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On the Extension of q-Hermite-Hadamard Inequalities for Strong Convexity

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Abstract. This study leverages q-calculus to establish q-Hermite-Hadamard type inequalities for strongly convex functions, showcasing possible extensions of well-known results in the field. Additionally, it enhances these findings by exploring the strong convexity of the function Φ. Furthermore, the q-midpoint and q-trapezoidal inequalities are unified within a comprehensive framework. Ultimately, the results suggest that the newly derived inequalities can be effectively applied in the context of special means.

2020 Mathematics Subject Classifications: 05A30, 26A51, 26D10, 26D15, 26D25, 52A01

Key Words and Phrases: H-H type inequalities, strongly convex functions, quantum calculus

1. Introduction

The Hermite-Hadamard inequality (H-H inequality) was proposed and explored by C. Hermite [13] and J. Hadamard [12]. This inequality offers an estimate for the mean value of a convex function and refines the Jensen inequality [9]. In 2018, N. Alp et al. [3] demonstrated a version of the q-H-H inequality for convex functions utilizing left q-integrals.

Theorem 1 ([3]). *Let $\Phi : [\alpha, \Upsilon] \to \mathbb{R}$ be a convex differentiable function defined on $[\alpha, \Upsilon]$ with $0 < q < 1$. The following inequalities then hold:*

$$\Phi\left(\frac{q\alpha + \Upsilon}{[2]_q}\right) \leq \frac{1}{\Upsilon - \alpha} \int_{\alpha}^{\Upsilon} \Phi(\omega) \ {}_{\alpha}d_q\omega \leq \frac{q\Phi(\alpha) + \Phi(\Upsilon)}{[2]_q}, \tag{1}$$

where $[2]_q = 1 + q$.

In 2020, S. Bermudo et al. [5] established the following q-H-H inequality applicable to convex functions.

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Theorem 2 ([5]). *Let $\Phi : [\alpha, \Upsilon] \to \mathbb{R}$ be a convex differentiable function defined on $[\alpha, \Upsilon]$ with $0 < q < 1$. The following inequalities are established:*

$$\Phi\left(\frac{\alpha + q\Upsilon}{[2]_q}\right) \leq \frac{1}{\Upsilon - \alpha}\int_{\alpha}^{\Upsilon} \Phi(\omega)\ {}^{\Upsilon}d_q\omega \leq \frac{\Phi(\alpha) + q\Phi(\Upsilon)}{[2]_q}. \tag{2}$$

By combining the inequalities (1) and (2), the result is presented in [5] as follows:

$$\Phi\left(\frac{\alpha + \Upsilon}{2}\right) \leq \frac{1}{2(\Upsilon - \alpha)}\left(\int_{\alpha}^{\Upsilon} \Phi(\omega)\ {}_{\alpha}d_q\omega + \int_{\alpha}^{\Upsilon} \Phi(\omega)\ {}^{\Upsilon}d_q\omega\right) \leq \frac{\Phi(\alpha) + \Phi(\Upsilon)}{2}. \tag{3}$$

In 2023, M.A. Ali et al. [1] and T. Sitthiwirattham et al. [23] formulated the following q-H-H-type inequalities.

Theorem 3 ([1, 23]). *Let $\Phi : [\alpha, \Upsilon] \to \mathbb{R}$ be a convex function. The following inequalities are valid:*

$$\Phi\left(\frac{\alpha + \Upsilon}{2}\right) \leq \frac{1}{\Upsilon - \alpha}\left(\int_{\alpha}^{\frac{(\alpha+\Upsilon)}{2}} \Phi(\omega)\ {}_{\frac{(\alpha+\Upsilon)}{2}}d_q\omega + \int_{\frac{(\alpha+\Upsilon)}{2}}^{\Upsilon} \Phi(\omega)\ {}^{\frac{(\alpha+\Upsilon)}{2}}d_q\omega\right) \leq \frac{\Phi(\alpha) + \Phi(\Upsilon)}{2} \tag{4}$$

and

$$\Phi\left(\frac{\alpha + \Upsilon}{2}\right) \leq \frac{1}{\Upsilon - \alpha}\left(\int_{\alpha}^{\frac{(\alpha+\Upsilon)}{2}} \Phi(\omega)\ {}_{\alpha}d_q\omega + \int_{\frac{(\alpha+\Upsilon)}{2}}^{\Upsilon} \Phi(\omega)\ {}^{\Upsilon}d_q\omega\right) \leq \frac{\Phi(\alpha) + \Phi(\Upsilon)}{2}. \tag{5}$$

The H-H inequality has garnered significant attention in the literature, particularly concerning various notions of convexity and extensions and refinements. For further details, interested readers may refer to [6–8] and the references therein.

In 1966, B.T. Polyak [19] introduced the principle of strongly convex functions. This idea is significant in mathematical programming and the application of mathematical models. It has been widely utilized in various studies, such as those mentioned in [4, 17, 18, 21, 22].

Definition 1 ([19]). *Let $\Phi : I \to \mathbb{R}$ be a function defined on an interval I. We say that Φ is a strongly convex function with modulus $\varphi > 0$ if*

$$\Phi(\varpi\omega + (1 - \varpi)\zeta) \leq \varpi\Phi(\omega) + (1 - \varpi)\Phi(\zeta) - \varphi\varpi(1 - \varpi)(\omega - \zeta)^2 \tag{6}$$

for all $\omega, \zeta \in I$ and $\varpi \in [0, 1]$.

Calculus is a vital branch of mathematics focused on the analysis of functions and their continuous changes. In the 17th century, significant advancements in calculus were made by I. Newton and G.W. Leibniz, laying the groundwork for its modern development. Subsequently, L. Euler (1707–1783) introduced the concept of quantum calculus (q-calculus), which operates without the traditional notion of limits and forges a link between mathematics and physics. In the 20th century, F.H. Jackson [14, 15] further expanded Euler's ideas by formalizing the principles of q-calculus. Later, in 2000, V.G. Kac and P. Cheung [16] provided a comprehensive overview of the foundational concepts of q-calculus in their publication; additional insights can be found in [10, 11].

In [24, 25], J. Tariboon and S.K. Ntouyas (2013, 2014) introduced the q-calculus for continuous functions defined on finite intervals and examined some of its properties, referred to as q_a-calculus.

Inspired by the earlier discussion of q-calculus, this study develops q-H-H inequalities for strongly convex functions using the principles of q-calculus. It refines existing findings by utilizing the characterization of strong convexity of Φ. Furthermore, the q-midpoint and q-trapezoidal inequalities are unified into one. The newly established inequalities also demonstrate applications to special means.

2. Preliminaries

In this section, we will review the definitions and key properties related to the concept of q-calculus that are pertinent to this study. For the purposes of this paper, we will consider $\alpha < \Upsilon$ and $0 < q < 1$ as fixed parameters.

$$[\eta]_q := \frac{1-q^\eta}{1-q} = 1 + q + q^2 + \cdots + q^{\eta-1}, \quad \eta \in \mathbb{N}.$$

This represents the q-analogue of η; for further information, please refer to [16].

Definition 2 ([16, 24]). *Let $\Phi : [\alpha, \Upsilon] \to \mathbb{R}$ be a continuous function. The q_α-derivative of Φ at $\omega \in [\alpha, \Upsilon]$ is, consequently, defined by the following expression*

$$_\alpha D_q \Phi(\omega) = \frac{\Phi(\omega) - \Phi(q\omega + (1-q)\alpha)}{(1-q)(\omega - \alpha)}, \quad \omega \neq \alpha. \tag{7}$$

If $\omega = \alpha$, we define

$$_\alpha D_q \Phi(\alpha) = \lim_{\omega \to a} {_\alpha D_q} \Phi(\omega),$$

whether it exists and is finite. If we take $\alpha = 0$ in (7), then we have $_0D_q\Phi(\omega) = D_q\Phi(\omega)$, which can be simplified to

$$D_q \Phi(\omega) = \frac{\Phi(\omega) - \Phi(q\omega)}{(1-q)\omega}, \quad \omega \neq 0.$$

This represents the q-Jackson derivative.

Definition 3 ([5]). *Let $\Phi : [\alpha, \Upsilon] \to \mathbb{R}$ be a continuous function. Then, the q^Υ-derivative of Φ at $\omega \in [\alpha, b]$ is defined by*

$$^\Upsilon D_q \Phi(\omega) = \frac{\Phi(q\omega + (1-q)\Upsilon) - \Phi(\omega)}{(1-q)(\Upsilon - \omega)}, \quad \omega \neq \Upsilon. \tag{8}$$

If $\omega = \Upsilon$, we define

$$^{\Upsilon}D_q\Phi(b) = \lim_{\omega\to\Upsilon} {}^{\Upsilon}D_q\Phi(\omega),$$

whether it exists and is finite.

Definition 4 ([20]). *Let $\Phi : [\alpha, \Upsilon] \to \mathbb{R}$ be a continuous function and $\omega \in [\alpha, \Upsilon]$. The q_α-integral of Φ on $[\alpha, \omega]$ is defined by*

$$\int_\alpha^\omega \Phi(\varpi)\ {}_\alpha d_q\varpi = (1-q)(\omega-\alpha)\sum_{n=0}^{\infty} q^n \Phi\left(q^n\omega + (1-q^n)\alpha\right). \tag{9}$$

Note that

$$\int_\alpha^\omega \Phi(\varpi)\ {}_\alpha d_q\varpi = (\omega-\alpha)\int_0^1 \Phi\left((1-\varpi)\alpha + \varpi\omega\right)\ {}_\alpha d_q\varpi,$$

and if $\alpha = 0$, then (9) reduces to

$$\int_0^\omega \Phi(\varpi)\ {}_0 d_q\varpi = \int_0^\omega \Phi(\varpi) d_q\varpi = (1-q)\omega\sum_{n=0}^{\infty} q^n\Phi(q^n\omega).$$

This represents the q_α-integral, for further details, refer to [16, 24].

Definition 5 ([5]). *Let $\Phi : [\alpha, \Upsilon] \to \mathbb{R}$ be a continuous function and $\omega \in [\alpha, \Upsilon]$. The q^Υ-integral of Φ on $[\omega, \Upsilon]$ is defined by*

$$\int_\omega^\Upsilon \Phi(\varpi)\ {}^{\Upsilon}d_q\varpi \ = \ (1-q)(\Upsilon-\omega)\sum_{n=0}^{\infty} q^n \Phi\left(q^n\omega + (1-q^n)\Upsilon\right). \tag{10}$$

Note that

$$\int_\omega^\Upsilon \Phi(\varpi)\ {}^{\Upsilon}d_q\varpi = (\Upsilon-\omega)\int_0^1 \Phi\left(\varpi\Upsilon + (1-\varpi)\omega\right)\ {}^{1}d_q\varpi,$$

and if $\omega = 0$, then (10) reduces to

$$\int_0^\Upsilon \Phi(\varpi)\ {}^{\Upsilon}d_q\varpi = (1-q)\Upsilon\sum_{n=0}^{\infty} q^n\Phi((1-q^n)\Upsilon).$$

This represents the q^Υ-integral.

3. Refinements of q-H-H-type inequalities

We begin by refining Theorem 3.1 in N. Alp et al. [2] using the characterization of strong convexity.

