EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 1, Article Number 5643
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Dynamics of Deposit and Loan Volumes Model with Non-Performing Loan and Deposit Withdrawal Factors

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Abstract. Non-performing loan (NPL) and deposit withdrawal are two factors that can affect loan and deposit volumes in the bank. The distribution of deposits into loans at banks can be modelled using the predator-prey model. In this research, we propose and analyse a mathematical model dealing with two loans, i.e. a loan for individuals and a loan for companies at one bank, which is developed from two predators and one prey model. We aim to study the dynamics and long-term behaviour of the proposed model, as well as to discuss the effects of the NPL and deposit withdrawal parameters associated with the model. The results of the analysis show that the model has five equilibrium points. We find that the equilibrium point without deposit and loan activities in the bank is always unstable, while the other equilibrium points are globally asymptotically stable if their certain conditions are satisfied. The theoretical results are verified by our numerical simulations.

2020 Mathematics Subject Classifications: 91B74, 37N40, 34C25, 92D25

Key Words and Phrases: Deposit and Loan Model, Non-Performing Loan, Withdrawal, Predator-Prey, Global Stability

1. Introduction

A bank is a business entity that collects deposits from the public and then lends them back to the public in the form of loans, otherwise known as credit. In Indonesia, OJK or the Financial Services Authority classifies commercial banks into four KBMI or Bank Groups based on Core Capital. Core capital is a component of capital sourced from the bank itself that can affect the bank's operations in a stable and safe manner, so the amount of core capital of a bank affects the management of deposits from the public [5, 12, 26]. Thus, the KBMI classification causes banks to have their limit on receiving deposits from the public. A bank will reward people who deposit their funds with interest on their deposits. The volume of deposits will be distributed into loans by adding loan interest

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DOI: https://doi.org/10.29020/nybg.ejpam.v18i1.5643

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[12]. The distribution of deposit volumes into loan volumes is a way for banks to obtain funds for operational costs and return deposit interest to their customers [27]. People who deposit their funds in the bank can also withdraw their deposits in cash or transfer them to another account according to the bank's requirements [10, 12, 14]. Thus, the bank's ability to meet withdrawal requests and other cash outflows serves as a crucial indicator to assess its financial health and overall viability [9]. If a bank cannot meet its financial obligations, it could threaten its stability and negatively impact the financial system as a whole.

Banks distribute deposit funds into loan funds to financial markets, such as individuals or companies [27] to finance various projects, investments, and economic activities [12]. The amount of loans given to individuals or companies also depends on the amount of deposits in the bank. Banks often have loan limits as a form of credit risk control [32]. Consequently, the more companies or individuals that request a loan, the higher competition rate, which may slow down the process or reduce their chances of successfully obtaining a loan. Before giving a loan, the bank will conduct a credit analysis to ensure that the potential credit recipient is truly trustworthy so that it reduces the possibility of bad credit or Non-Performing Loan (NPL). Credit analysis includes the background of potential credit recipients, business prospects, and guarantees that include various indemnification agreements if the loan is not returned on time [1, 34]. Loans that are not repaid on time or are not repaid at all are called to as NPL [10]. If NPL occurs, it will affect the availability of funds in the bank [21] as well as its profitability and bank operations [7, 13].

The distribution of deposits into loans is analogous to the predator-prey interaction model. The predator-prey interaction model was first introduced by the American biophysicist Lotka in 1925 and by the Italian mathematician Volterra in 1926 [19]. In a predator-prey interaction, the predator will eat the prey. This is similar to the behaviour of the distribution of deposit and loan funds where loans as predators and deposits as prey. In 2013, Haque et al. proposed a two predators-one prey model by adding the effects of intra-specific competition that occurs in each predator [11]. Furthermore, Mukhopadhyay and Bhattacharyya (2016) constructed a two predators-one prey model by adding the effect of inter-predator competition to the model [18]. Mukhopadhyay and Bhattacharyya also added the effect of harvesting on one of the predators. Harvesting occurs due to interactions with humans, when humans take or kill predator or prey for consumption or sale that have been used primarily in fishing, forestry, and wildlife conservation. In [8, 23, 24, 30], the effect of harvesting on predator-prey interactions was studied. Deposit withdrawals in banks exhibit behaviour similar to prey harvesting. Furthermore, Long et al. (2022) developed a two predators-one prey model using the effects of inter-specific and intra-specific competition [16]. This is also in accordance with the previous problem, where there is competition between loan volumes in the bank.

In 2014, Sumarti et al. proposed a dynamic model of deposit and loan volumes based on the Lotka-Volterra predator-prey model [29]. One of the deposit and loan volumes models of Sumarti et al. (2014) uses the Michaelis-Menten response function. Then, in 2018, Sumarti et al. proposed a bank balance sheet model [28]. After that, in 2021, Ansori et al. modified the parameters used in the deposit and loan fund variables of the Sumarti et al. (2018) balance sheet model [4]. In deposit volume, Ansori et al. (2021) considered the rate of deposit withdrawal, while for loan volume, the NPL rate and the loan repayment rate were added. The increase in deposit volume is assumed to follow logistic growth because of the bank's limitations in receiving funds from customers [3, 4, 29].

The deposit and loan model of Sumarti et al. (2014) has been developed into a model of two loans, individual and company at one bank that compete each other for getting loan volumes, similar to the two predators-one prey model of Long et al. (2022). Then, the deposit volume will be considered the effect of withdrawal by adding a withdrawal rate based on the research of Ansori et al. (2021). In addition, for both loan volumes, the effect of NPL on both loan volumes will also be considered by adding the NPL rate and the loan repayment rate based on the research of Ansori et al. (2021). Thus, this research proposes a model of two loans in one bank that considers the effects of NPL and deposit withdrawal.

We set the rest of the paper as follows: In Section 2, we develope the model, followed by the verification of the existence, uniqueness, non-negativity, and boundness solutions of the model in Section 3. The existence of equilibrium point and its local stability are discussed in Sections 4 and 5, while the global stability analysis of the equilibrium points is given in Section 6. Moreover, in Section 7, we demonstrate some numerical simulations with a set of hypothetical parameters to validate the theoretical results. Finally, the conclusion is presented in Section 8.

2. The Mathematical Model

The model in this paper is constructed by developing deposit and loan volumes model of Sumarti et al. (2014) into a model of two loans at one bank which consists of three variables, namely D which represents the volume of deposit, L_1 which represents the volume of individual loan, and L_2 which represents the volume of company loan. Then, in the deposit volume, a parameter of withdrawal rate (w) is added, while in the both loan volumes, parameters of NPL (n_i) and repayment rate of overdue loans ($c_i(1-\mu_i)$) are added according to the research of Ansori et al. (2021). The increase of deposit volume uses a logistic growth model because banks have limitations in managing deposit funds effectively. The distribution of deposit volume into loan volumes uses the Michaelis-Menten response function due to the limited deposit volumes that can be lent to customers. In this model, the deposit interest rate is the only factor that affects the increase of deposit and the depositor will not be able to withdraw more than their deposit volume and additional interest, so that the deposit interest rate (γ) are always greater than the rate of withdrawal (w).

The nonlinear system autonomous model of two loans at one bank is obtained as follows.

$$\frac{dD}{dt} = \gamma D\left(1-\frac{D}{k}\right)-\frac{p_1 D L_1}{b_1 D+1}-\frac{p_2 D L_2}{b_2 D+1}-wD,$$

$$\begin{aligned}\frac{dL_1}{dt} &= \frac{p_1 D L_1}{b_1 D+1} - (\sigma_1+n_1)L_1 - c_1(1-\mu_1)L_1 - \beta_1 L_1 L_2 - h_1 L_1^2, \\ \frac{dL_2}{dt} &= \frac{p_2 D L_2}{b_2 D+1} - (\sigma_2+n_2)L_2 - c_2(1-\mu_2)L_2 - \beta_2 L_1 L_2 - h_2 L_2^2,\end{aligned} \tag{1}$$

with non-negative initial condition $D(0), L_1(0)$, and $L_2(0)$. The description of the parameters in model (1) can be seen in Table 1 where $i = 1$ for individual loan and $i = 2$ for company loan. The proposed model resembles the two predators-one prey model of Long et al. (2022), and some analyses will follow their approach.

Table 1: Description of Parameters

Parameter	Description
b_i	The barriers to obtaining loans
c_i	Loan repayments rate
h_1	Competition rate between individual loans
h_2	Competition rate between company loans
k	The carrying capacity of deposits
n_i	Non-performing loan rate
p_i	Maximum rate of the mixture between deposit and loan volumes
w	Withdrawal deposit rate
β_1	Competition rate of two loans on individual loan
β_2	Competition rate of two loans on company loan
γ	Interest rate of deposit
μ_i	The average portion of NPL
σ_1	Interest rate of loan

3. Preliminaries Results

As model (1) represents the volume of funds in the bank, the solution of the model must exist and unique, non-negative, and ultimately bounded. The following theorems ensure that the solution of the model (1) satisfies them.

