1, y_2, \cdots, y_n$ for commodity. The demand at y_j is considered as $b(y_j)$. Let x_i and y_i, $a(x_i, y_i)$ represents the unit cost of shipment. Then, the question of finding flow satisfying the demand from supplies, and to minimize the cost was studied in [15, 16, 18].

Transportation problems can be considered as an optimization problems, particularly the linear optimization. In linear optimization the aim could be in finding effective techniques in order to distribute goods from many suppliers to many final destinations. The mathematical concepts dealing with transportation problems involves the analysis of structured and, unstructured matrices representing supply, demand, and transportation costs. The notable methods likewise Northwest Corner Rule, Least Cost technique, and Vogel's Approximation Method are developed to find initial feasible solutions corresponding to the optimization problems. On the other hand, the optimization methods for instance Modified Distribution (MODI) Method or stepping-stone method are used to analyze and refine obtained solutions for the purpose of achieving optimality conditions.

The transportation problems have an extensive amount of applications in many diverse range of research directions [2, 10]. In [4], the service network design problems were studied in order to measure the minimum cost with given constraints. To study the evolution of coalition over a given time with trust-related issues, an agent-based model was developed in [45]. The traditional transportation models deals with the problems like transportation costs, delivery routes, production places, and the reduction of carbon emissions [42, 43, 48, 54]. The transportation algorithms [17, 34, 51] were developed to solve practical nature of the problems.

A new method was developed in [31] to solve the transport problems on northwest corner method in order to reduce the number of steps to determine the number of iteration given in [29]. A new technique for solving balanced transport problem based on geometric average for transport costs was developed in [8].

A number of transportation problems can be solved by using numeric techniques implemented in Matlab software. The northwest corner method was used to analyze the transport of chemical substance of a pharmaceutical company [37]. The Vogel's approximation and modified distribution method were used to deal with large scale cost matrices [27].

The Vogel's Approximation Method primarily determine the number of penalties appearing in each row and column of the matrix by computing the difference among minimum and second minimum cost. Further, this method also allocate to the cell having minimum cost across each of the row or column with maximum penalty. Once compared with

Northwest Corner Rule, this technique generates an initial solution which is very much near to optimality condition. On the other hand, the Modified Distribution Method is being mainly used for the analysis of the optimization to an initial feasible solution which is first obtained by Vogel's approximation, and Northwest Corner techniques. The main advantage of Modified Distribution Method compared to Northwest Corner Method is the insurance that an optimal solution has reached up to a desired level.

The Vogel's Approximation Method (VAM), Modified Distribution Method (MODI), and Northwest Corner Method (NWC) are mainly used to deal with analysis and the solution of transportation problems appearing in the operations research, specifically for the optimization of the logistics. The NWC method is being used as an initial approach for the location of resources when the consideration of cost function are much smaller than critical values. VAM mainly does focus on penalties. It helps to finds applications once the minimization of transportation cost function is in a crucial stage. MODI is being used to ensure the efficiency of cost solution. It does makes its vital role to optimize the transportation networks at very large scale, to balance both supply and demands. Furthermore, it ensure the cost efficiencies across the industries, for instance, the manufacturing, retail industry.

A new mathematical technique [11] based on simplex algorithm was developed to solve transport problem. A bi-criterion transportation problem was solved by Aneya and Nair [1]. In [28, 52], the optimization of passenger transport problems and passenger control flow problems were studied and analyzed. An extensive amount of literature has been written to deal with time-minimization transport problems, we refer interested readers to see [5, 24, 40, 46, 47] and the references therein.

The structured singular values are non-negative numbers obtained by computing the singular values of perturbation matrix $\hat{\Delta}$ from the set of block-diagonal matrices Δ. The structured singular values were first introduced by Doyle [12] to study and investigate both stability and instability of feedback systems. The computation of structured singular is a NP-hard problem [7]. The global search numerical and analytical techniques were developed [19, 32] to deal with lower dimensional mathematical problems.

The concept of D-stability for the very first time was introduced by Arrow and McManus [3], and then Enthoven and Arrow [14] in their classical papers. The main aim in these classical papers was to study and investigate the dynamic models from the economics. The theory of D-stability plays an important role in a vast amount of application areas in economic analysis [20, 22, 25, 26, 33, 38, 53].

The main objective of this paper is two-fold: First to study and analyze the spectral properties of structured matrices appearing across the Hitchcock-Koopmans transportation models. The computation of various tools like eigenvalues, singular values, right and left singular vectors, structured singular values, and pseudo-inverse describes various spectral properties of structured matrices under consideration. On the other hand, we present new results on the interconnection between structured singular values, and D-stability of structured matrices across Hitchcock-Koopmans transportation problem.

1.1. Overview of the article.

In section 2, we provide new results on singular values of a matrix corresponding to Hitchcock-Koopmans transportation problem. We have employed some tools from linear algebra and matrix analysis to develop these results. Furthermore, we also provide numerical experimentation on the computation of singular values and pseudo-inverse of transportation matrices. In section 3 of this article, we provide some new results on the interconnection between structured singular values of $(M^*M)^{-1}M^*$ and $M^*(MM^*)^{-1}$. The numerical experimentation on comparison of bounds of structured singular values are presented in section 4, and finally we have presented conclusion in section 5.

2. Singular values for Hitchcock-Koopmans transportation problem

The Hitchcock-Koopmans transportation problem was first developed by Hitchcock in 1941, and then investigated and analyzed by Koopmans in 1947. This problem was applied to simplex algorithm by Dantzig in 1951. For m numbers of origins and n numebrs of destinations, the Hitchcock-Koopmans transportation is to study and solve following optimization problem:

$$min\{c^T x : Mx = g,\, 1_m a = 1_n b,\ x \geq 0\},$$

where min is taken over x, also

$$a^T = [a_1, a_2, \cdots, a_m],\ b^T = [b_1, b_2, \cdots, b_m],\ x^T = [x_{11}, x_{12}, \cdots, x_{mn}],\ c^T = [c_{11}, c_{12}, \cdots, c_{mn}],$$

and $g^T = [a^T : b^T]$. The coefficient matrix M is given as

$$M = \begin{pmatrix} 1_n \otimes I_m \\ I_n \otimes 1_n \end{pmatrix},$$

where I_m is m-dimensional identity matrix , 1_m is $1 \times m$ vector of all $1's$, and $\otimes$ denotes kronecker product.

