EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 1, Article Number 5645
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

The Superfluous Kernel Property in First Theorems on Generalized Hopficity through Hereditarily Hopfian Groups

Abderrahim Bouzendaga[1], Seddik Abdelalim[1], Ilias Elmouki[1,*]

[1] *Laboratory of Fundamental and Applied Mathematics (LMFA), Faculty of Sciences Ain Chock (FSAC), University Hassan II of Casablanca (UNIVH2C), Morocco*

Abstract. In this paper, we introduce the superfluous kernel property in order to characterize generalized hereditarily Hopfian groups. Then, we state our first theorems in this regard, through the study of two categories of Abelian groups, namely the reduced $p-$groups and the reduced torsion groups. In fact, we answer to the open question about the implication from generalized Hereditarily Hopficity to finiteness. Additionally, we succeed to prove a third theorem for the category of the divisible $p-$groups. Along all these results, we also try to benefit from the properties of Hereditarily Hopfian groups to easily reach the generalized hopficity property.

2020 Mathematics Subject Classifications: 20K10, 20K25, 20F24, 20E10

Key Words and Phrases: Superfluous kernel, Abelian group, Hopfian group, $p-$group, Torsion group, Divisible group, Hereditarily Hopfian group, Generalized Hopfian group.

---

1. Introduction

In modern Algebra, the concept of Hopficity plays an interesting role as it has been introduced to various algebraic systems, including Abelian groups, modules, rings, topological spaces, and functional spaces [18]. Nielsen used in 1921 [16], a purely algebraic method to show that a finitely generated free group can not be isomorphic to its proper factor group. Eleven years later, Hopf showed the same result through a topological method [2]. Then, in 1944, Baer extended the study of Hopfian groups through the names of Q-group and S-group [17]. In relation to Abelian groups, Baumslag showed in 1962, how the property of Hopficity can be applied to gain a deeper understanding of such groups [3]. Three years later, Corner presented concrete examples that illustrate that concept in the context of torsion-free Abelian groups [4]. Then, in 1969, Irwin and Takashi presented a

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v18i1.5645

Email addresses: bouzendaga1978@gmail.com (A. Bouzendaga), seddikabd@hotmail.com (S. Abdelalim), i.elmouki@gmail.com (I. Elmouki)

specific case of a quasi-decomposable Abelian group that possesses neither distinct isomorphic subgroups nor distinct isomorphic quotient groups [14]. In the 80's, Hirmath in 1986, studied the Hopficity in the context of rings and modules [13], then, after two years, Kaidi and Mamadou provided a characterization of Artinian rings with principal ideals [15]. We can also find the work of Haghany who explored in 1999, both Hopficity and co-Hopficity within the context of Morita and which play a significant construction in category theory as it establishes equivalences between categories of rings [11]. Ghorbani and Haghany also extended this work in the context of modules by introducing generalized Hopfian modules [12]. As for the 21st century, Gang and Zhongkui also explored in 2007, new criteria on Hopfian and co-Hopfian modules [19], then published a note in 2010 where they provided generalizations on these concepts [20]. More recently, Abdelalim et al. provided a characterization of strongly Hopfian Abelian groups [1, 5] using background from [6, 7].

In our present paper, we characterize generalized Hopfian groups within the category of Abelian groups, based on a series of results in the context of Hopfian hereditation. For that, we first demonstrate that generalized hereditarily Hopfian reduced $p-$groups are finite. Next, we prove that reduced torsion groups are generalized hereditarily Hopfian if and only if their p-components are finite. Finally, we prove that a divisible group is generalized hereditarily Hopfian group if it is a finite direct sum of quasi-cyclic group $\mathbb{Z}(p^{\infty})$.

The paper is organized as follows. In section 2, our first theorem states that generalized hereditarily Hopfian reduced $p-$groups are finite, while the proof uses properties of pure subgroups, basic subgroups, bounded groups and direct summand. As for section 3, our second theorem states that reduced torsion groups are generalized hereditarily Hopfian if and only if their p-components are finite, while the proof uses the previous result in addition to the fact that a torsion group is a direct sum of p-components. Then, in section 4, our third theorem provides the condition to characterize a divisible $p-$group as a generalized hereditarily Hopfian group., and this has been reached by using a characterization of divisible $p-$groups.

For the convenience of reading, here is some information regarding the notation and terminology. Throughout the paper, when we refer to a group, we are referring to an Abelian group with additive notation, also we use the following notations: p is a prime number, Q is a group of rational number, $\mathbb{Z}$ is a group of integer number, D is a divisible group, R is a reduced group, G_p is a p-component of $G, \mathbb{Z}(p^{\infty})$ is a quasi-cyclic group, G/H is a quotient group, B is a p-basic subgroup, $\langle x \rangle$ is a cyclic group, $\oplus$ is a direct sum, φ, α and ϕ are groups homomorphisms.