Theorem 1. *All solutions of system (1) exist and are unique in $\Omega \times [0,\infty)$ for any non-negative initial condition $D(0), L_1(0)$, and $L_2(0)$ where $\Omega = \{(D, L_1, L_2) \in \mathbb{R}^3_+ \cup \{\vec{0}\} : max\{|D|, |L_1|, |L_2|\} \leq M\}$ for sufficiently large of M.*

Proof: Let the existence of M be guaranteed by the boundedness of the solution which will be proved in the next theorem. It is also assumed that $D(t), L_1(t), L_2(t) \geq 0$ for every $t \geq 0$ which will also be proved in the next theorem. Let $X = (D, L_1, L_2)$, $\bar{X} = (\bar{D}, \bar{L_1}, \bar{L_2})$ and consider a mapping $F(X) = (F_1(X), F_2(X), F_3(X))$ such that

$$F_1(X) = \gamma D\left(1-\frac{D}{k}\right) - \frac{p_1 D L_1}{b_1 D+1} - \frac{p_2 D L_2}{b_2 D+1} - wD,$$

$$\begin{aligned} F_2(X) &= \frac{p_1 D L_1}{b_1 D + 1} - (\sigma_1 + n_1)L_1 - c_1(1-\mu_1)L_1 - \beta_1 L_1 L_2 - h_1 L_1^2, \\ F_3(X) &= \frac{p_2 D L_2}{b_2 D + 1} - (\sigma_2 + n_2)L_2 - c_2(1-\mu_2)L_2 - \beta_2 L_1 L_2 - h_2 L_2^2. \end{aligned}$$

Then, for any $X, \bar{X} \in \Omega$, we have

$$\begin{aligned} ||F(X) - F(\bar{X})|| &= |F_1(X) - F_1(\bar{X})| + |F_2(X) - F_2(\bar{X})| + |F_3(X) - F_3(\bar{X})| \\ &\leq \left(\gamma + \frac{2\gamma M}{k} + w + 2p_1 M + 2p_2 M\right) |D - \bar{D}| \\ &\quad + (2p_1 M + H_5 + \beta_1 M + 2h_1 M + \beta_2 M)\, |L_1 - \bar{L_1}| \\ &\quad + (2p_2 M + H_6 + \beta_1 M + 2h_2 M + \beta_2 M)\, |L_2 - \bar{L_2}|, \\ &= \mathcal{K}_1 |D - \bar{D}| + \mathcal{K}_2 |L_1 - \bar{L_1}| + \mathcal{K}_3 |L_2 - \bar{L_2}| \\ &\leq \mathcal{K}\left(|D - \bar{D}| + |L_1 - \bar{L_1}| + |L_2 - \bar{L_2}|\right) = \mathcal{K}||X - \bar{X}||, \end{aligned}$$

where

$$\begin{aligned} \mathcal{K}_1 &= \gamma + \frac{2\gamma M}{k} + w + 2p_1 M + 2p_2 M, \\ \mathcal{K}_2 &= H_5 + M(2p_1 + \beta_1 + 2h_1 + \beta_2), \\ \mathcal{K}_3 &= H_6 + M(2p_2 + \beta_1 + 2h_2 + \beta_2), \\ \mathcal{K} &= max\{\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3\}, \\ H_5 &= \sigma_1 + n_1 + c_1(1-\mu_1), \text{ and } H_6 = \sigma_2 + n_2 + c_2(1-\mu_2). \end{aligned}$$

Based on a Lemma in [22] (p.71), the function $F(X)$ satisfies the Lipschitz condition so that the system (1) has a unique solution in $\Omega \times [0, \infty)$.

Theorem 2. *All solutions of system (1) are non-negative and ultimately bounded for any non-negative initial condition $D(0), L_1(0)$, and $L_2(0)$.*

Proof: If we assume that $D(t)$ can be negative. This shows that there exists $t_1 > 0$ such that $D(t) > 0$ for $0 < t < t_1$, $D(t) = 0$ for $t = t_1$ and $D(t) < 0$ for $t > t_1$. Based on the system (1) obtained

$$\left.\frac{dD}{dt}\right|_{t=t_1} = 0,$$

meaning that there is no rate of increase in the deposit fund when $t = t_1$. This contradicts the statement that $D(t) < 0$ for $t > t_1$, so the supposition is incorrect, which implies that $D(t) \geq 0$ for every $t > 0$. In the same way, $L_1(t) \geq 0$ and $L_2(t) \geq 0$ for any $t > 0$.

Furthermore, suppose $V(t) = D(t) + L_1(t) + L_2(t)$, then the first derivative of $V(t)$ is as follows.

$$\frac{dV}{dt} = (\gamma - w)D - \frac{\gamma D^2}{k} + (c_1\mu_1 - c_1 - \sigma_1 - n_1)L_1 + (c_2\mu_2 - c_2 - \sigma_2 - n_2)L_2$$

$$
\begin{aligned}
& -(\beta_1+\beta_2)L_1L_2 - h_1L_1^2 - h_2L_2^2 \\
\leq\ & (\gamma - w)D - \frac{\gamma D^2}{k} + (c_1\mu_1 - c_1 - \sigma_1 - n_1)L_1 + (c_2\mu_2 - c_2 - \sigma_2 - n_2)L_2.
\end{aligned}
$$

Then, for every positive constant ξ is obtained

$$
\begin{aligned}
\frac{dV}{dt} + \xi V \ \leq\ & (\gamma + \xi - w)D - \frac{\gamma D^2}{k} + (\xi + c_1\mu_1 - c_1 - \sigma_1 - n_1)L_1 \\
& + (\xi + c_2\mu_2 - c_2 - \sigma_2 - n_2)L_2.
\end{aligned}
$$

by taking $\xi < \min\{c_1(1-\mu_1) + \sigma_1 + n_1, c_2(1-\mu_2) + \sigma_2 + n_2\}$, then we get

$$
\begin{aligned}
\frac{dV}{dt} + \xi V &\leq (\gamma + \xi - w)D - \frac{\gamma D^2}{k} \leq \frac{k(\gamma + \xi - w)^2}{4\gamma}, \\
\frac{dV}{dt} &\leq \frac{k(\gamma + \xi - w)^2}{4\gamma} - \xi V.
\end{aligned}
$$

Clearly, the solution to a first order differential inequality satisfies

$$
V(t) \leq \frac{k(\gamma + \xi - w)^2}{4\xi\gamma} + \left(V(0) - \frac{k(\gamma + \xi - w)^2}{4\xi\gamma}\right) e^{-\xi t}.
$$

It is obvious that $V(t) \to \dfrac{k(\gamma + \xi - w)^2}{4\xi\gamma}$ as $t \to \infty$. Thus, all solutions of system (1) are ultimately bounded to the region

$$
\Omega = \left\{ \{D(t), L_1(t), L_2(t)\} \in \mathbb{R}_+^3 \cup \{\vec{0}\} : V(t) \leq \left\{ \frac{k(\gamma + \xi - w)^2}{4\xi\gamma} \right\} \right\}.
$$

4. The Existence of The Equilibrium Points

The equilibrium points of the system (1) are obtained when

$$
\frac{dD}{dt} = \frac{dL_1}{dt} = \frac{dL_2}{dt} = 0. \tag{2}
$$

By solving the system (2), five equilibrium points are obtained, namely

(i) The equilibrium point without deposit and loan activities, $E_0 = (0, 0, 0)$, that always exists.

(ii) The loan-free equilibrium point, $E_1 = \left(\frac{k}{\gamma}(\gamma - w), 0, 0\right)$, that always exists since $\gamma > w$.

(iii) The individual loan-free equilibrium point, $E_2 = (D_{(2)}, 0, L_{2_{(2)}})$. $L_{2_{(2)}}$ exist if $H_{4_{(2)}} > H_6$ where

$$L_{2_{(2)}} = \frac{1}{h_2}\left(\frac{p_2 D_{(2)}}{1+b_2 D_{(2)}} - (\sigma_2 + n_2 + c_2(1-\mu_2))\right) \text{ and } H_{4_{(2)}} = \frac{p_2 D_{(2)}}{1+b_2 D_{(2)}}.$$

$L_{2_{(2)}}$ still contains $D_{(2)}$ while $D_{(2)}$ values are the roots of the following cubic equation

$$D_{(2)}^3 + 3B_1 D_{(2)}^2 + 3B_2 D_{(2)} + B_3 = 0, \tag{3}$$

where
$B_1 = -\dfrac{h_2 b_2[(\gamma - w)kb_2 - 2\gamma]}{3\gamma h_2 b_2^2}$,
$B_2 = \dfrac{-h_2[2(\gamma - w)kb_2 - \gamma] + kp_2(p_2 - H_6 b_2)}{3\gamma h_2 b_2^2}$,
$B_3 = -\dfrac{k[(\gamma - w)h_2 + p_2 H_6]}{\gamma h_2 b_2^2}$.
Let $\chi = D_{(2)} + B_1$, then (3) can be written as

$$\chi^3 + 3\varpi_1 \chi + \varpi_2 = 0, \tag{4}$$

where

$$\begin{aligned} \varpi_1 &= B_2 - B_1^2, \\ \varpi_2 &= B_3 - 3B_1 B_2 + 2B_1^3. \end{aligned}$$

Based on Cardan criterion in [6], the possible positive roots of equation (4) are as follows.

(a) If $\varpi_2 < 0$, then eq. (4) has a single positive root.

(b) Suppose $\varpi_2 > 0$, $\varpi_1 < 0$ and
   i. if $\varpi_2^2 + 4\varpi_1^3 = 0$, then eq. (4) has a positive root of multiplicity two,
   ii. if $\varpi_2^2 + 4\varpi_1^3 < 0$, then eq. (4) has two positive roots.