Theorem 4. *Let $\Phi : [\alpha, \Upsilon] \to \mathbb{R}$ be a strongly convex function for $\varphi > 0$. Then the following inequalities are established:*

$$\begin{aligned}\Phi\left(\frac{\alpha+\Upsilon}{2}\right) &\leq \Phi\left(\frac{\alpha+\Upsilon}{2}\right)+\frac{\varphi(\Upsilon-\alpha)^2}{4}\left(\frac{4\Theta^2}{[3]_q}-\frac{4\Theta}{[2]_q}+1\right)\\ &\leq \frac{1}{2\Theta(\Upsilon-\alpha)}\left(\int_{\alpha}^{\Theta\Upsilon+(1-\Theta)\alpha}\Phi(\varpi)_{\alpha}d_q\varpi+\int_{\Theta\alpha+(1-\Theta)\Upsilon}^{b}\Phi(\varpi)\ {}^{\Upsilon}d_q\varpi\right)\\ &\leq \frac{\Phi(\alpha)+\Phi(\Upsilon)}{2}-\varphi(\Upsilon-\alpha)^2\left(\frac{\Theta}{[2]_q}-\frac{\Theta^2}{[3]_q}\right)\\ &\leq \frac{\Phi(\alpha)+\Phi(\Upsilon)}{2}\end{aligned} \tag{11}$$

for all $\Theta \in (0, 1]$.

Proof. It follows from strong convexity of Φ on $[\alpha, \Upsilon]$ that

$$\Phi(\varpi s+(1-\varpi)s_1) \leq \varpi\Phi(s)+(1-\varpi)\Phi(s_1)-\varphi\varpi(1-\varpi)(s-s_1)^2, \tag{12}$$

for all $s, s_1 \in [\alpha, \Upsilon]$ and $\varpi \in [0, 1]$. Substituting $\varpi = 1/2$ into the inequality (12), then

$$\Phi\left(\frac{s+s_1}{2}\right) \leq \frac{\Phi(s)+\Phi(s_1)}{2}-\frac{\varphi(s-s_1)^2}{4}. \tag{13}$$

Let $\Theta \in (0, 1]$ and putting $s = \Theta\varpi\Upsilon + (1 - \Theta\varpi)\alpha$ and $s_1 = \Theta\varpi\alpha + (1 - \Theta\varpi)\Upsilon$ in the inequality (13), we have

$$\Phi\left(\frac{\alpha+\Upsilon}{2}\right) \leq \frac{\Phi(\Theta\varpi\Upsilon+(1-\Theta\varpi)\alpha)+\Phi(\Theta\varpi\alpha+(1-\Theta\varpi)\Upsilon)}{2}-\frac{\varphi(\Upsilon-\alpha)^2\left(2\Theta\varpi-1\right)^2}{4}. \tag{14}$$

It follows from the definitions 4 and 5, and by applying q-integration to both sides of the inequality (14), we find that

$$\begin{aligned}&\Phi\left(\frac{\alpha+\Upsilon}{2}\right)\\ &\leq \int_0^1\left(\frac{\Phi(\Theta\varpi\Upsilon+(1-\Theta\varpi)\alpha)+\Phi(\Theta\varpi\alpha+(1-\Theta\varpi)\Upsilon)}{2}-\frac{\varphi(\Upsilon-\alpha)^2\left(2\Theta\varpi-1\right)^2}{4}\right)d_q\varpi\\ &= \int_0^1\frac{\Phi(\Theta\varpi\Upsilon+(1-\Theta\varpi)\alpha)+\Phi(\Theta\varpi\alpha+(1-\Theta\varpi)\Upsilon)}{2}d_q\varpi\\ &\quad -\int_0^1\frac{\varphi(\Upsilon-\alpha)^2\left(4\Theta^2\varpi^2-4\Theta\varpi+1\right)}{4}d_q\varpi\\ &= (1-q)\sum_{n=0}^{\infty}q^n\frac{\Phi(\Theta q^n\Upsilon+(1-\Theta q^n)\alpha)+\Phi(\Theta q^n\alpha+(1-\Theta q^n)b)}{2}\end{aligned}$$

$$
\begin{aligned}
&- \frac{\varphi(\Upsilon-\alpha)^2}{4}\left(\frac{4\Theta^2}{[3]_q}-\frac{4\Theta}{[2]_q}+1\right) \\
&= \frac{\Theta(\Upsilon-\alpha)(1-q)}{2\Theta(\Upsilon-\alpha)}\left(\sum_{n=0}^{\infty} q^n \Phi\Big((\Theta\Upsilon+(1-\Theta)\alpha)q^n+(1-q^n)\alpha\Big)\right. \\
&\left.+\sum_{n=0}^{\infty} q^n \Phi\Big((\Theta\alpha+(1-\Theta)\Upsilon)q^n+(1-q^n)\Upsilon\Big)\right) - \frac{\varphi(\Upsilon-\alpha)^2}{4}\left(\frac{4\Theta^2}{[3]_q}\right. \\
&\left.-\frac{4\Theta}{[2]_q}+1\right) \\
&= \frac{1}{2\Theta(\Upsilon-\alpha)}\left(\int_{\alpha}^{\Theta\Upsilon+(1-\Theta)\alpha} \Phi(\varpi)\,{}_{\alpha}d_q\varpi + \int_{\Theta\alpha+(1-\Theta)\Upsilon}^{\Upsilon} \Phi(\varpi)\,{}^{\Upsilon}d_q\varpi\right) \\
&- \frac{\varphi(\Upsilon-\alpha)^2}{4}\left(\frac{4\Theta^2}{[3]_q}-\frac{4\Theta}{[2]_q}+1\right).
\end{aligned} \tag{15}
$$

Note that $4\Theta^2/[3]_q - 4\Theta/[2]_q + 1 \geq 0$. This implies that

$$
\begin{aligned}
\Phi\left(\frac{\alpha+\Upsilon}{2}\right) &\leq \Phi\left(\frac{\alpha+\Upsilon}{2}\right) + \frac{\varphi(\Upsilon-\alpha)^2}{4}\left(\frac{4\Theta^2}{[3]_q}-\frac{4\Theta}{[2]_q}+1\right) \\
&\leq \frac{1}{2\Theta(\Upsilon-\alpha)}\left(\int_{\alpha}^{\Theta\Upsilon+(1-\Theta)\alpha} \Phi(\varpi)\,{}_{\alpha}d_q\varpi + \int_{\Theta\alpha+(1-\Theta)\Upsilon}^{\Upsilon} \Phi(\varpi)\,{}^{\Upsilon}d_q\varpi\right).
\end{aligned}
$$

This gives the first and second inequalities of (11). From the inequality (14) and by strong convexity of Φ, we obtain

$$
\begin{aligned}
&\frac{\Phi(\Theta\varpi\Upsilon+(1-\Theta\varpi)\alpha)+\Phi(\Theta\varpi\alpha+(1-\Theta\varpi)\Upsilon)}{2} - \frac{\varphi(\Upsilon-\alpha)^2\,(2\Theta\varpi-1)^2}{4} \\
&\leq \frac{\Theta\varpi\Phi(\alpha)+(1-\Theta\varpi)\Phi(\Upsilon)-\varphi\Theta\varpi(1-\Theta\varpi)(\Upsilon-\alpha)^2}{2} \\
&\quad + \frac{\Theta\varpi\Phi(\Upsilon)+(1-\Theta\varpi)\Phi(\alpha)-\varphi\Theta\varpi(1-\Theta\varpi)(\Upsilon-\alpha)^2}{2} - \frac{\varphi(\Upsilon-\alpha)^2\,(2\Theta\varpi-1)^2}{4} \\
&= \frac{\Phi(\alpha)+\Phi(\Upsilon)}{2} - \frac{\varphi(\Upsilon-\alpha)^2}{4}.
\end{aligned}
$$

Applying q-integration to both sides of the above inequality yields

$$
\begin{aligned}
&\int_0^1 \left(\frac{\Phi(\Theta\varpi\Upsilon+(1-\Theta\varpi)\alpha)+\Phi(\Theta\varpi\alpha+(1-\Theta\varpi)\Upsilon)}{2} - \frac{\varphi(\Upsilon-\alpha)^2\,(2\Theta\varpi-1)^2}{4}\right) d_q\varpi \\
&\leq \int_0^1 \left(\frac{\Phi(\alpha)+\Phi(\Upsilon)}{2} - \varphi\Theta\varpi(1-\Theta\varpi)(\Upsilon-\alpha)^2 - \frac{\varphi(\Upsilon-\alpha)^2\,(2\Theta\varpi-1)^2}{4}\right) d_q\varpi \\
&= \frac{\Phi(\alpha)+\Phi(\Upsilon)}{2} - \frac{\varphi(\Upsilon-\alpha)^2}{4}.
\end{aligned} \tag{16}
$$

From the inequalities (15) and (16), we obtain

$$\int_0^1 \left(\frac{\Phi(\Theta\varpi\Upsilon + (1-\Theta\varpi)\alpha) + \Phi(\Theta\varpi\alpha + (1-\Theta\varpi)b)}{2} - \frac{\varphi(\Upsilon-\alpha)^2\,(2\Theta\varpi-1)^2}{4} \right) d_q\varpi$$
$$= \frac{1}{2\Theta(\Upsilon-\alpha)} \left(\int_\alpha^{\Theta\Upsilon+(1-\Theta)\alpha} \Phi(\varpi)_\alpha d_q\varpi + \int_{\Theta\alpha+(1-\Theta)\Upsilon}^{\Upsilon} \Phi(\varpi)\ {}^{\Upsilon}d_q\varpi \right)$$
$$- \frac{\varphi(\Upsilon-\alpha)^2}{4} \left(\frac{4\Theta^2}{[3]_q} - \frac{4\Theta}{[2]_q} + 1 \right)$$
$$\leq \frac{\Phi(\alpha)+\Phi(\Upsilon)}{2} - \frac{\varphi(\Upsilon-\alpha)^2}{4}.$$

Note that $\Theta/[2]_q - \Theta^2/[3]_q \geq 0$. This implies that

$$\frac{1}{2\Theta(\Upsilon-\alpha)} \left(\int_\alpha^{\Theta\Upsilon+(1-\Theta)\alpha} \Phi(\varpi)_\alpha d_q\varpi + \int_{\Theta\alpha+(1-\Theta)\Upsilon}^{\Upsilon} \Phi(\varpi)\ {}^{\Upsilon}d_q\varpi \right)$$
$$\leq \frac{\Phi(\alpha)+\Phi(\Upsilon)}{2} - \varphi(\Upsilon-\alpha)^2 \left(\frac{\Theta}{[2]_q} - \frac{\Theta^2}{[3]_q} \right)$$
$$\leq \frac{\Phi(\alpha)+\Phi(\Upsilon)}{2}.$$

This finalizes the proof.

Remark 1. *Setting $\Theta = 1$, in Theorem 4, then we obtain*

$$\Phi\left(\frac{\alpha+\Upsilon}{2}\right) \leq \Phi\left(\frac{\alpha+\Upsilon}{2}\right) + \frac{\varphi(\Upsilon-\alpha)^2}{4} \left(\frac{q^3-2q^2+2q-1}{[2]_q[3]_q} \right)$$
$$\leq \frac{1}{2(\Upsilon-\alpha)} \left(\int_\alpha^{\Upsilon} \Phi(\varpi)_\alpha d_q\varpi + \int_\alpha^{\Upsilon} \Phi(\varpi)\ {}^{\Upsilon}d_q\varpi \right)$$
$$\leq \frac{\Phi(\alpha)+\Phi(\Upsilon)}{2} - \frac{\varphi q^2(\Upsilon-\alpha)^2}{[2]_q[3]_q}$$
$$\leq \frac{\Phi(\alpha)+\Phi(\Upsilon)}{2},$$

which appeared in [21].

