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Analysis of Stability, D-Stability, and Pseudospectra in Economic Modeling

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Abstract. The analysis of dynamic stability is a fundamental and an important concept in system dynamics. Its focus is on the ability of a dynamical system to return to an equilibrium state under structured perturbations. The study of dynamic stability plays critical role in various fields, for instance, engineering, control theory, and economics. The analysis on the dynamic stability mostly involves computation of the eigenvalues of a system's state matrix. The D-stability is a particular and specialized form of dynamic stability, and its mainly focus dynamical systems subject to structured perturbations. In this paper, we present new results on dynamic stability, and D-stability of a class of linear economic model in the mathematical form

$$y_t = Ay_t + By_{t-1} + Cx_t,$$

with y_t, a vector of the endogenous variables, x_t is a vector of exogenous variables, and A, B, and C are the matrices having an appropriate dimensions. The new results are developed on both necessary and sufficient conditions on the interconnection between D-stable matrices and structured singular values. The numerical experimentation show the behaviour of structured singular values for matrices appearing across linear dynamic model.

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1. Introduction

A vast amount of literature has been written on the problem related to the control of economics. The main concentration was given to study and analyze the static economic models. The most of the economics models are dynamic in their nature. The study on the current state of an economic system involving a vast amount of policies are being used to shift system from current status to future state while dealing with such dynamic systems.

For the input-output economics, a vast amount of literature has been written in order to describe the real economics [25, 31]. The general linear dynamic systems are much easier to deal with compare to singular dynamic systems such as implicit dynamic systems, generalized dynamic systems, generalized state-space dynamic systems, semi-state dynamic systems [9, 24]. A vast amount of literature has been written on regular dynamic systems and descriptor systems, we refer interested reader to see [14, 15, 42] and references therein. The linear matrix inequalities techniques were developed to study the singular dynamic system in economics, see [10, 35, 36, 43, 45].

The new results on the interconnections between dynamic behavior of macroeconomics, continuous-time dynamical models and relationships between dynamic stability and dominant-diagonal structure was studied and analyzed in [30]. The most of macroeconomics continuous-time dynamic models appears to be non-stable. In [5, 6, 11, 37] it was shown that macro-economic continuous-time models are unstable. For general discrete-time dynamical models appearing in economics, the relationship between the dominant diagonals and the stability was developed by [16, 21, 26, 27, 41].

The D-stability for a class of real valued matrices to study the equilibrium in dynamic models of competitive market for the first time was studied by Arrow and McManus [2], and Enthoven and Arrow in [13]. The study of dynamic stability of **tatonnement process** for Walrsian model of general equilibrium attracted a major community of economists. The classical approach developed by Sanuelson describes the dynamic behaviour of the economic models.

The new results on D-stability, strong D-stability and structured singular values were developed by using various tools from linear algebra, matrix analysis, and system theory, see [34]. In [22], the most general general relationships between performance and robustness of dynamical system, and special type of matrix stabilities, that is, D stability and diagonal stability were studied and analyzed. The results on new stability conditions for second-order dynamical systems were presented and analyzed. An extension to D-stability for non-square matrices which are applicable to distributed and decentralized controllability analysis was recently studied in [40]. The μ-values or structured singular values first introduced and analyzed by J. C. Doyel [12] and Safonov [38] is a mathematical technique in order to investigate and test the stability of linear dynamical systems. In general problem aiming the determination of stability in the presence of structured or unstructured uncertainties is most fundamental issue in control and has attracted researchers from almost last three decades.

In this article, we present new results on stability and D-stability of linear dynamic models that appears in economics. We also present new results on necessary and sufficient conditions on interconnections between D-stability and structured singular values of matrices appearing across dynamic models. The numerical experimentation shows the comparison of structured singular values for matrices with various dimensions from dynamic models. Finally, the pseudo-spectrum of matrices across dynamic models is presented while making use of EigTool [28].

Overview of article: In Subsection 1.1, we give the preliminaries to recall important concepts, and definition to be used in this article. In Section 2, we provide new results to study dynamic stability and D-stability of economic models. We make use of various tools from linear algebra, system theory to derive these new results. In Section 3, necessary and sufficient conditions are derived for the D-stability of dynamical system. The numerical experimentation to support new results are presented in Section 4. We have made use of Eigtool to visualize the pseudo-spectrum of structured matrices across the dynamical systems. Finally, in Section 5, we conclude our paper.

1.1. Preliminaries.

For $M \in \mathbb{C}^{n,n}$, the largest singular value is denoted by σ_{max}, and is a non-negative real number. The smallest singular value is denoted by σ_{min}. The notation M^T denotes the transpose of a matrix, $\lambda_i(M)$ denotes all the eigenvalues of the matrix M. For $M > 0 (M \geq 0)$ means that matrix M is positive definite, and positive semi-definite, respectively. For $M < 0 \, (M \leq 0)$ means that matrix M is negative definite, and negative semi-definite, respectively. The symbol $\mathbb{C}_+$.