In [6], a relation between eigenvectors of M^+M and MM^T was developed. The Moore-Penrose inverse M^+ for M was obtained in [9] as

$$M^+ = \frac{1}{mn}\left(m1_n^T \otimes ((m+n)I_m - J_m)n((m+n)I_n) - J_n \otimes 1_m^T\right),$$

with J_m as an $m \times m$ matrix of all $1's$, means that, $J_m = 1_m^T 1_m$. Furthermore,

$$M^+M = I - \frac{1}{mn}\left(nI_n - J_n\right) \otimes \left(mI_m - J_m\right),$$

with $I - M^+M$, a symmetric and idempotent matrix. The relation between eigenvectors of J_n, J_m, and $I - M^+M$ was developed in the Theorem 2 [6].

The following theorem gives Moore-Penrose inverse of a matrix M.

Theorem 1. *Let $M \in \mathbf{R}^{n,n}$. Then, then M^+ is the Moore-Penrose inverse satisfying the matrix equation*

$$M = MM^+M.$$

Theorem 2. *Let v_j be all eigenvectors of J_m, and u_k, the eigenvectors corresponding to J_n. Then eigenvectors corresponding to simple eigenvalues 1 of $I - M^+M$ are*

$$\hat{v}_i = u_j \otimes x_k,$$

where $j = 1 : n-1$; $k = 1 : m-1$; $i = m+n, \cdots, (m-1)(n-1)$.

Definition 1. *The singular value of a square or rectangular matrix M are the positive square root of eigenvalue of M^TM, where T denotes the transpose of M.*

Theorem 3. *Let $M \in \mathbf{R}^{n,n}$. Then there exists unitary matrices U, V, and a diagonal matrix Σ such that M can be decomposed as*

$$M = U\Sigma V^T.$$

Further, the non-zero singular values of M are contained along the main diagonal of Σ.

Corollary 1 [6]. Let $\{a_1, a_2, \cdots, a_{m+n-1}\}$ and $\{b_1, b_2, \cdots, b_{m+n-1}\}$ are the sets of eigenvectors corresponding to eigenvalues of MM^T and M^TM, and consider that $\sigma_1, \sigma_2, \cdots, \sigma_{m+n-1}$ are singular values of M. Then,

$$MM^Ta_i = \sigma_i^2 a_i,\ i = 1 : m+n-1,$$

$$M^TMb_i = \sigma_i^2 b_i,\ i = 1 : m+n-1.$$

Corollary 2 [6]. The eigenvectors a_i corresponding to the eigenvalue 0 and $m+n$ are

$$J_m x_i = m x_i,\ J_n y_i = n y_i.$$

The eigenvectors corresponding to eigenvalues m and n are given by

$$J_m x_i = \vec{0},\ J_n y_i = \vec{0}$$

with x_i are $m \times 1$ vectors and y_i are $n \times 1$ vectors.

Let $M^+ = (M^*M)^{-1}M^*$ be the Moore-Penrose inverse matrix of the matrix M, $*$ denotes the complex conjugate transpose. Let $M^*(MM^*)^{-1}$ be the right inverse of M. The following Theorem 4 gives the orthogonal nature of the leading singular values.

Theorem 4. *Let $(M^*M)^{-1}M^*$ and $M^*(MM^*)^{-1}$ be n-dimensional matrices. Let $\{\sigma_i\}$, $\forall i = 1 : n$ be sequence of singular values with $\{v_i\}$, and $\{\hat{v}_i\}$, $\forall i = 1 : n$ as the left hand side and right hand side singular vectors such that $||v_i||_2 = 1 = ||\hat{v}_i||_2$, and $\{u_i\}$, and $\{\hat{u}_i\}$, $\forall i = 1 : n$ as the left hand side and right hand side singular vectors for $\{\hat{\sigma}_i\}$ such that $||u_i||_2 = 1 = ||\hat{u}_i||_2$. The leading singular vectors v_1 and u_1 are orthogonal to σ_1 and $\hat{\sigma}_1$. Then any non-zero vector $\tilde{v} = \{\tilde{v}_1, \tilde{v}_2, \cdots, \tilde{v}_n\}$ is an orthogonal to vector $e_n = \frac{1}{\sqrt{n}}(1, 1, \cdots, 1)^T$ and is not a singular vector σ_1 and $\hat{\sigma}_1$.*

Proof. For the prove, we consider singular value problem of form

$$M\tilde{v} = \sigma\tilde{v},$$

where σ is singular value of M corresponding to singular vector $\tilde{v}$. The vector $\tilde{v}$ doesn't act as singular vector corresponding to singular values σ_1 and $\hat{\sigma}_1$. The matrix-vector product $M\tilde{v}$, can be rewritten as

$$M\tilde{v} = \begin{pmatrix} m_{11}\tilde{v}_1 + m_{12}\tilde{v}_2 + \cdots m_{1n}\tilde{v}_n \\ \cdots \\ \cdots \\ \cdots \\ m_{n1}\tilde{v}_1 + m_{n2}\tilde{v}_2 + \cdots m_{nn}\tilde{v}_n \end{pmatrix}.$$