2. On generalized hereditarily Hopfian $p-$groups

In this section, we state our first theorem which says that every reduced generalized hereditarily Hopfian $p-$group is a finite group. But before that, we use three lemmas and one proposition to achieve the proof of this result.

We recall that a group G is Hopfian if every surjective endomorphism $\alpha : G \to G$ is an automorphism. This is also equivalent to say that G is not isomorphic to any of its proper quotient groups. Some examples of such groups are finite groups, finitely generated free groups, finitely generated residually finite groups, and torsion-free groups of finite rank.

We recall that groups needed in the proof of our theorems thereafter are about the following types.

Definition 1. *(i) Torsion group (An Abelian group G is a torsion group if $\forall x \in G : \circ(x) < \infty$).*

(ii) $p-$group (A group G is said to be a $p-$group when the order of every element is a power of p, with p a prime number).

(iii) Bounded group (A group G is bounded if there is $n \in \mathbb{N}^*$ such that $nG = 0$).

(iv) Divisible group (An Abelian group G is a divisible group if for any $a \in G$ and integer $n \geq 1$ there exists $b \in G$ such that $a = nb$. Otherwise expressed, G is divisible if $nG = G$ holds for every natural integer n).

(v) Reduced group A group (G is said to be a reduced group if it does not contain any proper divisible subgroups).

(vi) Direct summand (A subgroup B of A is called a direct summand of A, if there is a $C \leq A$ such that $A = B \oplus C$. In this case, C is a complementaray direct summand, or simply a complement of B in A).

(vii) Basic subgroup (A subgroup H of a torsion group G is basic if H is a direct sum of cyclic $p-$groups and it is pure in G, and G/H is divisible.)

(viii) Pure subgroup (A subgroup H of a group G is said to be a pure subgroup if $\forall n \in N \ H \cap nG = nH$).

(ix) Small or superfluous subgroup (A subgroup H of a group G is called small or superfluous (denoted $H \ll G$) if, for every subgroup K of G, the condition $H + K = G$ implies $K = G$).

(x) Small or superfluous homomorphism (An homomorphism φ of G is called small or superfluous if $\mathrm{Ker}(\varphi)$ is superfluous in G).

(xi) A Hopfian group (A group G is called a Hopfian group if, every surjective endomorphism is an automorphism).

(xii) Generalized Hopfian group (A group G is called a generalized Hopfian group if, for every surjective endomorphism φ, $\mathrm{Ker}(\varphi)$ is superfluous subgroup.

(xiii) Hereditarily Hopfian group (A group G is said to be a Hereditarily Hopfian group if every subgroup H of G is Hopfian group).

For more information about other properties of these groups, the reader could check the books [8, 9].

We start with the following two remarks that are important thereafter.

Remark 1. *If G is a finite group, then G is Hopfian and hereditarily Hopfian group. In fact, if we consider a finite group G, and $\varphi \in End(G)$ an epimorphism, then $G/ker(\varphi)$ and G are isomorphic. By using the Lagrange theorem, we get $|G/ker\varphi| \, . \, |ker\varphi| = |G|$. Therefore, $|ker\varphi| = 1$, and then φ is a monomorphism, thus G is a Hopfian group. Finally, since all subgroups of G are finite, so they are Hopfian, hence, G is hereditarily Hopfian.*

If we consider just the case of reduced $p-$groups, our motivation arises from the fact that Goldsmith in [10] proved that there is an equivalence between the hereditarily Hopficity (co-Hopficity as well) and finiteness, but could we claim that a generalized hereditarily Hopfian group is finite? One could just take $\mathbb{Z}(p^{\infty})$ as a counterexample.

We need this following proposition as well.

Proposition 1. *Let G be a hereditarily Hopfian group. If H is a subgroup of G, then H is hereditarily Hopfian.*

Proof. Let G be a hereditarily Hopfian group.
Suppose H is a subgroup of G.
Then, H is a Hopfian group. Now, let K be a subgroup of H.
Since K is a subgroup of G, it follows that K is Hopfian.
Therefore, every subgroup of H is Hopfian, which shows that H is hereditarily Hopfian.

Remark 2. *If G is hereditarily Hopfian group, then G is generalized hereditarily Hopfian group. In fact, if we consider a subgroup K of G, then K is hereditarily Hopfian by Proposition 1, thus K is generalized Hopfian subgroup, and therefore G is generalized hereditarily Hopfian group.*

Lemma 1. *If G is a generalized hereditarily Hopfian group, and H is a subgroup of G, then H is a generalized hereditarily Hopfian subgroup.*

Proof. Since G is a generalized hereditarily Hopfian group, then H is a generalized Hopfian group. and again, using the same method as in the proof of Proposition 1, we conclude that H is a generalized hereditarily Hopfian group.