Remark 2. *Setting $\Theta = 1/[2]_q$, in Theorem 4, leads us to the new q-H-H inequalities:*

$$\Phi\left(\frac{\alpha+\Upsilon}{2}\right) \leq \Phi\left(\frac{\alpha+\Upsilon}{2}\right) + \frac{\varphi(\Upsilon-\alpha)^2}{4([2]_q)^2[3]_q} \left(4 - 4[3]_q + ([2]_q)^2[3]_q\right)$$
$$\leq \frac{[2]_q}{2(\Upsilon-\alpha)} \left(\int_\alpha^{\frac{(q\alpha+\Upsilon)}{[2]_q}} \Phi(\varpi)_\alpha d_q\varpi + \int_{\frac{(\alpha+q\Upsilon)}{[2]_q}}^{\Upsilon} \Phi(\varpi)\ {}^{\Upsilon}d_q\varpi \right)$$

$$\leq \frac{\Phi(\alpha)+\Phi(\Upsilon)}{2} - \frac{\varphi(\Upsilon-\alpha)^2}{([2]_q)^2[3]_q}([3]_q-1)$$
$$\leq \frac{\Phi(\alpha)+\Phi(\Upsilon)}{2}.$$

Next, we refine Theorem 3.5 from N. Alp et al. [2] by leveraging the properties associated with strong convexity

Theorem 5. *Let $\Phi : [\alpha, \Upsilon] \to \mathbb{R}$ be a strongly convex function for $\varphi > 0$ and $\Theta \in (0, 1]$. Then the following inequalities are established:*

$$\begin{aligned}\Phi\left(\frac{\alpha+\Upsilon}{2}\right) &\leq \Phi\left(\frac{\alpha+\Upsilon}{2}\right) + \frac{\varphi}{4}\left(\frac{(1-q)(\Upsilon-\alpha)}{[2]_q}\right)^2 \\ &\leq \frac{1}{2}\left(\Phi\left(\frac{\alpha+q\Upsilon}{[2]_q}\right) + \Phi\left(\frac{q\alpha+\Upsilon}{[2]_q}\right)\right) \\ &\leq \frac{1+q^2}{2q\Theta[2]_q(\Upsilon-\alpha)}\left(\int_{\alpha}^{\Theta\Upsilon+(1-\Theta)\alpha}\Phi(\varpi)_{\alpha}d_q\varpi + \int_{\Theta\alpha+(1-\Theta)\Upsilon}^{\Upsilon}\Phi(\varpi)\ {}^{\Upsilon}d_q\varpi\right) \\ &\quad - \frac{1-q}{2q[2]_q}\left(\Phi(\Theta\alpha+(1-\Theta)\Upsilon) + \Phi(\Theta\Upsilon+(1-\Theta)\alpha)\right) \\ &\leq \frac{\Phi(\alpha)+\Phi(\Upsilon)}{2} - \frac{\varphi q\Theta(\Upsilon-a)^2}{[2]_q}\left(\frac{2}{[2]_q} - \frac{q\Theta}{[3]_q} - \frac{\Theta}{[3]_q}\right) \\ &\leq \frac{\Phi(\alpha)+\Phi(\Upsilon)}{2}. \end{aligned} \tag{17}$$

Proof. It follows from strong convexity of Φ on $[\alpha, \Upsilon]$ that

$$\begin{aligned}\Phi\left(\frac{\alpha+\Upsilon}{2}\right) &= \Phi\left(\frac{\alpha+q\Upsilon+q\alpha+\Upsilon}{2[2]_q}\right) \\ &\leq \frac{1}{2}\left(\Phi\left(\frac{\alpha+q\Upsilon}{[2]_q}\right) + \Phi\left(\frac{q\alpha+\Upsilon}{[2]_q}\right)\right) - \frac{\varphi}{4}\left(\frac{(1-q)(\Upsilon-\alpha)}{[2]_q}\right)^2,\end{aligned}$$

which implies that

$$\begin{aligned}\Phi\left(\frac{\alpha+\Upsilon}{2}\right) &\leq \Phi\left(\frac{\alpha+\Upsilon}{2}\right) + \frac{\varphi}{4}\left(\frac{(1-q)(\Upsilon-\alpha)}{[2]_q}\right)^2 \\ &\leq \frac{1}{2}\left(\Phi\left(\frac{\alpha+q\Upsilon}{[2]_q}\right) + \Phi\left(\frac{q\alpha+\Upsilon}{[2]_q}\right)\right).\end{aligned}$$

This proves the first and second inequalities of (17)*. By strong convexity of Φ, and $\varpi \in [0, 1]$ and $\Theta \in (0, 1]$, we can write*

$$\Phi\left(\frac{\alpha+q\Upsilon}{[2]_q}\right) = \Phi\left(\frac{(q\Theta\varpi\Upsilon+(1-q\Theta\varpi)\alpha)+q(\Theta\varpi\alpha+(1-\Theta\varpi)\Upsilon)}{[2]_q}\right)$$

$$
\begin{aligned}
&= \Phi\left(\frac{(q\Theta\varpi\Upsilon + (1-q\Theta\varpi)\alpha)}{[2]_q} + \frac{q(\Theta\varpi\alpha + (1-\Theta\varpi)\Upsilon)}{[2]_q}\right)\\
&\leq \frac{\Phi(q\Theta\varpi\Upsilon + (1-q\Theta\varpi)\alpha)}{[2]_q} + \frac{q\Phi(\Theta\varpi\alpha + (1-\Theta\varpi)b)}{[2]_q}\\
&\leq \frac{q\Theta\varpi\Phi(\Upsilon) + (1-q\Theta\varpi)\Phi(\alpha) - \varphi q\Theta\varpi(1-q\Theta\varpi)(\Upsilon-a)^2}{[2]_q}\\
&\quad + \frac{q\left(\Theta\varpi\Phi(\alpha) + (1-\Theta\varpi)\Phi(\Upsilon) - \varphi\Theta\varpi(1-\Theta\varpi)(\Upsilon-\alpha)^2\right)}{[2]_q}\\
&= \frac{\Phi(\alpha) + q\Phi(\Upsilon)}{[2]_q} - \frac{\varphi q\Theta(\Upsilon-\alpha)^2}{[2]_q}(2\varpi - q\Theta\varpi^2 - \Theta\varpi^2),
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\Phi\left(\frac{\alpha + q\Upsilon}{[2]_q}\right) &\leq \frac{\Phi(q\Theta\varpi\Upsilon + (1-q\Theta\varpi)\alpha)}{[2]_q} + \frac{q\Phi(\Theta\varpi\alpha + (1-\Theta\varpi)\Upsilon)}{[2]_q}\\
&\leq \frac{\Phi(\alpha) + q\Phi(\Upsilon)}{[2]_q} - \frac{\varphi q\Theta(\Upsilon-\alpha)^2}{[2]_q}(2\varpi - q\Theta\varpi^2 - \Theta\varpi^2).
\end{aligned}
$$

By performing q-integration on both sides of the inequality above, we derive

$$
\begin{aligned}
\Phi\left(\frac{\alpha + q\Upsilon}{[2]_q}\right) &\leq \int_0^1 \left(\frac{\Phi(q\Theta\varpi\Upsilon + (1-q\Theta\varpi)\alpha)}{[2]_q} + \frac{q\Phi(\Theta\varpi\alpha + (1-\Theta\varpi)b)}{[2]_q}\right) d_q\varpi \qquad (18)\\
&\leq \int_0^1 \left(\frac{\Phi(\alpha) + q\Phi(\Upsilon)}{[2]_q} - \frac{\varphi q\Theta(\Upsilon-\alpha)^2}{[2]_q}(2\varpi - q\Theta\varpi^2 - \Theta\varpi^2)\right) d_q\varpi\\
&= \frac{\Phi(\alpha) + q\Phi(\Upsilon)}{[2]_q} - \frac{\varphi q\Theta(\Upsilon-\alpha)^2}{[2]_q}\left(\frac{2}{[2]_q} - \frac{q\Theta}{[3]_q} - \frac{\Theta}{[3]_q}\right).
\end{aligned}
$$

The inequality of (18) *can be computed as*

$$
\begin{aligned}
\Phi\left(\frac{\alpha + q\Upsilon}{[2]_q}\right) &\leq \int_0^1 \left(\frac{\Phi(q\Theta\varpi\Upsilon + (1-q\Theta\varpi)\alpha)}{[2]_q} + \frac{q\Phi(\Theta\varpi\alpha + (1-\Theta\varpi)\Upsilon)}{[2]_q}\right) d_q\varpi\\
&= (1-q)\sum_{n=0}^{\infty} q^n \frac{\Phi(\Theta q^{n+1}\Upsilon + (1-q^{n+1}\Theta)\alpha) + q\Phi(q^n\Theta\alpha + (1-\Theta q^n)\Upsilon)}{[2]_q}\\
&= \frac{1}{\Theta[2]_q(\Upsilon-\alpha)}\left(\frac{1}{q}\int_{\alpha}^{\Theta\Upsilon+(1-\Theta)\alpha} \Phi(\varpi)_\alpha d_q\varpi + q\int_{\Theta\alpha+(1-\Theta)\Upsilon}^{\Upsilon} \Phi(\varpi)\ {}^{\Upsilon}d_q\varpi\right)\\
&\quad - \frac{(1-q)}{q[2]_q}\Phi(\Theta\Upsilon + (1-\Theta)\alpha).
\end{aligned}
$$

So, we can write

$$
\Phi\left(\frac{\alpha + q\Upsilon}{[2]_q}\right) \leq \frac{1}{\Theta[2]_q(\Upsilon-\alpha)}\left(\frac{1}{q}\int_{\alpha}^{\Theta\Upsilon+(1-\Theta)a} \Phi(\varpi)_\alpha d_q\varpi\right.
$$

$$+q\int_{\Theta\alpha+(1-\Theta)\Upsilon}^{\Upsilon}\Phi(\varpi)\ {}^{\Upsilon}d_q\varpi\Bigg)-\frac{(1-q)}{q[2]_q}f(\Theta\Upsilon+(1-\Theta)\alpha)$$

$$\leq\frac{\Phi(\alpha)+q\Phi(\Upsilon)}{[2]_q}-\frac{\varphi q\Theta(\Upsilon-\alpha)^2}{[2]_q}\left(\frac{2}{[2]_q}-\frac{q\Theta}{[3]_q}-\frac{\Theta}{[3]_q}\right). \tag{19}$$

Alternatively, by employing a comparable technique, we can formulate

$$\Phi\left(\frac{q\alpha+\Upsilon}{[2]_q}\right)\leq\left(\frac{q\Phi(\Theta\varpi\Upsilon+(1-\Theta\varpi)\alpha)+\Phi(q\Theta\varpi\alpha+(1-q\Theta\varpi)\Upsilon)}{[2]_q}\right)$$

$$\leq\frac{q\Phi(\alpha)+\Phi(\Upsilon)}{[2]_q}-\frac{\varphi q\Theta(\Upsilon-\alpha)^2}{[2]_q}(2\varpi-q\Theta\varpi^2-\Theta\varpi^2).$$

The process of q-integrating both sides of the inequality above yields

$$\Phi\left(\frac{q\alpha+\Upsilon}{[2]_q}\right)\leq\frac{1}{\Theta[2]_q(\Upsilon-\alpha)}\Bigg(q\int_{\alpha}^{\Theta\Upsilon+(1-\Theta)\alpha}\Phi(\varpi)_\alpha d_q\varpi$$

$$+\frac{1}{q}\int_{\Theta\alpha+(1-\Theta)\Upsilon}^{\Upsilon}\Phi(\varpi)\ {}^{\Upsilon}d_q\varpi\Bigg)-\frac{(1-q)}{q[2]_q}\Phi(\Theta\alpha+(1-\Theta)\Upsilon)$$

$$\leq\frac{q\Phi(\alpha)+\Phi(\Upsilon)}{[2]_q}-\frac{\varphi q\Theta(\Upsilon-\alpha)^2}{[2]_q}\left(\frac{2}{[2]_q}-\frac{q\Theta}{[3]_q}-\frac{\Theta}{[3]_q}\right). \tag{20}$$

Note that $2/[2]_q-q\Theta/[3]_q-\Theta/[3]_q\geq 0$. *By combinating the inequalities* (19) *and* (20), *we obtain*

$$\Phi\left(\frac{\alpha+q\Upsilon}{[2]_q}\right)+\Phi\left(\frac{q\alpha+\Upsilon}{[2]_q}\right)$$

$$\leq\frac{1}{\Theta[2]_q(\Upsilon-\alpha)}\left((\frac{1}{q}+q)\int_{\alpha}^{\Theta\Upsilon+(1-\Theta)\alpha}\Phi(\varpi)_\alpha d_q\varpi+(\frac{1}{q}+q)\int_{\Theta\alpha+(1-\Theta)\Upsilon}^{\Upsilon}\Phi(\varpi)\ {}^{\Upsilon}d_q\varpi\right)$$

$$-\frac{(1-q)}{q[2]_q}\left(\Phi(\Theta\Upsilon+(1-\Theta)\alpha)+\Phi(\Theta\alpha+(1-\Theta)\Upsilon)\right)$$

$$\leq\Phi(\alpha)+\Phi(\Upsilon)-\frac{2\varphi q\Theta(\Upsilon-\alpha)^2}{[2]_q}\left(\frac{2}{[2]_q}-\frac{q\Theta}{[3]_q}-\frac{\Theta}{[3]_q}\right)$$

$$\leq\Phi(\alpha)+\Phi(\Upsilon).$$

Multiplying the inequality above by $1/2$ *results in the third, fourth, and fifth inequalities of* (17).

