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Connected Hop Roman Dominating Functions in Graphs

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Abstract. Let G be a connected graph. A hop Roman dominating function $f : V(G) \to \{0, 1, 2\}$ is a connected hop Roman dominating function (CHRDF) on G if the set $\{u \in V(G) : f(u) \neq 0\}$ induces a connected subgraph of G. The weight of a CHRDF f is given by $\omega_G^{cRh}(f) = \sum_{v \in V(G)} f(v)$ and the minimum weight among all connected hop Roman dominating functions on G, denoted $\gamma_{cRh}(G)$, is the connected hop Roman domination number of G. In this paper, we show that the parameter lies between the connected hop domination number of G and twice this number. We characterize the graphs that attain small values of the parameter and determine the connected hop Roman domination number of some graphs.

2020 Mathematics Subject Classifications: 05C69

Key Words and Phrases: Connected, hop domination, hop Roman dominating function, connected hop Roman domination number

1. Introduction

Motivated by the ancient Roman Empire's military strategy, Cockayne et al. in [9] introduced and studied the parameter called Roman domination. In a sense, the concept is one of the numerous variants of the standard domination concept. Various studies have been done since the introduction of Roman domination. In particular, a significant number of variations of the parameter have already been defined and investigated (see [1], [3], [4], [7], [8], [10], [11], [12], [16], [20], [21], [24]).

The concept of hop domination, one that utilizes distance two rather than unit distance, has also been widely studied since the time it was introduced in [22]. Studies in [5], [6], [13],

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[14], [15], [17], [19], [18], [23], [26], and [27] considered further the concept and some of its constructs or variants. Recently, the parameter hop Roman domination was introduced and, just like Roman domination, various modifications of the concept have also been introduced and studied (see [2], [25], and [28]). This present study considers connected hop Roman domination. Since hop domination and Roman domination have both applications in many networks (for example, to model defense strategies, communication in social networks, and management problems), this newly defined parameter can easily find its own similar applications. This among others gives added motivation for introducing and studying the said parameter.

2. Terminology and Notations

Let $G = (V(G), E(G))$ be an undirected graph. The *open neighborhood* of $v \in V(G)$ is the set $N_G(v) = \{u \in V(G) : uv \in E(G)\}$ while its *closed neighborhood* is the set $N_G[v] = \{v\} \cup N_G(v)$. Vertex v is an *isolated vertex* if $N_G(v) = \varnothing$. The *open neighborhood* and *closed neighborhood* of set $S \subseteq V(G)$ are the sets $N_G(S) = \cup_{v\in S} N_G(v)$ and $N_G[S] = \cup_{v\in S} N_G[u]$, respectively. The degree of v, denoted by $deg_G(v)$, is equal to $|N_G(v)|$. Any shortest path connecting two vertices x and y of G is called an x-y geodesic and the length of an x-y geodesic in G is the distance $d_G(x, y)$ of x and y. The *diameter* of G, denoted $diam(G)$, is the maximum distance between the pair of vertices. A vertex v of G is a *leaf* if $deg_G(v) = 1$ and $w \in V(G)$ is a *support vertex* if $wz \in E(G)$ for some leaf $z \in V(G)$.

A set $S \subseteq V(G)$ is said to be a *dominating set* of G if $N_G[S] = V(G)$. The minimum cardinality of a dominating set, denoted by $\gamma(G)$, is called the *domination number* of G. A vertex v is a *dominating vertex* of G if $N_G[v] = V(G)$. Any dominating set of cardinality $\gamma(G)$ is referred to as a γ-*set* of G.

The *open hop neighborhood* of $v \in V(G)$ is the set $N^2_G(v) = \{u \in V(G) : d_G(u, v) = 2\}$ and its *closed hop neighborhood* is $N^2_G[v] = \{v\} \cup N^2_G(v)$. The *open hop neighborhood* and *closed hop neighborhood* of set $S \subseteq V(G)$ are the sets $N^2_G(S) = \cup_{v\in S} N^2_G(v)$ and $N^2_G[S] = \{v\} \cup N^2_G(u)$, respectively.

A set $S \subseteq V(G)$ is said to be a *hop dominating set* of G if $N^2_G[S] = V(G)$, i.e., for each $v \in V(G) \setminus S$, there exists $w \in S$ such that $d_G(v, w) = 2$. A hop dominating set S is *connected hop dominating* if the graph $\langle S\rangle$ induced by S is connected. The minimum cardinality among all hop dominating (resp. connected hop dominating) sets in G is called the *hop domination number* (resp. *connected hop domination number*) of G, and is denoted by $\gamma_h(G)$ (resp. $\gamma_{ch}(G)$. Any hop dominating (resp. connected hop dominating) set of cardinality $\gamma_h(G)$ (resp. $\gamma_{ch}(G)$) is called a γ_h-set (resp. γ_{ch}-set) of G.

A function $f : V(G) \to \{0, 1, 2\}$ is a *hop Roman dominating function* on G if for each $u \in V(G)$ for which $f(u) = 0$, there exists $v \in V(G)$ such that $f(v) = 2$ and $d_G(u, v) = 2$. The *weight* of f is given by $\omega^{Rh}_G(f) = \sum_{v\in V(G)} f(v)$. The *hop Roman domination number* of G, denoted by $\gamma_{Rh}(G)$, is the minimum weight of a hop Roman dominating function on G.