Theorem 1. *Let G be a Abelian reduced $p-$group. Then the following properties are equivalent,*

- *(a) G is finite group,*

- *(b) G is hereditarily Hopfian group,*
- *(c) G is generalized hereditarily Hopfian group.*

Proof.

- (a) $\Rightarrow$ (b) is obvious due to Remark 1.
- (b) $\Rightarrow$ (c) is obvious due to Remark 2.
- (c) $\Rightarrow$ (a). In fact, this is the right remaining question that is still open.

  Let G be a generalized hereditary Hopfian group. According to *Theorem 32.3 [8]*, G contains a basic subgroup B, and since G is a $p-$group (then of torsion), then

$$B = \bigoplus_{n=1}^{\infty} B_n, \quad B_n = \bigoplus_{i \in I_n} \langle x_{i,n} \rangle, \quad I_n \subseteq \mathbb{N}, \quad \mathrm{ord}(x_{i,n}) = p^n.$$

  To prove that G is finite, it suffices to show that $G = B$ and B is finite.

  Since G is reduced, it is necessary to prove that B is a direct summand of G. Being B is a basic subgroup of G, it implies that is pure. But according to Theorem 27.5 (Kulikov, [8]), it remains to prove that B is bounded to become a direct summand, which will be shown hereafter by contradiction.

  Assume that B is not bounded. Then, for every integer $n \geq 1$ and prime p, we have:

$$p^n B \neq 0 \iff p^n(B_1 \oplus B_2 \oplus \cdots \oplus B_n \oplus B_{n+1} \oplus \dots) \neq 0$$
$$\iff p^n(B_{n+1} \oplus B_{n+2} \oplus \dots) \neq 0, \quad .$$

  then there exists increasing sequence $m_k > n$ with $k > 0$, such that $B_{m_k} \neq 0$, and also we can write $B_{m_k} = \langle x_{m_k} \rangle) \bigoplus B'$ with $\mathrm{ord}(x_{m_k}) = p^{m_k}$ and $B' = \bigoplus_{k=1}^{\infty} \langle x_{m_k} \rangle$. It is clear that B' is generalized hereditary Hopfian subgroup of G, by lemma 1, because B' is subgroup of G.
  Let φ the following map defined by

$$\begin{array}{c} B' \longrightarrow B' \\ \sum_{k=1}^{n_0} m_k x_{m_k} \longmapsto \sum_{k=1}^{n_0-1} m_{k+1} x_{m_k}. \end{array}$$

  It is clear that φ is a surjective endomorphism, and we can write B' as:
  $B' = \langle x_{m_1} \rangle \oplus \bigoplus_{k=2}^{\infty} \langle x_{m_k} \rangle \subset ker(\varphi) \oplus \bigoplus_{k=2}^{\infty} \langle x_{m_k} \rangle \subset B'$
  Thus, $B' = ker(\varphi) \oplus \bigoplus_{k=2}^{\infty} \langle x_{m_k} \rangle$ and consequently $B' = \bigoplus_{k=2}^{\infty} \langle x_{m_k} \rangle$, which is absurd. Consequently, B is bounded group.

  Since also B is a pure subgroup of G, then B is a direct summand of G. Therefore, $G = B \oplus C$ by Property (a) of page 38 in [8], and thus G/B is isomorphic

to C, and also G/B is divisible .
Hence, C is a divisible subgroup of G, by Property (D) of page 98 [8].
Since G is reduced, then G contains no proper divisible subgroup, which implies that $C = 0$, and consequently $G = B$.

It remains to prove that G is finite.

To do this, we will show that B is finite and this holds true when $\text{card}(B_k) < \infty$, with $1 \leq k \leq n$.
To achieve this, let us suppose that there exists a nonzero positive integer k_0 such that $\text{card}\,(B_{k_0}) = \infty$, and $B_{k_0} = \oplus_{i_k \in I_{k_0}} \langle x_{i_k,k_0} \rangle$ with $\text{card}\,(I_{k_0}) = \infty$,
then $B_{k_0} = \oplus_{k\in\mathbb{N}} \langle x_{i_k,k_0} \rangle \bigoplus \oplus_{i_k\in T} \langle x_{i_k,k_0} \rangle$ with $T = I_{k_0} \setminus \{i_k,\ k \in \mathbb{N}\}$
Let $H = \oplus_{k\in\mathbb{N}} \langle x_{i_k,k_0} \rangle$, we remark that H is generalized hereditarily Hopfian group according to Lemma 1, and we define the following surjective endomorphism of H,