The following result gives the refinements of inequalities (5).

Corollary 1. *Setting* $\Theta=1/2$ *in Theorem 5, then we obtain*

$$\Phi\left(\frac{\alpha+\Upsilon}{2}\right)\leq\Phi_1(q)\leq\Phi_2(q)$$

$$\le \frac{1}{2(\Upsilon-\alpha)}\left(\int_{\alpha}^{\frac{\alpha+\Upsilon}{2}}\Phi(\varpi)_{\alpha}d_q\varpi+\int_{\frac{\alpha+\Upsilon}{2}}^{\Upsilon}\Phi(\varpi)\ ^{\Upsilon}d_q\varpi\right)$$
$$\le \Phi_3(q)\le \frac{\Phi(\alpha)+\Phi(\Upsilon)}{2}, \tag{21}$$

where

$$\Phi_1(q)=\frac{q[2]_q}{(1+q^2)}\left(\Phi\left(\frac{\alpha+\Upsilon}{2}\right)+\frac{\varphi}{4}\left(\frac{(1-q)(\Upsilon-\alpha)}{[2]_q}\right)^2\right)+\frac{(1-q)}{(1+q^2)}\Phi\left(\frac{\alpha+\Upsilon}{2}\right),$$

$$\Phi_2(q)=\frac{q[2]_q}{2(1+q^2)}\left(\Phi\left(\frac{\alpha+q\Upsilon}{[2]_q}\right)+\Phi\left(\frac{q\alpha+\Upsilon}{[2]_q}\right)\right)+\frac{(1-q)}{(1+q^2)}\Phi\left(\frac{\alpha+\Upsilon}{2}\right)$$

and

$$\Phi_3(q)=\frac{q[2]_q}{(1+q^2)}\left(\frac{\Phi(\alpha)+\Phi(\Upsilon)}{2}-\frac{\varphi q(\Upsilon-\alpha)^2}{4([2]_q)^2[3]_q}\left(3+2q+3q^2\right)\right)+\frac{(1-q)}{(1+q^2)}\Phi\left(\frac{\alpha+\Upsilon}{2}\right).$$

Proof. Setting $\Theta=1/2$ *in Theorem 5, gives us*

$$\Phi\left(\frac{\alpha+\Upsilon}{2}\right)\le \Phi\left(\frac{\alpha+\Upsilon}{2}\right)+\frac{\varphi}{4}\left(\frac{(1-q)(\Upsilon-\alpha)}{[2]_q}\right)^2$$
$$\le \frac{1}{2}\left(\Phi\left(\frac{\alpha+q\Upsilon}{[2]_q}\right)+\Phi\left(\frac{q\alpha+\Upsilon}{[2]_q}\right)\right)$$
$$\le \frac{1+q^2}{q[2]_q(\Upsilon-\alpha)}\left(\int_{\alpha}^{\frac{\alpha+\Upsilon}{2}}\Phi(\varpi)_{\alpha}d_q\varpi+\int_{\frac{\alpha+\Upsilon}{2}}^{\Upsilon}\Phi(\varpi)\ ^{\Upsilon}d_q\varpi\right)$$
$$-\frac{1-q}{q[2]_q}\Phi\left(\frac{\alpha+\Upsilon}{2}\right)$$
$$\le \frac{\Phi(\alpha)+\Phi(\Upsilon)}{2}-\frac{\varphi q(\Upsilon-a)^2}{4([2]_q)^2[3]_q}\left(3+2q+3q^2\right)$$
$$\le \frac{\Phi(\alpha)+\Phi(\Upsilon)}{2}.$$

This implies that

$$\left(\frac{1+q^2}{q[2]_q}\right)\Phi\left(\frac{\alpha+\Upsilon}{2}\right)$$
$$=\Phi\left(\frac{\alpha+\Upsilon}{2}\right)+\frac{(1-q)}{q[2]_q}\Phi\left(\frac{\alpha+\Upsilon}{2}\right)$$
$$\le \Phi\left(\frac{\alpha+\Upsilon}{2}\right)+\frac{(1-q)}{q[2]_q}\Phi\left(\frac{a+\Upsilon}{2}\right)+\frac{\varphi}{4}\left(\frac{(1-q)(\Upsilon-\alpha)}{[2]_q}\right)^2$$
$$\le \frac{1}{2}\left(\Phi\left(\frac{\alpha+q\Upsilon}{[2]_q}\right)+\Phi\left(\frac{q\alpha+\Upsilon}{[2]_q}\right)\right)+\frac{(1-q)}{q[2]_q}\Phi\left(\frac{\alpha+\Upsilon}{2}\right)$$

$$
\begin{aligned}
&\leq \frac{1+q^2}{q[2]_q(\Upsilon-\alpha)}\left(\int_{\alpha}^{\frac{\alpha+\Upsilon}{2}}\Phi(\varpi)_{\alpha}d_q\varpi+\int_{\frac{\alpha+\Upsilon}{2}}^{\Upsilon}\Phi(\varpi)\ ^{\Upsilon}d_q\varpi\right)-\frac{(1-q)}{q[2]_q}\Phi\left(\frac{\alpha+\Upsilon}{2}\right)\\
&+\frac{(1-q)}{q[2]_q}\Phi\left(\frac{\alpha+\Upsilon}{2}\right)\\
&\leq \frac{\Phi(\alpha)+\Phi(\Upsilon)}{2}-\frac{\varphi q(\Upsilon-\alpha)^2}{4([2]_q)^2[3]_q}\left(3+2q+3q^2\right)+\frac{(1-q)}{q[2]_q}\Phi\left(\frac{\alpha+b}{2}\right)\\
&\leq \frac{\Phi(\alpha)+\Phi(\Upsilon)}{2}+\frac{1-q}{q[2]_q}\left(\frac{\Phi(\alpha)+\Phi(\Upsilon)}{2}\right)\\
&=\frac{1+q^2}{q[2]_q}\left(\frac{\Phi(\alpha)+\Phi(b)}{2}\right). \qquad (22)
\end{aligned}
$$

Multiply the inequality (22) *by* $q[2]_q/(1+q^2)$ *leads us to the desired result.*

The result presented below provides refinements for the inequalities (3).

Corollary 2. *Setting* $\Theta = 1$ *in Theorem 5, then we obtain*

$$
\begin{aligned}
\Phi\left(\frac{\alpha+\Upsilon}{2}\right) &\leq \Phi_4(q)\leq\Phi_5(q)\\
&\leq \frac{1}{2(\Upsilon-\alpha)}\left(\int_{\alpha}^{\Upsilon}\Phi(\varpi)_{\alpha}d_q\varpi+\int_{\alpha}^{\Upsilon}\Phi(\varpi)\ ^{\Upsilon}d_q\varpi\right)\\
&\leq \Phi_6(q)\leq\frac{\Phi(\alpha)+\Phi(\Upsilon)}{2}, \qquad (23)
\end{aligned}
$$

where

$$
\Phi_4(q)=\frac{q[2]_q}{(1+q^2)}\left(\Phi\left(\frac{\alpha+\Upsilon}{2}\right)+\frac{\varphi}{4}\left(\frac{(1-q)(\Upsilon-\alpha)}{[2]_q}\right)^2\right)+\frac{(1-q)}{(1+q^2)}\Phi\left(\frac{\alpha+\Upsilon}{2}\right),
$$

$$
\Phi_5(q)=\frac{q[2]_q}{2(1+q^2)}\left(\Phi\left(\frac{\alpha+q\Upsilon}{[2]_q}\right)+\Phi\left(\frac{q\alpha+\Upsilon}{[2]_q}\right)\right)+\frac{(1-q)}{(1+q^2)}\Phi\left(\frac{\alpha+\Upsilon}{2}\right)
$$

and

$$
\Phi_6(q)=\frac{q[2]_q}{(1+q^2)}\left(\frac{\Phi(\alpha)+\Phi(\Upsilon)}{2}\right)-\frac{\varphi q^2(\Upsilon-\alpha)^2}{[2]_q[3]_q}+\frac{(1-q)}{(1+q^2)}\left(\frac{\Phi(\alpha)+\Phi(\Upsilon)}{2}\right).
$$

Proof. *Setting* $\Theta = 1$ *in Theorem 5 gives us*

$$
\begin{aligned}
\Phi\left(\frac{\alpha+\Upsilon}{2}\right) &\leq \Phi\left(\frac{\alpha+\Upsilon}{2}\right)+\frac{\varphi}{4}\left(\frac{(1-q)(\Upsilon-\alpha)}{[2]_q}\right)^2\\
&\leq \frac{1}{2}\left(\Phi\left(\frac{\alpha+q\Upsilon}{[2]_q}\right)+\Phi\left(\frac{q\alpha+\Upsilon}{[2]_q}\right)\right)
\end{aligned}
$$

$$\begin{aligned}
&\leq \frac{1+q^2}{2q[2]_q(\Upsilon-\alpha)}\left(\int_\alpha^\Upsilon \Phi(\varpi)_\alpha d_q\varpi + \int_\alpha^\Upsilon \Phi(\varpi)\ ^\Upsilon d_q\varpi\right) \\
&\quad - \frac{1-q}{2q[2]_q}\left(\Phi(\alpha)+\Phi(\Upsilon)\right) \\
&\leq \frac{\Phi(\alpha)+\Phi(\Upsilon)}{2} - \frac{\varphi q(1+q^2)(\Upsilon-\alpha)^2}{([2]_q)^2[3]_q} \\
&\leq \frac{\Phi(\alpha)+\Phi(\Upsilon)}{2}.
\end{aligned}$$

Which gives

$$\begin{aligned}
&\left(\frac{1+q^2}{q[2]_q}\right)\Phi\left(\frac{\alpha+\Upsilon}{2}\right) \\
&= \Phi\left(\frac{\alpha+\Upsilon}{2}\right) + \frac{(1-q)}{q[2]_q}\Phi\left(\frac{\alpha+\Upsilon}{2}\right) \\
&\leq \Phi\left(\frac{\alpha+\Upsilon}{2}\right) + \frac{\varphi}{4}\left(\frac{(1-q)(\Upsilon-\alpha)}{[2]_q}\right)^2 + \frac{(1-q)}{q[2]_q}\Phi\left(\frac{\alpha+\Upsilon}{2}\right) \\
&\leq \frac{1}{2}\left(\Phi\left(\frac{\alpha+q\Upsilon}{[2]_q}\right) + \Phi\left(\frac{q\alpha+\Upsilon}{[2]_q}\right)\right) + \frac{(1-q)}{q[2]_q}\Phi\left(\frac{\alpha+\Upsilon}{2}\right) \\
&\leq \frac{1+q^2}{2q[2]_q(\Upsilon-\alpha)}\left(\int_\alpha^\Upsilon f(\varpi)_\alpha d_q\varpi + \int_\alpha^\Upsilon \Phi(\varpi)\ ^\Upsilon d_q\varpi\right) - \frac{(1-q)}{q[2]_q}\left(\frac{\Phi(\alpha)+\Phi(\Upsilon)}{2}\right) \\
&\quad + \frac{(1-q)}{q[2]_q}\left(\frac{\Phi(\alpha)+\Phi(\Upsilon)}{2}\right) \\
&\leq \frac{\Phi(\alpha)+\Phi(\Upsilon)}{2} - \frac{\varphi q(1+q^2)(\Upsilon-\alpha)^2}{([2]_q)^2[3]_q} + \frac{(1-q)}{q[2]_q}\left(\frac{\Phi(\alpha)+\Phi(\Upsilon)}{2}\right) \\
&\leq \frac{\Phi(\alpha)+\Phi(\Upsilon)}{2} + \frac{(1-q)}{q[2]_q}\left(\frac{\Phi(\alpha)+\Phi(\Upsilon)}{2}\right) \\
&= \left(\frac{1+q^2}{q[2]_q}\right)\left(\frac{\Phi(\alpha)+\Phi(\Upsilon)}{2}\right). \qquad (24)
\end{aligned}$$

Multiplying the inequality (24) *by* $q[2]_q/(1+q^2)$ *yields the desired result.*

4. Parameterized q-integral inequalities

We prove the H-H inequalities (11) of Theorem 4 by utilizing the q-differentiability of the function.