Definition 1. *A block diagonal matrix D is defined as $D = diag(D_{11}, D_{22}, \cdots, D_{nn})$, where $D_{11}, \cdots, D_{nn}$, are the matrices.*

Definition 2. *The block-diagonal structure Δ represents the uncertainty set and is defined as*

$$\Delta =: \{D = diag(D_{11}, D_{22}, \cdots, D_{nn})\}, \; D \in \mathbb{R}^{n,n}.$$

Definition 3. *The set Δ_+ is defined as*

$$\Delta_+ =: \{D \in \Delta : D_{ii} > 0, \; \forall i = 1 : n\}.$$

Definition 4. *The singular values $\sigma_i \; \forall \, i = 1 : n$ are the non-negative numbers appearing in the diagonal matrix Σ in the singular value decomposition of a matrix $M = U\Sigma V^T$, with U, V being as real orthogonal matrices.*

Definition 5. *For a given matrix $M \in \mathbb{C}^{n,n}$, and an underlying set Δ, the structured singular value is defined [33] as*

$$\mu_\Delta(M) := \frac{1}{min\{||\hat{\Delta}|| : det(I_n - M\hat{\Delta}) = 0, \; \forall \hat{\Delta} \in \Delta\}},$$

otherwise $\mu_\Delta(M) = 0$ if $det(I_n - M\hat{\Delta}) \neq 0, \; \forall \hat{\Delta} \in \Delta$.

Note that **min** is taken over $\hat{\Delta} \in \Delta$, and **min** over an empty set is $+\infty$.

Definition 6. *[4, 17]. The n-dimensional real valued matrix $A \in \mathbb{R}^{n,n}$ is continuous-time stable if $\exists\ D \in \Delta_+$ such that $(A^T DA - D) < 0$.*

Theorem 1. *[23] The continuous-time linear system $\frac{dx(t)}{dt} = Ax(t),\ \ x(t) \in \mathbb{R}^{n,1}$ is asymptotically stable iff*

$$Re(\lambda_i) < 0 \Leftrightarrow \frac{\pi}{2} < \phi < \frac{3\pi}{2},\ \ \forall\, i = 1 : n,$$

with $\lambda_i = |\lambda_i| e^{\iota \phi_i},\ \forall\, i = 1 : n$, the eigenvalues of A.

Note that the matrix $\hat{A} \in \mathbb{R}^{n,n}$ is discrete-time diagonal stable if $\exists\ D \in \Delta_+$ such that $(\hat{A}^T D\hat{A} - D) < 0$.

Definition 7. *[8, 44]. The n-dimensional real valued matrix $A \in \mathbb{R}^{n,n}$ is continuous-time D-stable if the product DA is such that $Re(\lambda_i(DA)) < 0,\ \forall i,\ \ D \in \Delta_+$, a positive diagonal matrix.*

Definition 8. *[3]. The n-dimensional real valued matrix $\hat{A} \in \mathbb{R}^{n,n}$ is discrete-time D-stable if the product $D\hat{A}$ is such that $\rho(D\hat{A}) < 1$, for all real valued diagonal and norm bounded matrices D.*

Theorem 2. *[23] The discrete-time linear system $x_{i+1} = \bar{A}x_i,\ \ x_i \in \mathbb{R}^{n,1},\ i = \mathbb{Z}_+$ is asymptotically stable iff*

$$|\lambda_i| < 1$$

with $\lambda_i,\ \forall\, i = 1 : n$, are the eigenvalues of $\bar{A}$.

2. Dynamic stability and D-stability of economic models

In the section, we present new results to study the dynamic stability and D-stability of a class of linear economic model given in the mathematical form

$$y_t = Ay_t + By_{t-1} + Cx_t.$$

Here, y_t is a vector of the endogenous variables, and x_t is a vector of exogenous variables. A, B, and C are the matrices having an appropriate dimensions. The reduced mathematical form of the above dynamic model is given as

$$y_t = (I_n - A)^{-1} By_{t-1} + Ex_t.$$

If the spectral radius, that is, $\rho((I_n - A)^{-1}B) < 1$, then dynamic model is said to be dynamically stable. The following Theorem 3 show that the reduced dynamic model is dynamically stable.

Theorem 3. *Let $A, B \in \mathbb{R}^{n,n}$. Then dynamical model $y_t = (I_n - A)^{-1}By_{t-1} + Ex_t$ is dynamically stable if $\rho((I_n - A)^{-1}B) < 1$.*

Proof. We aim to prove that $\rho(I_n - A)^{-1}B) < 1$ by using inequality

$$\rho((I_n - A)^{-1}B)) \leq ||(I_n - A)^{-1}B)||_2.$$

In order to prove our result, we assume that inequality holds true for strict inequality, that is,

$$\rho((I_n - A)^{-1}B)) < ||(I_n - A)^{-1}B||_2.$$

For the matrix $(I_n - A)^{-1}B$, there exists unitary matrices $U \in \mathbb{C}^{m,n}, V \in \mathbb{C}^{m,n}$ such that

$$(I_n - A)^{-1}B = U \left(\begin{array}{c|c} \sigma_1 & 0 \\ \hline 0 & T \end{array} \right) V^H.$$

Next, we consider σ_1 and $\theta_1 \in \mathbb{C}^{n,1}$ so that

$$\sigma_1 = \|(I_n - A)^{-1}B\theta_1\|_2 = \|(I_n - A)^{-1}B\|_2,$$

while $\|\theta_1\|_2 = 1$. Further, we let $u_1 = \frac{(I_n - A)^{-1}B\theta_1}{\sigma_1}$ such that

$$\|u_1\|_2 = \frac{\|(I_n - A)^{-1}B\theta_1\|_2}{\sigma_1} = \frac{\|(I_n - A)^{-1}B\theta_1\|_2}{\|(I_n - A)^{-1}B\|_2} = 1.$$