Let G be a connected graph. A function $f : V(G) \longrightarrow \{0,1,2\}$ is a connected hop Roman dominating function on G provided that it satisfies the following properties:

($P1$) For each $v \in V(G)$ with $f(v) = 0$, there exists $w \in V(G)$ with $f(w) = 2$ and $d_G(w,v) = 2$ (i.e., f is a hop Roman dominating function on G).

($P2$) The set $\{u \in V(G) : f(u) \neq 0\}$ induces a connected subgraph of G.

The weight of f is given by $\omega_G^{cRh}(f) = \sum_{v \in V(G)} f(v)$. The minimum weight among all connected hop Roman dominating functions on G is the connected hop Roman domination number $\gamma_{cRh}(G)$ of G. If f is a connected hop Roman dominating function on G and $\omega_G^{cRh}(f) = \gamma_{cRh}(G)$, then f is called a γ_{cRh}-function on G. If f is a (connected) hop Roman dominating function on G, then we may write $f = (V_0, V_1, V_2)$ where $V_j = \{x \in V(G) : f(x) = j\}$ for $j \in \{0,1,2\}$.

Consider graph G in Figure 1. Let $V_0 = \{a,b,c,f,g,h\}$, $V_1 = \{i\}$, and $V_2 = \{d,e\}$. Then $f = (V_0, V_1, V_2)$ is a γ_{Rh}-function on G. On the other hand, the function $g = (V_0', V_1', V_2')$, where $V_0' = \{a,b,c,f,g,h,i\}$, $V_1' = \varnothing$, and $V_2' = \{d,e,f\}$, is a γ_{cRh}-function on G. Therefore, $\gamma_{Rh}(G) = 5$ and $\gamma_{cRh}(G) = 6$.

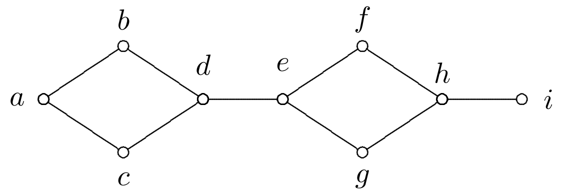

Figure 1: Graph G with $\gamma_{Rh}(G) = 5$ and $\gamma_{cRh}(G) = 6$

Henceforth, the family $CHRDF(G)$ refers to the set containing all the connected hop Roman dominating functions on G.

3. Results

Proposition 1. *Let G be a nontrivial connected graph and let $f = (V_0, V_1, V_2)$ be a γ_{cRh}-function on G. Then each of the following holds:*

(i) $|V_0| = 0$ *if and only if* $|V_2| = 0$.

(ii) $|V_1| = 0$ *if and only if* V_2 *is a* γ_{ch}*-set in* G.

Proof. (i) The sufficiency part is clear by property ($P1$).

Suppose $|V_0| = 0$ and suppose $|V_2| \neq \varnothing$. Let $V_0' = V_0$, $V_1' = V_1 \cup V_2$ and $V_2' = \varnothing$. Then $g = (V_0', V_1', V_2') \in CHRDF(G)$ and so,

$$\omega_G^{cRh}(g) = |V_1'| = |V_1 \cup V_2| = |V_1| + |V_2| < |V_1| + 2|V_2| = \omega_G^{cRh}(f),$$

a contradiction. Thus, $|V_2| = \varnothing$.

(ii) Suppose $|V_1| = 0$. Then V_2 is a connected hop dominating set in G. Suppose that V_2 is not a γ_{ch}-set in G. Let $V_2' \subseteq V(G)$ be a γ_{ch}-set in G. Then $|V_2'| < |V_2|$. Let $V_2^* = V_2'$, $V_1^* = \varnothing$ and $V_0^* = V(G) \setminus V_2^*$. Then $h = (V_0^*, V_1^*, V_2^*) \in CHRDF(G)$. Thus,

$$\omega_G^{cRh}(h) = 2|V_2^*| = 2|V_2'| < 2|V_2| = \omega_G^{cRh}(f).$$

This is a contradiction to the assumption that f is a γ_{cRh}-function on G. Hence, V_2 is a γ_{ch}-set in G.

Conversely, suppose that V_2 is a γ_{ch}-set in G. Suppose further that $|V_1| \neq 0$. Then $\gamma_{cRh}(G) = |V_1| + 2|V_2| > 2|V_2|$. Let $V_0'' = V_0 \cup V_1$, $V_1'' = \varnothing$ and $V_2'' = V_2$. Then $V_0'' \subseteq N_G^2(V_2'')$ and $V_1'' \cup V_2'' = V_2$ is a connected hop dominating set in G. Thus, $h = (V_0'', V_1'', V_2'') \in CHRDF(G)$ and

$$\omega_G^{cRh}(h) = 2|V_2''| = 2|V_2| < |V_1| + 2|V_2| = \omega_G^{cRh}(f),$$

a contradiction. Hence, $|V_1| = 0$. □

Proposition 2. *For any graph G of order n, it holds that*

$$\gamma_{ch}(G) \leq \gamma_{cRh}(G) \leq \min\{n, 2\gamma_{ch}(G)\}.$$

Proof. Let $f = (V_0, V_1, V_2)$ be a γ_{cRh}-function of G. Since $V_1 \cup V_2$ is a connected hop dominating set of G, we have