Take $\sum$ on the components of $M\tilde{v}$, we have

$$\sum(M\tilde{v}) = \sum \begin{pmatrix} m_{11}\tilde{v}_1 + m_{12}\tilde{v}_2 + \cdots m_{1n}\tilde{v}_n \\ \cdots \\ \cdots \\ \cdots \\ m_{n1}\tilde{v}_1 + m_{n2}\tilde{v}_2 + \cdots m_{nn}\tilde{v}_n \end{pmatrix} = v_1 \sum(m_i 1) + \cdots + v_n \sum(m_i n) = v_1(\sigma_1) + \cdots + v_n(\sigma_1).$$

Thus,

$$\sum(Mv) = \sigma_1 \sum(v).$$

Taking $\sum$ of right hand side of $Mv = \sigma v$, we get

$$\sum(\sigma v) = \sigma \sum(v).$$

We conclude from last two equations that

$$(\sigma - \sigma_1) = \sum(v) = 0.$$

This implies that $\sigma \neq \sigma_1$, and $\sum(v) = 0$. This proves that $v = (v_1, v_2, \cdots, v_n)$ is orthogonal to vector $e_n = \frac{1}{\sqrt{n}}(1, 1, \cdots, 1)^T$. The vector e_n is not a singular vector corresponding to singular values σ_1 and $\hat{\sigma}_1$.

Theorem 5. *Let $(M^*M)^{-1}M^*$ and $M^*(MM^*)^{-1}$ be n-dimensional matrices. Let σ_i and $\hat{\sigma}_i$ are leading singular values corresponding to singular vectors v_i and $\hat{v}_i$, respectively. Let*

$$\hat{M} = \begin{pmatrix} (M^*M)^{-1}M^* + \alpha I_n & 2\alpha\hat{v}_2 v_1^T \\ 2\alpha v_1\hat{v}_2^T & M^*(MM^*)^{-1} + \alpha I_n \end{pmatrix},$$

with I_n is an identity matrix, α is any scalar. The singular values of $\hat{M}$ does not have leading singular values of matrices $(M^*M)^{-1}M^*$ and $M^*(MM^*)^{-1}$. The leading singular values σ_1 and $\hat{\sigma}_1$ are along the main diagonal of $\hat{M}_1$ with

$$\hat{M}_1 = \begin{pmatrix} 3\alpha + \sigma_1 & 0 \\ 0 & \alpha - \hat{\sigma}_1 \end{pmatrix}.$$

Proof. Consider the singular value problem of the form

$$\hat{M}v_i = (\sigma_1 + \alpha)v_i, \ \forall i = 1 : n.$$

Let $\beta_1, \sigma_2+\alpha, \sigma_3+\alpha, \cdots, \sigma_n+\alpha$, and $\hat{\beta}_2, \hat{\sigma}_2+\alpha, \hat{\sigma}_3+\alpha, \cdots, \hat{\sigma}_n+\alpha$ with $\beta, \hat{\beta} \in \{3\alpha+\sigma_1, \alpha-\hat{\sigma}_1\}$ are the singular values of matrix

$$\begin{pmatrix} 3\alpha + \sigma_1 & 0 \\ 0 & \alpha - \hat{\sigma}_1 \end{pmatrix}.$$

For the singular values $\sigma_i, \ \forall i = 2 : n$, the singular vectors can be written as $[v_i \ \ 0]^T, \ \forall i = 2 : m$. For the singular values $\hat{\sigma}_i, \ \forall i = 2 : n$, the singular vectors can be written as $[0 \ \ \hat{v}_i]^T, \ \forall i = 2 : n$. This ensures that singular vectors corresponding to $\hat{M}$ can be expressed as $[\beta_i v_i \ \ \hat{\beta}_i\hat{v}_i]^T$. Thus, $[\beta_i \ \ \hat{\beta}_i], \ \forall i = 2 : n$ are the singular vectors corresponding to singular values $\sigma_1 + 3\alpha$ and $\alpha - \hat{\sigma}$.

The following Theorem shows that singular values of a matrix does not depend continuously on the entries of that matrix.

Theorem 6. *Let $M_1 = (M^*M)^{-1}M^*$ and $M_2 = M^*(MM^*)^{-1}$ be n-dimensional matrices. Let $\lim_{k\to\infty}(M_k) = M$, and let $q = minm, n$. Consider that $\sigma_1(M) \geq \sigma_2(M) \geq \cdots \geq \sigma_q(M)$, and $\sigma_1(M_k) \geq \cdots \geq \sigma_q(M_k)$ be non-increasing ordered singular values of M and M_k, respectively for $k = 1, 2, \cdots$. Then the $\lim_{k\to\infty}\sigma_i(M_k) = \sigma_i(M)$ for all $i = 1 : q$.*

Proof. Let $k_1 < k_2 < \cdots$, be the sequence of positive integers, and let $\epsilon > 0$, we have that

$$max|\sigma_i(M_{kj}) - \sigma_i(M)| > \epsilon,$$

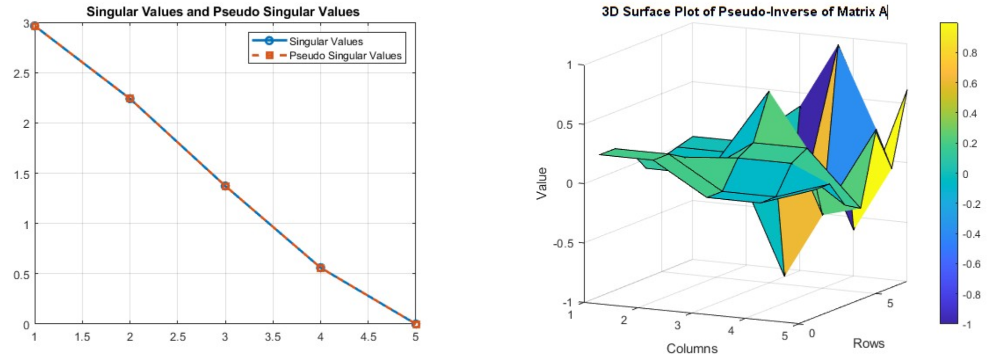
where max is taken over all $i = 1 : q$.