$$\begin{array}{ccc} \phi : H & \longrightarrow & H \\ \sum_{k=1}^{n_0} m_k x_{i_k,k_0} & \longmapsto & \sum_{k=1}^{n_0-1} m_{k+1} x_{i_k,k_0} \end{array}$$

we have $H = \oplus_{k\in\mathbb{N}} \langle x_{i_k,k_0} \rangle$ or
$H = \langle x_{i_0,k_0} \rangle \bigoplus \oplus_{k\in\mathbb{N}^*} \langle x_{i_k,k_0} \rangle \subset \ker\phi \bigoplus \oplus_{k\in\mathbb{N}^*} \langle x_{i_k,k_0} \rangle \subset H$, then
$H = \ker\phi \bigoplus \oplus_{k\in\mathbb{N}^*} \langle x_{i_k,k_0} \rangle$, and since H is generalized hereditarily Hopfian group, then $H = \oplus_{k\in\mathbb{N}^*} \langle x_{i_k,k_0} \rangle$, witch is absurd. Therefore, B is finite, and consequently, G is finite.

We have just showed that an Abelian reduced $p-$group, is generalized hereditarily Hopfian, when it is finite. Now, let us expand our research and ask the question: What are the generalized hereditarily Hopfian groups within the category of reduced torsion groups? The answer to this question will be provided in the following section.

3. On reduced torsion generalized hereditarily Hopfian groups

We recall that groups needed in the proof of our next theorem hereafter are about the following types,

- Group G_p *(Let p be a prime number, the p-component of an Abelian group G is the group G_p defined as, $G_p = \{a \in G/ \circ (a) = p^n,\ n \in N\}$).*

Lemma 2. *Let G be an Abelian group. If G is a generalized hereditarily Hopfian group then the direct summand of G is a generalized hereditarily Hopfian group.*

Proof. Let $G = A \oplus B$, let $\varphi_1 : A \longrightarrow A$ be a surjective endomorphism and let ϕ :

$$\begin{array}{ccc} \phi : G = A \oplus B & \longrightarrow & A \oplus B \\ x_1 + x_2 & \longmapsto & \varphi_1\,(x_1) + x_2 \end{array}$$

It is clear that ϕ is a surjective endomorphism because for all $x = x_1 + x_2$ and $y = y_1 + y_2$, where $x_1, y_1 \in A$ and $y_1 + y_2 \in B$, we have,
$\phi(x + y) = \phi(x_1 + x_2 + y_1 + y_2)$
$= \phi(x_1 + y_1 + x_2 + y_2)$
$= \varphi_1(x_1 + y_1) + x_2 + y_2$
$= \varphi_1(x_1) + x_2 + \varphi_1(y_1) + y_2$
$= \phi(x) + \phi(y)$.
Then ϕ is an endomorphism. And we have $\phi(G) \subset G$.
Let us check that $G \subset \phi(G)$.
Let $x \in G$, there exists $(x_1, x_2) \in A \times B$ such that $x = x_1 + x_2$, then $x = \varphi_1(x_1') + x_2$ (because φ_1 is surjective) and also $x = \phi(x_1' + x_2)$, then $x \in \phi(G)$, therefore $G \subset \phi(G)$, and thus $G = \phi(G)$, hence ϕ is surjective.

Let us now show that A is generalized hereditarily Hopfian.
Assume that H as a subgroup of A such that $H + \ker(\varphi_1) = A$, then $G = H + \ker(\varphi_1) \oplus B$.
Since $\ker(\varphi_1) \subset \ker(\phi)$, and G is a generalized hereditarily Hopfian group , it follows that $G = H \oplus B$.
Since $H < A$ and $G = A \oplus B$, we conclude that $H = A$, hence A is generalized hereditarily Hopfian group.

Theorem 2. *Let G a reduced torsion group, then G is generalized hereditarily Hopfian if and only if G_p is finite for every prime number p.*

Proof.

$\Rightarrow$) Let G be a reduced torsion group.
Suppose that G is generalized hereditarily Hopfian, and let us show that G_p is finite for every prime number p.
Since G is a reduced torsion group, then according to Theorem 8.4 of page 43 in [8], $G = \bigoplus_p G_p$ and with Lemma 2, we therefore have, G_p is a $p-$group that is generalized hereditarily Hopfian, and also a reduced group, which implies by Theorem 1, that G_p is finite. Therefore, G_p is finite for every prime number p.