$$\gamma_{ch}(G) \leq |V_1| + |V_2| \leq |V_1| + 2|V_2| = \gamma_{cRh}(G).$$

Next, set $V_0' = V_2' = \varnothing$ and $V_1' = V(G)$. Then $g_1 = (V_0', V_1', V_2') \in CHRDF(G)$. Thus,

$$\gamma_{cRh}(G) \leq \omega_G^{cRh}(g_1) = |V_1'| = |V_1| = n.$$

Finally, let S be a γ_{ch}-set in G and let $V_1'' = \varnothing$, $V_0'' = V(G) \setminus S$, and $V_2'' = S$. Then $g_2 = (V_0'', V_1'', V_2'') \in CHRDF(G)$. This implies that

$$\gamma_{cRh}(G) \leq \omega_G^{cRh}(g_2) = 2|V_2''| = 2\gamma_{ch}(G).$$

Hence,

$$\gamma_{cRh}(G) \leq \min\{n, 2\gamma_{ch}(G)\}.$$

Therefore,

$$\gamma_{ch}(G) \leq \gamma_{cRh}(G) \leq \min\{n, 2\gamma_{ch}(G)\}. \quad \square$$

It is worth noting that the bounds given in Proposition 2 are sharp. Indeed, it can be verified easily that $\gamma_{cRh}(K_4) = 4 = \gamma_{ch}(K_4)$, $\gamma_{ch}(P_3) = 2 < 3 = |V(P_3)| = \gamma_{cRh}(P_3)$, and $\gamma_{cRh}(P_5) = 4 = 2\gamma_{ch}(P_5) < |V(P_5)|$.

Remark 1. *Let G be a connected graph. If $f = (V_0, V_1, V_2)$ is a γ_{cRh}-function in G, then $V_1 \cup V_2$ need not be a γ_{ch}-set in G.*

Remark 2. *Let G be a connected graph of order n and $f = (V_0, V_1, V_2)$ a γ_{cRh}-function on G. If $v \in V(G)$ and $deg_G(v) = n - 1$ (that is, v is a dominating vertex in G), then $v \in V_1$.*

Proposition 3. *Let G be a connected graph of order n. Then each the following statements holds:*

(i) $\gamma_{cRh}(G) = 1$ *if and only if* $G = K_1$.

(ii) $\gamma_{cRh}(G) = 2$ *if and only if* $G = K_2$.

(iii) $\gamma_{cRh}(G) = 3$ *if and only if* $n \geq 3$ *and* $G = K_3$ *or* $G = K_1 + H$, *where H is a (disconnected) graph of order $n - 1$ and has at least one isolated vertex.*

(iv) $\gamma_{cRh}(G) = 4$ *if and only if* $n \geq 4$ *and* $G \in \{K_4, P_4, C_4, K_1 + P_3\}$ *or one of the following conditions holds:*

- (a) *There exists $w \in V(G)$ such that $|N_G(w)| = 2$, $N_G^2(w) = V(G) \setminus N_G[w]$, and x is not a support vertex whenever $x \in N_G(w)$ and is a dominating vertex of G.*
- (b) *There exist adjacent vertices $u, v \in V(G)$ such that $|N_G(u)| \geq 2$, $|N_G(v)| \geq 2$, $N_G^2(u) \cup N_G^2(v) = V(G) \setminus \{u, v\}$, $N_G(u) \cap N_G(v) = \varnothing$, and $\{u, v\}$ is a γ_{ch}-set in G.*

Proof. (i) Assume that $\gamma_{cRh}(G) = 1$ and $f = (V_0, V_1, V_2)$ is a γ_{cRh}-function on G. Then $|V_1| + 2|V_2| = 1$. This implies that $V_2 = \varnothing$. By Proposition 1(i), $V_0 = \varnothing$. Thus, $|V(G)| = |V_1| = 1$. Hence, $G = K_1$. The converse is clear.

(ii) Suppose $\gamma_{cRh}(G) = 2$. Let $f = (V_0, V_1, V_2)$ be a γ_{cRh}-function in G. Then $|V_1| + 2|V_2| = 2$. Thus, $|V_2| \leq 1$. Assume that $|V_2| = 1$. Then $V_1 = \varnothing$ and, by Proposition 1(i), $V_0 \neq \varnothing$. Let $V_2 = \{x\}$ and let $y \in V_0$. Then $d_G(x, y) = 2$. Let $z \in N_G(x) \cap N_G(y)$. Then $z \in V_1$, a contradiction. Therefore, $|V_2| = 0$. It follows that $|V_1| = 2$, i.e., $G = K_2$.

Conversely, suppose $G = K_2 = \langle \{u, v\} \rangle$. Let $V_1 = \{u, v\}$ and $V_0 = V_2 = \varnothing$. Then $f = (V_0, V_1, V_2) \in CHRDF(G)$. Thus, $\gamma_{cRh}(G) \leq \omega_G^{cRh}(f) = |V_1| = 2$. Since $G \neq K_1$, $\gamma_{cRh}(G) \geq 2$. Hence, $\gamma_{cRh}(G) = 2$.