Consider the singular value decomposition of M_{kj} as

$$M_{kj} = U_{kj}\Sigma_{kj}V_{kj}^*,$$

where U_{kj} and V_{kj} are the unitary matrices and Σ_{kj} is a diagonal matrix with structure

$$\Sigma_{kj} = [\sigma_1(M_{kj}) \cdots \sigma_q(M_{kj})]^T.$$



(a) Singular values and pseudo singular values

(b) Surface plot of pseudo-spectrum

Figure 1: The graphs of singular values and pseudo-inverse of M in Example-1

Then,

$$\lim_{r\to\infty} \Sigma_{kjr} = \lim_{r\to\infty} U^*_{kjr} M_{kjr} V_{kjr} = \left(\lim_{r\to\infty} U^*_{kjr}\right)\left(\lim_{r\to\infty} M_{kjr}\right)\left(\lim_{r\to\infty} V_{kjr}\right) = U^* M V.$$

The matrix U^*MV is a non-negative diagonal matrix. The uniqueness of the singular values of M implies that Diag $\Sigma = [\sigma_1(M), \sigma_2(M), \cdots, \sigma_q(M)]^T$, a contradiction with inequality

$$max|\sigma_i(M_{kj}) - \sigma_i(M)| > \epsilon.$$

This proves required result.

Next, we give numerical examples on the computation of singular values and pseudo inverse of transportation matrices.

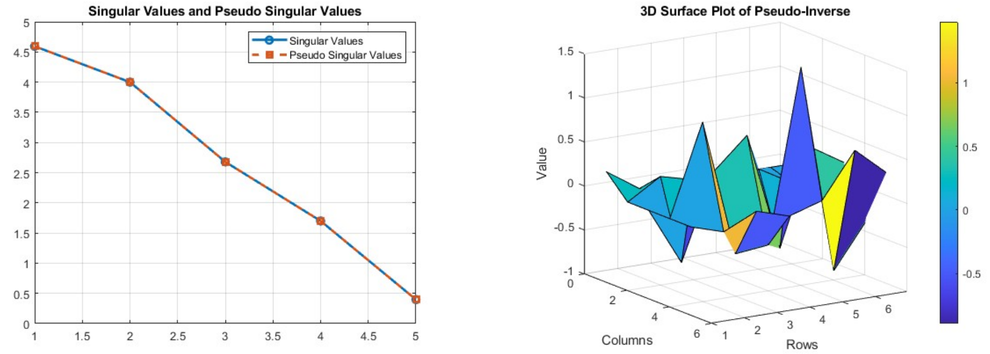
Example 1. Consider 7×5 transportation matrix (staircase matrix) taken from [16].

$$M = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

The computation of singular values and the graphs of the pseudo-inverse are shown in Figure 1.

Example 2. Consider 7×5 transportation matrix (distribution matrix) taken from [16].

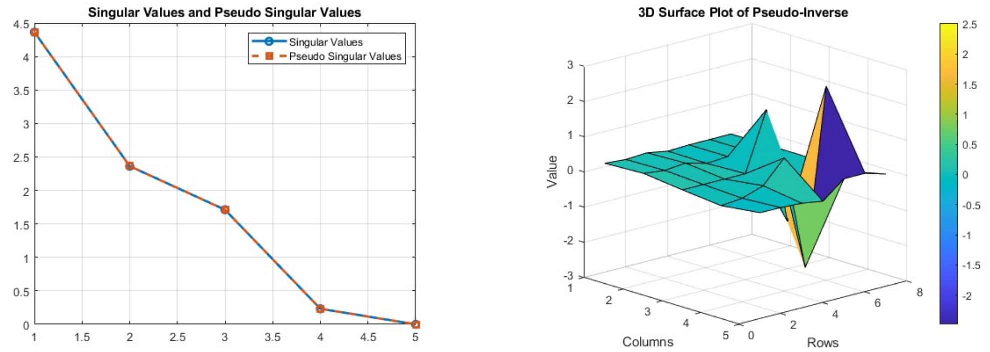
$$M = \begin{pmatrix} 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1.4 & 0.6 \end{pmatrix}.$$



(a) Singular values and pseudo singular values

(b) Surface plot of pseudo-spectrum

Figure 2: The graphs of singular values and pseudo-inverse of M in Example-2



(a) Singular values and pseudo singular values

(b) Surface plot of pseudo-spectrum

Figure 3: The graphs of singular values and pseudo-inverse of M in Example-3

The computation of singular values and the graphs of the pseudo-inverse are shown in Figure 2.

Example 3. Consider 7×5 transportation matrix (distribution matrix) taken from [16].

$$M = \begin{pmatrix} 1.6 & 0.8 & 1.6 & 0 & 0 & 0 & 0 \\ 2.4 & 1.2 & 2.4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.6 & 1.8 & 0.6 & 0 & 0 \\ 0 & 0 & 0.4 & 1.2 & 0.2 & 0.2 & 0 \\ 0 & 0 & 0 & 0 & 1.2 & 1.2 & 0.6 \end{pmatrix}.$$

The computation of singular values and the graphs of the pseudo-inverse are shown in Figure 3.

3. Structured singular values for Hitchcock-Koopmans transportation problem

In this section, we present new results on the computation of structured singular values for matrices appearing in Hitchcock-Koopmans transportation problems. We make use of mathematical tools from linear algebra, system theory and matrix theory to provide and analyzed the results on structured singular values.