Consider that $U_2 \in \mathbb{C}^{m,m-1}, V_2 \in \mathbb{C}^{n,n-1}$. Thus, U and V takes the form $U = (u_1|U_2)$, and $V = (v_1|V_2)$ with U, V being unitary matrices. The matrix product $U^H(I_n - A)^{-1}BV$ can be rewritten as

$$(u_1|U_2)(I_n - A)^{-1}B(v_1|V_2) =$$

$$\left(\begin{array}{c|c} u_1^H(I_n - A)^{-1}B\theta_1 & u_1^H(I_n - A)^{-1}BV_2 \\ \hline U_2(I_n - A)^{-1}B\theta_1 & U_2^H(I_n - A)^{-1}BV_2 \end{array} \right) = \left(\begin{array}{c|c} \sigma_1 u_1^H u1 & u_1^H(I_n - A)^{-1}BV_2 \\ \hline \sigma_1 U_2^H u_1 & U_2^H(I_n - A)^{-1}BV_2 \end{array} \right) = \left(\begin{array}{c|c} \sigma_1 & w^H \\ \hline 0 & B \end{array} \right),$$

where $u_1^H u_1 = 1, U_2^H u_1 = 0, w = V_2^H((I_n - A)^{-1}B)^H u_1$, and $C = U_2^H(I_n - A)^{-1}BV_2$. Let $w = 0$, we get

$$\sigma_1^2 = \|(I_n - A)^{-1}B\|_2^2 = \|U^H(I_n - A)^{-1}BV\|_2^2 = \underset{x \neq 0}{max} \frac{\|U^H(I_n - A)^{-1}BVx\|_2^2}{\|x\|_2^2} = \underset{x \neq 0}{max} \frac{\| \left(\begin{array}{c|c} \sigma_1 & u^H \\ \hline 0 & C \end{array} \right) x\|_2^2}{\|x\|_2^2}.$$

Take $x \longrightarrow w$, we have that

$$\sigma_1^2 > \frac{(\sigma_1^2 + w^H w)^2}{(\sigma_1^2 + w^H w)} = \sigma_1^2 + w^H w.$$

Thus, this yield that $w = 0$, and

$$U^H(I_n - A)^{-1}BV = \left(\begin{array}{c|c} \sigma_1 & 0 \\ \hline 0 & C \end{array} \right) \quad \text{or} \quad (I_n - A)^{-1}B = U \left(\begin{array}{c|c} \sigma_1 & 0 \\ \hline 0 & C \end{array} \right) V^H.$$

We see that $\mu_s((I_n - A)^{-1}B) < \sigma_1((I_n - A)^{-1}B)$, we have that

$$(I_n - A)^{-1}B := S^{-1}(I_n - A)^{-1}B = \left(\begin{array}{c|c} ((I_n - A)^{-1}B)_{11} & ((I_n - A)^{-1}B)_{12} \\ \hline \frac{1}{\nu}((I_n - A)^{-1}B)_{21} & \frac{1}{\nu}((I_n - A)^{-1}B)_{22} \end{array}\right).$$

Furthermore,

$$\left(\begin{array}{c|c} I & (I_n - A)^{-1}B \\ \hline ((I_n - A)^{-1}B)^H & I \end{array}\right) > 0 \Longleftrightarrow I - (I_n - A)^{-1}BI^{-1}((I_n - A)^{-1}B)^H > 0.$$

From this we follow that

$$\lambda_i(I - (I_n - A)^{-1}B((I_n - A)^{-1}B)^H(\nu)) > 0, \ \forall i$$

or

$$1 - \lambda_i((I_n - A)^{-1}B((I_n - A)^{-1}B)^H(\nu) > 0, \forall i$$

or

$$\lambda_i((I_n - A)^{-1}B((I_n - A)^{-1}B)^H(\nu)) < 1, \ \forall i.$$

Thus finally, we get

$$\sigma_1((I_n - A)^{-1}B) < 1 \Rightarrow ||(I_n - A)^{-1}B||_2 < 1 \Rightarrow \rho((I_n - A)^{-1}B)) < 1.$$

Theorem 4 gives the dynamic D-stability of dynamical model $y_t = (I_n - A)^{-1}B + Ex_t$. We make use of results on interconnection between structured singular value, and D-stability.

Theorem 4. *Let the dynamical system be $y_t = (I_n - A)^{-1}B + Ex_t$. Then, for dynamic D-stability the matrix $(I_n - A)^{-1}B$ is D-stable iff $(I_n - A)^{-1}B$ is stable, and $0 \leq \mu_\Delta(M) < 1$, where $M := \left(\boldsymbol{i}I_n + (I_n - A)^{-1}B\right)^{-1}\left(\boldsymbol{i}I_n - (I_n - A)^{-1}B\right)$.*

Proof. The matrix $(I_n - A)^{-1}B$ is D-stable iff $0 \leq \mu_\Delta(M) < 1$. Assume that matrix M is D-stable, means that,

$$\lambda_i\left(I_n - (I_n - A)^{-1}B + \mathbf{i}P\right) \neq 0, \ \forall\, i = 1 : n,$$

and for some $P = Diag(Re(p_{ii})) > 0, \ \forall\, i = 1 : n$. In order to prove that $(I_n - A)^{-1}B$ is D-stable matrix iff $0 \leq \mu_\Delta(M) < 1$, we let $(I_n - A)^{-1}B$ is D-stable matrix, that is,

$$\lambda_i\left((I_n - A)^{-1}B + \mathbf{i}P\right) \neq 0, \ \forall\, i = 1 : n.$$

