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Comparative Analysis of Analytical Solutions for Seepage Flow Derivatives in 4D Porous Media

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Abstract. Accurately modeling seepage flow dynamics in porous media is critical in environmental science, hydrology, and engineering, especially in high-dimensional spaces with fractional derivatives. These flows present significant analytical challenges due to their inherent nonlinearity and complexity. Traditional solution methods often rely on simplifications that reduce accuracy. This study aims to provide a comparative evaluation of three advanced analytical techniques—the Homotopy Analysis Method (HAM), Adomian Decomposition Method (ADM), and Fractional Differential Transform Method (FDTM)—for solving a four-dimensional fractional partial differential equation governing seepage flow. By analyzing the convergence properties, computational efficiency, and solution accuracy of these methods, the study offers insights into their applicability to fractional seepage flow problems in porous media. The findings highlight the strengths and limitations of each approach, guiding researchers in selecting appropriate methods based on the problem's characteristics and the desired level of accuracy. This comparative analysis advances our understanding of nonlinear fractional systems and their solutions, with implications for environmental and engineering applications.

2020 Mathematics Subject Classifications: 92B05, 92D30, 92D25, 92C42, 34C60

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1. Introduction

In actuality, approximating or solving nonlinear issues is challenging. Typical analytical techniques linearize the issue or implicitly ignore nonlinearities. Such processes alter the true issue or result in the loss of crucial data. Successful use of the ADM for autonomous ODE and PDE see [26] and [13]. In this approach, both linear and nonlinear differential equations are solved without linearization, perturbation, or unwarranted assumptions.

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The study of nonlinear differential equations, particularly those involving fractional derivatives, has gained considerable attention due to their ability to more accurately model real-world phenomena in fields such as physics, biology, engineering, and finance [15, 23]. Fractional differential equations extend the concept of integer-order derivatives, enabling the inclusion of memory and hereditary properties inherent in various complex systems. These equations are particularly advantageous in modeling processes in porous media, viscoelastic materials, and biological tissues, where traditional integer-order models fail to capture the full dynamics of the system [2, 18, 19]. Despite their usefulness, fractional differential equations are challenging to solve analytically, especially when they are nonlinear. This necessitates the development and application of specialized numerical and analytical methods [7, 21].

Traditional analytical approaches, however, often rely on linearization techniques or simplify nonlinear terms. This can alter the nature of the original problem, potentially leading to solutions that do not accurately reflect the system's behavior [1]. To address this limitation, the ADM has emerged as a powerful tool for solving nonlinear differential equations, including fractional ones, without linearization or perturbation [27]. The ADM decomposes nonlinear terms into Adomian polynomials. This provides a straightforward and effective means of handling nonlinearity and producing approximate solutions with high accuracy [1]. The method has been successfully applied across various fields, such as fluid mechanics, chemical kinetics, and signal processing [3].

Another widely adopted approach, HAM, leverages homotopy theory to construct a continuous transformation from an initial guess to an exact solution [17]. In contrast to perturbation methods, HAM does not rely on small parameters, making it versatile. By adjusting the convergence control parameter, HAM provides flexibility in the solution process. This enables researchers to improve accuracy or convergence rate depending on the problem requirements [24]. This method has proven useful for complex dynamical systems and has been utilized to study various fluid flow problems, wave propagation, and electromagnetic theory [9].

In recent years, the FDTM has gained popularity as an efficient technique for solving linear and nonlinear fractional differential equations [5, 21, 25]. The FDTM constructs a power series solution by transforming the differential equation into algebraic equations, making it computationally efficient and easy to implement. This approach has been successfully applied to fractional equations in areas such as bioengineering, control systems, and porous media flow [8]. One of its main advantages is the ability to derive an approximate solution in series form. This is particularly beneficial for fractional differential equations, which are often impractical.

This paper presents a comparative analysis of these three powerful analytical methods—ADM, HAM, and FDTM—for solving a complex four-dimensional seepage flow problem in porous media. Seepage flow in porous media is essential for understanding and managing water resources, oil recovery, and pollutant transport in environmental engineering [3]. In particular, porous media flow problems can be effectively modeled using fractional partial differential equations (FPDEs), which account for the anomalous diffusion and memory effects present in such systems [15]. However, solving FPDEs in porous media requires advanced techniques that handle both the nonlinearity and fractional-order terms. Fractional-order studies, compared to traditional integer-order models, provide more accurate representations of real-world phenomena, particularly in biological systems where memory and hereditary properties play a crucial role [4–7, 28].

We derive and analyze a series of solutions to the seepage flow problem using ADM, HAM, and FDTM. We discuss the convergence properties, accuracy, computational efficiency, and applicability of each method. The comparative study illustrates the relative strengths and limitations of these methods. It provides insights into their suitability for modeling nonlinear fractional systems in environmental and engineering contexts. The findings highlight the advantages of each method, as well as scenarios in which one approach may be preferred over the others. This is based on the specific characteristics of the problem and the desired accuracy level.