Definition 2. *The n-dimensional matrix M is stable if all the real parts of the eigenvalues are strictly positive, that is, $Re(\lambda_i(M)) > 0$.*

Definition 3. *The n-dimensional matrix M is D-stable if all the real parts of the eigenvalues of MD are strictly positive, that is, $Re(\lambda_i(MD)) > 0$, where $D = diag(d_{ii}) > 0$, for all $i = 1 : n$.*

The structured singular value of a given matrix M with respect to set of block-diagonal matrices Δ, where

$$\Delta := \{diag\,(\delta_1 I_{r_1}, \delta_2 I_{r_2}, \cdots, \delta_S I_{r_S}; \Delta_1, \Delta_2, \cdots, \Delta_F) : \delta_i \in \mathbb{R}(\mathbb{C}),\ \Delta_j \in \mathbb{K}^{m_j, m_j},\ i = 1 : S;\ J = 1 : F\},$$

with $\mathbb{K} = \mathbb{R}(\mathbb{C})$, is the computation of largest singular value of $\hat{\Delta} \in \Delta$. The structured singular value is denoted by μ, and for a given matrix M and Δ, it is defined as (see [12]):

$$\mu_\Delta(M) := \Big(min\{||\hat{\Delta}||_2 : det(I_n - M\hat{\Delta}) = 0,\ \forall\, \hat{\Delta} \in \Delta\}\Big)^{-1},$$

where **min** is taken over $\hat{\Delta} \in \Delta$, and $\mu_\Delta(M) = 0$ if $det(I_n - M\hat{\Delta}) \neq 0,\ \forall\, \hat{\Delta} \in \Delta$.

Remark 1. The block-diagonal structure Δ can be associated with multi-index of the positive integers.

Remark 2. The full blocks in Δ can be taken as either pure real blocks, pure complex blocks or a mixture of both. These blocks can be chosen as rank-1 matrices.

Remark 3. For any $\alpha \in \mathbb{C}$, $\mu_\Delta(\alpha M) = |\alpha| \mu_\Delta(M)$.

Corollary 1. *[12] The structured singular value $\mu_\Delta(M)$ is equal to the computation of spectral radius ρ of $V^T MW$, that is,*

$$\mu_\Delta(M) = \rho(V^T MV),$$

with V, W having the block-diagonal structure.

The $\mu_\Delta(M)$ can be considered as the computation of spectral radius ρ of $M\hat{\Delta}$, $\hat{\Delta} \in \Delta$.

Lemma 1. *[35] For M, and a block-diagonal structure Δ,*

$$\mu_\Delta(M) = max\,\rho(M\hat{\Delta}),$$

where max is taken over $\hat{\Delta} \in \Delta$.

Theorem 7 is a well-known result on D-stability of a given matrix while computing the strictly positive real part of the spectrum of the product of given matrix with a positive diagonal matrix.

Theorem 7. *[14] If M has all negative diagonal elements, and no negative off diagonal elements, for $D = diag(d_{ii})$, the matrices M, and DM are stable, then $D = diag(d_{ii}) > 0,\ \forall i$.*

The following theorem 8 is the characterization of D-stability and bridge a link between μ-values and D-stable matrices.

Theorem 8. *Consider M be a n-dimensional matrix. Then M is D-stable matrix if and only if M is stable and*

$$0 \leq \mu_{\Delta}(\boldsymbol{i}I_n + M)^{-1}(\boldsymbol{i}I_n - M) < 1.$$

Theorem 9. *Let $(MM^*)^{-1}M^*$ be $n-$dimensional complex valued matrix. Then*

$$0 \leq \mu_{\Delta}((I_n + (MM^*)^{-1}M^*)^{-1}(I_n - (MM^*)^{-1}M^*)) < 1.$$

Proof. To prove this result, we use the concept of $D-$stability of a matrix. That is, the given $(MM^*)^{-1}M$ is $D-$stable, means that, $\lambda_k(I_n + (MM^*)^{-1}Mp) \neq 0, \forall k = 1 : n$, with $P = diag(p_{11}, p_{22}, \ldots, p_{nn}) > 0$. Let $P = (I_n + R)^{-1}(I_n - R), \forall R \in \Delta$, then

$$\lambda_k(I_n + (M^*M)^{-1}M^*(I_n + R)^{-1}(I_n - R)) \neq 0, \forall k = 1 : n, \forall R \in \Delta.$$

This further implies that

$$\lambda_k((I_n + (M^*M)^{-1}M^*) + (I_n - (M^*M)^{-1}M^*R)) \neq 0, \forall k = 1 : n, \forall R \in \Delta.$$

Since,

$$\lambda_k(I_n+(M^*M)^{-1}M^*) \neq 0 \sim \lambda_k((I_n+(M^*M)^{-1}M^*)+(I_n-(M^*M)^{-1}M^*R)) \neq 0, \forall k = 1 : n, \forall R \in \Delta$$

Finally, this yields that

$$0 \leq \mu_{\Delta}((I_n + (MM^*)^{-1}M^*)^{-1}(I_n - (MM^*)^{-1}M^*)) < 1,$$

which is the required prove.

Theorem 10. *Let $(M^*M)^{-1}M^*$ be $n-$dimensional complex valued matrix. Then $0 \leq \mu_{\Delta}(A) < 1$ if $(M^*M)^{-1}M^*$ is $D-$stable, with*

$$A = (\boldsymbol{i}I_n + P(M^*M)^{-1}M^* + M((M^*M)^{-1})^*)(\boldsymbol{i}I_n - P(M^*M)^{-1}M^* - M((M^*M)^{-1})^*P)$$

with $P = diag(p_{11}, p_{22}, \ldots, p_{nn})$, $p_{ii} > 0, \forall i = 1 : n,\ \boldsymbol{i} = \sqrt{-1}$.