(iii) Suppose $\gamma_{cRh}(G) = 3$. Then $n \geq 3$ by (i) and (ii). Let $f = (V_0, V_1, V_2)$ be a γ_{ch}-function on G. Then $|V_1| + 2|V_2| = 3$. Thus, $|V_2| \leq 1$. Consider the following cases:

Case 1. $|V_2| = 0$.
Then $|V_0| = 0$ and $|V_1| = |V(G)| = 3$. This implies that $G \in \{K_3, P_3\}$.

Case 2. $|V_2| = 1$.
Then $|V_0| \neq \varnothing$ and $|V_1| = 1$. Let $V_1 = \{v\}$ and $V_2 = \{w\}$. Since $\langle V_1 \cup V_2 \rangle$ is connected, $uw \in E(G)$. Also, $V_0 = V(G) \setminus \{v, w\}$. Now, let $u \in V_0$. By (P1), $u \in N_G^2(w)$. Let $[u, y, w]$

be a $u-w$ geodesic. Since $yw \in E(G)$, $y \notin V_0$. Hence, $y \in V_1$, implying that $y = v$. Let $H_1 = \langle V_0 \rangle$ and $H = \{w\} \cup H_1$. Then $G = \{v\} + H$ or, equivalently, $G = K_1 + H$, where H is a disconnected graph with isolated vertex $\{w\}$.

Conversely, suppose $n \geq 3$ and G satisfies the given conditions. Then $\gamma_{cRh}(G) \geq 3$ by (i) and (ii). Clearly, $\gamma_{cRh}(K_3) = 3$. So suppose $G = K_1 + H$ and u is an isolated vertex of H. Let $K_1 = \{v\}$. Then $vu \in E(G)$. Since u is an isolated vertex of H, $[u, v, w]$ is a $u-w$ geodesic in G for every $w \in V(G) \setminus \{u, v\}$. This implies that $d_G(u, w) = 2$ for every $w \in V(G) \setminus \{u, v\}$. Define the function $f = (V_0, V_1, V_2)$ where $V_2 = \{u\}$, $V_1 = \{v\}$, and $V_0 = V(G) \setminus \{u, v\}$. Then $f \in CHRDF(G)$ and

$$\gamma_{cRh}(G) \leq \omega_G^{cRh}(f) = |V_1| + 2|V_2| = 3.$$

Therefore, $\gamma_{cRh}(G) = 3$. □

(iv) Suppose $\gamma_{cRh}(G) = 4$. Then $n \geq 4$ by (i), (ii), and (iii). Let $f = (V_0, V_1, V_2)$ be a γ_{cRh}-function on G. Then $|V_1| + 2|V_2| = 4$. It follows that $|V_2| \leq 2$. Consider the following cases:

Case 1. $|V_2| = 0$.
Then $V_0 = \varnothing$ and $|V_1| = |V(G)| = 4$. Therefore, with reference to (iii), we must have $G \in \{K_4, P_4, C_4, K_1 + P_3\}$.

Case 2. $|V_2| = 1$.
Then $|V_1| = 2$ and $|V_0| \neq 0$. Let $V_1 = \{u, v\}$ and $V_2 = \{w\}$. Suppose $v \notin N_G(w)$. Then $[v, u, w]$ is v-w geodesic by (P2). This and (P1) would imply that $[z, u, w]$ is a z-w geodesic for all $z \in V(G) \setminus \{u, w\}$. Let $V_2' = V_2$, $V_1' = \{u\}$, $V_0' = V_0 \cup \{v\} = V(G) \setminus \{u, w\}$. Then $g = (V_0', V_1', V_2') \in CHRDF(G)$. Thus, $\omega_G^{cRh}(g) = |V_1'| + 2|V_2'| = 3$, a contradiction. Thus, $v \in N_G(w)$. Similarly, $u \in N_G(w)$. Hence, $u, v \in N_G(w)$. This implies that $deg_G(w) = 2$, that is, $|N_G(w)| = 2$. Moreover, $N_G^2(w) = V_0 = V(G) \setminus N_G[w]$. Suppose $z \in N_G(w)$ is a dominating vertex of G. Suppose further that z is a support vertex of some leaf p. Then $G = \langle z \rangle + H$, where $H = \langle V(G) \setminus \{z\} \rangle$ is a graph with isolated vertex p. By part (iii), it follows that $\gamma_{cRh}(G) = 3$, contrary to the assumption that $\gamma_{cRh}(G) = 4$. This shows that (a) holds.

Case 3. $|V_2| = 2$.
Then $|V_1| = 0$. Let $V_2 = \{u, v\}$. Then $uv \in E(G)$ by (P2). Suppose $|N_G(u)| = 1$. Then $N_G(u) = \{v\}$. Since $G \neq K_2$, $G = \langle v \rangle + H$ where $H = \langle V(G) \setminus \{v\} \rangle$ is a disconnected graph having u as an isolated vertex. By (iii), $\gamma_{cRh}(G) = 3$, a contradiction. Thus, $|N_G(u)| \geq 2$. Similarly, $|N_G(v)| \geq 2$. Suppose $z \in N_G(u) \cap N_G(v)$. Then $z \in V_0$, which is not possible. Hence, $N_G(u) \cap N_G(v) = \varnothing$. Let $y \in V(G) \setminus V_2 = V_0$. By (P1), $y \in N_G^2(u) \cup N_G^2(v)$. Hence, $N_G^2(u) \cup N_G^2(v) = V(G) \setminus \{u, v\}$. Cleary, $\{u, v\}$ is a γch-set in G. This shows that (b) holds.