Lemma 1 ([2]). *Let* $\Phi : [\alpha, \Upsilon] \to \mathbb{R}$ *be a* q*-differentiable function. If* ${}_\alpha D_q\Phi$ *and* ${}^\Upsilon D_q\Phi$ *are two continuous and integrable functions on* $[\alpha, \Upsilon]$*, then we have:*

$$\frac{\Theta(\Upsilon-\alpha)}{2}\int_0^1 q\varpi\left({}^\Upsilon D_q\Phi(\Theta\varpi\alpha+(1-\Theta\varpi)\Upsilon) - {}_\alpha D_q\Phi(\Theta\varpi\Upsilon+(1-\Theta\varpi)\alpha)\right) d_q\varpi$$

$$= \frac{1}{2\Theta(\Upsilon-\alpha)}\left(\int_{\Theta\alpha+(1-\Theta)\Upsilon}^{\Upsilon}\Phi(\varpi)\ {}^{\Upsilon}d_q\varpi+\int_{\alpha}^{\Theta\Upsilon+(1-\Theta)\alpha}\Phi(\varpi)_{\alpha}d_q\varpi\right)$$
$$-\frac{\Phi(\Theta\alpha+(1-\Theta)\Upsilon)+\Phi(\Theta\Upsilon+(1-\Theta)\alpha)}{2}. \tag{25}$$

Proof. By Definition 3, we have

$$\begin{aligned}
I_1 &:= \int_0^1 q\varpi\left(\,{}^{\Upsilon}D_q\Phi(\Theta\varpi\alpha+(1-\Theta\varpi)\Upsilon)\right)d_q\varpi \\
&= \int_0^1 q\varpi\frac{\Phi(q\Theta\varpi\alpha+(1-q\Theta\varpi)\Upsilon)-\Phi(\Theta\varpi\alpha+(1-\Theta\varpi)\Upsilon)}{(1-q)\Theta\varpi(\Upsilon-\alpha)}d_q\varpi \\
&= \frac{q}{(\Upsilon-\alpha)\Theta}\sum_{n=0}^{\infty}q^n\Phi(\Theta q^{n+1}\alpha+(1-\Theta q^{n+1})\Upsilon) \\
&\quad -\frac{q}{(\Upsilon-\alpha)\Theta}\sum_{n=0}^{\infty}q^n\Phi(\Theta q^{n}\alpha+(1-\Theta q^{n})\Upsilon) \\
&= \frac{1}{(\Upsilon-\alpha)\Theta}\sum_{n=0}^{\infty}q^n\Phi(\Theta q^{n}\alpha+(1-\Theta q^{n})\Upsilon)-\frac{1}{(\Upsilon-\alpha)\Theta}\Phi(\Theta\alpha+(1-\Theta)\Upsilon) \\
&\quad -\frac{q}{(\Upsilon-\alpha)\Theta}\sum_{n=0}^{\infty}q^n\Phi(\Theta q^{n}\alpha+(1-\Theta q^{n})\Upsilon) \\
&= \frac{1-q}{(\Upsilon-\alpha)\Theta}\sum_{n=0}^{\infty}q^n\Phi(\Theta q^{n}\alpha+(1-\Theta q^{n})\Upsilon)-\frac{1}{(\Upsilon-\alpha)\Theta}\Phi(\Theta\alpha+(1-\Theta)b) \\
&= \frac{1}{(\Upsilon-\alpha)^2\Theta^2}\int_{\Theta\alpha+(1-\Theta)\Upsilon}^{\Upsilon}\Phi(\varpi)\ {}^{\Upsilon}d_q\varpi-\frac{1}{(\Upsilon-\alpha)\Theta}\Phi(\Theta\alpha+(1-\Theta)\Upsilon).
\end{aligned}$$

Similarly, by Definition 2, we have

$$\begin{aligned}
I_2 &:= \int_0^1 q\varpi\left(\,{}_{\alpha}D_q\Phi(\Theta\varpi\Upsilon+(1-\Theta\varpi)\alpha)\right)d_q\varpi \\
&= \frac{1}{(\Upsilon-\alpha)\Theta}\Phi(\Theta\Upsilon+(1-\Theta)\alpha)-\frac{1}{(\Upsilon-\alpha)^2\Theta^2}\int_{\alpha}^{\Theta\Upsilon+(1-\Theta)\alpha}\Phi(\varpi)\ {}_{\alpha}d_q\varpi.
\end{aligned}$$

Then it follows that

$$\begin{aligned}
&\frac{\Theta(\Upsilon-\alpha)}{2}(I_1-I_2) \\
&= \frac{\Theta(\Upsilon-\alpha)}{2}\Bigg(\frac{1}{(\Upsilon-\alpha)^2\Theta^2}\int_{\Theta\alpha+(1-\Theta)\Upsilon}^{\Upsilon}\Phi(\varpi)\ {}^{\Upsilon}d_q\varpi-\frac{1}{(\Upsilon-\alpha)\Theta}\Phi(\Theta\alpha+(1-\Theta)\Upsilon) \\
&\quad -\frac{1}{(\Upsilon-\alpha)\Theta}\Phi(\Theta\Upsilon+(1-\Theta)\alpha)+\frac{1}{(\Upsilon-\alpha)^2\Theta^2}\int_{\alpha}^{\Theta\Upsilon+(1-\Theta)\alpha}\Phi(\varpi)\ {}_{\alpha}d_q\varpi\Bigg)
\end{aligned}$$

$$= \frac{\Theta(\Upsilon-\alpha)}{2}\left(\int_{\Theta\alpha+(1-\Theta)\Upsilon}^{\Upsilon}\Phi(\varpi)\ {}^{\Upsilon}d_q\varpi+\int_{\alpha}^{\Theta\Upsilon+(1-\Theta)\alpha}\Phi(\varpi)\ {}_{\alpha}d_q\varpi\right)$$
$$-\frac{\Phi(\Theta\alpha+(1-\Theta)\Upsilon)+\Phi(\Theta\Upsilon+(1-\Theta)\alpha)}{2}.$$

This finalizes the proof.

Theorem 6. *Let $\Phi:[\alpha,\Upsilon]\to\mathbb{R}$ be a q-differentiable function. If $|{}_{\alpha}D_q\Phi|$ and $|{}^{\Upsilon}D_q\Phi|$ are strongly convex functions on $[\alpha,\Upsilon]$ for $\varphi>0$. Then the following inequalities are established:*

$$\left|\frac{\Theta(\Upsilon-\alpha)}{2}\left(\int_{\alpha}^{\Theta\Upsilon+(1-\Theta)\alpha}\Phi(\varpi)\ {}_{\alpha}d_q\varpi+\int_{\Theta\alpha+(1-\Theta)\Upsilon}^{\Upsilon}\Phi(\varpi)\ {}^{\Upsilon}d_q\varpi\right)\right.$$
$$\left.-\frac{\Phi(\Theta\alpha+(1-\Theta)\Upsilon)+\Phi(\Theta\Upsilon+(1-\Theta)\alpha)}{2}\right|$$
$$\leq\frac{\Theta q(\Upsilon-\alpha)}{2[2]_q[3]_q}\Bigg([2]_q\Theta\left(|{}^{\Upsilon}D_q\Phi(\alpha)|+|{}_{\alpha}D_q\Phi(\Upsilon)|\right)+([3]_q-[2]_q\Theta)\left(|{}^{\Upsilon}D_q\Phi(\Upsilon)|\right.$$
$$\left.+|{}_{\alpha}D_q\Phi(\alpha)|\right)\Bigg)-\Theta^2q\varphi(\Upsilon-\alpha)^3\left(\frac{[4]_q-[3]_q\Theta}{[3]_q[4]_q}\right)$$
$$\leq\frac{\Theta q(\Upsilon-\alpha)}{2[2]_q[3]_q}\Bigg([2]_q\Theta\left(|{}^{\Upsilon}D_q\Phi(\alpha)|+|{}_{\alpha}D_q\Phi(\Upsilon)|\right)+([3]_q-[2]_q\Theta)\left(|{}^{\Upsilon}D_q\Phi(\Upsilon)|\right.$$
$$\left.+|{}_{\alpha}D_q\Phi(\alpha)|\right)\Bigg). \tag{26}$$

Proof. It follows from Lemma 1 and $|{}_{\alpha}D_q\Phi|$ and $|{}^{\Upsilon}D_q\Phi|$ are strongly convex functions that

$$\left|\frac{\Theta(\Upsilon-\alpha)}{2}\left(\int_{\Theta\alpha+(1-\Theta)\Upsilon}^{\Upsilon}\Phi(\varpi)\ {}^{\Upsilon}d_q\varpi+\int_{\alpha}^{\Theta\Upsilon+(1-\Theta)\alpha}\Phi(\varpi)\ {}_{\alpha}d_q\varpi\right)\right.$$
$$\left.-\frac{\Phi(\Theta\alpha+(1-\Theta)\Upsilon)+\Phi(\Theta\Upsilon+(1-\Theta)\alpha)}{2}\right|$$
$$\leq\frac{\Theta(\Upsilon-\alpha)}{2}\int_0^1 q\varpi\left|{}^{\Upsilon}D_q\Phi(\Theta\varpi\alpha+(1-\Theta\varpi)\Upsilon)\right|d_q\varpi$$
$$+\frac{\Theta(\Upsilon-\alpha)}{2}\int_0^1 q\varpi\left|{}_{\alpha}D_q\Phi(\Theta\varpi\Upsilon+(1-\Theta\varpi)\alpha)\right|d_q\varpi$$
$$\leq\frac{\Theta(\Upsilon-\alpha)}{2}\int_0^1 q\varpi\Bigg(\Theta\varpi|{}^{\Upsilon}D_q\Phi(\alpha)|+(1-\Theta\varpi)|{}^{\Upsilon}D_qf(\Upsilon)|$$
$$-\varphi\Theta\varpi(1-\Theta\varpi)(\Upsilon-\alpha)^2\Bigg)d_q\varpi+\frac{\Theta(\Upsilon-\alpha)}{2}\int_0^1 q\varpi\Bigg(\Theta\varpi|{}_{\alpha}D_q\Phi(\Upsilon)|$$

$$
\begin{aligned}
&+ (1-\Theta\varpi)|\,{}_{\alpha}D_q\Phi(\alpha)| - \varphi\Theta\varpi(1-\Theta\varpi)(\Upsilon-\alpha)^2\Bigg) d_q\varpi \\
&= \frac{\Theta q(\Upsilon-\alpha)}{2[2]_q[3]_q}\Bigg([2]_q\Theta\left(|\,{}^{\Upsilon}D_q\Phi(\alpha)| + |\,{}_{\alpha}D_q\Phi(\Upsilon)|\right) + ([3]_q - [2]_q\Theta)\left(|\,{}^{\Upsilon}D_q\Phi(\Upsilon)|\right. \\
&\left. + |\,{}_{\alpha}D_q\Phi(\alpha)|\right)\Bigg) - \Theta^2 q\varphi(\Upsilon-\alpha)^3\left(\frac{[4]_q - [3]_q\Theta}{[3]_q[4]_q}\right) \\
&\leq \frac{\Theta q(\Upsilon-\alpha)}{2[2]_q[3]_q}\Bigg([2]_q\Theta\left(|\,{}^{\Upsilon}D_q\Phi(\alpha)| + |\,{}_{\alpha}D_q\Phi(\Upsilon)|\right) + ([3]_q - [2]_q\Theta)\left(|\,{}^{\Upsilon}D_q\Phi(\Upsilon)|\right. \\
&\left. + |\,{}_{\alpha}D_q\Phi(\alpha)|\right)\Bigg).
\end{aligned}
$$

This finalizes the proof.