Proof. The given matrix $(M^*M)^{-1}M^*$ is $D-$stable if $Re(\lambda_k(P(M^*M)^{-1}M^*+M(M^*M^{-1})^*)) > 0, \forall k = 1:n$, see [13]. To prove that $0 \leq \mu_\Delta(A) < 1$, we consider a block-diagonal structure $\hat{\Delta} = (\mathbf{i}I_n - P)(\mathbf{i}I_n + P)^{-1}$ with $\hat{\Delta} \in \Delta$. For all P, the diagonal positive definite matrices, we have

$$\lambda_k(P(M^*M)^{-1}M^* + M((M^*M)^{-1})^*P + \mathbf{i}(\mathbf{i}I_n + \hat{\Delta})^{-1}(\mathbf{i}I_n - \hat{\Delta})) \neq 0 \forall k = 1:n.$$

In turn, the expression for $\lambda_k, \forall k = 1:n$ takes the form

$$\lambda_k((\mathbf{i}I_n+P(M^*M)^{-1}M^*+M((M^*M)^{-1})^*P)-(\mathbf{i}I_n-P(M^*M)^{-1}M^*-M((M^*M)^{-1})^*P)\hat{\Delta}) \neq 0, \forall\hat{\Delta} \in \Delta.$$

Thus,

$$\lambda_k(I_n-(\mathbf{i}I_n+P(M^*M)^{-1}M^*+M((M^*M)^{-1})^*P))(\mathbf{i}I_n-P(M^*M)^{-1}M^*-M((M^*M)^{-1})^*P)\hat{\Delta}) \neq 0, \forall\hat{\Delta} \in \Delta.$$

The last expression for $\lambda_k, \forall k = 1:n$ implies that $0 \leq \mu_\Delta(A) < 1$

Theorem 11. *Let $(M^*M)^{-1}M^*$ be $n-$dimensional complex valued matrix. Then, $0 \leq \mu_\Delta(A) < 1$ if*

$$x^*(M((M^*M)^{-1})^*P^2 + P^2(M^*M)^{-1}M^*)x > 0$$

for $x \in \mathbb{C}^{n,1}$, and for all $P = diag(p_{11}, p_{22}, \ldots, p_{nn}) > 0$, with

$$A = (\boldsymbol{i}I_n + (M^*M)^{-1}M^*)^{-1}(\boldsymbol{i}I_n - (M^*M)^{-1}M^*).$$

Proof. The structured singular value is the computation of $\alpha_{max} \geq 0$ such that for each P, the matrix inequality

$$\frac{\|P(M^*M)^{-1}M^*x\|)}{\|Px\|} \geq \alpha_{max}.$$

For given A, $0 \leq \mu_\Delta(A) < 1$ if $\|PAx\| < \|Px\|$ for every $x \in \mathbb{C}^{n,1}$ and positive diagonal matrix P. The above inequality holds true for $\mathbf{i}I_n + (M^*M)^{-1}M^*$, means that,

$$\|PA(\mathbf{i}I_n + (M^*M)^{-1}M^*)x\| < \|P(\mathbf{i}I_n + (M^*M)^{-1}M^*)x\|.$$

Also,

$$\|PA(\mathbf{i}I_n + (M^*M)^{-1}M^*)x\|^2 < \|P(\mathbf{i}I_n + (M^*M)^{-1}M^*)x\|^2.$$

This further implies that

$$x^*((\mathbf{i}I_n + (M^*M)^{-1}M^*)^*A^*P^*PA(\mathbf{i}I_n + (M^*M)^{-1}M^*))x$$

$$<$$

$$x^*((\mathbf{i}I_n + (M^*M)^{-1}M^*)^*P^*P(\mathbf{i}I_n + (M^*M)^{-1}M^*))x.$$

This inequality further reduces to

$$x^*((\mathbf{i}I_n+(M^*M)^{-1}M^*)^*A^*P^2A(\mathbf{i}I_n+(M^*M)^{-1}M^*))x < x^*(\mathbf{i}I_n+(M^*M)^{-1}M^*P^2(\mathbf{i}I_n+(M^*M)^{-1}M^*))x.$$

Since,

$$A = (\mathbf{i}I_n + (M^*M)^{-1}M^*)^{-1}(\mathbf{i}I_n - (M^*M)^{-1}M^*).$$

Thus, above inequality rewritten as

$$x^*((\mathbf{i}I_n{+}(M^*M)^{-1}M^*)^*(\mathbf{i}I_n{-}(M^*M)^{-1}M^*)^*(\mathbf{i}I_n{+}(M^*M)^{-1}M^*)^{-1}P^2(\mathbf{i}I_n{+}(M^*M)^{-1}M^*)^{-1}$$

$$(\mathbf{i}I_n{-}(M^*M)^{-1}M^*)(\mathbf{i}I_n{+}(M^*M)^{-1}M^*)x{-}x^*((\mathbf{i}I_n{+}(M^*M)^{-1}M^*)^*P^2(\mathbf{i}I_n{+}(M^*M)^{-1}M^*))x < 0.$$

or

$$x^*(\mathbf{i}I_n{-}(M^*M)^{-1}M^*)^*P^2(\mathbf{i}I_n{-}(M^*M)^{-1}M^*)x{-}x^*(\mathbf{i}I_n{+}(M^*M)^{-1}M^*)^*P^2(\mathbf{i}I_n{+}(M^*M)^{-1}M^*)x < 0.$$

In turn, we have that

$$x^*((\mathbf{i}I_n{-}(M^*M)^{-1}M^*)^*P^2(\mathbf{i}I_n{-}(M^*M)^{-1}M^*){-}(\mathbf{i}I_n{+}(M^*M)^{-1}M^*)^*P^2(\mathbf{i}I_n{+}(M^*M)^{-1}M^*))\kappa < 0.$$

Thus,

$$x^*((\mathbf{i}I_n{-}(M^*M)^{-1}M^*)^*(\mathbf{i}P^2{-}P^2(M^*M)^{-1}M^*){-}(\mathbf{i}I_n{+}(M^*M)^{-1}M^*)^*(\mathbf{i}P^2{+}P^2(M^*M)^{-1}M^*))x < 0.$$

Also,

$$x^*(-2\mathbf{i}P^2(M^*M)^{-1}M^* - 2\mathbf{i}M(M^*M)^{-1}P^2)x < 0$$

or

$$x^*(-2\mathbf{i}(M(M^*M)^{-1})^*P^2 + P^2(M^*M)^{-1}M^*)x < 0.$$

Finally,

$$x^*(M(M^*M)^{-1})^*P^2 + P^2(M^*M)^{-1}M^*)x > 0,$$

which completes required proof.