Conversely, suppose $n \geq 4$ and suppose $G \in \{K_4, P_4, C_4, K_1 + P_3$. Let $V(G) = \{w, x, y, z\}$ and set $V_1 = \{w, x, y, z\}$ and $V_0 = V_2 = \varnothing$. Then $f = (V_0, V_1, V_2) \in$

$CHRDF(G)$. Thus, $\gamma_{cRh}(G) \leq \omega_G^{cRh}(f) = |V_1| = 4$. Since $G \neq K_1 + H$ for any graph H having an isolated vertex, it follows from (iii) that $\gamma_{cRh}(G) \geq 4$. Accordingly, $\gamma_{cRh}(G) = 4$.

Now, suppose (a) holds, i.e., there exists $w \in V(G)$ such that $|N_G(w)| = 2$, $N_G^2(w) = V(G) \setminus N_G[w]$, and x is not a support vertex whenever $x \in N_G(w)$ and is a dominating vertex of G. Let $V_0' = V(G) \setminus N_G[w]$, $V_1' = N_G(w)$, and $V_2' = \{w\}$. Then $g = (V_0', V_1', V_2') \in CHRDF(G)$ and $\gamma_{cRh}(G) \leq \omega_G^{cRh}(g) = |V_1'| + 2|V_2'| = 4$. The assumption that x is not a support vertex whenever $x \in N_G(w)$ and is a dominating vertex of G, implies that $G \neq K_1 + H$ for any graph H having an isolated vertex. Therefore, $\gamma_{cRh}(G) = 4$.

Lastly, suppose there exist adjacent vertices $u, v \in V(G)$ such that $|N_G(u)| \geq 2$ and $|N_G(v)| \geq 2$, $N_G^2(u) \cup N_G^2(v) = V(G) \setminus \{u, v\}$, $N_G(u) \cap N_G(v) = \varnothing$ and $\{u, v\}$ is a γ_{ch}-set in G. Define a function $h : V(G) \to \{0, 1, 2\}$ by

$$h(x) = \begin{cases} 2, & \textit{if } x \in \{u, v\}, \\ 0, & \textit{if } x \in V(G) \setminus \{u, v\}. \end{cases}$$

Then $h \in CHRDF(G)$ and $\gamma_{cRh}(G)(h) \leq \omega_G^{cRh} = h(u) + h(v) = 4$. Now, suppose $\gamma(G) = 1$, say q is a dominating vertex of G. Then clearly, $q \notin V(G) \setminus \{u, v\}$ since $N_G^2(u) \cup N_G^2(v) = V(G) \setminus \{u, v\}$. This implies that $q \in \{u, v\}$. This, however, is not possible because $N_G(u) \cap N_G(v) = \varnothing$. Thus, $\gamma(G) \neq 1$. Therefore, $G \neq K_1 + H$ for any graph H. Therefore, $\gamma_{cRh}(G) = 4$. □

The following result follows from Proposition 3.

Corollary 1. *Let n, m be positive integers. Then*

- (i) $\gamma_{cRh}(K_n) = n$ *for all* $n \geq 1$;
- (ii) $\gamma_{cRh}(K_{1,n}) = 3$ *for all* $n \geq 2$;
- (iii) $\gamma_{cRh}(K_{n,m}) = 4$ *for all* $n, m \geq 2$;
- (iv) $\gamma_{cRh}(P) = 4$, *where P is the Petersen graph; and*
- (v) $\gamma_{cRh}(F_n) = 4$ *for all* $n \geq 3$.

Proposition 4. *For any wheel graph W_n with $n \geq 4$, $\gamma_{cRh}(W_n) = 5$.*

Proof. Clearly, $\gamma_{cRh}(W_n) \geq 4$. Let v_0 be the hub (central) vertex of the wheel graph $W_n = K_1 + C_n$ and let $V(W_n) \setminus \{v_0\} = V(C_n) = \{v_1, v_2, \ldots, v_n\}$, where $C_n = [v_1, v_2, \ldots, v_n, v_1]$. Define a function $f : V(W_n) \to \{0, 1, 2\}$ by

$$f(v) = \begin{cases} 2, & \textit{if } v \in \{v_1, v_2\} \\ 1, & \textit{if } v = v_0 \\ 0, & \textit{if } v \in V(W_n) \setminus \{v_0, v_1, v_2\}. \end{cases}$$

Then $f \in CHRDF(W_n)$ and

$$\gamma_{cRh}(W_n) \leq \omega^{cRh}_{W_n}(f) = f(v_0) + f(v_1) + f(v_2) = 5.$$

Since $\gamma_{ch}(W_n) = 3$ and $|N_{W_n}(x)| \geq 3$ for every $x \in V(W_n)$, W_n does not satisfy (a) and (b) of Proposition 3(iv). Consequently, $\gamma_{cRh}(W_n) = 5$. □

Proposition 5. *For any path P_n of order $n \geq 1$,*

$$\gamma_{cRh}(P_n) = \begin{cases} 4, & \text{if } n = 5, 6 \\ 6, & \text{if } n = 7 \\ n, & \text{if } n \neq 5, 6, 7. \end{cases}$$