(c) If $\varpi_2 = 0$ and $\varpi_1 < 0$, then Eq. (4) has a unique positive root.

Thus, the third equilibrium point $E_2 = (D_{(2)}, 0, L_{2_{(2)}})$ exists if $H_{4_{(2)}} > H_6$ and it satisfies one of the Cardan criterion.

(iv) The company loan-free equilibrium point, $E_3 = (D_{(3)}, L_{1_{(3)}}, 0)$. The existence of this equilibrium point also uses the Cardan criterion [6] in the same way as E_2. So, E_3 exists if $H_{3_{(3)}} > H_5$ and it satisfies one of the Cardan criterion, where

$$L_{1_{(3)}} = \frac{1}{h_1}\left(\frac{p_1 D_{(3)}}{1+b_1 D_{(3)}} - (\sigma_1 + n_1 + c_1(1-\mu_1))\right) \text{ and } H_{3_{(3)}} = \frac{p_1 D_{(3)}}{1+b_1 D_{(3)}}.$$

(v) The equilibrium point with deposit and loan activities, $E^* = (D^*, L_1^*, L_2^*)$, where

$$L_1^* = \frac{p_1 h_2(D^* + b_2 D^{*2}) - \beta_1 p_2(D^* + b_1 D^{*2}) + (1+b_1 D^*)(1+b_2 D^*)[\beta_1 H_6 - H_5 h_2]}{(1+b_1 D^*)(1+b_2 D^*)(h_1 h_2 - \beta_1 \beta_2)}$$

and

$$L_2^* = \frac{p_2h_1(D^* + b_1D^{*2}) - \beta_2p_1(D^* + b_2D^{*2}) + (1 + b_1D^*)(1 + b_2D^*)[\beta_2H_5 - H_6h_1]}{(1 + b_1D^*)(1 + b_2D^*)(h_1h_2 - \beta_1\beta_2)}.$$

L_1^* and L_2^* exist if they satisfy one of these conditions.

(a) If $h_1h_2 > \beta_1\beta_2$, then it must satisfy

$$\frac{h_2}{\beta_1} > \frac{H_4^* - H_6}{H_3^* - H_5} > \frac{\beta_2}{h_1}. \tag{5}$$

(b) If $h_1h_2 < \beta_1\beta_2$, then it must satisfy

$$\frac{h_2}{\beta_1} < \frac{H_4^* - H_6}{H_3^* - H_5} < \frac{\beta_2}{h_1}. \tag{6}$$

Further, D^* values are the roots of the following equation

$$G_0D^{*5} + G_1D^{*4} + G_2D^{*3} + G_3D^{*2} + G_4D^* + G_5 = 0 \tag{7}$$

where

$$\begin{aligned}
G_0 &= -\gamma(h_1h_2 - \beta_1\beta_2)b_1^2b_2^2,\\
G_1 &= (h_1h_2 - \beta_1\beta_2)b_1b_2[k(\gamma - w)b_1b_2 - 2\gamma(b_1 + b_2)],\\
G_2 &= (h_1h_2 - \beta_1\beta_2)[2kb_1b_2(\gamma - w)(b_1 + b_2) - \gamma(b_1^2 + b_2^2 + 4b_1b_2)]\\
&\quad +kp_1b_2[b_1b_2(h_2H_5 - \beta_1H_6) + b_1p_2(\beta_1 + \beta_2) - p_1b_2h_2]\\
&\quad +kp_2b_1[b_1b_2(h_1H_6 - \beta_2H_5) - p_2b_1h_1],\\
G_3 &= (h_1h_2 - \beta_1\beta_2)[k(\gamma - w)(b_1^2 + b_2^2 + 4b_1b_2) - 2\gamma(b_1 + b_2)]\\
&\quad +kp_1[(2b_1b_2 + b_2^2)(h_2H_5 - \beta_1H_6) + p_2(\beta_1 + \beta_2)(b_1 + b_2) - 2p_1b_2h_2]\\
&\quad +kp_2[(2b_1b_2 + b_1^2)(h_1H_6 - \beta_2H_5) - 2p_2b_1h_1],\\
G_4 &= (h_1h_2 - \beta_1\beta_2)[2k(\gamma - w)(b_1 + b_2) - \gamma]\\
&\quad +kp_1[(b_1 + 2b_2)(h_2H_5 - \beta_1H_6) + p_2(\beta_1 + \beta_2) - p_1h_2]\\
&\quad +kp_2[(2b_1 + b_2)(h_1H_6 - \beta_2H_5) - p_2h_1],\\
G_5 &= k(\gamma - w)(h_1h_2 - \beta_1\beta_2) + k[p_1(h_2H_5 - \beta_1H_6) + p_2(h_1H_6 - \beta_2H_5)].
\end{aligned}$$

By Descartes' Rule of Signs in [33], equation (7) will have at least one positive real root. Thus, E^* exists if it satisfies (5) or (6).

5. Local Stability

The Jacobian matrix at the equilibrium point is obtained as follows.

$$J(D, L_1, L_2) = \begin{pmatrix} J_1 & -H_3 & -H_4 \\ H_1 & J_2 & -\beta_1L_1 \\ H_2 & -\beta_2L_2 & J_3 \end{pmatrix}, \tag{8}$$

where

$$
\begin{aligned}
J_1 &= \gamma\left(1-\frac{2D}{k}\right)-H_1-H_2-w,\\
J_2 &= H_3-H_5-\beta_1L_2-2h_1L_1,\\
J_3 &= H_4-H_6-\beta_2L_1-2h_2L_2,\\
H_1 &= \frac{p_1L_1}{(b_1D+1)^2}, H_2=\frac{p_2L_2}{(b_2D+1)^2}, H_3=\frac{p_1D}{b_1D+1}, H_4=\frac{p_2D}{b_2D+1},\\
H_5 &= \sigma_1+n_1+c_1(1-\mu_1),\ \text{and}\ H_6=\sigma_2+n_2+c_2(1-\mu_2).
\end{aligned}
$$

The next theorems give the local stability of the equilibrium points by evaluating the real part of all eigenvalues of the Jacobian matrix (8).

Theorem 3. *The equilibrium point without deposit and loan activities E_0 of the system (1) is always unstable.*

Proof: The Jacobian matrix at E_0 is given by

$$
J(0,0,0)=\begin{pmatrix} \gamma-w & 0 & 0\\ 0 & -\sigma_1-n_1-c_1(1-\mu_1) & 0\\ 0 & 0 & -\sigma_2-n_2-c_2(1-\mu_2)\end{pmatrix}.
$$

It is clear that the eigenvalues are

$$
\begin{aligned}
\lambda_1 &= \gamma-w,\\
\lambda_2 &= -\sigma_1-n_1-c_1(1-\mu_1)<0,\\
\lambda_3 &= -\sigma_2-n_2-c_2(1-\mu_2)<0.
\end{aligned}
$$

Since $\gamma>w$, then $\lambda_1>0$. Hence, the equilibrium point without deposit and loan activities E_0 is always unstable.

Theorem 4. *The loan-free equilibrium point E_1 of the system (1) is locally asymptotically stable if $\frac{p_1D_{(1)}}{b_1D_{(1)}+1}<H_5$ and $\frac{p_2D_{(1)}}{b_2D_{(1)}+1}<H_6$.*

Proof: The Jacobian matrix at E_1 is given by

$$
J(D_{(1)},0,0)=\begin{pmatrix} -(\gamma-w) & -\frac{p_1D_{(1)}}{b_1D_{(1)}+1} & -\frac{p_2D_{(1)}}{b_2D_{(1)}+1}\\ 0 & \frac{p_1D_{(1)}}{b_1D_{(1)}+1}-H_5 & 0\\ 0 & 0 & \frac{p_2D_{(1)}}{b_2D_{(1)}+1}-H_6\end{pmatrix}.
$$

It is clear that the eigenvalues are $\lambda_1=-(\gamma-w)<0, \lambda_2=\frac{p_1D_{(1)}}{b_1D_{(1)}+1}-H_5$, and $\lambda_3=\frac{p_2D_{(1)}}{b_2D_{(1)}+1}-H_6$. If $\frac{p_1D_{(1)}}{b_1D_{(1)}+1}<H_5$ and $\frac{p_2D_{(1)}}{b_2D_{(1)}+1}<H_6$, then $\lambda_2<0$ and $\lambda_3>0$

consequently E_1 is locally asymptotically stable. Otherwise, it is an unstable point. So, if the maximum loanable funds are too low relative to the three main factors reducing loan funds, lending activity will cease, potentially harming the bank and preventing further loans.