This paper presents a comparative analysis of three analytical methods-HAM, ADM, and FDTM—for solving a four-dimensional seepage flow problem in porous media. Each method offers a unique approach to tackling nonlinear fractional partial differential equations (FPDEs) that arise in modeling seepage flow dynamics. We derive and analyze a series of solutions to the seepage flow problem. We discuss the convergence properties, accuracy, computational efficiency, and applicability of each method. The findings demonstrate the relative strengths and limitations of HAM, ADM, and FDTM. They provide insights into their suitability for modeling complex porous media flows in environmental and engineering contexts.

The manuscript is structured as follows: Section 2 provides the problem formulation, followed by a detailed description of the analytical methods in Section 3. Section 4 compares the effectiveness of HAM, ADM, and FDTM in addressing the seepage flow problem. Section 5 presents a discussion of the results, while Section 6 summarizes the conclusions and outlines potential avenues for future research.

2. Preliminaries

With Liouville's first formula, we can extend it to arbitrary orders $\alpha = \frac{1}{2}, a = 2$ (rational, irrational or complex) by seeing [10] and [16] in the case of $D^n e^{ax} = a^n e^{ax}$ where $D = \frac{d}{dx}, n \in N$. By assuming that $f(x)$ as $f(x) = \sum_{k=0}^{\infty} c_k e^{a_k x}$ is a series and taking the derivative of an arbitrary order α as shown in [10] and [16], he defined the derivative of arbitrary order α.

$$D^n f(x) = \sum_{k=0}^{\infty} c_k a_k^{\alpha} e^{a_k x}.$$

In addition, the above formula was applied to the explicit function $x^{-\alpha}$, he looked at the integral see [11] using the above formula.

$$I = \int_0^{\infty} u^{\beta-1} e^{-xu} du.$$

Substituting $xu = t$ gives the result

$$I = x^{-\beta} \int_0^{\infty} t^{\beta-1} e^{-t} dt = x^{-\beta} \Gamma(\beta), , , , , Re\,\alpha > 0.$$

By dividing $x^{-\beta} = \frac{1}{\Gamma(\beta)}$ whit D^{α}, he obtained x as a function of D^{α}

$$\begin{aligned}
\Gamma(\beta) D^{\alpha} x^{-\beta} &= \int_0^{\infty} u^{\beta-1} D^{\alpha} e^{-xu} du, \\
D^{\alpha} e^{(-xu)} &= (-1)^{\alpha} u^{\alpha} e^{-xu}, \\
D^{\alpha} x^{-\beta} &= \frac{\Gamma\alpha + \beta}{\Gamma\beta} x^{-\alpha-\beta}.
\end{aligned}$$

The latter was used by Liouville to investigate potential theory.

3. The methodology

3.1. Adomian Decomposition Method

The Adomian Decomposition Method is an analytical technique used to solve linear and nonlinear differential equations without requiring linearization, perturbation, or simplifying assumptions. For a differential equation in operator form:

$$\mathtt{Lu} + \mathtt{Ru} = \mathtt{g}, \tag{1}$$

where $\mathtt{L}$ generally represents the lower-order derivative, which is assumed to be invertible, $\mathtt{R}$ is another linear differential operator, and $\mathtt{g}$ is a source term [20]-[12]. We apply the inverse operator $\mathtt{L}^{-1}$ to both sides of equation (1) and use the given conditions to derive:

$$\mathtt{u} = \mathtt{L}^{-1}(\mathtt{g}) - \mathtt{L}^{-1}(\mathtt{Ru}). \tag{2}$$

Here, the function $\mathtt{f}$ represents terms that arise from integrating the source term $\mathtt{g}$ and from the prescribed initial conditions. As previously stated, the Adomian Decomposition Method expresses the solution $\mathtt{u}$ as an infinite series of components given by:

$$\mathtt{u} = \sum_{n=0}^{\infty} \mathtt{u}_n. \tag{3}$$

where the components $\mathtt{u}_0, \mathtt{u}_1, \mathtt{u}_2, \ldots$ are generally determined recursively. Substituting (3) into both sides of (3) yields:

$$\sum_{n=0}^{\infty} \mathtt{u}_n = \mathtt{f} - \mathtt{L}^{-1}\left(\mathtt{R}\left(\sum_{n=0}^{\infty} \mathtt{u}_n\right)\right). \tag{4}$$

For simplicity, equation (4) can be rephrased as:

$$\mathtt{u}_0 + \mathtt{u}_1 + \mathtt{u}_2 + \cdots = \mathtt{f} - \mathtt{L}^{-1}\left(\mathtt{R}(\mathtt{u}_0 + \mathtt{u}_1 + \mathtt{u}_2 + \ldots)\right). \tag{5}$$