Conversely: $\Leftarrow$)
Let us assume that G_p is finite for every prime number p.
Let $G_1 < G$, $G_1 = \bigoplus_p G_{1_p}$, $G_{1_p} < G_p$, $\varphi : G_1 \longrightarrow G_1$ be a surjective endomorphism, and let $\varphi_p : G_{1_p} \longrightarrow G_{1_p}$ be the restriction of φ.
Let us check that φ_p is surjective for every prime p.
Suppose that $y \in G_{1_p}$, then $y \in G_1$, implying the existence of $x \in G_1$ such that $\varphi(x) = y$.

Since $G_1 = \bigoplus_p G_{1_p} = G_{1_p} \oplus G'$ where $G' = \bigoplus_{q \in P} G_{1_q}$ with $p \neq q$, we have $x \in G_1$ implies $x = x_p + x'$ where $x_p \in G_{1_p}$ and $x' \in G'$.
Therefore, $\varphi(x_p + x') = y$ implies $\varphi(x_p) + \varphi(x') = y$, and thus $\varphi_p(x_p) + \varphi(x') = y$.
Since $\varphi_p(x_p) \in G_{1_p}$ and $y \in G_{1_p}$, then $\varphi(x') = 0$, leading to $\varphi(x_p) = y$.
This implies that $\varphi_p(x_p) = y$.

Therefore, we have the existence of $x_p \in G_{1_p}$ and then φ_p is surjective for every prime p. Since G_{1_p} is finite because it is a subgroup of a finite group G_p, then φ_p is bijective, and therefore, G_{1_p} is Hopfian for every prime p.
Hence, all subgroups of G_1 are Hopfian, and then G_1 is hereditarily Hopfian group.
Thus, according to Remark 2, G_1 is generalized Hopfian group, and therefore G is generalized hereditarily Hopfian group.

In the previous sections, we have examined the property of generalized hereditarily Hopficity within the categories of reduced $p-$groups and reduced torsion groups. Now, we expand upon those findings to study the case of divisible $p-$groups.

4. On generalized Hopfian divisible $p-$groups

In this section, we characterize generalized Hopfian groups in the category of divisible $p-$groups.
We note that the group needed in the proof of our final theorem hereafter is about the divisible group as recalled among the classical definitions listed in Section 2.
In addition, we will also need the following property, see [8] and which states that G is a divisible group if and only if G can be expressed as $G = \left(\oplus_{r_0(G)}\mathbb{Q}\right) \bigoplus \left[\bigoplus_{p\in P}\left(\oplus_{r_p(G)}\mathbb{Z}\left(p^\infty\right)\right)\right]$, such that $\mathbb{Q}$ is the rational group, while we recall $\mathbb{Z}\left(p^\infty\right)$ is the Prufer group to be defined as follows.

$\mathbb{Z}(p^\infty)$ is a fundamental example of a quasi-cyclic group or a group of type p^∞. It is commonly used in the theory of Abelian groups and the classification of finite groups.
It is generated by a sequence of elements $a_1, a_2, a_3, \ldots, a_n, \ldots$, or $\mathbb{Z}(p^\infty) = \bigcup_{i=1}^{\infty}\langle a_i\rangle$ where each element a_n satisfies the condition $\circ(a_n) = p^n$ which means that a_n has order p^n with p a fixed prime number.
The relations between these generators are given by the following equations,

$$pa_1 = 0, \quad pa_2 = a_1, \quad pa_{n+1} = a_n, \quad \forall n \in \mathbb{N}.$$

This means that a_1 is annihilated by multiplication by p, and each subsequent generator a_n is related to the previous generator a_{n-1} by multiplication by p.
The elements of $\mathbb{Z}(p^\infty)$ are of order p^n for increasingly larger values of n and its structure is such that each subgroup $\langle a_n\rangle$ is contained in the next as,

$$\langle a_1\rangle \subseteq \langle a_2\rangle \subseteq \langle a_3\rangle \subseteq \cdots \subseteq \mathbb{Z}(p^\infty).$$

This means that each subgroup generated by a_n is contained within the subgroup generated by a_{n+1}, and every element of $\mathbb{Z}(p^\infty)$ can be expressed as a linear combination of the elements $a_1, a_2, \ldots$, with integer coefficients.
If $x \in \mathbb{Z}(p^\infty)$, then x is a multiple of some a_n, which can be expressed as,

$$x \in \mathbb{Z}(p^\infty) \Longleftrightarrow \exists m \in \mathbb{Z}, \exists a_k \in \{a_1, a_2, a_3, \ldots\} \textit{ such that } x = ma_k \textit{ and } m \wedge p = 1.$$

This means that each element x of $\mathbb{Z}(p^\infty)$ can be written as a multiple of a_k by some integer m coprime with p.