Theorem 5. *By letting the individual loan-free equilibrium point E_2 of the system (1) exists. If $H_{3_{(2)}} - H_5 < \beta_1 L_{2_{(2)}}$ and $\frac{\gamma}{k} > H_{2_{(2)}} b_2$, then E_2 is locally asymptotically stable, where $H_{2_{(2)}}$ and $H_{3_{(2)}}$ are stated in the proof.*

Proof: The Jacobian matrix at E_2 is given by

$$J(D_{(2)}, 0, L_{2_{(2)}}) = \begin{pmatrix} D_{(2)}\left(H_{2_{(2)}} b_2 - \frac{\gamma}{k}\right) & -H_{3_{(2)}} & -H_{4_{(2)}} \\ 0 & H_{3_{(2)}} - H_5 - \beta_1 L_{2_{(2)}} & 0 \\ H_{2_{(2)}} & -\beta_2 L_{2_{(2)}} & -h_2 L_{2_{(2)}} \end{pmatrix},$$

where

$$H_{2_{(2)}} = \frac{p_2 L_{2_{(2)}}}{(b_2 D_{(2)} + 1)^2}, H_{3_{(2)}} = \frac{p_1 D_{(2)}}{b_1 D_{(2)} + 1}, \text{ and } H_{4_{(2)}} = \frac{p_2 D_{(2)}}{b_2 D_{(2)} + 1}.$$

The characteristic equation of the Jacobian matrix $J(E_2)$ is given by

$$\left(H_{3_{(2)}} - H_5 - \beta_1 L_{2_{(2)}} - \lambda\right) \left|J_{(2)} - \lambda I\right| = 0$$

where

$$J_{(2)} = \begin{pmatrix} D_{(2)}\left(H_{2_{(2)}} b_2 - \frac{\gamma}{k}\right) & -H_{4_{(2)}} \\ H_{2_{(2)}} & -h_2 L_{2_{(2)}} \end{pmatrix}.$$

It is clear that the first eigenvalue is $\lambda_1 = H_{3_{(2)}} - H_5 - \beta_1 L_{2_{(2)}}$ which will negative if $H_{3_{(2)}} - H_5 < \beta_1 L_{2_{(2)}}$. Then, the other two eigenvalues λ_2 and λ_3 will be negative if $tr(J_{(2)}) < 0$ and $|J_{(2)}| > 0$. If $\frac{\gamma}{k} > H_{2_{(2)}} b_2$, then $tr(J_{(2)}) < 0$ and $|J_{(2)}| > 0$. Thus, if $H_{3_{(2)}} - H_5 < \beta_1 L_{2_{(2)}}$ and $\frac{\gamma}{k} > H_{2_{(2)}} b_2$, then E_2 is locally asymptotically stable. So, if the maximum loanable funds for individuals are too low compared to factors reducing them, individual lending will cease, leaving the bank without profit from such loans.

Theorem 6. *Assume that the company loan-free equilibrium point E_3 of the system (1) exists. If $H_{4_{(3)}} - H_6 < \beta_2 L_{1_{(3)}}$ and $\frac{\gamma}{k} > H_{1_{(3)}} b_1$, then E_3 is locally asymptotically stable, where $H_{4_{(3)}}$ and $H_{1_{(3)}}$ are stated in the proof.*

Proof: The Jacobian matrix at E_3 is given by

$$J(D_{(3)}, L_{1_{(3)}}, 0) = \begin{pmatrix} D_{(3)}\left(H_{1_{(3)}} b_1 - \frac{\gamma}{k}\right) & -H_{3_{(3)}} & -H_{4_{(3)}} \\ H_{1_{(3)}} & -h_1 L_{1_{(3)}} & -\beta_1 L_{1_{(3)}} \\ 0 & 0 & H_{4_{(3)}} - H_6 - \beta_2 L_{1_{(3)}} \end{pmatrix},$$

where

$$H_{1_{(3)}} = \frac{p_1 L_{1_{(3)}}}{(b_1 D_{(3)} + 1)^2}, H_{3_{(3)}} = \frac{p_1 D_{(3)}}{b_1 D_{(3)} + 1}, \text{ and } H_{4_{(3)}} = \frac{p_2 D_{(3)}}{b_2 D_{(3)} + 1}.$$

And we have the characteristic equation of the Jacobian matrix $J(E_3)$ as follows.

$$\left(H_{4_{(3)}} - H_6 - \beta_2 L_{1_{(3)}} - \lambda\right) \left|J_{(3)} - \lambda I\right| = 0,$$

where

$$J_{(3)} = \begin{pmatrix} D_{(3)} \left(H_{1_{(3)}} b_1 - \dfrac{\gamma}{k}\right) & -H_{3_{(3)}} \\ H_{1_{(3)}} & -h_1 L_{1_{(3)}} \end{pmatrix}.$$

It is obvious that the first eigenvalue is $\lambda_1 = H_{4_{(3)}} - H_6 - \beta_2 L_{1_{(3)}}$. If $H_{4_{(3)}} - H_6 < \beta_2 L_{1_{(3)}}$, then $\lambda_1 < 0$. Next, the other two eigenvalues will be negative if $tr(J_{(3)}) < 0$ and $|J_{(3)}| > 0$. If $\dfrac{\gamma}{k} > H_{1_{(3)}} b_1$, then $tr(J_{(3)}) < 0$ and $|J_{(3)}| > 0$. Hence, if $H_{4_{(3)}} - H_6 < \beta_2 L_{1_{(3)}}$ and $\dfrac{\gamma}{k} > H_{1_{(3)}} b_1$, then E_3 is locally asymptotically stable. Thus, if the maximum amount of funds available for the company loan is too low relative to the factors that reduce lending, the company lending will stop, preventing the bank from making profit from this loan.

Theorem 7. *If E^* exists, $a_1 > 0, a_3 > 0$ and $a_1 a_2 - a_3 > 0$ where a_1, a_2 and a_3 are stated in the proof, then E^* is locally asymptotically stable.*

Proof: The Jacobian matrix at E^* is given by

$$J(D^*, L^*, L^*) = \begin{pmatrix} D^* \left(H_1^* b_1 + H_2^* b_2 - \dfrac{\gamma}{k}\right) & -H_3^* & -H_4^* \\ H_1^* & -h_1 L_1^* & -\beta_1 L_1^* \\ H_2^* & -\beta_2 L_2^* & -h_2 L_2^* \end{pmatrix},$$

where

$$H_1^* = \frac{p_1 L_1^*}{(b_1 D^* + 1)^2}, H_2^* = \frac{p_2 L_2^*}{(b_2 D^* + 1)^2}, H_3^* = \frac{p_1 D^*}{b_1 D^* + 1}, \text{ and} H_4^* = \frac{p_2 D^*}{b_2 D^* + 1}$$

By evaluating $|J(E^*) - \lambda I| = 0$, we obtain the characteristic equation of the matrix $J(E^*)$ as follows.

$$\lambda^3 + a_1 \lambda^2 + a_2 \lambda + a_3 = 0, \tag{9}$$

where

$$\begin{aligned} a_1 &= j_2 + j_5 - j_1, \\ a_2 &= j_2 j_5 + H_2^* H_4^* + H_1^* H_3^* - j_1 j_2 - j_1 j_5 - j_3 j_4, \\ a_3 &= j_1 j_3 j_4 + j_2 H_2^* H_4^* + j_5 H_1^* H_3^* - j_1 j_2 j_5 - j_3 H_2^* H_3^* - j_4 H_1^* H_4^*. \\ j_1 &= D^* \left(H_1^* b_1 + H_2^* b_2 - \frac{\gamma}{k}\right), \\ j_2 &= h_1 L_1^*, j_3 = \beta_1 L_1^*, j_4 = \beta_2 L_2^*, j_5 = h_2 L_2^*. \end{aligned}$$

Based on the Routh-Hurwitz criterion in [19], the solution of equation (9) has a negative real part if and only if $a_1 > 0, a_3 > 0$ and $a_1 a_2 - a_3 > 0$. Therefore, E^* is locally asymptotically stable if $a_1 > 0, a_3 > 0$ and $a_1 a_2 - a_3 > 0$.

6. Global Stability

6.1. Global Stability of E_1

Theorem 8. *E_1 is globally asymptotically stable (GAS) if $p_1 D_{(2)} \leq H_5$ and $p_2 D_{(1)} \leq H_6$.*

Proof: We first consider a Lyapunov function.

$$\mathcal{U} = D - D_{(1)} - D_{(1)} \ln \frac{D}{D_{(1)}} + L_1 + L_2$$

The first derivative of $\mathcal{U}$ with respect to t is given by

$$\begin{aligned}
\frac{d\mathcal{U}}{dt} &= (D - D_{(1)}) \left(\frac{\gamma D_{(1)}}{k} - \frac{\gamma D}{k} - \frac{p_1 L_1}{b_1 D + 1} - \frac{p_2 L_2}{b_2 D + 1} \right) \\
&\quad + \left(\frac{p_1 D L_1}{b_1 D + 1} - H_5 L_1 - \beta_1 L_1 L_2 - h_1 L_1^2 \right) + \left(\frac{p_2 D L_2}{b_2 D + 1} - H_6 L_2 - \beta_2 L_1 L_2 - h_2 L_2^2 \right) \\
&= -\frac{\gamma}{k}(D - D_{(1)})^2 + \left(\frac{p_1 D_{(1)}}{b_1 D + 1} - H_5 \right) L_1 + \left(\frac{p_2 D_{(1)}}{b_2 D + 1} - H_6 \right) L_2 \\
&\quad -(\beta_1 + \beta_2) L_1 L_2 - h_1 L_1^2 - h_2 L_2^2 \\
&\leq -\frac{\gamma}{k}(D - D_{(1)})^2 + \left(p_1 D_{(1)} - H_5\right) L_1 + \left(p_2 D_{(1)} - H_6\right) L_2 \\
&\quad -(\beta_1 + \beta_2) L_1 L_2 - h_1 L_1^2 - h_2 L_2^2.
\end{aligned}$$

If $p_1 D_{(1)} \leq H_5$ and $p_2 D_{(1)} \leq H_6$, then $\frac{d\mathcal{U}}{dt} \leq 0$ for all $(D, L_1, L_2) \in \mathbb{R}^3_+ \cup \{\vec{0}\}$ and $\frac{d\mathcal{U}}{dt} = 0$ for $(D, L_1, L_2) = (D_{(1)}, 0, 0)$. According to the LaSalle invariance principle in [2], E_1 is GAS.