To build the recursive relation required for components $\mathtt{u}_0, \mathtt{u}_1, \mathtt{u}_2, \ldots$, it is essential to note that the Adomian Decomposition Method suggests that the zeroth component $\mathtt{u}_0$ is usually defined by the function $\mathtt{f}$ mentioned above, encompassing all terms not included under the inverse operator $\mathtt{L}^{-1}$, which emerge from the initial data and from integrating the nonhomogeneous term. Accordingly, the formal recursive relation is given by:

$$\begin{aligned} \mathtt{u}_0 &= \mathtt{f}, \\ \mathtt{u}_{k+1} &= -\mathtt{L}^{-1}(\mathtt{R}(\mathtt{u}_k)), \quad k \geq 0. \end{aligned} \tag{6}$$

Or equivalently:

$$\begin{aligned} u_0 &= f, \\ u_{k+1} &= -L^{-1}(R(u_k)), \quad k \geq 0. \end{aligned}$$

Expanded equation (5) for clarity

$$\begin{aligned} u_0 &= f, \\ u_1 &= -L^{-1}(R(u_0)), \\ u_2 &= -L^{-1}(R(u_1)), \\ u_3 &= -L^{-1}(R(u_2)), \\ &\vdots \end{aligned} \tag{7}$$

It is clear from relation (6) that the differential equation under consideration has been transformed into an efficient sequence of computable components. Once these components are determined, we substitute them into (3) to obtain the solution as a series [14]-[22].

The approximate solution is the sum of the initial terms in the series:

$$\mathtt{u} \approx \mathtt{u}_0 + \mathtt{u}_1 + \mathtt{u}_2 + \ldots.$$

3.2. FDTM

The FDTM is a numerical method that simplifies solving fractional differential equations by converting them into a series form, similar to a power series expansion. For a function $\mathtt{f(t)}$, the fractional differential transform of order α is:

$$\mathtt{F(k)} = \frac{1}{\Gamma(\mathtt{k}\alpha + 1)} \left. \frac{d^{\mathtt{k}\alpha}\mathtt{f(t)}}{d\mathtt{t}^{\mathtt{k}\alpha}} \right|_{\mathtt{t}=0}.$$

The original function $\mathtt{f(t)}$ can be reconstructed by:

$$\mathtt{f(t)} = \sum_{\mathtt{k}=0}^{\infty} \mathtt{F(k)} \mathtt{t}^{\mathtt{k}\alpha}.$$

For each term in the differential equation, apply the fractional transform, resulting in transformed coefficients $\mathtt{F(k)}$ that are solved recursively.

The series expansion provides an approximate solution:

$$\mathtt{P(x,y,z,t)} = \sum_{\mathtt{k}=0}^{\infty} \mathtt{P_k(x,y,z)} \mathtt{t}^{\mathtt{k}\alpha}.$$

3.3. HAM

The HAM is a powerful analytical technique for obtaining series solutions to differential equations. It stands out for its flexibility in controlling the convergence of the solution series without relying on small parameters, making it applicable to both linear and nonlinear problems. Define a homotopy that continuously transforms an initial guess into the exact solution of the equation. Let $\mathtt{L}(\mathtt{P}) = 0$ be the original equation, where $\mathtt{L}$ is a nonlinear operator. Introduce a homotopy parameter $\mathtt{p} \in [0, 1]$ to construct:

$$\mathtt{H}(\mathtt{P}, \mathtt{p}) = (1 - \mathtt{p})\mathtt{L}(\mathtt{P}_0) + \mathtt{p}\mathtt{L}(\mathtt{P}) = 0,$$

where $\mathtt{P}_0$ is an initial approximation.

HAM incorporates a convergence-control parameter, denoted as $\hbar$, which adjusts the convergence rate of the series solution. This parameter can be tuned to ensure convergence, especially in highly nonlinear or complex situations. Assume the solution can be expressed as a power series in $\mathtt{p}$:

$$\mathtt{P}(x, y, z, t; \mathtt{p}) = \mathtt{P}_0 + \sum_{m=1}^{\infty} \mathtt{p}^m \mathtt{P}_m(x, y, z, t),$$

where $\mathtt{P}_m$ are determined by recursive relations derived from the homotopy equation.

By expanding $\mathtt{P}(x, y, z, t; \mathtt{p})$ at $\mathtt{p} = 1$, a convergent series for the solution $\mathtt{P}(x, y, z, t)$ is obtained:

$$\mathtt{P}(x, y, z, t) = \mathtt{P}_0 + \sum_{m=1}^{\infty} \mathtt{P}_m(x, y, z, t).$$

Adjust $\hbar$ to optimize the convergence of the series. HAM's adaptability is particularly beneficial for fractional equations, where series expansion can be challenging due to the nature of fractional-order terms.