Remark 3. *When $\Theta = 1$ in Theorem 6, we derive trapezoid-type inequalities:*

$$
\begin{aligned}
&\left|\frac{(\Upsilon-\alpha)}{2}\left(\int_{\alpha}^{\Upsilon}\Phi(\varpi)\ {}_{\alpha}d_q\varpi + \int_{\alpha}^{\Upsilon}\Phi(\varpi)\ {}^{\Upsilon}d_q\varpi\right) - \frac{\Phi(\alpha)+\Phi(\Upsilon)}{2}\right| \\
&\leq \frac{q(\Upsilon-\alpha)}{2[2]_q[3]_q}\Bigg([2]_q\left(|\,{}^{\Upsilon}D_q\Phi(\alpha)| + |\,{}_{\alpha}D_q\Phi(\Upsilon)|\right) + q^2\left(|\,{}^{\Upsilon}D_q\Phi(\Upsilon)| + |\,{}_{\alpha}D_q\Phi(\alpha)|\right)\Bigg) \\
&\quad - \frac{q^4\varphi(\Upsilon-\alpha)^3}{[3]_q[4]_q} \\
&\leq \frac{q(\Upsilon-\alpha)}{2[2]_q[3]_q}\Bigg([2]_q\left(|\,{}^{\Upsilon}D_q\Phi(\alpha)| + |\,{}_{\alpha}D_q\Phi(\Upsilon)|\right) + q^2\left(|\,{}^{\Upsilon}D_q\Phi(\Upsilon)| + |\,{}_{\alpha}D_q\Phi(\alpha)|\right)\Bigg).
\end{aligned}
$$

Remark 4. *When $\Theta = 1/2$ in Theorem 6, we derive Midpoint-type inequalities:*

$$
\begin{aligned}
&\left|\frac{(\Upsilon-\alpha)}{2}\left(\int_{\alpha}^{\frac{(\alpha+\Upsilon)}{2}}\Phi(\varpi)\ {}_{\alpha}d_q\varpi + \int_{\frac{(\alpha+\Upsilon)}{2}}^{\Upsilon}\Phi(\varpi)\ {}^{\Upsilon}d_q\varpi\right) - \Phi\left(\frac{\alpha+\Upsilon}{2}\right)\right| \\
&\leq \frac{q(\Upsilon-\alpha)}{8[2]_q[3]_q}\Bigg([2]_q\left(|\,{}^{\Upsilon}D_q\Phi(\alpha)| + |\,{}_{\alpha}D_q\Phi(\Upsilon)|\right) + ([3]_q + q^2)\left(|\,{}^{\Upsilon}D_q\Phi(\Upsilon)|\right. \\
&\left. + |\,{}_{\alpha}D_q\Phi(\alpha)|\right)\Bigg) - q^4\varphi(\Upsilon-\alpha)^3\left(\frac{q^3+[4]_q}{[3]_q[4]_q}\right) \\
&\leq \frac{q(\Upsilon-\alpha)}{8[2]_q[3]_q}\Bigg([2]_q\left(|\,{}^{\Upsilon}D_q\Phi(\alpha)| + |\,{}_{\alpha}D_q\Phi(\Upsilon)|\right) + ([3]_q + q^2)\left(|\,{}^{\Upsilon}D_q\Phi(\Upsilon)|\right. \\
&\left. + |\,{}_{\alpha}D_q\Phi(\alpha)|\right)\Bigg).
\end{aligned}
$$

Theorem 7. *Let* $\Phi : [\alpha, \Upsilon] \to \mathbb{R}$ *be a q-differentiable function. If* $|{}_{\alpha}D_q\Phi|^c$ *and* $|\,{}^{\Upsilon}D_q\Phi|^c$, $c > 1$ *are strongly convex functions on* $[\alpha, \Upsilon]$ *for* $\varphi > 0$ *, then the following inequalities are established:*

$$\begin{aligned}
&\left|\frac{\Theta(\Upsilon-\alpha)}{2}\left(\int_{\alpha}^{\Theta\Upsilon+(1-\Theta)\alpha}\Phi(\varpi)\ {}_{\alpha}d_q\varpi+\int_{\Theta\alpha+(1-\Theta)\Upsilon}^{\Upsilon}\Phi(\varpi)\ {}^{\Upsilon}d_q\varpi\right)\right.\\
&\quad\left.-\frac{\Phi(\Theta\alpha+(1-\Theta)\Upsilon)+\Phi(\Theta\Upsilon+(1-\Theta)\alpha)}{2}\right|\\
&\le\frac{q\Theta(\Upsilon-\alpha)}{2}\left(\frac{1}{[d+1]_q}\right)^{\frac{1}{d}}\left(\left(\frac{\Theta|\ {}_{\alpha}D_q\Phi(\Upsilon)|^c}{[2]_q}+\left(1-\frac{\Theta}{[2]_q}\right)|\ {}_{\alpha}D_q\Phi(\alpha)|^c\right.\right.\\
&\quad\left.-\varphi\Theta\left(\frac{1}{[2]_q}-\frac{\Theta}{[3]_q}\right)(\Upsilon-\alpha)^2\right)^{\frac{1}{c}}+\left(\frac{\Theta|\ {}^{\Upsilon}D_q\Phi(\alpha)|^c}{[2]_q}+\left(1-\frac{\Theta}{[2]_q}\right)|\ {}^{\Upsilon}D_q\Phi(\Upsilon)|^c\right.\\
&\quad\left.\left.-\varphi\Theta\left(\frac{1}{[2]_q}-\frac{\Theta}{[3]_q}\right)(\Upsilon-\alpha)^2\right)^{\frac{1}{c}}\right)\\
&\le\frac{q\Theta(\Upsilon-\alpha)}{2}\left(\frac{1}{[d+1]_q}\right)^{\frac{1}{d}}\left(\left(\frac{\Theta|\ {}_{\alpha}D_qf(\Upsilon)|^c}{[2]_q}+\left(\frac{[2]_q-\Theta}{[2]_q}\right)|\ {}_{\alpha}D_qf(\alpha)|^c\right)^{\frac{1}{c}}\right.\\
&\quad\left.+\left(\frac{\Theta|\ {}^{\Upsilon}D_q\Phi(\alpha)|^c}{[2]_q}+\left(\frac{[2]_q-\Theta}{[2]_q}\right)|\ {}^{\Upsilon}D_q\Phi(\Upsilon)|^c\right)^{\frac{1}{c}}\right),
\end{aligned}\tag{27}$$

where $1/c + 1/d = 1$.

Proof. It follows from Lemma 1, Hölder's inequality, and $|{}_{\alpha}D_q\Phi|^c$ and $|{}^{\Upsilon}D_q\Phi|^c$ are strongly convex functions that

$$\begin{aligned}
&\left|\frac{\Theta(\Upsilon-\alpha)}{2}\left(\int_{\alpha}^{\Theta\Upsilon+(1-\Theta)\alpha}\Phi(\varpi)\ {}_{\alpha}d_q\varpi+\int_{\Theta\alpha+(1-\Theta)\Upsilon}^{\Upsilon}\Phi(\varpi)\ {}^{\Upsilon}d_q\varpi\right)\right.\\
&\quad\left.-\frac{\Phi(\Theta\alpha+(1-\Theta)\Upsilon)+\Phi(\Theta\Upsilon+(1-\Theta)\alpha)}{2}\right|\\
&\le\frac{\Theta(\Upsilon-\alpha)}{2}\left(\int_0^1(q\varpi)^d d_q\varpi\right)^{\frac{1}{d}}\left(\int_0^1|\ {}_{\alpha}D_q\Phi(\Theta\varpi b+(1-\Theta\varpi)\alpha)|^c\,d_q\varpi\right)^{\frac{1}{c}}\\
&\quad+\frac{\Theta(\Upsilon-\alpha)}{2}\left(\int_0^1(q\varpi)^d d_q\varpi\right)^{\frac{1}{d}}\left(\int_0^1\left|\ {}^{\Upsilon}D_q\Phi(\Theta\varpi\alpha+(1-\Theta\varpi)\Upsilon)\right|^c d_q\varpi\right)^{\frac{1}{c}}\\
&\le\frac{\Theta(\Upsilon-\alpha)}{2}\left(\int_0^1(q\varpi)^d d_q\varpi\right)^{\frac{1}{d}}\left(\int_0^1(\Theta\varpi|\ {}_{\alpha}D_q\Phi(\Upsilon)|^c+(1-\Theta\varpi)|\ {}_{\alpha}D_q\Phi(\alpha)|^c\right.
\end{aligned}$$

$$- \varphi\Theta\varpi(1-\Theta\varpi)(\Upsilon-\alpha)^2)d_q\varpi\Bigg)^{\frac{1}{c}} + \frac{\Theta(\Upsilon-\alpha)}{2}\left(\int_0^1 (q\varpi)^d d_q\varpi\right)^{\frac{1}{d}}\Bigg(\int_0^1 (\Theta\varpi$$

$$| \,{}^bD_q\Phi(\alpha)|^c + (1-\Theta\varpi)| \,{}^\Upsilon D_q\Phi(\Upsilon)|^c - \varphi\Theta\varpi(1-\Theta\varpi)(b-\alpha)^2)d_q\varpi\Bigg)^{\frac{1}{c}}$$

$$\leq \frac{q\Theta(\Upsilon-\alpha)}{2}\left(\frac{1}{[d+1]_q}\right)^{\frac{1}{d}}\Bigg(\left(\frac{\Theta|\,{}_\alpha D_q\Phi(\Upsilon)|^c}{[2]_q} + \left(1-\frac{\Theta}{[2]_q}\right)|\,{}_\alpha D_q\Phi(\alpha)|^c\right)^{\frac{1}{c}}$$

$$+ \left(\frac{\Theta|\,{}^\Upsilon D_q\Phi(\alpha)|^c}{[2]_q} + \left(1-\frac{\Theta}{[2]_q}\right)|\,{}^\Upsilon D_q\Phi(\Upsilon)|^c\right)^{\frac{1}{c}}\Bigg)$$

$$\leq \frac{q\Theta(\Upsilon-\alpha)}{2}\left(\frac{1}{[d+1]_q}\right)^{\frac{1}{d}}\Bigg(\left(\frac{\Theta|\,{}_\alpha D_q\Phi(\Upsilon)|^c}{[2]_q} + \left(\frac{[2]_q-\Theta}{[2]_q}\right)|\,{}_\alpha D_q\Phi(\alpha)|^c\right)^{\frac{1}{c}}$$

$$+ \left(\frac{\Theta|\,{}^\Upsilon D_q\Phi(\alpha)|^c}{[2]_q} + \left(\frac{[2]_q-\Theta}{[2]_q}\right)|\,{}^\Upsilon D_q\Phi(\Upsilon)|^c\right)^{\frac{1}{c}}\Bigg).$$

This concludes the proof.