Proof. For $n = 1, 2, 3$, the result follows from Propositon 3 $(i), (ii), (iii)$. By Proposition 3 (iv), $\gamma_{cRh}(P_n) = 4$ for $n = 4, 5, 6$. Let $n \geq 7$ and let $P_n = [x_1, x_2, ..., x_n]$. Let $f = (V_0, V_1, V_2)$ be a γ_{cRh}-function on P_n. Since $V_1 \cup V_2$ is a connected hop dominating set, $V_0 \subseteq \{x_1, x_2, x_{n-1}, x_n\}$. Suppose first that $n = 7$. Since the function $g = (\{x_1, x_2, x_6, x_7\}, \varnothing, \{x_3, x_4, x_5\})$ is a connected hop dominating function on P_7, and P_7 does not satisfy any of the conditions given in Proposition 3, it follows that $5 \leq \gamma_{cRh}(P_7) \leq 6$. Now, suppose that $\gamma_{cRh}(P_5) = |V_1| + 2|V_2| = 5$. If $V_0 = \varnothing$, then $|V_2| = 0$ and $|V_1| = 7$. This implies that $\gamma_{cRh}(P_7) = 7$ which is not possivle. Thus, $|V_0| \neq 0$ and $1 \leq |V_2| \leq 2$. Suppose $|V_2| = 1$. Then $|V_0| = 3$ and $|V_1| = 3$. If $x_1 \in V_1 \cup V_2$, then $V_1 \cup V_2 = \{x_1, x_2, x_3, x_4\}$ since $\langle V_1 \cup V_2 \rangle$ is connected. Hence, $x_7 \in V_0$ and $d_{P_7}(x_7, y) \neq 2$ for all $y \in V_2$, a contradiction. Thus, $x_1 \in V_0$. Similarly, $x_7 \in V_0$. It follows that $V_1 \cup V_2$ is $\{x_2, x_3, x_4, x_5\}$ or $\{x_3, x_4, x_5, x_6\}$. This implies that $x_1 \notin N^2_{P_7}(V_2)$ or $x_7 \notin N^2_{P_7}(V_2)$, a contradiction. This forces $|V_2| = 2$ and $|V_1| = 1$. It is routine to show that this also leads to a contradiction. Therefore, $\gamma_{cRh}(P_7) = 6$. Next, suppose that $n \geq 8$. If $|V_0| = 0$, then $|V_2| = 0$ and $|V_1| = n$. Hence, $\gamma_{cRh}(P_n) = n$. Suppose $|V_0| = 1$. Then $V_0 = \{x_1\}$ or $V_0 = \{x_n\}$. Assume that $V_0 = \{x_1\}$. Then $V_2 = \{x_3\}$ and $V_1 = V(P_n) \setminus \{x_1, x_3\}$. This yields

$$\gamma_{cRh}(P_n) = \omega^{cRh}_{P_n}(f) = |V_1| + 2|V_2 = (n-2) + 2 = n.$$

If $|V_0| = 2$, then $V_0 = \{x_1, x_2\}$ or $V_0 = \{x_1, x_n\}$ or $V_0 = \{x_{n-1}, x_n\}$. It follows that $V_2 = \{x_3, x_4\}$ or $V_2 = \{x_3, x_{n-2}\}$ or $V_2 = \{x_{n-3}, x_{n-2}\}$, respectively. If $|V_0| = 3$, then $V_0 = \{x_1, x_2, x_n\}$ or $V_0 = \{x_1, x_{n-1}, x_n\}$. Hence, $V_2 = \{x_3, x_4, x_{n-2}\}$ or $V_2 = \{x_3, x_{n-3}, x_{n-2}\}$, respectively. Finally, if $V_0 = \{x_1, x_2, x_{n-1}, x_n\}$, then $V_2 = \{x_3, x_4, x_{n-3}, x_{n-2}\}$. It can easily be shown that any of these cases will imply that

$$\gamma_{cRh}(P_n) = \omega^{cRh}_{P_n}(f) = |V_1| + 2|V_2 = (n-2) + 2 = n.$$

This proves the assertion. □

Proposition 6. *For any cycle C_n of order $n \geq 3$,*

$$\gamma_{cRh}(C_n) = \begin{cases} 4, & \text{if } n = 4,5,6 \\ 6, & \text{if } n = 7 \\ n, & \text{if } n \neq 5,6,7. \end{cases}$$

Proof. By Proposition 3(iii), $\gamma_{cRh}(C_3) = 3$ and by Proposition 3(iv), $\gamma_{cRh}(C_n) = 4$ for $n = 4, 5, 6$. Next, let $n \geq 7$ and let $C_n = [v_1, v_2, ..., v_n, v_1]$. Let $f = (V_0, V_1, V_2)$ be a γ_{cRh}-function on C_n. Since $V_1 \cup V_2$ is a connected hop dominating set, $\langle V_0 \rangle$ is connected and $|V_0| \leq 4$. Suppose first that $n = 7$. Since the function $g = (\{v_1, v_2, v_3, v_4\}, \varnothing, \{v_5, x_6, v_7\})$ is a connected hop dominating function on C_7, $\gamma_{cRh}(C_7) \leq 6$. If $|V_0| = 0$, then $|V_2| = 0$ and $|V_1| = 7$. If $|V_0| = 1$, then $|V_2| = 1$ and $|V_1| = 5$. If $|V_0| = 2$, then $|V_2| = 2$ and $|V_1| = 3$ and if $|V_0| = 3$, then $|V_2| = 3$ and $|V_1| = 1$. Any of these four cases will yield $\gamma_{cRh}(C_7) = 7$ which is not possible. Therefore, $|V_0| = 4$. Hence, $|V_2| = 3$ and $|V_1| = 0$. It follows that $\gamma_{cRh}(C_7) = 6$. Finally, suppose that $n \geq 8$. Clearly, if $|V_0| = j$, then $|V_2| = j$ and $|V_1| = n - 2j$. Therefore, $\gamma_{cRh}(C_n) = n - 2j + 2j = n$. □