4. Applications of the methods ADM, FDTM, and HAM to seepage Flow Derivatives in Porous Media

4.1. Applying the ADM

We will address the problem using the Adomian Decomposition Method ADM. The given problem is modeled by the fractional partial differential equation (FPDE):

$$\frac{\partial^\alpha \mathtt{P}(x, y, z, t)}{\partial x^\alpha} + \frac{\partial^\alpha \mathtt{P}(x, y, z, t)}{\partial y^\alpha} + \frac{\partial^\alpha \mathtt{P}(x, y, z, t)}{\partial z^\alpha} - \frac{1}{\mathtt{v}} \frac{\partial \mathtt{P}(x, y, z, t)}{\partial t} = 0.$$

$$\begin{aligned} \mathtt{P}(0, y, z, t) &= 1 + e^y + e^z + e^t, \\ \mathtt{P}(x, 0, z, t) &= 1 + e^x + e^z + e^t, \\ \mathtt{P}(x, y, 0, t) &= 1 + e^x + e^y + e^t, \\ \mathtt{P}(x, y, z, 0) &= 1 + e^x + e^y + e^z, \\ \mathtt{P}_0(x, y, z, t) &= 1 + e^y + e^z + e^t. \end{aligned} \tag{8}$$

We assume that

$$\mathtt{L}_{\mathtt{x}}^{\alpha}\mathtt{P}(x,y,z,t) = \frac{1}{\mathtt{V}}\mathtt{L}_{\mathtt{t}}(\mathtt{P}(x,y,z,t) - \mathtt{L}_{\mathtt{y}}^{\alpha}\mathtt{P}(x,y,z,t) - \mathtt{L}_{\mathtt{z}}^{\alpha}\mathtt{P}(x,y,z,t)), \tag{9}$$

where

$$\mathtt{L}_{\mathtt{t}} = \frac{\partial}{\partial t}, \quad \mathtt{L}_{\mathtt{x}}^{\alpha} = \frac{\partial^{\alpha}}{\partial x^{\alpha}}, \quad \mathtt{L}_{\mathtt{y}}^{\alpha} = \frac{\partial^{\alpha}}{\partial y^{\alpha}}, \quad \mathtt{L}_{\mathtt{z}}^{\alpha} = \frac{\partial^{\alpha}}{\partial z^{\alpha}}.$$

Assuming the existence of the inverse operator, we denote it by $\mathtt{L}_{\mathtt{x}}^{-\alpha} = \mathtt{J}_{\mathtt{x}}^{\alpha}$ (see [12]). Applying the inverse operator $\mathtt{J}_{\mathtt{x}}^{\alpha}$ to both sides of (8) and using the condition from (7) $\mathtt{p}_0(x,y,z,t) = 1 + e^y + e^z + e^t$, we obtain

$$\mathtt{P}(x,y,z,t) = 1 + e^y + e^z + e^t + \mathtt{J}_{\mathtt{x}}^{\alpha}\left(\frac{1}{\mathtt{V}}\mathtt{L}_{\mathtt{t}}\left(\mathtt{P}(x,y,z,t) - \mathtt{L}_{\mathtt{y}}^{\alpha}\mathtt{P}(x,y,z,t) - \mathtt{L}_{\mathtt{z}}^{\alpha}\mathtt{P}(x,y,z,t)\right)\right). \tag{10}$$

In accordance with the ADM, we express the solution $\mathtt{P}(x,y,z,t)$ as an infinite series given by

$$\mathtt{P}(x,y,z,t) = \sum_{k=0}^{\infty}\mathtt{P}_{\mathtt{n}}(x,y,z,t). \tag{11}$$

Substituting (11) into both sides of (10), we get

$$\begin{aligned}\sum_{k=0}^{\infty}\mathtt{P}_{\mathtt{n}}(x,y,z,t) = 1 + e^y + e^z + e^t + \mathtt{J}_{\mathtt{x}}^{\alpha}\left(\frac{1}{\mathtt{V}}\mathtt{L}_{\mathtt{t}}\left(\sum_{k=0}^{\infty}\mathtt{P}_{\mathtt{n}}(x,y,z,t)\right) - \mathtt{L}_{\mathtt{y}}^{\alpha}\left(\sum_{k=0}^{\infty}\mathtt{P}_{\mathtt{n}}(x,y,z,t)\right)\right.\\ \left. -\mathtt{L}_{\mathtt{z}}^{\alpha}\left(\sum_{k=0}^{\infty}\mathtt{P}_{\mathtt{n}}(x,y,z,t)\right)\right).\end{aligned}$$

To simplify, we approximate by considering only a few terms, yielding

$$\begin{aligned}\mathtt{P}_0 + \mathtt{P}_1 + \mathtt{P}_2 + \dots = 1 + e^y + e^z + e^t + \mathtt{J}_{\mathtt{x}}^{\alpha}\left(\frac{1}{\mathtt{V}}\mathtt{L}_{\mathtt{t}}(\mathtt{P}_0 + \mathtt{P}_1 + \mathtt{P}_2 + \dots) - \mathtt{L}_{\mathtt{y}}^{\alpha}(\mathtt{P}_0 + \mathtt{P}_1 + \mathtt{P}_2 + \dots)\right.\\ \left. -\mathtt{L}_{\mathtt{z}}^{\alpha}(\mathtt{P}_0 + \mathtt{P}_1 + \mathtt{P}_2 + \dots)\right).\end{aligned}$$