Pseudo-spectrum: The pseudo-spectrum of a matrix M is the set of which contains the spectrum, that is, all the eigenvalues of M. The important question one can raise is about the singularity of M which does not appear as a robust in the sense that a small perturbation ϵ may vary the answer from yes to no in a dramatic way. This helps to think that either $||M^{-1}||$ is large enough or not?

For λ, an eigenvalue of M, a much better question is to ask: Does $||(\lambda I_n - M)^{-1}||$ is large or not? such a pattern allows following definitions and results [49] of pseudo-spectrum.

Definition 4. *Let M be a given n-dimensional matrix, $\epsilon > 0$, a small perturbation. The ϵ-pseudospectrum $\sigma_\epsilon(M)$ is the set of eigenvalues $\lambda \in \mathbb{C}$ such that*

$$||(\lambda I_n - M)^{-1}|| > \frac{1}{\epsilon}.$$

Remark 4. For $\lambda \in \sigma(M)$, $\sigma(M)$ being as the set of eigenvalues of M, $||(\lambda I_n - M)^{-1}|| = \infty$. The second definition of pseudo-spectrum is given as follows.

Definition 5. *Let M be a given n-dimensional matrix, $\epsilon > 0$, a small perturbation. The ϵ-pseudospectrum $\sigma_\epsilon(M)$ is the set of eigenvalues $\lambda \in \mathbb{C}$ such that*

$$\lambda \in \sigma(M + E),$$

for some E with $||E|| < \epsilon$.

The third characterization of pseudo-spectrum is given as bellow.

Definition 6. *Let M be a given n-dimensional matrix, $\epsilon > 0$, a small perturbation. The ϵ-pseudo spectrum $\sigma_\epsilon(M)$ is the set of eigenvalues $\lambda \in \mathbb{C}$ such that*

$$||(\lambda I_n - M)v|| < \epsilon$$

for some $v \in \mathbb{C}^{n,1}$, $||v|| = 1$.

The following Theorem gives an equivalence of all above definitions of pseudo-spectrum.

Theorem 12. *[13] Let $||\cdot||$ denotes a matrix norm for a matrix M which is induced by a vector norm. Then following are equivalent:*

$$(i)\ \Lambda_\epsilon(M) = \{z \in \mathbf{C} : ||(zI_n - M)^{-1}|| \geq \frac{1}{\epsilon}\}.$$

$$(ii)\ \Lambda_\epsilon(M) = \{z \in \mathbf{C} : z \in \Lambda(M + E),\ ||E|| \leq \epsilon\}.$$

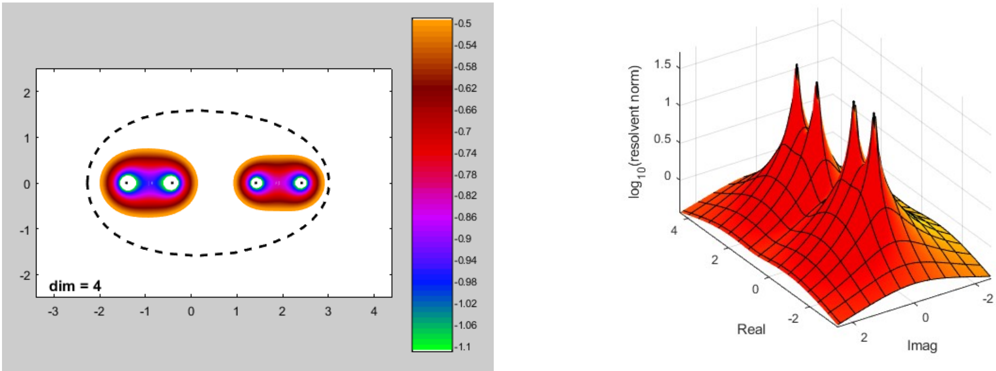
$$(iii)\ \Lambda_\epsilon(M) = \{z \in \mathbf{C} : \exists\ v \in \mathbf{C}^{n,1}\ s.t\ ||(M - zI_n)v|| \leq \epsilon\}.$$

Remark 1. *In above Theorem 12, the second statement is true for some matrix E. Also, in the last statement the column vector v is such that $||v||_2 = 1$.*

4. Numerical Experimentation

We present a comparison on the numerical approximation of structured singular values. The numerical algorithms are: The Matlab function **mussv**, the power algorithm (PA) [36], Gain Based Algorithm (GBA) [44], Poles migration Algorithm (PMA) [30], Non-linear optimization Algorithm (NLA) [23], and the Low-rank ODE's based Algorithm (LRA) given by first author [21]. We consider structured matrices appearing across transportation models. The Matlab EigTool [13] is being used for graphical interpretation of the pseudo-spectrum.

In each example, we have shown different graphics. The graphical representations in a 2-dimensional space presents the spectrum and pseudo-spectrum of structured matrices. The black dots around the spectrum and pseudo-spectrum denotes the field of values enclosing spectrum and pseudo-spectrum. The visualization of the pseudo-spectrum in the complex plane are level sets of the resolvent norm $||(M - zI_n)^{-1}||$ which indicates various ϵ-values. These visualizations of the level sets helps to analyze the stability analysis and robustness in various applications like control systems and fluid dynamics.