Proposition 7. *Let $G = K_{n_1,n_2,...,n_k}$ be the complete k-partite graph with $2 \leq n_1 \leq n_2 \leq \cdots \leq n_k$, where $k \geq 2$. Then $\gamma_{cRh}(G) = 2k$.*

Proof. Let $S_{n_1}, S_{n_2}, \ldots, S_{n_k}$ be the partite sets in G. Pick $v_{n_j} \in S_{n_j}$ for each $j \in [k]$, where $[k] = \{1, 2, \ldots, k\}$, and let $S = \{v_{n_j} : j \in [k]\}$. Set $V_0 = V(G) \setminus S$, $V_1 = \varnothing$, and $V_2 = S$. Then $f = (V_0, V_1, V_2) \in CHRDF(G)$ and $\gamma_{cRh}(G) \leq 2|V_2| = 2k$.

Now let $g = (W_0, W_1, W_2)$ be a γ_{cRh}-function on G. Let $M = \{j \in [k] : n_j \geq 3\}$. Suppose $|W_2| = 0$. Then $|W_0| = 0$ and $|W_1| = V(G)$. It follows that $\gamma_{cRh}(G) = |V(G)| = \sum_{j=1}^{k} n_j$. The value is $2k$ if $M = \varnothing$ and strictly greater than $2k$ if $M \neq \varnothing$. Hence, $|W_2| \neq 0$ if $M \neq \varnothing$. So suppose $|W_2| \neq 0$. Let $R = \{j \in [k] : W_2 \cap S_{n_j} \neq \varnothing\}$. Since g is a Roman dominating function on G, it follows that $W_0 = \cup_{j \in R}[S_{n_j} \setminus \ [(W_2 \cap S_{n_j}) \cup (W_1 \cap S_{n_j})]$ and $W_1 = [\cup_{j \in [k] \setminus R} S_{n_j}] \cup [\cup_{j \in R}(S_{n_j} \cap W_1)]$. It follows that

$$\gamma_{cRh}(G) = \sum_{j \in [k] \setminus R} S_{n_j} + \sum_{j \in R} (S_{n_j} \cap W_1) + 2 \sum_{j \in R} (S_{n_j} \cap W_2) \geq 2(k - |R|) + 2|R| = 2k.$$

This establishes the desired equality. □

Lemma 1. *Let G be a connected graph of order n. Then $\gamma_{ch}(G) = n$ if and only if $G = K_n$.*

Proof. Suppose $\gamma_{ch}(G) = n$ and suppose for a contradiction that $G \neq K_n$. Then $diam(G) \geq 2$. Pick any two vertices x and y of G such that $d_G(x, y) = diam(G)$. Then x and y are non-cut vertices of G. Let $[x_1, x_2, \ldots, x_k]$, where $x = x_1$ and $y = x_k$, be an x-y geodesic in G. Then $k \geq 3$ and $d_G(x_1, x_3) = 2$. Put $S = V(G) \setminus \{x\}$. Since x is a non-cut vertex, it follows that $\langle S \rangle$ is connected. Therefore, S is a connected hop dominating set in G and $\gamma_{ch}(G) \leq |S| = n - 1$, contrary to our assumption. Thus, $G = K_n$.

For the converse, suppose $G = K_n$ and let S be a $gamma_{ch}$-set in G. Since S is a hop dominating set, $V(K_n) \subseteq S$. Therefore, $S = V(G)$ and $\gamma_{ch}(G) = n$. □

Proposition 8. *Let G be any graph of order n. Then $\gamma_{ch}(G) = \gamma_{cRh}(G)$ if and only if $G = K_n$.*

Proof. Suppose $\gamma_{ch}(G) = \gamma_{cRh}(G)$ and let $f = (V_0, V_1, V_2)$ be a γ_{cRh}-function in G. Since $\gamma_{ch}(G) \leq |V_1| + |V_2| \leq |V_1| + 2|V_2| = \gamma_{cRh}(G)$ and $\gamma_{ch}(G) = \gamma_{cRh}(G)$, $|V_1| + |V_2| = |V_1| + 2|V_2|$. This implies that $V_2 = \varnothing$ and thus, $V_0 = \varnothing$. Hence, $\gamma_{cRh}(G) = |V_1| = n = \gamma_{ch}(G)$. By Lemma 1, $G = K_n$.

For the converse, suppose that $G = K_n$. By Lemma 1, $\gamma_{ch}(G) = n$. By Proposition 2, $\gamma_{cRh}(G) = n$. This shows that $\gamma_{ch}(G) = \gamma_{cRh}(G)$. □

The next result is immediate from Proposition 3.