Theorem 3. *Let G be an Abelian divisible $p-$group. G is generalized hereditarily Hopfian group if and only if $G = \oplus_{i\in I_p}\mathbb{Z}(p^\infty)$ such that $card(I_p) < \infty$.*

Proof.

- $\Rightarrow$)
  Let G be a generalized hereditarily Hopfian group.
  Let us show that $G = \oplus_{i\in I_p} Z(p^\infty)$ where $card(I_p) < \infty$.
  Suppose that $card(I_p) = \infty$, then we can write $G = G_1 \bigoplus G_1'$ with $G_1' = (\oplus_{i=1}^\infty Z(p^\infty))_i$ and $G_1 = (\oplus_{k=1}^\infty Z(p^\infty))_k$

  and we consider now,

$$\begin{array}{ccc} \phi : G_1 & \longrightarrow & G_1 \\ x = \sum_{k=1}^{n_0} x_k & \longmapsto & \sum_{k=1}^{n_0-1} y_k \end{array}$$

  with $x_k,\ y_k \in (Z(p^\infty))_k$, $y_k = x_{k+1}$, $y_{n_0} = 0$.
  It is clear that ϕ is a surjective endomorphism and $ker(\phi) = Z(p^\infty)_1 = Z(p^\infty)$,
  We can write $G_1 = Z(p^\infty) \bigoplus (\oplus_{i=2}^\infty Z(p^\infty))_i \subset ker(\phi) \bigoplus (\oplus_{i=2}^\infty Z(p^\infty))_i \subset G_1$, then $ker(\phi) \bigoplus (\oplus_{i=2}^\infty Z(p^\infty)_i) = G_1$
  And since G is a generalized hereditarily Hopfian group, then by Lemma 1, G_1 is also generalized hereditarily Hopfian group, then $ker(\phi) \ll G$, thus $G_1 = (\oplus_{i=2}^\infty Z(p^\infty))_i$, which is absurd, therefore $card(I_p) < \infty$.

- Conversely: $\Leftarrow$)
  Let $G = \bigoplus_{i=1}^n \mathbb{Z}(p^\infty)$ be a divisible $p-$group, and let G_1 be a subgroup of G.
  Now, we aim to show that G_1 is generalized Hopfian, and for this, we need to discuss the two possible following cases.

  First case. Let assume that $G_1 = \bigoplus_{i=1}^{n_0} \mathbb{Z}(p^\infty)$, where $n_0 \leq n$.

  Consider the epimorphism $\varphi : G_1 \to G_1$
  As we have recalled in the introduction of this section, it is known that $\mathbb{Z}(p^\infty) = \bigcup_{n=1}^\infty \langle c_{i,n}\rangle$, where $\mathrm{ord}(c_{i,n}) = p^n$, and $p(c_{i,n+1}) = c_{i,n}$.

  For $c_{i,1} \in G_1$, there exists $t_{i,1} \in G_1$ such that $\varphi(t_{i,1}) = c_{i,1}$, because φ is an epimorphism of G_1. Since $p(t_{i,2}) = t_{i,1}$, the subgroup $\bigcup_{k=1}^\infty \langle t_{i,k}\rangle$ is a divisible $p-$group.

  This implies that,

$$\bigoplus_{i=1}^{n_0} \bigcup_{k=1}^{\infty} \langle t_{i,k}\rangle = \bigoplus_{i=1}^{n_0} \mathbb{Z}(p^\infty) = G_1.$$

  Now, let us check that $\ker(\varphi) \ll G_1$ or $G_1 = G_2$, such that $G_1 = \ker(\varphi) + G_2$.

We have $\ker(\varphi) = \bigoplus_{i=1}^{n_0} \langle pt_{i,1} \rangle$ because,

$$\varphi\left(\sum_{i=1}^{n_0} m_i t_{i,k}\right) = \sum_{i=1}^{n_0} m_i \varphi(t_{i,k}) = \sum_{i=1}^{n_0} m_i c_{i,k} = 0.$$

This implies that $m_i c_{i,k} = 0$ for $1 \leq i \leq n_0$. Hence, $p^k \mid m_i$, so $m_i = p^k m'_i$.

Therefore, $m_i t_{i,k} = m'_i p^k t_{i,k} = m'_i p t_{i,1}$.

We define $p^M = \max(\text{ord}(pt_{i,1}), 1 \leq i \leq n_0)$. Then, for every element $h \in \ker(\varphi)$, we have $p^M h = 0$.
Now let $t_{i,k} \in G_1$, then $t_{i,k} = p^M t_{i,k+M}$, and also $t_{i,k+M} = h + g_2$, where $h \in \ker(\varphi)$ and $g_2 \in G_2$.