Consider a block-diagonal matrix $\hat{\Delta} = (\mathbf{i}I_n - P)(\mathbf{i}I_n + P)^{-1}, \ \hat{\Delta} \in \Delta$. This allow us $P = (\mathbf{i}I_n + \hat{\Delta})^{-1}(\mathbf{i}I_n - \hat{\Delta})$ is a positive diagonal matrix if $\hat{\Delta} \in \Delta$. Since, $\lambda_i\left((I_n - A)^{-1}B + \mathbf{i}P\right) \neq 0, \ \forall\, i = 1 : n$. Thus, it shows that

$$\lambda_i\left((I_n - A)^{-1}B + \mathbf{i}(\mathbf{i}I_n + \hat{\Delta})^{-1}(\mathbf{i}I_n - \hat{\Delta})\right) \neq 0, \ \forall\, i = 1 : n, \ \forall\, \hat{\Delta} \in \Delta.$$

As, the rank of matrix $\left((I_n - A)^{-1}B + \mathbf{i}(\mathbf{i}I_n + \hat{\Delta})^{-1}(\mathbf{i}I_n - \hat{\Delta})\right)$ is exactly equal to the rank of matrix $\left((\mathbf{i}I_n + (I_n - A)^{-1}B) - (\mathbf{i}I_n - (I_n - A)^{-1}B)\hat{\Delta}\right)$, $\forall\, \hat{\Delta} \in \Delta$. Also,

$$\left((I_n - A)^{-1}B + \mathbf{i}(\mathbf{i}I_n + \hat{\Delta})^{-1}(\mathbf{i}I_n - \hat{\Delta})\right) \sim \left((\mathbf{i}I_n + (I_n - A)^{-1}B) - (\mathbf{i}I_n - (I_n - A)^{-1}B)\hat{\Delta}\right),\ \forall\, \hat{\Delta} \in \Delta.$$

Furthermore,

$$\lambda_i\left(I_n - (\mathbf{i}I_n + (I_n - A)^{-1}B)^{-1}(\mathbf{i}I_n - (I_n - A)^{-1}B)\hat{\Delta}\right) \neq 0,\ \forall \hat{\Delta} \in \Delta.$$

This is a necessary condition that $0 \leq \mu_\Delta(M) < 1$. Since, our aim is to show that $(I_n - A)^{-1}B$ is D-stable matrix. This means that we need to show

$$\lambda_i\left((I_n - A)^{-1}B + \mathbf{i}P\right) \neq 0,\ \forall\, i = 1 : n.$$

Since, $0 \leq \mu_\Delta(M) < 1$, means that $\lambda_i(\mathbf{i}I_n - M\hat{\Delta}) \neq 0$, $\forall\, \hat{\Delta} \in \Delta$. In turn this implies that $\lambda_i\left(I_n - (\mathbf{i}I_n + (I_n - A)^{-1}B)^{-1}(\mathbf{i}I_n - (I_n - A)^{-1}B)\hat{\Delta}\right) \neq 0$, $\forall \hat{\Delta} \in \Delta$ which further reduces to the fact that $\lambda_i\left((I_n - A)^{-1}B + \mathbf{i}P\right) \neq 0$, and this shows that $(I_n - A)^{-1}B$ is a D-stable matrix.

Theorem 5. *Let the dynamical system be $y_t = (I_n - A)^{-1}B + Ex_t$. Then, for dynamic D-stability the matrix $(I_n - A)^{-1}B$ is D-stable iff $Re\left(\lambda_i(P(I_n - A)^{-1}B + ((I_n - A)^{-1}B)^T P)\right) > 0$, $\forall\ i = 1 : n$, and $0 \leq \mu_\Delta(M) < 1$, with*

$$M := \left(\boldsymbol{i}I_n + P(I_n - A)^{-1}B + ((I_n - A)^{-1}B)^T P\right)^{-1}\left(\boldsymbol{i}I_n - P(I_n - A)^{-1}B - ((I_n - A)^{-1}B)^T P\right).$$

Proof. We follow the same procedure as given in Theorem 4 to prove Theorem 5. We aim to show that $(I_n - A)^{-1}B$ is D-stable iff $Re\left(\lambda_i(P(I_n - A)^{-1}B + ((I_n - A)^{-1}B)^T P)\right) > 0$, $\forall\ i = 1 : n$. Let $\hat{\Delta} \in \Delta$ be a block-diagonal structure, and defined as $\hat{\Delta} := (\mathbf{i}I_n - P)(\mathbf{i}I_n + P)^{-1}$, a diagonal matrix. As $\lambda_i(P(I_n - A)^{-1}B + ((I_n - A)^{-1}B)^T P) \neq 0$, $\forall i = 1 : n$. This implies that $\lambda_i(P(I_n - A)^{-1}B + ((I_n - A)^{-1}B)^T P + \mathbf{i}P) \neq 0$, $\forall i = 1 : n$ iff $\lambda_i(P(I_n - A)^{-1}B + ((I_n - A)^{-1}B)^T P + \mathbf{i}(\mathbf{i}I_n + \hat{\Delta})^{-1}(\mathbf{i}I_n - \hat{\Delta})) \neq 0$, $\forall i = 1 : n$. This further reduces to

$$\lambda_i\left(\mathbf{i}I_n + P(I_n - A)^{-1}B + ((I_n - A)^{-1}B)^T P) - (\mathbf{i}I_n - P(I_n - A)^{-1}B - ((I_n - A)^{-1}B)^T P)\hat{\Delta}\right) \neq 0.$$

This, finally we have that

$$\lambda_i\left(I_n - (\mathbf{i}I_n + P(I_n - A)^{-1}B + ((I_n - A)^{-1}B)^T P)^{-1}(\mathbf{i}I_n - P(I_n - A)^{-1}B - ((I_n - A)^{-1}B)^T P)\hat{\Delta}\right) \neq 0.$$

This last inequality is the necessary condition that structured singular values is strictly less than 1, means that, $0 \leq \mu_\Delta(M) < 1$.