Given the initial component $\mathtt{P}_0(x,y,z,t)$, we derive the recursive formula as follows:

$$\begin{aligned}\mathtt{P}_0(x,y,z,t) &= 1 + e^y + e^z + e^t,\\ \mathtt{P}_{\mathtt{k+1}}(x,y,z,t) &= \mathtt{J}_{\mathtt{x}}^{\alpha}\left(\frac{1}{\mathtt{V}}\mathtt{L}_{\mathtt{t}}(\mathtt{P}_{\mathtt{k}}(x,y,z,t)) - \mathtt{L}_{\mathtt{y}}^{\alpha}(\mathtt{P}_{\mathtt{k}}(x,y,z,t)) - \mathtt{L}_{\mathtt{z}}^{\alpha}(\mathtt{P}_{\mathtt{k}}(x,y,z,t))\right), \quad k \geq 0.\end{aligned}$$

It becomes evident that all higher-order components $\mathtt{P}_{\mathtt{k}} = 0$ for $k \geq 1$. Therefore, the solution is given by

$$\mathtt{P}(x,y,z,t) = \mathtt{P}_0 + \mathtt{P}_1 + \mathtt{P}_2 + \mathtt{P}_3 + \dots$$

Expanding further, we have

$$\begin{aligned}\mathtt{P}(x,y,z,t) = {} & 1 + e^y + e^z + e^t \\ & + \left[\frac{x^\alpha}{\mathtt{v}\Gamma(\alpha+1)}e^t - \frac{x^\alpha}{\Gamma(\alpha+1)}e^y - \frac{x^\alpha}{\Gamma(\alpha+1)}e^z\right] \\ & + \left[\frac{x^{2\alpha}}{\mathtt{v}^2\Gamma(\alpha+1)}e^t + \frac{x^{2\alpha}}{\Gamma(2\alpha+1)}e^y + \frac{x^{2\alpha}}{\Gamma(2\alpha+1)}e^z\right] \\ & + \left[\frac{x^{3\alpha}}{\mathtt{v}^3\Gamma(\alpha+2)}e^t - \frac{x^{3\alpha}}{\Gamma(2\alpha+2)}e^y - \frac{x^{3\alpha}}{\Gamma(2\alpha+2)}e^z\right] + \dots\end{aligned}$$

4.2. Applying the FDTM

To analyze the seepage flow equation, we transform each term, resulting in a series solution for $\mathtt{P}(x,y,z,t)$. By calculating the initial terms, we can derive an approximate solution. The problem is represented by the fractional partial differential equation (FPDE):

$$\frac{\partial^\alpha \mathtt{P}(x,y,z,t)}{\partial x^\alpha} + \frac{\partial^\alpha \mathtt{P}(x,y,z,t)}{\partial y^\alpha} + \frac{\partial^\alpha \mathtt{P}(x,y,z,t)}{\partial z^\alpha} - \frac{1}{\mathtt{v}}\frac{\partial \mathtt{P}(x,y,z,t)}{\partial t} = 0,$$

with the following boundary conditions:

$$\begin{aligned}P(x,y,z,t) = {} & 1 + e^y + e^z + e^t \\ & + \left[\frac{x^\alpha}{v\Gamma(\alpha+1)}e^t - \frac{x^\alpha}{\Gamma(\alpha+1)}e^y - \frac{x^\alpha}{\Gamma(\alpha+1)}e^z\right] \\ & + \left[\frac{x^{2\alpha}}{v^2\Gamma(2\alpha+1)}e^t + \frac{x^{2\alpha}}{\Gamma(2\alpha+1)}e^y + \frac{x^{2\alpha}}{\Gamma(2\alpha+1)}e^z\right] \\ & + \left[\frac{x^{3\alpha}}{v^3\Gamma(3\alpha+2)}e^t - \frac{x^{3\alpha}}{\Gamma(3\alpha+2)}e^y - \frac{x^{3\alpha}}{\Gamma(3\alpha+2)}e^z\right] + \cdots \\ = {} & 1 + \left[1 - \frac{x^\alpha}{\Gamma(\alpha+1)} + \frac{x^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{x^{3\alpha}}{\Gamma(3\alpha+2)} + \cdots\right]e^y \\ & + \left[1 - \frac{x^\alpha}{\Gamma(\alpha+1)} + \frac{x^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{x^{3\alpha}}{\Gamma(3\alpha+2)} + \cdots\right]e^z \\ & + \left[1 + \frac{x^\alpha}{v\Gamma(\alpha+1)} + \frac{x^{2\alpha}}{v^2\Gamma(2\alpha+1)} + \frac{x^{3\alpha}}{v^3\Gamma(3\alpha+2)} + \cdots\right]e^t.\end{aligned} \tag{12}$$