6.2. Global Stability of E_2

Theorem 9. *E_2 is GAS if $\frac{\gamma}{k} > p_2 b_2 L_{2_{(2)}}$ and $p_1 D_{(2)} + v_1 \beta_2 L_{2_{(2)}} \leq H_5$.*

Proof: By considering a Lyapunov function as follows.

$$\mathcal{V}_1 = D - D_{(2)} - D_{(2)} \ln \frac{D}{D_{(2)}} + L_1 + v_1 \left(L_2 - L_{2_{(2)}} - L_{2_{(2)}} \ln \frac{L_2}{L_{2_{(2)}}} \right),$$

where $v_1 = b_2 D_{(2)} + 1$. Then, the first derivative of $\mathcal{V}_1$ is given by

$$\begin{aligned}
\frac{d\mathcal{V}_1}{dt} &= (D - D_{(2)}) \left(\frac{\gamma D_{(2)}}{k} + \frac{p_2 L_{2_{(2)}}}{b_2 D_{(2)} + 1} + w - \frac{\gamma D}{k} - \frac{p_1 L_1}{b_1 D + 1} - \frac{p_2 L_2}{b_2 D + 1} - w \right) \\
&\quad + \left(\frac{p_1 D L_1}{b_1 D + 1} - H_5 L_1 - \beta_1 L_1 L_2 - h_1 L_1^2 \right) \\
&\quad + v_1 (L_2 - L_{2_{(2)}}) \left(\frac{p_2 D}{b_2 D + 1} - \frac{p_2 D_{(2)}}{b_2 D_{(2)} + 1} + h_2 L_{2_{(2)}} - \beta_2 L_1 - h_2 L_2 \right),
\end{aligned}$$

$$
\begin{aligned}
\frac{d\mathcal{V}_1}{dt} &= -\frac{\gamma}{k}(D-D_{(2)})^2 - v_1h_2(L_2-L_{2_{(2)}})^2 + \frac{p_2b_2L_{2_{(2)}}(D-D_{(2)})^2}{(b_2D_{(2)}+1)(b_2D+1)} \\
&\quad + \frac{p_2(D-D_{(2)})(L_2-L_{2_{(2)}})[v_1-(b_2D_{(2)}+1)]}{(b_2D_{(2)}+1)(b_2D+1)} \\
&\quad - (v_1\beta_2+\beta_1)\,L_1L_2 + \left(\frac{p_1D_{(2)}}{b_1D+1} - H_5 + v_1\beta_2L_{2_{(2)}}\right)L_1 - h_1L_1^2 \\
&\leq -\frac{\gamma}{k}(D-D_{(2)})^2 - v_1h_2(L_2-L_{2_{(2)}})^2 + p_2b_2L_{2_{(2)}}(D-D_{(2)})^2 \\
&\quad + \frac{p_2(D-D_{(2)})(L_2-L_{2_{(2)}})[v_1-(b_2D_{(2)}+1)]}{(b_2D_{(2)}+1)(b_2D+1)} \\
&\quad - (v_1\beta_2+\beta_1)\,L_1L_2 + \left(p_1D_{(2)} - H_5 + v_1\beta_2L_{2_{(2)}}\right)L_1 - h_1L_1^2,
\end{aligned}
$$

since $v_1 = b_2D_{(2)} + 1$, it can be written as

$$
\begin{aligned}
\frac{d\mathcal{V}_1}{dt} &\leq -(\frac{\gamma}{k} - p_2b_2L_{2_{(2)}})(D-D_{(2)})^2 - v_1h_2(L_2-L_{2_{(2)}})^2 \\
&\quad - (v_1\beta_2+\beta_1)\,L_1L_2 + \left(p_1D_{(2)} - H_5 + v_1\beta_2L_{2_{(2)}}\right)L_1 - h_1L_1^2.
\end{aligned}
$$

If $\frac{\gamma}{k} > p_2b_2L_{2_{(2)}}$ and $p_1D_{(2)} + v_1\beta_2L_{2_{(2)}} \leq H_5$, then $\frac{d\mathcal{V}_1}{dt} \leq 0$ for all $(D, L_1, L_2) \in \mathbb{R}^3_+ \cup \{\vec{0}\}$ and $\frac{d\mathcal{V}_1}{dt} = 0$ for $(D, L_1, L_2) = (D_{(2)}, 0, L_{2_{(2)}})$. Applying the LaSalle invariance principle in [2], E_2 is GAS.

6.3. Global Stability of E_3

Theorem 10. *E_3 is GAS if $\frac{\gamma}{k} > p_1b_1L_{1_{(3)}}$ and $p_2D_{(3)} + v_2\beta_1L_{1_{(3)}} \leq H_6$.*

Proof: We consider a Lyapunov function as follows.

$$
\mathcal{V}_2 = D - D_{(3)} - D_{(3)}\ln\frac{D}{D_{(3)}} + v_2\left(L_1 - L_{1_{(3)}} - L_{1_{(3)}}\ln\frac{L_1}{L_{1_{(3)}}}\right) + L_2,
$$

where $v_2 = b_1D_{(3)} + 1$ The first order derivative of the Lyapunov function $\mathcal{V}_2$ is given by

$$
\begin{aligned}
\frac{d\mathcal{V}_2}{dt} &= (D-D_{(3)})\left(\gamma\left(1-\frac{D}{k}\right) - \frac{p_1L_1}{b_1D+1} - \frac{p_2L_2}{b_2D+1} - w\right) \\
&\quad + v_2(L_1-L_{1_{(3)}})\left(\frac{p_1D}{b_1D+1} - (\sigma_1+n_1+c_1(1-\mu_1)) - \beta_1L_2 - h_1L_1\right) \\
&\quad + \left(\frac{p_2DL_2}{b_2D+1} - (\sigma_2+n_2+c_2(1-\mu_2))L_2 - \beta_2L_1L_2 - h_2L_2^2\right),
\end{aligned}
$$

$$\begin{aligned}
\frac{d\mathcal{V}_2}{dt} &= -\frac{\gamma}{k}(D-D_{(3)})^2 - v_2h_1(L_1-L_{1_{(3)}})^2 + \frac{p_1b_1L_{1_{(3)}}(D-D_{(3)})^2}{(b_1D_{(3)}+1)(b_1D+1)} \\
&\quad + \frac{p_1(D-D_{(3)})(L_1-L_{1_{(3)}})[v_2-(b_1D_{(3)}+1)]}{(b_1D_{(3)}+1)(b_1D+1)} \\
&\quad - (v_2\beta_1+\beta_2)L_1L_2 + \left(\frac{p_2D_{(3)}}{b_2D+1} - H_6 + v_2\beta_1L_{1_{(3)}}\right)L_2 - h_2L_2^2 \\
&\leq -\frac{\gamma}{k}(D-D_{(3)})^2 - v_2h_1(L_1-L_{1_{(3)}})^2 + p_1b_1L_{1_{(3)}}(D-D_{(3)})^2 \\
&\quad + \frac{p_1(D-D_{(3)})(L_1-L_{1_{(3)}})[v_2-(b_1D_{(3)}+1)]}{(b_1D_{(3)}+1)(b_1D+1)} \\
&\quad - (v_2\beta_1+\beta_2)L_1L_2 + \left(p_2D_{(3)} - H_6 + v_2\beta_1L_{1_{(3)}}\right)L_2 - h_2L_2^2, \\
&= -(\frac{\gamma}{k} - p_1b_1L_{1_{(3)}})(D-D_{(3)})^2 - v_2h_1(L_1-L_{1_{(3)}})^2 \\
&\quad - (v_2\beta_1+\beta_2)L_1L_2 + \left(p_2D_{(3)} - H_6 + v_2\beta_1L_{1_{(3)}}\right)L_2 - h_2L_2^2,
\end{aligned}$$

If $\frac{\gamma}{k} > p_1b_1L_{1_{(3)}}$ and $p_2D_{(3)} + v_2\beta_1L_{1_{(3)}} \leq H_6$, then $\frac{d\mathcal{V}_2}{dt} \leq 0$ for all $(D, L_1, L_2) \in \mathbb{R}^3_+ \cup \{\vec{0}\}$ and $\frac{d\mathcal{V}_2}{dt} = 0$ for $(D, L_1, L_2) = (D_{(3)}, L_{1_{(3)}}, 0)$. According to the LaSalle invariance principle in [2], E_3 is GAS.