3. Necessary and sufficient conditions for D-stability

In this section, we present some new results on necessary and sufficient conditions for D-stability of a given matrix in term of its structured singular values.

Lemma 1. *[20]. $A \in \mathbb{R}^{n,n}$ which is continuous-time diagonal stable matrix is a D-stable matrix.*

Lemma 2. *[20]. $A \in \mathbb{R}^{n,n}$ which is discrete-time diagonal stable matrix is a D-stable matrix.*

The following Theorem 6 show that Hurwitz-stable matrix is also a continuous-time D-stable matrix matrix.

Theorem 6. *[20]. Let $A \in \mathbb{R}^{n,n}$ be a Hurwitz-stable matrix. Then A is continuous-time D-stable only if*

$$0 \leq \mu_\Delta \left((sI_n + A)(sI_n - A)^{-1}\right) \leq 1, \ \forall\, s \in \mathbb{C}_+.$$

From above Theorem 6, it is clear that the definition of structured singular value holds true for all the values of parameter $s \in \mathbb{C}_+$, that is, in closed right-half of complex plane. The following lemma show that structured singular value can be determined at a single value of $s \in \mathbb{C}_+$ rather than evaluating at entire closed right-half of complex plane.

Lemma 3. *[20]. Let $A \in \mathbb{R}^{n,n}$ be a Hurwitz-stable matrix. Then, A is continuous-time D-stable matrix iff*

$$0 \leq \mu_\Delta \left((\boldsymbol{i}I_n + A)(\boldsymbol{i}I_n - A)^{-1}\right) \leq 1, \ \boldsymbol{i} = \sqrt{-1}.$$

Theorem 7 present an interesting relation between a continuous-time D-stable matrix $A \in \mathbb{R}^{n,n}$ and structured singular values of a perturbed matrix obtained from $A \in \mathbb{R}^{n,n}$.

Theorem 7. *Let $A \in \mathbb{R}^{n,n}$ such that $Re(\lambda_i(A)) > 0, \ \forall\ i$ and is continuous-time D-stable matrix, then*

$$0 \leq \mu_\Delta \left((\alpha I_n + A)^{-1}(\alpha I_n - A)\right) \leq 1, \ \alpha \in \mathbf{C}_+.$$

Proof. Let $A = e^H$ be a stable matrix. The matrix $H \geq 0$, a positive semi-definite matrix. Let $P > 0$, a positive definite such that $\lambda_i(\alpha I_n + e^H P) \neq 0$, $\forall i$, and $P = (\alpha I_n + \hat{\Delta})^{-1}(\alpha I_n - \hat{\Delta})$ for all $\hat{\Delta} \in \Delta$. This formulation allows as to have that

$$\lambda_i \left(\alpha I_n + e^H(\alpha I_n + \hat{\Delta})^{-1}(\alpha I_n - \hat{\Delta})\right) \neq 0 \quad \forall i, \ \forall \hat{\Delta} \in \Delta.$$

The above expression for λ_i reduces to

$$\lambda_i \left((\alpha I_n + e^H)^{-1}(\alpha I_n - e^H)\hat{\Delta}\right) \neq 0 \quad \forall i, \ \forall \hat{\Delta} \in \Delta.$$

In turn this yields

$$\lambda_i \left((\alpha I_n + A)^{-1}(\alpha I_n - A)\hat{\Delta}\right) \neq 0 \quad \forall i, \ \forall \hat{\Delta} \in \Delta.$$

Finally, we conclude that

$$0 \leq \mu_{\Delta}\left((\alpha I_n + A)^{-1}(\alpha I_n - A)\right) \leq 1.$$

The following Theorem 8 gives an interconnection between continuous-time D-stable matrix and structured singular values of a perturbed matrix.

Theorem 8. *Let $A \in \mathbb{R}^{n,n}$ such that $Re(\lambda_i(A)) > 0, \ \forall\ i$ and is continuous-time D-stable matrix, then*

$$0 \leq \mu_{\Delta}\left((\boldsymbol{i}I_n + A)^{-1}(\boldsymbol{i}I_n - A)\right) < 1, \ \ \boldsymbol{i} = \sqrt{-1}.$$

Proof. We aim to show that $\mathrm{Re}(\lambda_i(A)) > 0, \forall i$ if $\mathrm{Re}(\lambda_i(AH)) = \mathrm{Re}(\lambda_i(H)), \forall i, \forall H \geq 0$. If $\mathrm{Re}(\lambda_i(A)) \geq 0, \forall i$ and $A \in \mathbb{C}^{n\times n}$ is $n \times n$-singular matrix, then there exists a unitary matrix U such that

$$U^*AU = \begin{pmatrix} M_{11} + \mathbf{i}N_{11} & \mathbf{i}N_{12} \\ \mathbf{i}N_{21} & \cdot \end{pmatrix},$$

with $M_{11} > 0$, and for $U^*AU = \begin{pmatrix} \cdot & \cdot \\ \cdot & I_n \end{pmatrix} \geq 0$. In turn, this yields

$$\mathrm{Re}(\lambda_i(AH)) = \mathrm{Re}(\lambda_i(H)), \forall i.$$

Secondly, we prove that

$$0 \leq \mu_{\Delta}\left((\mathbf{i}I_n + A)^{-1}(\mathbf{i}I_n - A)\right) < 1.$$