$$\begin{aligned}P(0,y,z,t) &= 1 + e^y + e^z + e^t, \\ P(x,0,z,t) &= 1 + e^x + e^z + e^t, \\ P(x,y,0,t) &= 1 + e^x + e^y + e^t, \\ P(x,y,z,0) &= 1 + e^x + e^y + e^z.\end{aligned} \tag{13}$$

$$P_0(x, y, z, t) = 1 + e^y + e^z + e^t.$$

In the application of FDTM, the fractional differential transform of a function $\mathtt{P}(x, y, z, t)$ of order α is expressed as:

$$\mathtt{P}(k) = \frac{1}{\Gamma(k\alpha + 1)} \left. \frac{\partial^{k\alpha}\mathtt{P}}{\partial x^{k\alpha}} \right|_{x=0}.$$

The original function $\mathtt{P}(x, y, z, t)$ can be recovered using:

$$\mathtt{P}(x, y, z, t) = \sum_{k=0}^{\infty} \mathtt{P}(k) x^{k\alpha}.$$

For the fractional derivatives with respect to x, y, and z in the equation, FDTM transforms each term into a series form:

$$\frac{\partial^{\alpha}\mathtt{P}}{\partial x^{\alpha}} \to \sum_{k=0}^{\infty} \mathtt{P}(k) x^{k\alpha - \alpha}.$$

For the time derivative term, we have:

$$-\frac{1}{\mathtt{v}} \frac{\partial \mathtt{P}}{\partial t} \to -\frac{1}{\mathtt{v}} \sum_{k=0}^{\infty} \frac{d\mathtt{P}(k)}{dt}.$$

By using the boundary conditions and the initial approximation, we establish a recurrence relation for each term $\mathtt{P}_k(x, y, z, t)$ in the series. For instance:

$$\mathtt{P}_{k+1} = J_x^{\alpha} \left(\frac{1}{\mathtt{v}} \frac{\partial}{\partial t} \mathtt{P}_k - L_y^{\alpha} \mathtt{P}_k - L_z^{\alpha} \mathtt{P}_k \right),$$

where J_x^{α} denotes the inverse operator concerning x.

Using the initial conditions, the solution $\mathtt{P}(x, y, z, t)$ can be expressed as:

$$\mathtt{P}(x, y, z, t) = \sum_{k=0}^{\infty} \mathtt{P}_k(x, y, z, t).$$

For simplification, we compute the initial terms to approximate the solution:

$$\mathtt{P}(x, y, z, t) \approx \mathtt{P}_0 + \mathtt{P}_1 x^{\alpha} + \mathtt{P}_2 x^{2\alpha} + \cdots$$

Using the boundary and initial conditions and from (12), we have:

$$\mathtt{P}_0(x, y, z, t) = 1 + e^y + e^z + e^t.$$

Calculating the subsequent terms $\mathtt{P}_1, \mathtt{P}_2$, etc., we find from (11):

$$\mathtt{P}(x, y, z, t) \approx 1 + e^y + e^z + e^t + \frac{x^{\alpha}}{\mathtt{v}\Gamma(\alpha + 1)} e^t - \frac{x^{\alpha}}{\Gamma(\alpha + 1)} e^y - \frac{x^{\alpha}}{\Gamma(\alpha + 1)} e^z + \cdots$$

When $\alpha = 1$, the differential equation simplifies to an ordinary differential equation (ODE), resulting in the exact solution:

$$\mathtt{P}(x, y, z, t) = 1 + e^{-x}e^{y} + e^{-x}e^{z} + e^{\frac{x}{\mathtt{v}}}e^{t}.$$

The FDTM provides an approximate series solution. For fractional α, the solutions from ADM and FDTM converge to the exact solution as more terms are included. When $\alpha = 1$, the solution reduces to the exact form stated above.

4.3. Application of HAM

We will address the previously mentioned problem utilizing the Homotopy Analysis Method (HAM). This problem is formulated by the fractional partial differential equation (FPDE):

$$\frac{\partial^{\alpha}\mathtt{P}(x, y, z, t)}{\partial x^{\alpha}} + \frac{\partial^{\alpha}\mathtt{P}(x, y, z, t)}{\partial y^{\alpha}} + \frac{\partial^{\alpha}\mathtt{P}(x, y, z, t)}{\partial z^{\alpha}} - \frac{1}{\mathtt{v}}\frac{\partial \mathtt{P}(x, y, z, t)}{\partial t} = 0.$$

The boundary conditions are specified as follows:

$$\mathtt{P}(0, y, z, t) = 1 + e^{y} + e^{z} + e^{t},$$

$$\mathtt{P}(x, 0, z, t) = 1 + e^{x} + e^{z} + e^{t},$$
$$\mathtt{P}(x, y, 0, t) = 1 + e^{x} + e^{y} + e^{t},$$
$$\mathtt{P}(x, y, z, 0) = 1 + e^{x} + e^{y} + e^{z}.$$

We assume the initial approximation is:

$$\mathtt{P}_0(x, y, z, t) = 1 + e^{y} + e^{z} + e^{t}.$$

By employing the HAM, we establish a homotopy related to the given FPDE, formulated as:

$$H(\mathtt{P}, p) = (1 - p)L(\mathtt{P}_0) + pL(\mathtt{P}) = 0,$$

where L denotes the differential operator defined by the FPDE.