Substituting, we have

$$p^M t_{i,k+M} = p^M h + p^M g_2.$$

which implies that

$$t_{i,k} = p^M t_{i,k+M} = p^M h + p^M g_2 = p^M g_2 \in G_2,$$

because $p^M h = 0$. Therefore $t_{i,k} \in G_2$.

Since $t_{i,k} \in G_2$, it follows that $G_1 \subseteq G_2$. Conversely, by construction, $G_2 \subseteq G_1$.

Thus, $G_1 = G_2$. Consequently, G_1 is generalized Hopfian.

Second case: $G_1 = D \oplus C$.

Now assume $G_1 < G$. Then, by Theorem 21.3 [8], we can write:

$$G_1 = D \oplus C,$$

where D is a maximal divisible subgroup and C is a reduced subgroup.

Consider the surjective endomorphism $\varphi : G_1 \to G_1$
Denote ϕ_D as the restriction of φ to D, that is, $\varphi_D : D \to G_1$
φ_C as the restriction of φ to C, namely $\varphi_C : C \to G_1$
Take the projection p_1 as,

$$\begin{aligned} p_1 : G_1 &\to C \\ c + d &\mapsto c \end{aligned}$$

and the projection p_2 as,

$$\begin{aligned} p_2 : G_1 &\to D \\ c + d &\mapsto d \end{aligned}$$

We have $\mathrm{p}_1\varphi_C : C \to C$ is bijective.
In fact, for $y \in C$, there exists $x = c + d \in G_1$, with $c \in C$ and $d \in D$, such that $\varphi(x) = y$ $(*)$.

Expanding, we have $\varphi(c + d) = y,\ so\ \varphi(c) + \varphi(d) = y.$
Applying the projection p_1, this gives $p_1(\varphi(c)) + p_1(\varphi(d)) = p_1(y)$.
Since $p_1(\varphi(d)) = 0$ (as $\varphi(d) \in D$ and p_1 maps D to 0) and $p_1(y) = y$, it follows that $p_1(\varphi(c)) = y\ or\ p_1(\varphi_C)(c) = y.$
Thus, there exists an element $c \in C$ such that $p_1\varphi_C(c) = y$, showing that $p_1\varphi_C$ is surjective.

Since C is reduced $p-$group and of finite rank, it is finite, and hence, $p_1\varphi_C$ is bijective.

We have $p_2\varphi_D$ is surjective, because if we take $y \in D$, then there exists $x \in G_1$, with $x = c + d$, $c \in C$, and $d \in D$, such that $\varphi(x) = y$.

Applying the projection p_1, we get $p_1(\varphi(c + d)) = p_1(y)$.
Expanding, this gives $p_1(\varphi(c)) + p_1(\varphi(d)) = p_1(y)$.
Since $p_1(\varphi(d)) = 0$ (as $\varphi(d) \in D$ and p_1 maps D to 0), and $p_1(y) = 0$ (because $y \in D$), it follows that $p_1(\varphi(c)) = 0\ or\ p_1(\varphi_C(c)) = 0$.
Because now $p_1\varphi_C$ is bijective, $p_1(\varphi_C(c)) = 0$ implies $c = 0$.

Thus, $x = d$, and applying the projection p_2 to $(*)$, we obtain $p_2\varphi(d) = p_2(y)$, or $p_2\varphi_D(d) = y$, so there exists $x = d \in D$ such that $p_2\varphi_D(d) = y$, therefore $p_2\varphi_D$ is surjective.

Let $x \in \ker(\varphi)$, with $x = c + d$. Then $\varphi(c + d) = 0$, or $\varphi(c) + \varphi(d) = 0$, also $\varphi_C(c) + \varphi(d) = 0$ and applying p_1, we have

$$\begin{aligned} p_1\varphi_C(c) + p_1\varphi(d) = 0 \\ &\iff p_1\varphi_C(c) = 0 \\ &\iff c = 0,\ then\ \ x = d\,(*) \end{aligned}$$

Let $x \in \ker(\varphi)$, then $\varphi(x) = 0$, or $\varphi(d) = 0$, because $x = d$ by $(*)$ and applying p_2, we have $p_2\varphi(d) = 0\ or\ p_2\varphi(x) = 0.$
Thus $x \in ker(p_2\varphi)$, and therefore $\ker(\varphi) = \ker(p_2\phi) \subseteq D$.
Now, let us show that G_1 is generalized Hopfian. For this, let $G_1 = G' + \ker\varphi$, where $G' = C' \oplus D'$.
We have

$$p^M C \oplus p^M D = p^M C' \oplus p^M D' + p^M(\ker\varphi).$$

Since $p^M(\ker\varphi) = 0$, and D and D' are divisible subgroups, it follows that $p_M D = D$ and $p^M D' = D'$. Thus,

$$p^M C \oplus D = p^M C' \oplus D'.$$

Because D and D' are maximal divisible subgroups, we have $D = D'$. Therefore,

$$G_1 = G' + \ker\varphi, \quad \text{with } \ker\varphi \subseteq D.$$

This implies $G_1 = G' + \ker\varphi \subseteq G' + D \subseteq G'$, as $D = D' \subseteq G'$. Hence, $G_1 \subseteq G'$, which implies $G_1 = G'$.