To prove the above result, we take the given matrix $A = e^H$, a stable matrix where $H \geq 0$, a positive semi-definite matrix. Let $P > 0$, a positive definite such that $\lambda_i(\mathbf{i}I_n + e^H P) \neq 0$, $\forall i$ where $P = (\mathbf{i}I_n + \hat{\Delta})^{-1}(\mathbf{i}I_n - \hat{\Delta})$ for all $\hat{\Delta} \in \Delta$. This allows as to have that

$$\lambda_i\left(\mathbf{i}I_n + e^H(iI_n + \hat{\Delta})^{-1}(\mathbf{i}I_n - \hat{\Delta})\right) \neq 0 \quad \forall i, \forall \hat{\Delta} \in \Delta.$$

The above expression takes the form

$$\lambda_i\left((\mathbf{i}I_n + e^H)^{-1}(\mathbf{i}I_n - e^H)\hat{\Delta}\right) \neq 0 \quad \forall i, \forall \hat{\Delta} \in \Delta.$$

Finally, this implies that

$$\lambda_i\left((\mathbf{i}I_n + A)^{-1}(\mathbf{i}I_n - A)\hat{\Delta}\right) \neq 0 \quad \forall i, \forall \hat{\Delta} \in \Delta.$$

Thus,

$$0 \leq \mu_{\mathbb{B}}\left((\mathbf{i}I_n + A)^{-1}(\mathbf{i}I_n - A)\right) < 1.$$

Theorem 9. *Let $A \in \mathbb{C}^{n\times n}$ satisfies $Re(\lambda_i(A)) > 0, \ \forall\, i$ is a continuous-time diagonal stable matrix and its dual matrix $\hat{A} := (A - I)^{-1}(A + I)$ is discrete-time diagonal stable, then $\sigma_{max}(\hat{A}_1) < 1$, with $\hat{A}_1 := D^{\frac{1}{2}}\hat{A}D^{\frac{-1}{2}}$.*

Proof. The matrix $A \in \mathbb{C}^{n\times n}$ is continuous-time diagonal matrix. This means that for a positive diagonal matrix D, the matrix $DA + A^T D$ is negative definite, that is,

$$DA + A^T D < 0 \iff D(\hat{A} + I)(\hat{A} - I)^{-1} + ((\hat{A} + I)(\hat{A} - I)^{-1})^T D < 0$$

$$\iff D^{\frac{1}{2}}(\hat{A} + I)(\hat{A} - I)^{-1} D^{\frac{-1}{2}} + ((\hat{A} + I)(\hat{A} - I)^{-1})^T D < 0$$

$$\iff (\hat{A}_1 + I)(\hat{A}_1 - I)^{-1} + ((\hat{A} + I)(\hat{A} - I))^{-1} D < 0$$

$$\iff \hat{A_1}^T \hat{A}_1 - I < 0 \iff \lambda_{max}((\hat{A_1}^T)\hat{A}_1 - I) < 0 \iff \rho((\hat{A_1}^T)\hat{A}_1) < 1 \iff \sigma_{max}(\hat{A}_1) < 1.$$

3.1. Pseudo-spectrum

The computation of pseudo-spectrum for a given matrix (say) M is the set containing the all eigenvalues of M. One may raise an important question about the singularity of given matrix M as it does not appear as a small perturbation ϵ which may completely change the answer from a **yes** to a **no**. This further implies that either matrix-norm $||M^{-1}||$ is large enough or not?

For an eigenvalue λ corresponding to given matrix M, a much important question one may ask: Is the matrix $||(\lambda I_n - M)^{-1}||$ is large or not? this pattern allows to have definitions and results of pseudo-spectrum given as below:

Definition 9. *For matrix n-dimensional matrix M, and for $\epsilon > 0$, a small perturbation level. The ϵ-pseudospectrum $\sigma_\epsilon(M)$ is the set of eigenvalues $\lambda \in \mathbb{C}$ so that*

$$||(\lambda I_n - M)^{-1}|| > \frac{1}{\epsilon}.$$

Remark 1. *For quantity $\lambda \in \sigma(M)$, $\sigma(M)$, denotes the set of eigenvalues of M, $||(\lambda I_n - M)^{-1}|| = \infty$.*

The second definition of pseudo-spectrum is given as follows.

Definition 10. *For an n-dimensional matrix, and for a given $\epsilon > 0$, a small perturbation level. The ϵ-pseudospectrum $\sigma_\epsilon(M)$ is the set of eigenvalues $\lambda \in \mathbb{C}$ so that*

$$\lambda \in \sigma(M + E),$$

for some E having $||E|| < \epsilon$.

The third characterization of the computation of pseudo-spectrum for given matrix M is given as bellow.

Definition 11. *For a given n-dimensional matrix M, and $\epsilon > 0$, a small perturbation level. The ϵ-pseudo spectrum $\sigma_\epsilon(M)$ is the set of eigenvalues $\lambda \in \mathbb{C}$ so that*

$$||(\lambda I_n - M)v|| < \epsilon$$

for some $v \in \mathbb{C}^{n,1}$, $||v|| = 1$.

The following Theorem gives an equivalence of all above definitions of pseudo-spectrum.