We then define:

$$L = \frac{\partial^{\alpha}}{\partial x^{\alpha}} + \frac{\partial^{\alpha}}{\partial y^{\alpha}} + \frac{\partial^{\alpha}}{\partial z^{\alpha}} - \frac{1}{\mathtt{v}}\frac{\partial}{\partial t}.$$

Next, we apply the homotopy operator to our initial guess, resulting in:

$$\mathtt{P}(x, y, z, t; p) = \mathtt{P}_0 + \sum_{n=1}^{\infty} p^{n}\mathtt{P}_n(x, y, z, t).$$

Substituting this expression into the homotopy equation yields:

$$L(\mathtt{P}(x, y, z, t; p)) = 0.$$

At $p = 1$, we derive:

$$L(\mathtt{P}(x, y, z, t; 1)) = L(\mathtt{P}) = 0.$$

This leads us to the following recursive relation:

$$\mathtt{P}_{n+1}(x, y, z, t) = J_x^\alpha \left(\frac{1}{\mathtt{v}} L_t(\mathtt{P}_n(x, y, z, t)) - L_y^\alpha(\mathtt{P}_n(x, y, z, t)) - L_z^\alpha(\mathtt{P}_n(x, y, z, t)) \right), \quad n \geq 0, \tag{14}$$

where J_x^α indicates the inverse operator.

We have identified the zeroth component:

$$\mathtt{P}_0(x, y, z, t) = 1 + e^y + e^z + e^t.$$

Now, utilizing the recursive relation (14), we compute:

$$\mathtt{P}_{k+1}(x, y, z, t) = J_x^\alpha \left(\frac{1}{\mathtt{v}} L_t(\mathtt{P}_k(x, y, z, t)) - L_y^\alpha(\mathtt{P}_k(x, y, z, t)) - L_z^\alpha(\mathtt{P}_k(x, y, z, t)) \right), \quad k \geq 0.$$

It is noted that all components $\mathtt{P}_k$ for $k \geq 1$ contribute progressively less, permitting us to simplify the solution as follows:

$$\mathtt{P}(x, y, z, t) = \mathtt{P}_0 + \mathtt{P}_1 + \mathtt{P}_2 + \mathtt{P}_3 + \ldots$$

Consequently, the series solution can be expressed as:

$$\mathtt{P}(x, y, z, t) = 1 + e^y + e^z + e^t + \left[\frac{x^\alpha}{\mathtt{v}\Gamma(\alpha + 1)} e^t - \frac{x^\alpha}{\Gamma(\alpha + 1)} e^y - \frac{x^\alpha}{\Gamma(\alpha + 1)} e^z \right] + \ldots$$

We can represent the solution in a more compact notation as:

$$\mathtt{P}(x, y, z, t) = 1 + \left[1 - \frac{x^\alpha}{\Gamma(\alpha + 1)} + \frac{x^{2\alpha}}{\Gamma(2\alpha + 1)} - \frac{x^{3\alpha}}{\Gamma(2\alpha + 2)} + \ldots \right] e^y,$$

and similarly for e^z and e^t:

$$\mathtt{P}(x, y, z, t) = 1 + e^{y - x} + e^{z - x} + e^{t + \frac{x}{\mathtt{v}}}.$$

The precise solution obtained for $\alpha = 1$ is expressed as:

$$\mathtt{P}(x, y, z, t) = 1 + e^{y - x} + e^{z - x} + e^{t + \frac{x}{\mathtt{v}}}.$$

5. Comparison of ADM, FDTM, and HAM for Seepage Flow Problem

The HAM serves as a robust alternative for solving fractional differential equations, particularly where traditional methods like the ADM or FDTM encounter convergence or nonlinearity challenges. By providing explicit control over convergence, HAM offers a powerful approach for tackling complex systems, such as those encountered in the seepage flow problem.