6.4. Global Stability of E^*

Theorem 11. *E^* is GAS if $k > p_1b_1L_1^* + p_2b_2L_2^*$ and $0 < \frac{\beta_1}{2h_2-\beta_2} < \frac{v_4}{v_3} < \frac{2h_1-\beta_1}{\beta_2}$ where v_3 and v_4 are stated in the proof.*

Proof: Define a Lyapunov function as

$$\mathcal{L} = D - D^* - D^* \ln\frac{D}{D^*} + v_3\left(L_1 - L_1^* - L_1^*\ln\frac{L_1}{L_1^*}\right) + v_4\left(L_2 - L_2^* - L_2^*\ln\frac{L_2}{L_2^*}\right),$$

where $v_3 = b_1D^* + 1$ and $v_4 = b_2D^* + 1$. The first order derivative of $\mathcal{L}$ is

$$\begin{aligned}
\frac{d\mathcal{L}}{dt} &= (D-D^*)\Bigg(-\frac{\gamma}{k}(D-D^*) - \frac{p_1(L_1(b_1D^*+1) - L_1^*(b_1D+1))}{(b_1D+1)(b_1D^*+1)} \\
&\quad - \frac{p_2(L_2(b_2D^*+1) - L_2^*(b_2D+1))}{(b_2D+1)(b_2D^*+1)}\Bigg) \\
&\quad + v_3(L_1-L_1^*)\left(\frac{p_1(D-D^*)}{(b_1D+1)(b_1D^*+1)} - \beta_1(L_2-L_2^*) - h_1(L_1-L_1^*)\right) \\
&\quad + v_4(L_2-L_2^*)\left(\frac{p_2(D-D^*)}{(b_2D+1)(b_2D^*+1)} - \beta_2(L_1-L_1^*) - h_2(L_2-L_2^*)\right),
\end{aligned}$$

$$
\begin{aligned}
\frac{d\mathcal{L}}{dt} &= -\frac{\gamma}{k}(D-D^*)^2 + (D-D^*)\left(\frac{p_1[b_1L_1^*(D-D^*)-(L_1-L_1^*)(b_1D^*+1)]}{(b_1D+1)(b_1D^*+1)}\right) \\
&\quad +(D-D^*)\left(\frac{p_2[b_2L_2^*(D-D^*)-(L_2-L_2^*)(b_2D^*+1)]}{(b_2D+1)(b_2D^*+1)}\right) \\
&\quad +\frac{v_3p_1(D-D^*)(L_1-L_1^*)}{(b_1D+1)(b_1D^*+1)} + \frac{v_4p_2(D-D^*)(L_2-L_2^*)}{(b_2D+1)(b_2D^*+1)} \\
&\quad -v_3h_1(L_1-L_1^*)^2 - v_4h_2(L_2-L_2^*)^2 - (v_3\beta_1+v_4\beta_2)(L_1-L_1^*)(L_2-L_2^*) \\
&= -\frac{\gamma}{k}(D-D^*)^2 + \frac{p_1b_1L_1^*(D-D^*)^2}{(b_1D+1)(b_1D^*+1)} + \frac{p_2b_2L_2^*(D-D^*)^2}{(b_2D+1)(b_2D^*+1)} \\
&\quad +\frac{p_1(D-D^*)(L_1-L_1^*)[v_3-(b_1D^*+1)]}{(b_1D+1)(b_1D^*+1)} + \frac{p_2(D-D^*)(L_2-L_2^*)[v_4-(b_2D^*+1)]}{(b_2D+1)(b_2D^*+1)} \\
&\quad -v_3h_1(L_1-L_1^*)^2 - v_4h_2(L_2-L_2^*)^2 - (v_3\beta_1+v_4\beta_2)(L_1-L_1^*)(L_2-L_2^*) \\
&\leq -\frac{\gamma}{k}(D-D^*)^2 + p_1b_1L_1^*(D-D^*)^2 + p_2b_2L_2^*(D-D^*)^2 \\
&\quad +\frac{p_1(D-D^*)(L_1-L_1^*)[v_3-(b_1D^*+1)]}{(b_1D+1)(b_1D^*+1)} + \frac{p_2(D-D^*)(L_2-L_2^*)[v_4-(b_2D^*+1)]}{(b_2D+1)(b_2D^*+1)} \\
&\quad -v_3h_1(L_1-L_1^*)^2 - v_4h_2(L_2-L_2^*)^2 - (v_3\beta_1+v_4\beta_2)(L_1-L_1^*)(L_2-L_2^*), \\
&= -\left(\frac{\gamma}{k} - p_1b_1L_1^* - p_2b_2L_2^*\right)(D-D^*)^2 - v_3h_1(L_1-L_1^*)^2 - v_4h_2(L_2-L_2^*)^2 \\
&\quad -(v_3\beta_1+v_4\beta_2)(L_1-L_1^*)(L_2-L_2^*).
\end{aligned}
$$

Let

$$P = \sqrt{\frac{v_3\beta_1+v_4\beta_2}{2}}(L_1-L_1^*) \text{ and } Q = \sqrt{\frac{v_3\beta_1+v_4\beta_2}{2}}(L_2-L_2^*),$$

then we have

$$
\begin{aligned}
P^2+Q^2-(P+Q)^2 &= -2PQ \\
&= -(v_3\beta_1+v_4\beta_2)(L_1-L_1^*)(L_2-L_2^*),
\end{aligned}
$$

so that

$$
\begin{aligned}
\frac{d\mathcal{L}}{dt} &\leq -\left(\frac{\gamma}{k} - p_1b_1L_1^* - p_2b_2L_2^*\right)(D-D^*)^2 - v_3h_1(L_1-L_1^*)^2 - v_4h_2(L_2-L_2^*)^2 \\
&\quad +\frac{v_3\beta_1+v_4\beta_2}{2}(L_1-L_1^*)^2 + \frac{v_3\beta_1+v_4\beta_2}{2}(L_2-L_2^*)^2 \\
&\quad -\left(\sqrt{\frac{v_3\beta_1+v_4\beta_2}{2}}(L_1-L_1^*) + \sqrt{\frac{v_3\beta_1+v_4\beta_2}{2}}(L_2-L_2^*)\right)^2 \\
&\leq -\left(\frac{\gamma}{k} - p_1b_1L_1^* - p_2b_2L_2^*)(D-D^*)^2 - v_3h_1(L_1-L_1^*\right)^2 - v_4h_2(L_2-L_2^*)^2 \\
&\quad +\frac{v_3\beta_1+v_4\beta_2}{2}(L_1-L_1^*)^2 + \frac{v_3\beta_1+v_4\beta_2}{2}(L_2-L_2^*)^2,
\end{aligned}
$$

$$\frac{d\mathcal{L}}{dt} = -\left(\frac{\gamma}{k} - p_1 b_1 L_1^* - p_2 b_2 L_2^*\right)(D - D^*)^2 - \left(v_3 h_1 - \frac{v_3\beta_1 + v_4\beta_2}{2}\right)(L_1 - L_1^*)^2 - \left(v_4 h_2 - \frac{v_3\beta_1 + v_4\beta_2}{2}\right)(L_2 - L_2^*)^2.$$

If $\frac{\gamma}{k} > p_1 b_1 L_1^* + p_2 b_2 L_2^*$ and $0 < \frac{\beta_1}{2h_2 - \beta_2} < \frac{v_4}{v_3} < \frac{2h_1 - \beta_1}{\beta_2}$, then $\frac{d\mathcal{L}}{dt} \leq 0$ for all $(D, L_1, L_2) \in \mathbb{R}_+^3 \cup \{\vec{0}\}$ and $\frac{d\mathcal{L}}{dt} = 0$ for $(D, L_1, L_2) = (D^*, L_1^*, L_2^*)$. By applying the Lasalle invariance principle in [2], E^* is GAS.

7. Numerical Simulation

Table 2: Parameters Values

Parameter	Simulation 1	Simulation 2	Simulation 3	Simulation 4
γ	0.4	0.4	0.4	0.4
w	0.2	0.2	0.2	0.2
k	3	3	3	3
p_1	0.25	0.35	0.8	0.85
p_2	0.35	0.8	0.35	0.9
n_1	0.17	0.17	0.17	0.17
n_2	0.23	0.23	0.23	0.2
b_1	0.8	0.8	0.8	0.9
b_2	0.9	0.7	0.9	0.95
σ_1	0.1	0.2	0.1	0.2
σ_2	0.15	0.1	0.15	0.15
c_1	0.2	0.2	0.2	0.15
c_2	0.25	0.2	0.25	0.2
β_1	0.5	0.5	0.5	0.1
β_2	0.4	0.4	0.4	0.15
h_1	0.1	0.1	0.1	0.3
h_2	0.1	0.1	0.1	0.25
μ_1	0.15	0.15	0.15	0.15
μ_2	0.2	0.1	0.2	0.2

In this section, we perform some numerical simulation results of the model (1) using the fourth-order Runge-Kutta scheme in [15], to confirm our theoretical results. In [17, 20], the fourth-order Runge-Kutta method is used to solve the first-order ordinary differential equation corresponding to the model in this paper. This method is also widely used for the numerical simulation of dynamical system models such as in [25, 31]. All parameter values are chosen hypothetically because of the unavailability of the real data work.

The parameter values of model (1) are shown in Table 2 with the initial condition $D(0) = 10$, $L_1(0) = 4$, $L_2(0) = 5$ are used for simulation 1,2,3, while $L_1(0) = 6$, $L_2(0) = 3$ are used for simulation 4. The first simulation is conducted using the values of the parameters in the simulation 1 column of the Table 2. After the calculation, we find that the solution of the system is asymptotically stable to the loan-free equilibrium point, $E_1 = (1.5, 0, 0)$. The numerical simulation shown in Figure 1, which also shows that the solution converges to E_1.