Theorem 10. *Consider that* $||\cdot||$ *denotes a matrix norm for a given matrix* M. *Following statement are equivalent:*

$$(i)\ \Lambda_\epsilon(M) = \{z \in \mathbf{C} : ||(zI_n - M)^{-1}|| \geq \frac{1}{\epsilon}\}.$$

$$(ii)\ \Lambda_\epsilon(M) = \{z \in \mathbf{C} : z \in \Lambda(M+E),\ ||E|| \leq \epsilon\}.$$

$$(iii)\ \Lambda_\epsilon(M) = \{z \in \mathbf{C} : \exists\ v \in \mathbf{C}^{n,1}\ s.t\ ||(M - zI_n)v|| \leq \epsilon\}.$$

Remark 2. *The second statement in the above theorem is true for some matrix* E. *Furthermore, in last statement the column vector* v *has a unit 2-norm, that is,* $||v||_2 = 1$.

4. Numerical Experimentation

In this section, we a present a comparison on the numerical computation of lower bounds of structured singular values. The numerical algorithms under consideration for approximation of lower bounds of structured singular values are: The Matlab function **mussv**, the power algorithm (PA) [32], Gain Based Algorithm (GBA) [39], Poles migration Algorithm (PMA) [29], Non-linear optimization Algorithm (NLA) [19], and the Low-rank ODE's based Algorithm (LRA) given by first author [18]. The matrices are taken from various models of economy and finance. Furthermore, we use EigTool [28] for the computation of the pseudo-spectrum of each matrix.

Example 1. Consider macroeconomic model of the trade cycle [7]

$$\begin{cases} DK = \gamma_1(\hat{K} - K)\, with\ \hat{K} = \beta_1 Y \\ DC = \gamma_2(\hat{C} - C)\, with\ \hat{C} = \beta_2 Y + \beta_3 \\ DY = \gamma_3(\hat{Y} - Y)\, with\ \hat{Y} = C + DK \\ \end{cases}$$

Here, K, C, Y denotes stock of capital, consumption, and output less replacement, respectively.

Case-I: For $\gamma_1 < 0.0625,\ \gamma_2 = 0.6,\ \gamma_3 = 4.0,\ \beta_1 = 2.0,\ \beta_2 = 0.75$, the matrix C has the structure:

$$C = \begin{bmatrix} -0.05 & 0 & 0.1 \\ 0 & -0.6 & 0.45 \\ -0.2 & 4.0 & -3.6 \end{bmatrix}.$$

We present the comparison on numerical approximation of the lower bounds of structured singular values in following Table 1.

The numerical approximation of lower bounds of structured singular values					
mussv	PA	GBA	PMA	NLA	LRA
5.4369	5.4369	5.4389	5.4391	5.4371	5.4370

Case-II: For $\gamma_1 = 0.4$, the matrix C has the structure:

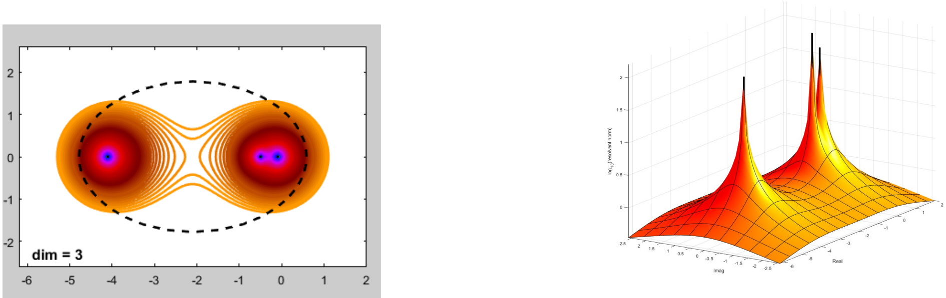

Figure 1: The graphs of singular values and pseudo-inverse of M in Example-1 (Case-I)

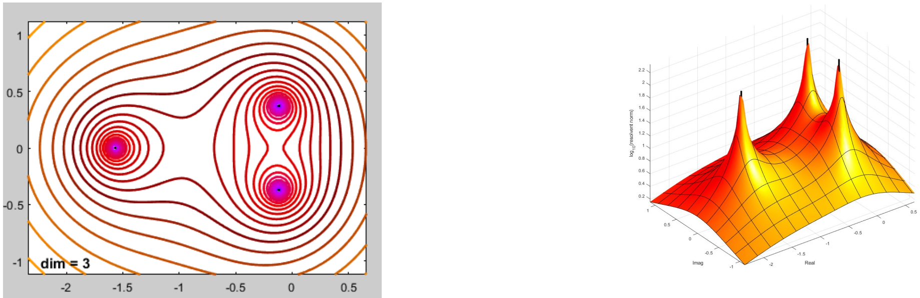

Figure 2: The graphs of singular values and pseudo-inverse of M in Example-1 (Case-II)

$$C = \begin{bmatrix} -0.4 & 0 & 0.8 \\ 0 & -0.6 & 0.45 \\ -1.6 & 4.0 & -0.8 \end{bmatrix}.$$

We present the comparison on numerical approximation of the lower bounds of structured singular values in following Table 2.

The numerical approximation of lower bounds of structured singular values					
mussv	PA	GBA	PMA	NLA	LRA
4.4272	4.4274	4.4285	4.4293	4.4290	4.4272

Example 2. Consider linear dynamical model

$$y_t = (I_n - A)^{-1}B + Ex_t.$$

For

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0.5 \\ 0 & 0 \end{bmatrix}.$$

The matrix $(I_n - A)^{-1}B$ for $n = 2$, has the following structure:

$$(I_2 - A)^{-1}B = \begin{bmatrix} 0 & 0.5 \\ 0 & 0 \end{bmatrix}.$$

We present the comparison on numerical approximation of the lower bounds of structured singular values in following Table 3.