(a) Field of values enclosing spectrum and pseudo-spectrum

(b) Resolvent norm

Figure 4: pseudo-spectrum of A in Example-1

Example 1. Consider 4-dimensional matrix for Traveling Salesman Problem [39].

$$A = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 3 & 0 & 2 & 1 \end{pmatrix}.$$

We present the comparison on numerical approximation of the lower bounds of structured singular values in following Table-1.

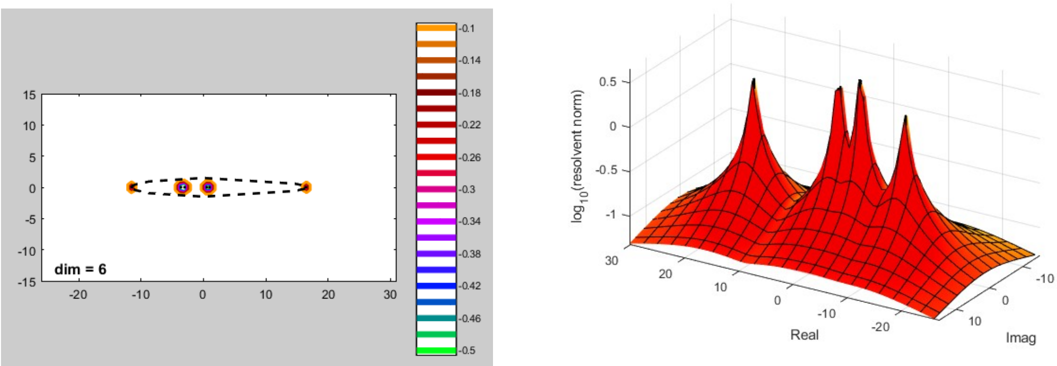
The numerical approximation of lower bounds of structured singular values					
mussv	PA	GBA	PMA	NLA	LRA
3.9984	3.9992	3.9987	3.9990	3.9988	3.9985

Example 2. Consider 6-dimensional symmetric circulant matrix for Travelling Salesman Problem [50].

$$A = \begin{pmatrix} 0 & 4 & 1 & 6 & 1 & 4 \\ 4 & 0 & 4 & 1 & 6 & 1 \\ 1 & 4 & 0 & 4 & 1 & 6 \\ 6 & 1 & 4 & 0 & 4 & 1 \\ 1 & 6 & 1 & 4 & 0 & 4 \\ 4 & 1 & 6 & 1 & 4 & 0 \end{pmatrix}.$$

We present the comparison on numerical approximation of the lower bounds of structured singular values in following Table-2.

The numerical approximation of lower bounds of structured singular values					
mussv	PA	GBA	PMA	NLA	LRA
3.9984	3.9992	3.9987	3.9990	3.9988	3.9985



(a) Field of values enclosing spectrum and pseudo-spectrum

(b) Resolvent norm

Figure 5: pseudo-spectrum of A in Example-2

Example 3. Consider 4-dimensional matrix taken from [41].

$$A = \begin{pmatrix} 3 & 5 & 1 & -2 \\ 0 & -1 & 5 & 10 \\ 3 & 5 & 1 & 9 \\ -2 & 1 & 6 & 6 \end{pmatrix}.$$

We present the comparison on numerical approximation of the lower bounds of structured singular values in following Table-3.

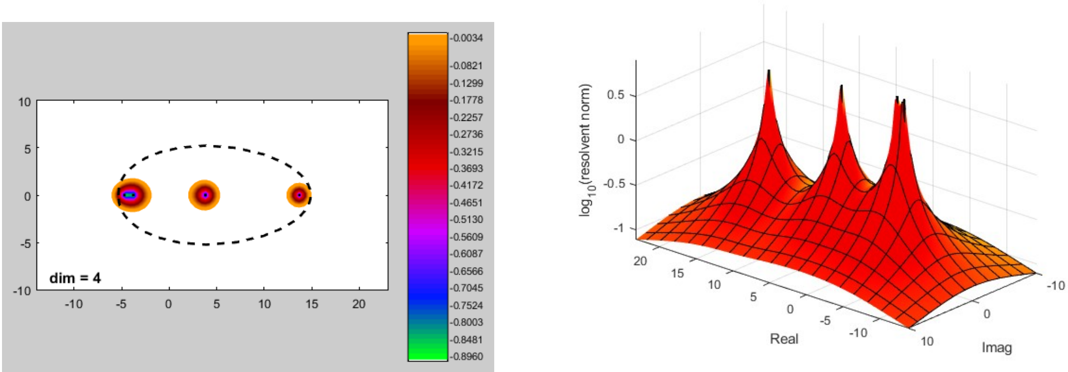
The numerical approximation of lower bounds of structured singular values					
mussv	PA	GBA	PMA	NLA	LRA
16.4123	16.4176	16.4198	16.4183	16.4142	16.4130

5. Conclusion

In this article we have presented new results on the spectral properties (singular values) of structured matrices appearing across Hitchcock-Koopmans transportation models. The interconnection between D-stable matrices and structured singular values of pseudo-inverse are analyzed. The numerical experimentation carried with Matlab shows the dynamics of singular values, structured singular values, and pseudo-spectrum. The advantages of the proposed methodology are listed as:

1. The proposed methodology helps to study the spectral properties via the computation of eigenvalues, eigenvectors, singular values, right and left handed singular vectors, and structured singular values.

2. The proposed methodology has an advantage in the sense that it allow us to establish new interconnections between structured singular values and D-stability analysis of structured matrices appearing across the Hitchcock-Koopmans transportation model.



(a) Field of values enclosing spectrum and pseudo-spectrum

(b) Resolvent norm

Figure 6: pseudo-spectrum of A in Example-3

3. The computation and graphical representation of spectrum and pseudo-spectrum gives an advantage to exploit the hidden structures and properties of structured matrices.
4. The proposed methodology is well-established in the sense that it has strong theoretical foundations and also numerical experimentation to support the theoretical construction.

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