Proposition 9. *Let G and H be connected graphs. Then $\gamma_{cRh}(G + H) \geq 2$ and each of the following holds:*

(i) $\gamma_{cRh}(G + H) = 2$ *if and only if G and H are trivial graphs.*

(ii) $\gamma_{cRh}(G + H) = 3$ *if and only if $G = K_1$ and $H = K_2$ (or $H = K_1$ and $G = K_2$) or $G = K_1$ and H is a graph with at least one isolated vertex (or $H = K_1$ and G is a graph with at least one isolated vertex).*

(iii) If G and H are non-trivial graphs and each contains an isolated vertex, then $\gamma_{cRh}(G + H) = 4$.

Proposition 10. *Let G and H be connected graphs. Then*

$$2 \leq \gamma_{cRh}(G + H) \leq \gamma_{cRh}(G) + \gamma_{cRh}(H).$$

Proof. Since $G + H$ is nontrivial, $\gamma_{cRh}(G + H) \geq 2$. Let $f = (V_0, V_1, V_2)$ and $g = (V_0', V_1', V_2')$ be γ_{cRh}-functions on G and H, respectively. Define a function $h = (V_0'', V_1'', V_2'')$ for which $V_0'' = V_0 \cup V_0'$, $V_1'' = V_1 \cup V_1'$ and $V_2'' = V_2 \cup V_2'$. Then $h \in CHRDF(G + H)$ and

$$\begin{aligned} \gamma_{cRh}(G + H) \leq \omega_{G+H}^{cRh} &= |V_1''| + 2|V_2''| \\ &= |V_1 \cup V_1'| + 2|V_2 \cup V_2'| \\ &= |V_1| + |V_1'| + 2|V_2| + 2|V_2'| \\ &= |V_1| + 2|V_2| + |V_1'| + 2|V_2'| \\ &= \gamma_{cRh}(G) + \gamma_{cRh}(H). \quad \square \end{aligned} \tag{1}$$

Remark 3. *The upper bound given in Proposition 10 is tight. However, strict inequality is attainable.*

To see this, let $G = P_3 = [a, b, c]$ and $H = P_2 = [p, q]$. Then $\gamma_{cRh}(G) = 3$ by Proposition 3(*iii*) and $\gamma_{cRh}(H) = 2$ by Proposition 3(*ii*). Let $f = (V_0, V_1, V_2)$ be a γ_{cRh}-function on $G + H$. Since $b, p,$ and q are dominating vertices of $G + H$, it follows from Remark 1 that $b, p, q \in V_1$. If $a, c \in V_1 \cup V_2$, then $V_1 = \{a, b, c, p, q\}$. Hence, $\gamma_{cRh}(G + H) = 5$. Suppose one of a and c is in V_0, say $V_0 = \{a\}$. Then necessarily, $c \in V_2$. It follows that $V_1 = \{b, p, q\}$ and $\gamma_{cRh}(G + H) = |V_1| + 2|V_2| = 5$. Therefore, $\gamma_{cRh}(G + H) = \gamma_{cRh}(G) + \gamma_{cRh}(H) = 5$.

Clearly, the upper bound is also attained when $G = K_m$ and $H = K_n$.

Next, to show that strict inequality is also possible, consider $G = P_4 = [a, b, c, d]$ and $H = P_4 = [x, y, z, w]$. Then $\gamma_{cRh}(G) = \gamma_{cRh}(H) = 4$ by Proposition 3(iv). Now set $V_0 = \{a, b, x, y\}$, $V_1 = \{c, z\}$, and $V_2 = \{d, w\}$. Then $f = (V_0, V_1, V_2) \in CHRDF(G + H)$. It is easy to see that f is a γ_{cRh}-function on $G + H$. Therefore, $\gamma_{cRh}(G + H) = 6 < 8 = \gamma_{cRh}(G) + \gamma_{cRh}(H)$.

Proposition 11. *There are infinitely many graphs G and H such that $\gamma_{cRh}(G+H) = 5$. In particular, if $G = P_n$, where $n \geq 4$, and H is any non-trivial graph with at least one trivial component, then $\gamma_{cRh}(G + H) = 5$.*

Proof. Let $n \geq 4$ and let $G = P_n = [x_1, x_2, \ldots, x_n]$. Let $H_1, H_2, \ldots, H_k$ be the components of H and suppose that $H_1 = \langle p \rangle$. Let $V_0 = (V(P_n) \setminus \{x_{n-1}, x_n\}) \cup (\cup_{j=2}^{k} V(H_j))$, $V_1 = \{x_{n-1}\}$ and $V_2 = \{p, x_n\}$. Then $g = (V_0, V_1, V_2)$ is a connected hop Roman dominating function on $G + H$. It follows that $\gamma_{cRh}(G + H) \leq \omega_G^{cRh}(f) = 5$. Since $G + H$ does not satisfy any of the conditions given in Proposition 3, it follows that $\gamma_{cRh}(G + H) = 5$. □

4. Conclusion

Connected hop Roman domination has been defined and investigated for some graphs. There are still a lot of aspects that can be explored and studied for this parameter. For any two graphs G and H, this initial study has only provided sharp lower and upper bounds for the join $G + H$. It was, however, shown that these bounds may not be attained. It is conjectured that the exact value of the parameter for the join $G + H$ can be described by defining yet another parameter. Furthermore, connected hop Roman domination can also be investigated for other graphs under binary operations and for the complexity of the connected hop Roman dominating function problem.

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