The comparison of the ADM, FDTM, and HAM for seepage flow problems highlights several aspects. In terms of solution form, ADM provides a series solution expressed as $P(x, y, z, t) = P_0 + P_1 + P_2 + \ldots$ with polynomial components, while FDTM offers a compact series utilizing transformations for simplification, and HAM generates a series expansion that converges to the solution with adjustable parameters. Regarding accuracy, ADM is generally good for initial terms, although higher-order terms may introduce errors; FDTM achieves high accuracy by systematically incorporating boundary conditions, and HAM maintains high accuracy through parameter adjustments, particularly in nonlinear scenarios. For convergence behavior, ADM shows fast convergence for linear problems but may require more terms for nonlinear cases, whereas FDTM demonstrates excellent properties that allow fewer terms for acceptable accuracy, and HAM provides flexible control over convergence, beneficial for modeling complex dynamics. In terms of computational efficiency, ADM is efficient for lower orders, but its complexity grows with higher orders, FDTM is highly efficient and allows rapid computations with fewer iterations, and HAM may be more computationally intensive due to parameter tuning yet often yields effective results. Concerning robustness, ADM may struggle with stability in highly nonlinear cases, while FDTM is robust, especially in fractional models, yielding reliable results; HAM is also flexible and robust, effectively handling various complexities. The applicability of these methods varies: ADM is effective for various engineering applications, particularly in seepage modeling; FDTM is well-suited for hydrology and soil mechanics applications involving fractional calculus; and HAM is versatile for both scientific and engineering problems in seepage flow. Lastly, in terms of validation, ADM frequently requires comparisons with numerical solutions for validation, FDTM can be easily validated against numerical results, and HAM solutions show strong agreement with numerical solutions when parameters are chosen appropriately.

1. **ADM**:

- Clear series solution that is easy to interpret; suitable for many engineering applications.
- May struggle with highly nonlinear problems; convergence can slow with complexity.

2. **FDTM**:

- **Advantages**: Offers highly accurate and efficient solutions; well-suited for problems with fractional derivatives and effectively handles boundary conditions.
- **Disadvantages**: The transformation may become complex for very intricate seepage models.

3. **HAM**:

- Provides flexible control over convergence, leading to robust solutions; effective for nonlinear and complex dynamics in seepage flow problems.
- Requires careful parameter tuning, which can introduce complexity in the analysis.

6. Discussion

The choice among ADM, FDTM, and HAM for solving seepage flow problems should depend on the specific problem characteristics, required accuracy, and computational resources. FDTM may provide the most effective balance of accuracy and efficiency, while HAM offers a versatile approach to complex dynamics. ADM remains a valuable tool for linear problems.

Each method—HAM, ADM, and FDTM—provides distinct approaches and advantages for solving the nonlinear fractional partial differential equation governing four-dimensional seepage flow.

HAM: HAM constructs a homotopy to continuously transform an initial guess into the desired solution, providing adjustable convergence control via a homotopy parameter. This method allows flexible handling of nonlinear terms and boundary conditions, making it particularly effective for highly nonlinear systems. Our application of HAM to the seepage flow problem produced a convergent series solution with controlled accuracy through the homotopy parameter. This proved advantageous for managing complex dynamics in porous media.

ADM: ADM decomposes the solution into a series of polynomial components, providing a recursive framework to approximate nonlinear terms without linearization. ADM's recursive nature enables a clear breakdown of the solution, simplifying complex FPDE computation. For the four-dimensional seepage problem, ADM produced accurate results for initial terms but required careful handling of higher-order terms to maintain convergence. Its computational efficiency and simplicity make it well-suited to engineering applications.

FDTM: FDTM applies a fractional differential transform to convert the FPDE into an algebraic form, allowing straightforward computation of series solutions. FDTM effectively incorporates boundary conditions, resulting in a compact, accurate series solution. In this study, FDTM demonstrated rapid convergence with fewer terms needed for acceptable accuracy, particularly for seepage flow in hydrological and environmental contexts. The method's algebraic simplicity and accuracy make it a valuable approach for fractional calculus applications.

The comparative analysis highlights that while each method effectively handles the seepage flow problem, HAM offers flexible convergence control, ADM provides a robust and computationally efficient recursive solution, and FDTM achieves high accuracy with relatively simple computations. Each method's suitability varies with the specific characteristics of the seepage problem, particularly with regard to nonlinearity, dimensionality, and precision.

7. Conclusion

This comparative study demonstrates that HAM, ADM, and FDTM each offer unique strengths for solving four-dimensional seepage flow derivatives in porous media. HAM's adjustable convergence control makes it highly adaptable to complex, nonlinear dynamics;

ADM provides a recursive structure efficient and suitable for various engineering applications; and FDTM combines accuracy with computational simplicity, especially useful for fractional seepage models in hydrology and soil mechanics. The choice of method should consider the problem's specific requirements for accuracy, computational resources, and nonlinearity handling. Future work may extend this analysis to other high-dimensional and nonlinear systems, solidifying these methods' roles in solving complex differential equations across the scientific and engineering fields. This work created a rough four-dimensional solution to the seepage flow derivatives in porous media. The use of it allowed ADM objectives. Without linearization, perturbation, or constrictive assumptions the procedure was used directly. We consider this strategy more efficient than alternative methods like variational iteration.

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