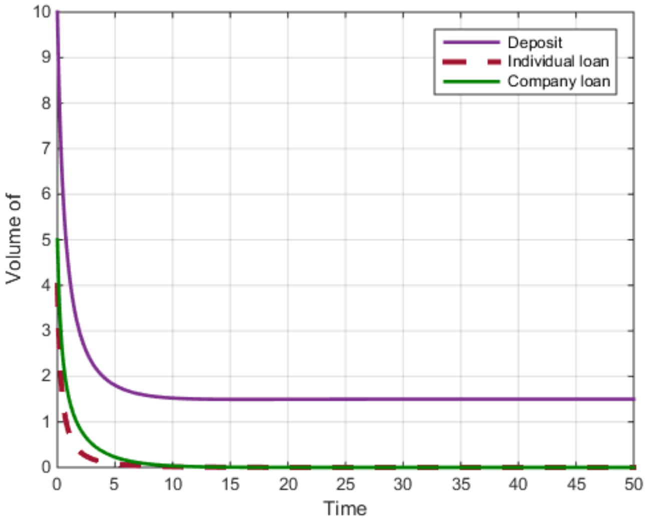

Figure 1: Numerical Simulation 1

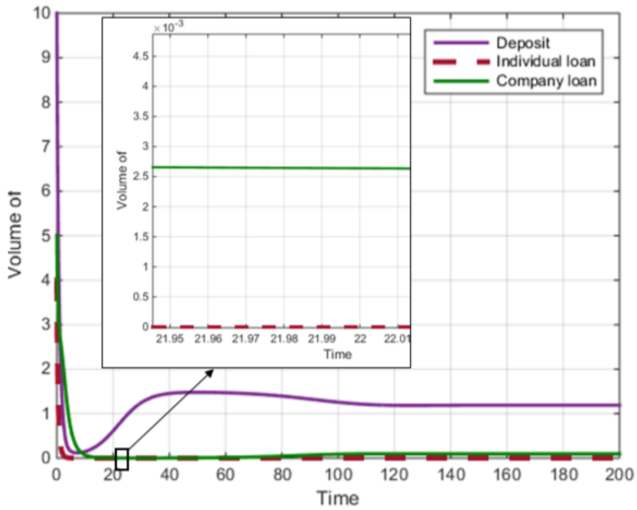

Figure 2: Numerical Simulation 2

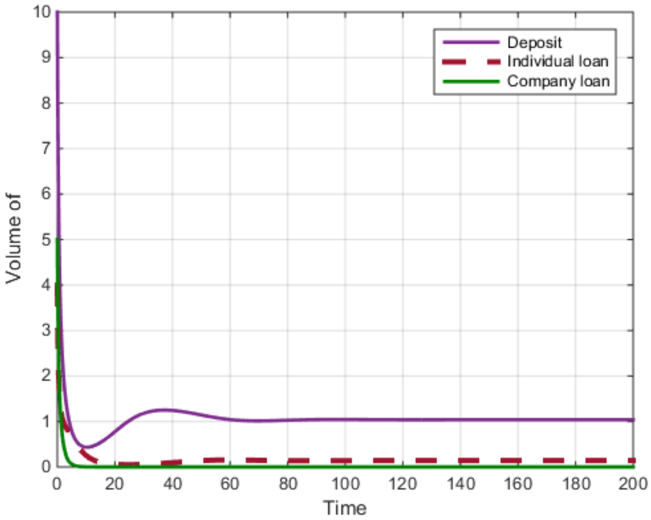

Figure 3: Numerical Simulation 3

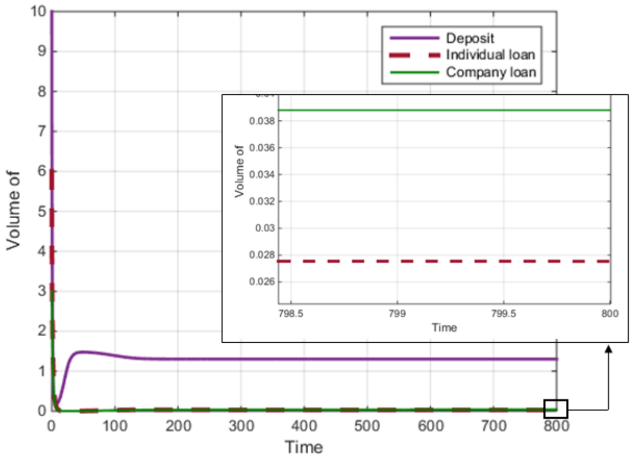

Figure 4: Numerical Simulation 4

The second simulation is conducted using the parameters values in the simulation 2 column of the Table 2. We find that the system solution is asymptotically stable to the individual loan-free equilibrium point $E_2 = (1.1904, 0, 0.0946)$, which also shown in the numerical simulation in Figure 2. In simulation 3, we see that the solution of the system will be asymptotically stable to the company loan-free equilibrium point, $E_3 = (1.0396, 0.1405, 0)$ which also shown in the numerical simulation in Figure 3. Lastly, the simulation uses the parameter values in the simulation 4 column of the Table 2. Using these parameters, we find that the solution of the system is asymptotically stable to the equilibrium point with deposit and loan activities, $E^* = (1.3021, 0.0273, 0.0391)$. The numerical simulation shown in Figure 4 also shows that the solution converges to E^*. The simulation shows that if the maximum loanable funds for individual and company exceed all factors reducing them, lending activity will continue.

In reality, the loan-free equilibrium point, E_1, is an equilibrium point that is highly avoided by a bank because in the absence of loans, a banks will not have the profit of interest loan that used for operational costs and return interest on depositors. By observing the parameter changes in the previous numerical simulations, it can be seen that if the maximum rate of mixture between deposit and both loan volumes or loanable volume is very high, there will eventually be individual loan and company loan. The amount of loanable volume depends on the amount of deposit volume. If the deposit volume can be properly distributed into the loan, then the banking operation will be stable so that the equilibrium point with the deposit and loan activities, E^*, is the most desirable condition for a bank.

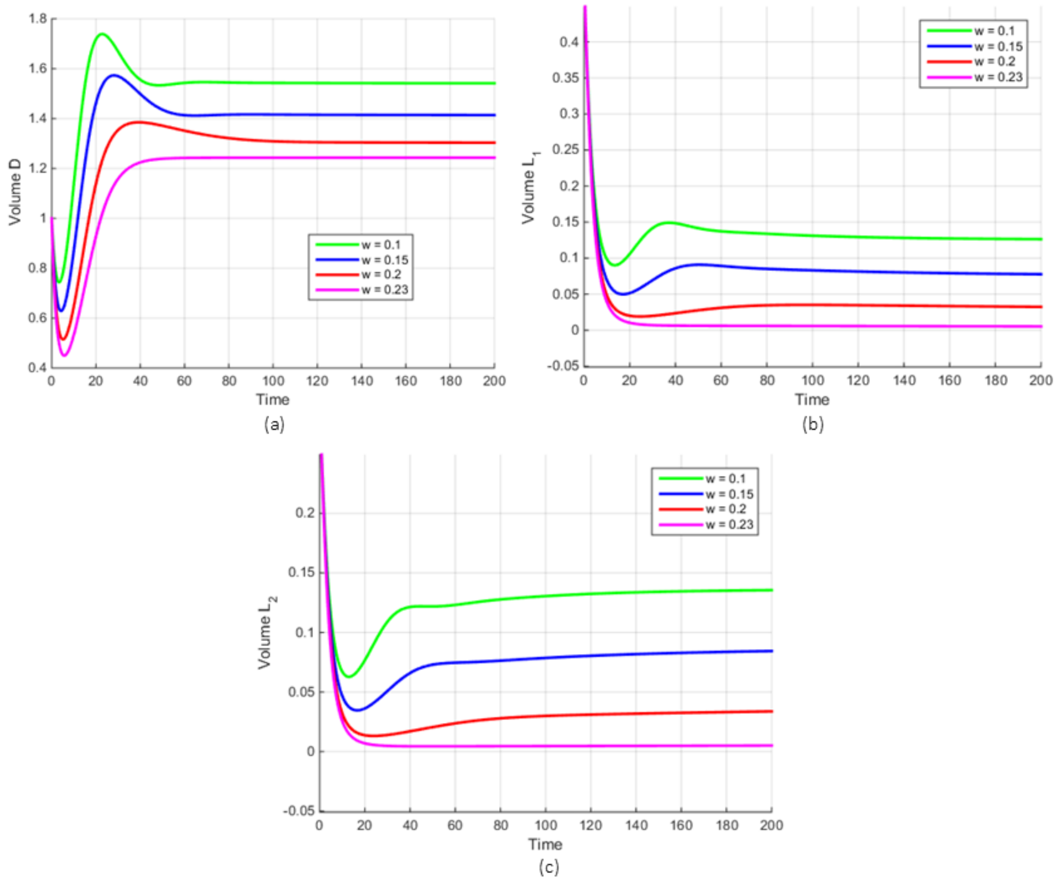

Figure 5: The Effect of Changes in Withdrawal Rate on The Volume of (a) Deposit (b) Individual Loan (c) Company Loan