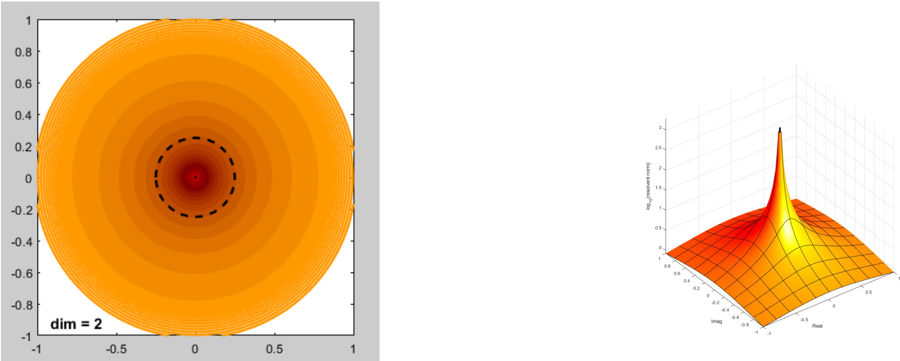

Figure 3: Matlab interface for computing pseudo-spectrum of matrix $(I_2 - A)^{-1}B$.

The numerical approximation of lower bounds of structured singular values					
mussv	PA	GBA	PMA	NLA	LRA
0.5000	0.5123	0.5341	0.5001	0.5012	0.5000

Example 3. Consider the non-linear model of macro-econometrics presented in [6]. The matrix $C = (I_n - A)^{-1}B$ and has the following structure for $n = 13$:

$$\begin{bmatrix}
-0.08 & 0 & 0 & 0 & 0 & 0 & 0 & -0.01 & 0.01 & 0 \\
0 & 0 & 0 \\
-1.27 & 0 & 0 & 0 & 0 & 0 & 0 & -0.01 & 0.01 & 0 \\
0 & 0 & 0 \\
0 & 0 & -0.24 & 0 & 0.24 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 & -0.09 & 0.10 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 \\
7.63 & 0 & 0.48 & 0 & -0.65 & 0 & 0.11 & 0 & 0 & 0 \\
0 & 0.05 & 0 \\
8.14 & 0 & 0.51 & 0 & 0 & -0.35 & 0.12 & 0 & 0 & 0 \\
-0.33 & 0.05 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -0.29 & -0.29 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.02 & 0 & 0 & -0.13 & 0.13 & 0 \\
0 & -0.02 & 0 \\
0 & 0 & 0 & 0 & 0.11 & 0 & 0 & 0 & 0 & 0 \\
0 & -0.01 & 0 \\
0 & 0 & 0 & 0 & 0.19 & 0 & 0 & 0.19 & 0 & -0.16 \\
0 & 0 & -0.19 \\
-8.60 & 0 & -0.54 & 0 & 0.60 & 0.13 & -0.13 & 0 & 0 & 0 \\
0 & -0.05 & 0 \\
1.00 & 0 & 0 & 0 & -0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.$$

We present the comparison on numerical approximation of the lower bounds of structured

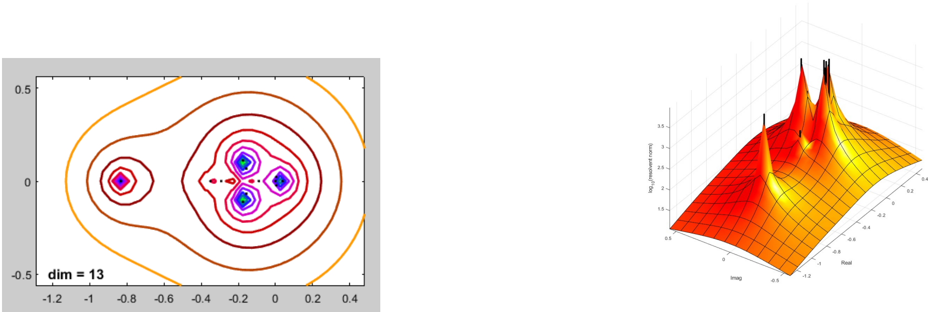

Figure 4: Matlab interface for computing pseudo-spectrum of matrix $(I_2 - A)^{-1}B$.

singular values in following Table 4.

The numerical approximation of lower bounds of structured singular values					
mussv	PA	GBA	PMA	NLA	LRA
14.2303	14.2309	14.2312	14.2332	14.2398	14.2306

5. Conclusion

In this article, we have developed new results on stability, and D-stability of linear economic models. The new results are obtained by using tools from linear algebra, matrix analysis, and system theory. Some novel results are also presented on necessary and sufficient conditions on the interconnection between D-stable matrices and structured singular values. The numerical experimentation show the comparison of structured singular values by various numerical techniques, the EigTool is used to present the pseudo-spectrum of matrices across linear dynamic models. The main advantages of the proposed methodology in the present study:
1. The proposed methodology helps to study and analyze many spectral properties of structured matrices. It contains the properties like eigenvalues, singular values, structured singular values.
2. The proposed methodology based on theoretical results link the bridge between stability, D-stability, and structured singular values for structured matrices corresponding to dynamical systems. **3.** The geometrical interpretation gives an advantage to exploit the hidden structures of structured matrices.
4. The proposed methodology has strong theoretical foundations and also numerical experimentation to support the theoretical construction.

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