Remark 5. *When $\Theta = 1$ in Theorem 7, we derive trapezoid-type inequalities:*

$$\left|\frac{(\Upsilon-\alpha)}{2}\left(\int_\alpha^\Upsilon \Phi(\varpi)\,{}_\alpha d_q\varpi + \int_\alpha^\Upsilon \Phi(\varpi)\,{}^\Upsilon d_q\varpi\right) - \frac{\Phi(\alpha)+\Phi(\Upsilon)}{2}\right|$$

$$\leq \frac{q(\Upsilon-\alpha)}{2}\left(\frac{1}{[d+1]_q}\right)^{\frac{1}{d}}\Bigg(\Bigg(\frac{|\,{}_\alpha D_q\Phi(\Upsilon)|^c}{[2]_q} + \left(1-\frac{1}{[2]_q}\right)|\,{}_\alpha D_q\Phi(\alpha)|^c$$

$$-\varphi\left(\frac{1}{[2]_q}-\frac{1}{[3]_q}\right)(\Upsilon-\alpha)^2\Bigg)^{\frac{1}{c}} + \Bigg(\frac{|\,{}^\Upsilon D_q\Phi(\alpha)|^c}{[2]_q} + \left(1-\frac{1}{[2]_q}\right)|\,{}^\Upsilon D_q\Phi(\Upsilon)|^c$$

$$-\varphi\left(\frac{1}{[2]_q}-\frac{1}{[3]_q}\right)(\Upsilon-\alpha)^2\Bigg)^{\frac{1}{c}}\Bigg)$$

$$\leq \frac{q(\Upsilon-\alpha)}{2}\left(\frac{1}{[d+1]_q}\right)^{\frac{1}{d}}\Bigg(\left(\frac{|\,{}_\alpha D_q\Phi(\Upsilon)|^c + q|\,{}_\alpha D_q\Phi(\alpha)|^c}{[2]_q}\right)^{\frac{1}{c}}$$

$$+\left(\frac{|\,{}^\Upsilon D_q\Phi(\alpha)|^c + q|\,{}^\Upsilon D_q\Phi(\Upsilon)|^c}{[2]_2}\right)^{\frac{1}{c}}\Bigg).$$

Remark 6. *When $\Theta = 1/2$ in Theorem 7, we derive Midpoint-type inequalities:*

$$\left|\frac{(\Upsilon-\alpha)}{2}\left(\int_\alpha^{\frac{(\alpha+\Upsilon)}{2}} \Phi(\varpi)\,{}_\alpha d_q\varpi + \int_{\frac{(\alpha+\Upsilon)}{2}}^\Upsilon \Phi(\varpi)\,{}^\Upsilon d_q\varpi\right) - \Phi\left(\frac{\alpha+\Upsilon}{2}\right)\right|$$

$$\leq \frac{q(\Upsilon-\alpha)}{2}\left(\frac{1}{[d+1]_q}\right)^{\frac{1}{d}}\left(\left(\frac{|\ _{\alpha}D_q\Phi(\Upsilon)|^c}{2[2]_q}+\left(1-\frac{1}{2[2]_q}\right)|\ _{\alpha}D_q\Phi(\alpha)|^c\right.\right.$$

$$\left.-\frac{\varphi}{2}\left(\frac{1}{[2]_q}-\frac{1}{2[3]_q}\right)(\Upsilon-\alpha)^2\right)^{\frac{1}{c}}+\left(\frac{|\ ^{\Upsilon}D_q\Phi(\alpha)|^c}{2[2]_q}+\left(1-\frac{1}{2[2]_q}\right)|\ ^{\Upsilon}D_q\Phi(\Upsilon)|^c\right.$$

$$\left.\left.-\frac{\varphi}{2}\left(\frac{1}{[2]_q}-\frac{1}{2[3]_q}\right)(\Upsilon-\alpha)^2\right)^{\frac{1}{c}}\right)$$

$$\leq \frac{q(\Upsilon-\alpha)}{4}\left(\frac{1}{[d+1]_q}\right)^{\frac{1}{d}}\left(\left(\frac{|\ _{\alpha}D_q\Phi(\Upsilon)|^c+([2]_q+q)|\ _{\alpha}D_q\Phi(\alpha)|^c}{[2]_q}\right)^{\frac{1}{c}}\right.$$

$$\left.+\left(\frac{|\ ^{\Upsilon}D_q\Phi(\alpha)|^c+([2]_q+q)|\ ^{\Upsilon}D_q\Phi(\Upsilon)|^c}{[2]_2}\right)^{\frac{1}{c}}\right).$$

Theorem 8. *Let* $\Phi:[\alpha,\Upsilon]\to\mathbb{R}$ *be a* q*-differentiable function. If* $|_{\alpha}D_q\Phi|^c$ *and* $|\ ^{\Upsilon}D_q\Phi|^c$, $c\geq 1$ *are strongly convex functions on* $[\alpha,\Upsilon]$ *for* $\varphi>0$ *, then the following inequalities are established:*

$$\left|\frac{\Theta(\Upsilon-\alpha)}{2}\left(\int_{\alpha}^{\Theta\Upsilon+(1-\Theta)\alpha}\Phi(\varpi)\ _{\alpha}d_q\varpi+\int_{\Theta\alpha+(1-\Theta)\Upsilon}^{\Upsilon}\Phi(\varpi)\ ^{\Upsilon}d_q\varpi\right)\right.$$

$$\left.-\frac{\Phi(\Theta\alpha+(1-\Theta)\Upsilon)+\Phi(\Theta\Upsilon+(1-\Theta)\alpha)}{2}\right|$$

$$\leq \frac{q\Theta(\Upsilon-\alpha)}{2}\left(\frac{1}{[2]_q}\right)^{1-\frac{1}{c}}\left(\left(\frac{\Theta|\ _{\alpha}D_qf(\Upsilon)|^c}{[3]_q}+\left(\frac{1}{[2]_q}-\frac{\Theta}{[3]_q}\right)|\ _{\alpha}D_q\Phi(\alpha)|^c\right.\right.$$

$$\left.-\varphi\Theta\left(\frac{1}{[3]_q}-\frac{\Theta}{[4]_q}\right)(\Upsilon-\alpha)^2\right)^{\frac{1}{c}}+\left(\frac{\Theta|\ ^{\Upsilon}D_q\Phi(\alpha)|^c}{[3]_q}+\left(\frac{1}{[2]_q}-\frac{\Theta}{[3]_q}\right)|\ ^{\Upsilon}D_q\Phi(\Upsilon)|^c\right.$$

$$\left.\left.-\varphi\Theta\left(\frac{1}{[3]_q}-\frac{\Theta}{[4]_q}\right)(\Upsilon-\alpha)^2\right)^{\frac{1}{c}}\right)$$

$$\leq \frac{q\Theta(\Upsilon-\alpha)}{2[2]_q}\left(\left(\frac{[2]_q\Theta|\ _{\alpha}D_q\Phi(\Upsilon)|^c+([3]_q-\Theta[2]_q)|\ _{\alpha}D_q\Phi(\alpha)|^c}{[3]_q}\right)^{\frac{1}{c}}\right.$$

$$\left.+\left(\frac{[2]_q\Theta|\ ^{\Upsilon}D_q\Phi(\alpha)|^c+([3]_q-\Theta[2]_q)|\ ^{\Upsilon}D_q\Phi(\Upsilon)|^c}{[3]_q}\right)^{\frac{1}{c}}\right). \tag{28}$$

Proof. It can be deduced from Lemma 1, the power mean inequality and and the strong convexity of $|_{\alpha}D_q\Phi|^c$ and $|^{\Upsilon}D_q\Phi|^c$ that

$$\left|\frac{\Theta(\Upsilon-\alpha)}{2}\left(\int_{\alpha}^{\Theta\Upsilon+(1-\Theta)\alpha}\Phi(\varpi)\ _{\alpha}d_q\varpi+\int_{\Theta\alpha+(1-\Theta)\Upsilon}^{\Upsilon}\Phi(\varpi)\ ^{\Upsilon}d_q\varpi\right)\right.$$

$$
\begin{aligned}
&-\frac{\Phi(\Theta\alpha+(1-\Theta)\Upsilon)+\Phi(\Theta\Upsilon+(1-\Theta)\alpha)}{2}\Bigg| \\
&\leq \frac{\Theta(\Upsilon-\alpha)}{2}\left(\int_0^1 q\varpi\ d_q\varpi\right)^{1-\frac{1}{c}}\left(\int_0^1 q\varpi\,|\ {}_{\alpha}D_q\Phi(\Theta\varpi\Upsilon+(1-\Theta\varpi)\alpha)|^c\, d_q\varpi\right)^{\frac{1}{c}} \\
&+\frac{\Theta(\Upsilon-\alpha)}{2}\left(\int_0^1 q\varpi\ d_q\varpi\right)^{1-\frac{1}{c}}\left(\int_0^1 q\varpi\left|\ {}^{\Upsilon}D_q\Phi(\Theta\varpi\alpha+(1-\Theta\varpi)\Upsilon)\right|^c d_q\varpi\right)^{\frac{1}{c}} \\
&\leq \frac{\Theta(\Upsilon-\alpha)}{2}\left(\int_0^1 q\varpi\ d_q\varpi\right)^{1-\frac{1}{c}}\left(\int_0^1 q\varpi(\Theta\varpi|\ {}_{\alpha}D_q\Phi(\Upsilon)|^c+(1-\Theta\varpi)|\ {}_{\alpha}D_q\Phi(\alpha)|^c\right. \\
&\left.-\varphi\Theta\varpi(1-\Theta\varpi)(\Upsilon-\alpha)^2)d_q\varpi\right)^{\frac{1}{c}}+\frac{\Theta(\Upsilon-\alpha)}{2}\left(\int_0^1 q\varpi\ d_q\varpi\right)^{1-\frac{1}{c}}\left(\int_0^1 q\varpi\right. \\
&\left.(\Theta\varpi|\ {}^{\Upsilon}D_q\Phi(\alpha)|^c+(1-\Theta\varpi)|\ {}^{\Upsilon}D_q\Phi(\Upsilon)|^c-\varphi\Theta\varpi(1-\Theta\varpi)(\Upsilon-\alpha)^2)d_q\varpi\right)^{\frac{1}{c}} \\
&=\frac{\Theta(\Upsilon-\alpha)}{2}\left(\frac{q}{[2]_q}\right)^{1-\frac{1}{c}}\left(\frac{q\Theta|\ {}_{\alpha}D_q\Phi(\Upsilon)|^c}{[3]_q}+\left(\frac{q}{[2]_q}-\frac{q\Theta}{[3]_q}\right)|\ {}_{\alpha}D_q\Phi(\alpha)|^c\right. \\
&\left.-\varphi q\Theta\left(\frac{1}{[3]_q}-\frac{\Theta}{[4]_q}\right)(\Upsilon-\alpha)^2\right)^{\frac{1}{c}}+\frac{\Theta(\Upsilon-\alpha)}{2}\left(\frac{q}{[2]_q}\right)^{1-\frac{1}{c}}\left(\frac{q\Theta|\ {}^{\Upsilon}D_q\Phi(\alpha)|^c}{[3]_q}\right. \\
&\left.+\left(\frac{q}{[2]_q}\frac{q\Theta}{[3]_q}\right)|\ {}^{\Upsilon}D_q\Phi(\Upsilon)|^c-\varphi q\Theta\left(\frac{1}{[3]_q}-\frac{\Theta}{[4]_q}\right)(\Upsilon-\alpha)^2\right)^{\frac{1}{c}} \\
&\leq \frac{q\Theta(\Upsilon-\alpha)}{2[2]_q}\left(\left(\frac{[2]_q\Theta|\ {}_{\alpha}D_q\Phi(\Upsilon)|^c+([3]_q-\Theta[2]_q)|\ {}_{\alpha}D_q\Phi(\alpha)|^c}{[3]_q}\right)^{\frac{1}{c}}\right. \\
&\left.+\left(\frac{[2]_q\Theta|\ {}^{\Upsilon}D_q\Phi(\alpha)|^c+([3]_q-\Theta[2]_q)|\ {}^{\Upsilon}D_q\Phi(\Upsilon)|^c}{[3]_q}\right)^{\frac{1}{c}}\right).
\end{aligned}
$$

This concludes the proof.