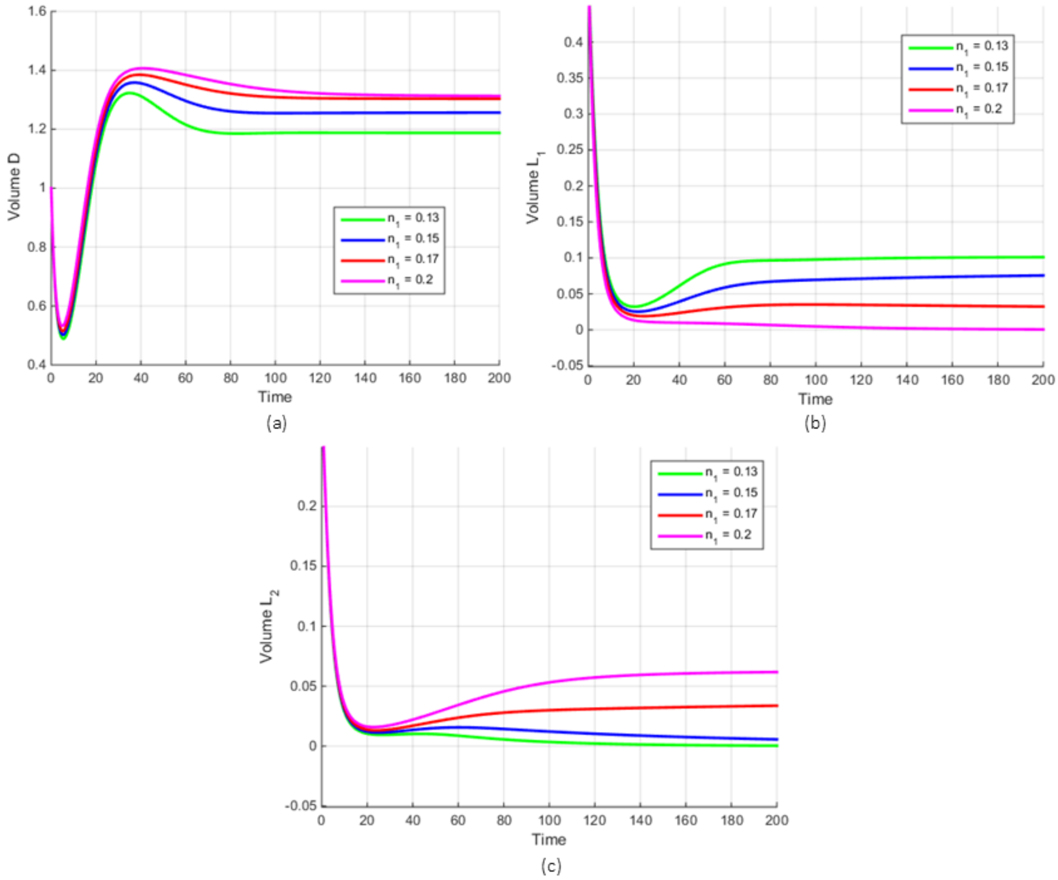

Figure 6: The Effect of Changes in Individual's NPL Rate on The Volume of (a) Deposit (b) Individual Loan (c) Company Loan

The effect of changes in withdrawal and NPL rates on deposit and both loan volumes is presented in Figures 5, 6, and 7. Figure 5 shows the effect of withdrawal on the deposit volume. If the rate of withdrawal is getting bigger, then the volume of deposit volume will decrease. It can lead to a reduced volume of loan that can be lent to individuals and companies. Moreover, Figure 6 shows the effect of individual's NPL on deposit and both loan volumes. The higher individual's NPL rate, the greater volumes of deposit and company loan, while the volume of individual loan decreases. This is in accordance with what happens in the bank, if someone has a high rate of NPL, then that person will get less loan volume which causes the volume of loan can be lent to the companies and there is more deposit volume left. Furthermore, Figure 7 shows the effect of the company's NPL rate on the deposit volume and both loan volumes. The higher rate of company's NPL, the higher deposit and individual loan volumes, but the company loan volume is less. This is also in accordance with what actually happens in the bank, if a company has a high NPL rate, it will receive fewer loans, which allows more deposit volume to be lent to individuals.

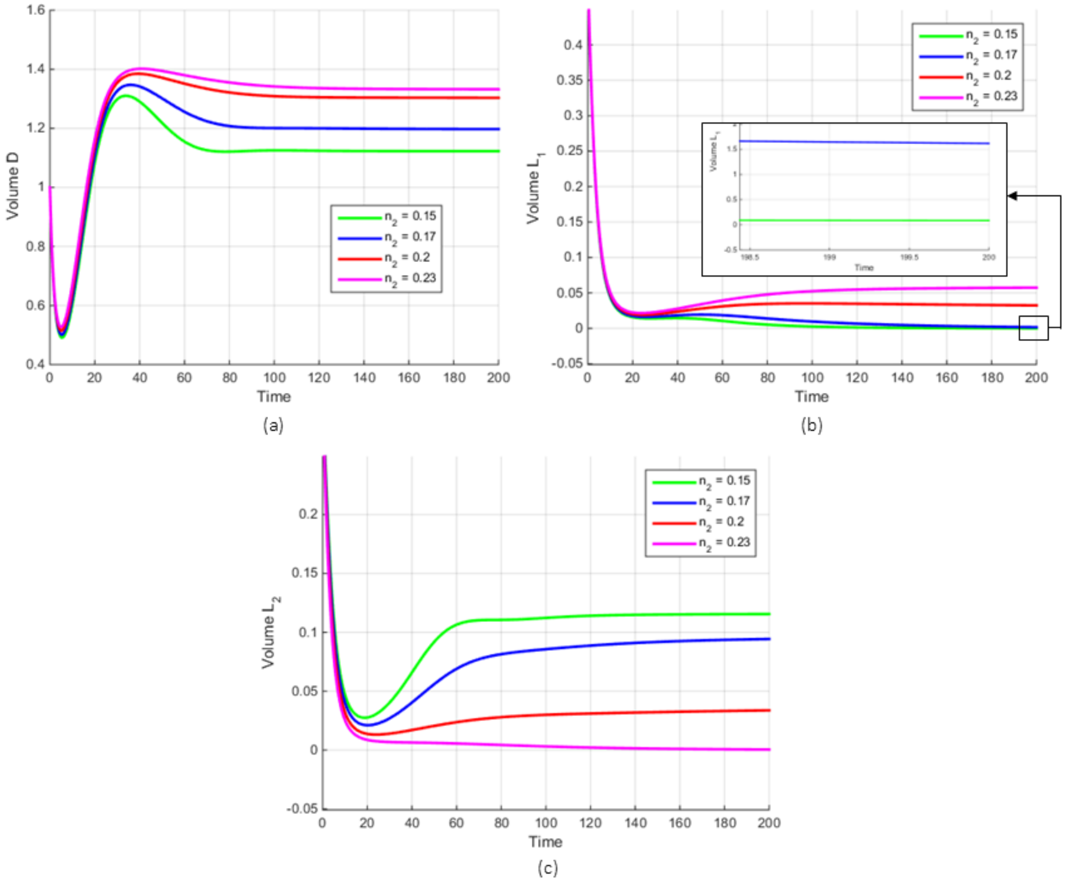

Figure 7: The Effect of Changes in Company's NPL Rate on The Volume of (a) Deposit (b) Individual Loan (c) Company Loan

The results of this study cannot be directly implemented in banking policies. However, they can serve as a consideration for banks in formulating policies or operational strategies. For instance, banks may adjust their strategies to increase deposits through product promotion or to reduce the volume of loans. These actions could indirectly influence the stability of the banking system.

8. Conclusion

In this paper, we construct a model of two loans in one bank analogous to the two predators-one prey model and consider withdrawal and NPL rates. The existence and uniqueness solution of the model (1) have been proved. Moreover, the non-negativity and boundedness of the solution have also been proved. The model (1) has five equilibrium points, namely the equilibrium point without deposit and loan activities, the loan-free equilibrium point, the individual loan-free equilibrium point, the company loan-free equilibrium point, and the equilibrium point with deposit and loan activities. The equilibrium point without deposit and loan activities always exists and is unstable. The loan-free equilibrium point always exists and is globally asymptotically stable if it satisfies two certain conditions. The individual loan-free equilibrium point and the company loan-free equilibrium point are asymptotically stable, both locally and globally, if they satisfy their certain conditions. The equilibrium point with deposit and loan activities is globally asymptotically stable if some certain conditions are satisfied. Our theoretical results have been confirmed by numerical solutions of the model. The results of the model presented in this paper can be implemented if all assumptions are met. Therefore, for future research, we recommend that the parameters used are derived from real-world data.

Acknowledgements

This work was supported by Faculty of Mathematics and Natural Science (FMIPA) through Public Funds DPA (Dokumen Pelaksanaan Anggaran) Perguruan Tinggi Berbadan Hukum Pelaksanaan Anggaran) Perguruan Tinggi Berbadan Hukum (PTNBH) University of Brawijaya and based on FMIPA Professor Grant, with contract number: 02172.22/UN10. F0901/B/KS/2024, dated July 12, 2024.

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