Thus, G_1 is generalized Hopfian.

Finally, G is generalized hereditarily Hopfian.

5. Conclusion

In the first two theorems of our work, we extended the results of the generalized hereditarily Hopficity property on Abelian groups within the categories of reduced $p-$groups and reduced torsion groups. As for the final part of our work, we succeeded in showing in a third theorem that if a group is generalized hereditarily Hopfian and p-divisible, it is a finite direct sum of $\mathbb{Z}(p^{\infty})$. As a perspective, one would think about the study of such hopficity properties but now in the case of torsion-free group.

Acknowledgements

A special thanks to the editors Professors Eyup Cetin and Baris Kiremitci. We would also like to thank all the anonymous referees for their time, effort and help for improving the content of our paper.

References

[1] S. Abdelalim. Characterization the strongly co-hopfian abelian groups in the category of abelian torsion groups. *Journal of Mathematical analysis, 6(4), 1-10*, 2015.

[2] R. Baer. Groups without proper isomorphic quotient groups. *Bulletin of the American Mathematical Society, 50(4), 267-278*, 1944.

[3] G. Baumslag. On abelian hopfian groups. *I. Mathematische Zeitschrift, 78(1), 53-54*, 1962.

[4] A. Corner. Three examples on hopficity in torsion-free abelian groups. *Acta Mathematica Hungarica, 16(3-4), 303-310*, 1965.

[5] A. Chillali S. Abdelalim H. Essannouni. The strongly hopfian abelian groups. *ulf Journal of Mathematics, 3(2)*, 2015.

[6] S. Abdelalim H. Essannouni. Characterization of the automorphisms of an abelian group having the extension property. *Portugaliae Math Vol.59, p 325-333*, 2002.

[7] S. Abdelalim H. Essannouni. Characterization of the inessential endomorphisms in the category of abelian groups. *Pub. Mat. 47 (2003) 359-372*, 2003.

[8] L. Fuchs. *Infinite Abelian groups.* vol. 1 Academic press., 1970.

[9] L. Fuchs. *Infinite Abelian groups.* vol. 2 Academic press., 1973.

[10] B. Goldsmith K. Gong. On super and hereditarily hopfian and co-hopfian abelian groups. *Archiv der Mathematik, 99, 1-8*, 2012.

[11] A. Haghany. Hopficity and co-hopficity for morita contexts. *Communications in Algebra, 27(1), 477-492*, 1999.

[12] A. Ghorbani A. Haghany. Generalized hopfian modules. *Journal of Algebra, 255(2), 324-341*, 2002.

[13] V.A. Hiremath. Hopfian rings and hopfian modules. *Indian J. Pure Appl.Math. 17(7), 895-900*, 1986.

[14] J. Irwin I. Ito. A quasi-decomposable abelian group without proper isomorphic quotient groups and proper isomorphic subgroups. *Pacific Journal of Mathematics, 29(1), 151-160*, 1969.

[15] K. El Amin Mokhtar S. Mamadou. Une caracterisation des anneaux artiniens a ideaux principaux. *In Ring Theory: Proceedings of a Conference held in Granada, Spain, Sept. 1–6, 1986 (pp. 245-254). Springer Berlin Heidelberg.*, 1988.

[16] J. Nielsen. Om regning med ikkekommutative faktorer og dens anvendelse i gruppenteorien. *Matematisk Tidsskrift B pp. 77-94*, 1921.

[17] X. Magnus A. Karrass D. Solitar. Combinatorial group theory: Presentations of groups in terms of generators and relations. *Courier Corporation*, 2004.

[18] K. Varadarajan. Some recent results on hopficity, co-hopficity and related properties. *In International Symposium on ring theory (pp. 371-392)*, 2001.

[19] Y. Gang L. Zhongkui. On hopfian and co-hopfian modules. *Vietnam J. Math, 35(1), 73-80*, 2007.

[20] Y. Gang L. Zhongkui. Notes on generalized hopfian and weakly co-hopfian modules. *Communications in Algebra, 38(10), 3556-3566*, 2010.