Remark 7. *When $\Theta=1$ in Theorem 8, we derive trapezoid-type inequalities:*

$$
\begin{aligned}
&\left|\frac{(\Upsilon-\alpha)}{2}\left(\int_{\alpha}^{\Upsilon}\Phi(\varpi)\ {}_{\alpha}d_q\varpi+\int_{\alpha}^{\Upsilon}\Phi(\varpi)\ {}^{\Upsilon}d_q\varpi\right)-\frac{\Phi(\alpha)+\Phi(\Upsilon)}{2}\right| \\
&\leq \frac{q(\Upsilon-\alpha)}{2}\left(\frac{1}{[2]_q}\right)^{1-\frac{1}{c}}\left(\left(\frac{|\ {}_{\alpha}D_q\Phi(\Upsilon)|^c}{[3]_q}+\left(\frac{1}{[2]_q}-\frac{1}{[3]_q}\right)|\ {}_{\alpha}D_q\Phi(\alpha)|^c\right.\right.
\end{aligned}
$$

$$
\begin{aligned}
&-\varphi\left(\frac{1}{[3]_q}-\frac{1}{[4]_q}\right)(\Upsilon-\alpha)^2\Bigg)^{\frac{1}{c}}+\left(\frac{|\ ^{\Upsilon}D_q\Phi(\alpha)|^c}{[3]_q}+\left(\frac{1}{[2]_q}-\frac{1}{[3]_q}\right)|\ ^{\Upsilon}D_q\Phi(\Upsilon)|^c\right.\\
&\left.-\varphi\left(\frac{1}{[3]_q}-\frac{1}{[4]_q}\right)(\Upsilon-\alpha)^2\right)^{\frac{1}{c}}\Bigg)\\
\leq&\ \frac{q(\Upsilon-\alpha)}{2[2]_q}\left(\left(\frac{[2]_q\Theta|\ _{\alpha}D_q\Phi(\Upsilon)|^c+q^2|\ _{\alpha}D_q\Phi(\alpha)|^c}{[3]_q}\right)^{\frac{1}{c}}\right.\\
&\left.+\left(\frac{[2]_q|\ ^{\Upsilon}D_q\Phi(\alpha)|^c+q^2|\ ^{\Upsilon}D_q\Phi(\Upsilon)|^c}{[3]_q}\right)^{\frac{1}{c}}\right).
\end{aligned}
$$

Remark 8. *When $\Theta = 1/2$ in Theorem 8, we derive Midpoint-type inequalities:*

$$
\begin{aligned}
&\left|\frac{(\Upsilon-\alpha)}{2}\left(\int_{\alpha}^{\frac{(\alpha+\Upsilon)}{2}}\Phi(\varpi)\ _{\alpha}d_q\varpi+\int_{\frac{(\alpha+\Upsilon)}{2}}^{\Upsilon}\Phi(\varpi)\ ^{\Upsilon}d_q\varpi\right)-\Phi\left(\frac{\alpha+\Upsilon}{2}\right)\right|\\
\leq&\ \frac{q(\Upsilon-\alpha)}{2}\left(\frac{1}{[2]_q}\right)^{1-\frac{1}{c}}\left(\left(\frac{|\ _{\alpha}D_q\Phi(b)|^c}{2[3]_q}+\left(\frac{1}{[2]_q}-\frac{1}{2[3]_q}\right)|\ _{\alpha}D_q\Phi(\alpha)|^c\right.\right.\\
&-\frac{\varphi}{2}\left(\frac{1}{[3]_q}-\frac{1}{2[4]_q}\right)(\Upsilon-\alpha)^2\Bigg)^{\frac{1}{c}}+\left(\frac{|\ ^{\Upsilon}D_q\Phi(\alpha)|^c}{2[3]_q}+\left(\frac{1}{[2]_q}-\frac{1}{2[3]_q}\right)|\ ^{\Upsilon}D_q\Phi(\Upsilon)|^c\right.\\
&\left.-\frac{\varphi}{2}\left(\frac{1}{[3]_q}-\frac{1}{2[4]_q}\right)(\Upsilon-\alpha)^2\right)^{\frac{1}{c}}\Bigg)\\
\leq&\ \frac{q(\Upsilon-\alpha)}{2[2]_q}\left(\left(\frac{[2]_q|\ _{\alpha}D_q\Phi(\Upsilon)|^c+([3]_q+q^2)|\ _{\alpha}D_q\Phi(\alpha)|^c}{2[3]_q}\right)^{\frac{1}{c}}\right.\\
&+\left(\left(\frac{[2]_q|\ ^{\Upsilon}D_q\Phi(\alpha)|^c+([3]_q+q^2)|\ ^{\Upsilon}D_q\Phi(\Upsilon)|^c}{2[3]_q}\right)^{\frac{1}{c}}.
\end{aligned}
$$

5. Applications

We show how special means can be used to prove the results in Theorems 6, 7, and 8. To establish arbitrary positive numbers W_1 and W_2 ($W_1 \neq W_2$), we define the means as follows:

(i) The arithmetic mean

$$\mathcal{A} = \mathcal{A}(W_1, W_2) = \frac{W_1 + W_2}{2}.$$

(ii) The logarithmic mean

$$L_P = L_P(W_1, W_2) = \frac{W_2^{P+1} - W_1^{P+1}}{(P+1)(W_1 - W_2)}.$$

Proposition 1. *Given that* $0 < \alpha < \Upsilon$*, the following inequalities are valid:*

$$\begin{aligned}
&\left|\frac{1}{c+1}\left(\Theta^2(\Upsilon-\alpha)^2\mathcal{A}(k_1,k_2) - \mathcal{A}(W_1^{c+1}, W_2^{c+1})\right)\right| \\
&\leq \frac{q\Theta(\Upsilon-\alpha)}{2[2]_q[3]_q}\Bigg([2]_q\Theta\left(L_c(q\alpha+(1-q)\Upsilon,\alpha) + L_c(q\Upsilon+(1-q)\alpha,\Upsilon)\right) \\
&\quad + ([3]_q - [2]_q\Theta)(\alpha^c + \Upsilon^c)\Bigg) - \frac{\Theta^2 q\varphi(\Upsilon-\alpha)^3([4]_q - [3]_q\Theta)}{[3]_q[4]_q},
\end{aligned}$$

where

$$k_1 = (1-q)\sum_{n=0}^{\infty} q^n \left(q^n\Theta\Upsilon + (1-q^n\Theta)\alpha\right)^{c+1}$$

$$k_2 = (1-q)\sum_{n=0}^{\infty} q^n \left(q^n\Theta\alpha + (1-q^n\Theta)\Upsilon\right)^{c+1}$$

and

$$W_1 = \Theta\alpha + (1-\Theta)\Upsilon, \quad W_2 = \Theta\Upsilon + (1-\Theta)\alpha.$$

Proof. *By substituting* $\Phi(\varpi) = \varpi^{c+1}/(c+1)$*, where* $\varpi > 0$ *in the inequalities* (26) *of Theorem 6, then we obtain the result.*

Proposition 2. *Given that* $0 < \alpha < \Upsilon$*, the following inequalities are valid:*

$$\begin{aligned}
&\left|\frac{1}{c+1}\left(\Theta^2(\Upsilon-\alpha)^2\mathcal{A}(k_1,k_2) - \mathcal{A}(W_1^{c+1}, W_2^{c+1})\right)\right| \\
&\leq \frac{q\Theta(\Upsilon-\alpha)}{2}\left(\frac{1}{[d+1]_q}\right)^{\frac{1}{d}}\Bigg(\Bigg(\frac{\Theta|L_c(q\Upsilon+(1-q)\alpha,\Upsilon)|^c}{[2]_q} + \left(1 - \frac{\Theta}{[2]_q}\right)\alpha^c \\
&\quad - \varphi\Theta(\frac{1}{[2]_q} - \frac{\Theta}{[3]_q})(\Upsilon-\alpha)^2\Bigg)^{\frac{1}{c}} + \Bigg(\frac{\Theta|L_c(q\alpha+(1-q)\Upsilon,\alpha)|^c}{[2]_q} + \left(1 - \frac{\Theta}{[2]_q}\right)\Upsilon^c \\
&\quad - \varphi\Theta\left(\frac{1}{[2]_q} - \frac{\Theta}{[3]_q}\right)(\Upsilon-\alpha)^2\Bigg)^{\frac{1}{c}}\Bigg).
\end{aligned}$$

Proof. *By substituting* $\Phi(\varpi) = \varpi^{c+1}/(c+1)$*, where* $\varpi > 0$ *in the inequalities* (27) *of Theorem 7, we obtain the result.*

Proposition 3. *Given that* $0 < \alpha < \Upsilon$*, the following inequalities are valid:*

$$\begin{aligned}
&\left|\frac{1}{c+1}\left(\Theta^2(\Upsilon-\alpha)^2\mathcal{A}(k_1,k_2)-\mathcal{A}(W_1^{c+1},W_2^{c+1})\right)\right| \\
&\leq \frac{q\Theta(\Upsilon-\alpha)}{2}\left(\frac{1}{[2]_q}\right)^{1-\frac{1}{c}}\left(\left(\frac{\Theta|L_c(q\Upsilon+(1-q)\alpha,\Upsilon)|^c}{[3]_q}+\left(\frac{1}{[2]_q}-\frac{\Theta}{[3]_q}\right)\alpha^c\right.\right. \\
&\quad \left. -\varphi\Theta\left(\frac{1}{[3]_q}-\frac{\Theta}{[4]_q}\right)(b-\alpha)^2\right)^{\frac{1}{c}}+\left(\frac{\Theta|L_c(q\alpha+(1-q)\Upsilon,\alpha)|^c}{[2]_q}+\left(\frac{1}{[2]_q}-\frac{\Theta}{[3]_q}\right)\Upsilon^c\right. \\
&\quad \left.\left. -\varphi\Theta\left(\frac{1}{[3]_q}-\frac{\Theta}{[4]_q}\right)(\Upsilon-\alpha)^2\right)^{\frac{1}{c}}\right).
\end{aligned}$$

Proof. By substituting $\Phi(\varpi) = \varpi^{c+1}/(c+1)$*, where* $\varpi > 0$ *in the inequalities* (28) *of Theorem 8, we obtain the result.*

6. Conclusion

This study focused on using the principles of q-calculus to prove varieties of q-H-H type inequalities for strongly convex functions. Our findings indicate the potential for generalizing existing comparable results in the literature. In addition, we also refine the results in the literature of N. Alp et al. [2] by using the characterization of strong convexity of Φ. The methodologies from this study can be widely utilized to establish q-calculus and similar inequalities for various types of convexities.

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