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A Bipolar Fermatean Fuzzy Hamacher Approach to Group Decision-Making for Electric Waste

Aliya Fahmi[1,*], Muhammad Arshad Shehzad Hassan[2], Aziz Khan[3], Thabet Abdeljawad[3,4,5,6], D.K.Almutairi[7]

[1] *Department of Mathematics and Statistics, Faculty of Sciences, The University of Faisalabad, Pakistan*
[2] *Department of Electrical Engineering, The University of Faisalabad, Faisalabad, Pakistan*
[3] *Department of Mathematics and Sciences, Prince Sultan University, P.O.Box 66833, 11586 Riyadh, Saudi Arabia*
[4] *Department of Mathematics, Saveetha School of Engineering, Saveetha Institute of Medical and Technical Sciences, Saveetha University, Chennai 602105, Tamil Nadu, India*
[5] *Department of Mathematics and Applied Mathematics, Sefako Makgatho Health Sciences University, Garankuwa, Medusa 0204, South Africa*
[6] *Center for Applied Mathematics and Bioinformatics (CAMB), Gulf University for Science and Technology, Hawally, 32093, Kuwait*
[7] *Department of Mathematics, College of Science Al-Zulfi, Majmaah University, 11952 Al-Majmaah, Saudi Arabia*

Abstract. The paper introduces advanced aggregation operators based on Hamacher operational laws for Bipolar Fermatean Fuzzy sets, addressing complex decision-making problems. In addition, a score function is utilized to evaluate and rank alternatives within the Bipolar Fermatean Fuzzy decision framework, offering a quantitative measure of each alternative's performance based on the aggregation results. This score function helps simplify decision-making, particularly in cases involving complex uncertainty and multiple criteria. The proposed operators include the Bipolar Fermatean Fuzzy-Hamacher weighted average, the Bipolar Fermatean Fuzzy-Hamacher Ordered Weighted Average, and the Bipolar Fermatean Fuzzy-Hamacher Hybrid Weighted Average. These operators are designed to integrate membership, nonmembership, and hesitation degrees, making them highly effective in Bipolar Fermatean Fuzzy environments. To demonstrate the practicality of the proposed approach, a real-world group decision-making scenario is applied to managing electronic waste. The results show the proposed methodology's superior efficiency, flexibility, and precision compared to existing approaches. The findings highlight the robustness of operators in handling uncertainty and enhancing the accuracy of decision-making.

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*Corresponding author.
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Email address: aliyafahmi@gmail.com (A. Fahmi),
tabdeljawad@psu.edu.sa (T. Abdeljawad), dk.almutairi@mu.edu.sa (D.K. Almutairi)

1. Introduction

The expansion of the competencies has the effect of reducing the electric plans' customer lifetime. This is the cause of a large river of electrical excess leftover from obsolete electronic devices. These existing methods have several drawbacks and challenges, including several characteristic removal procedures for e-waste of the later ecologically and financially interpreted. Careful reprocessing was therefore necessary for effective waste organization decisions. The hard substructure was lacking in the area. Burning and disposing of it in landfills expected to be the primary method for removing e-waste from porcelain. A thoughtful cargo within our vicinities may be the basis for the increase in landfill requirements [22, 41]. However, the realization of original plans depends on distinct fundamentals. The resulting appearances are covered by the existing policy requirement. Preserving and restoring EOL microelectronic designs was the most important aspect of an established technique. Nowadays, it is well known that e-waste recycling has a thorough foundation in the area and a little historical significance in porcelain. illustrating the approximate global significance of Microchip technology Over 700 workers in the US unbiased recyclers in the e-waste salvage industry. This implies to salvage EOL microchip technology, meetings and skills must be complex [30]. The greatest option fixed of limited choices that match to specific qualities is provided by MADM, which is regarded as the unpack-aged faster, and emerging field of study. When assessing applicants' information, certain problems because decision experts' perspectives are not always clear. These kinds of issues are resolved by using the indication of fuzzy sets, which was projected by Zadeh in 1965. Atanassov [7]] expanded the concept of the intuitionistic fuzzy sets as the membership and non-belonging degrees. A versatile and successful framework for handling complex MADM scenarios is probability fuzzy sets. Yager [45] expanded this approach by introducing q-rung orthopair fuzzy sets.

1.1. Literature review

Zadeh [47] introduced the fuzzy sets. Fuzzy sets are widely applied in modeling uncertainty across fields such as decision-making and artificial intelligence, while intuitionistic fuzzy sets and others extend this framework by incorporating hesitation degrees, enhancing applications in complex decision-making scenarios [31–33]. Pythagorean fuzzy sets, and q-rung orthopair fuzzy sets introduced by Yager [44], as well as Fermatean fuzzy sets introduced by Senapati and Yager [34]. Notably, FFSs provide a higher level of generalization and applicability than both intuitionistic fuzzy sets and PFSs, due to their enhanced ability to capture and represent relevant information effectively. Additionally, he developed four distinct cosine similarity measures for fuzzy sets, demonstrating the utility of his approach through a numerical example evaluating third-party logistics firms in cold chain management.

Ahmad et al. [2] conducted an analysis of the performance of various graph operations within these frameworks. Following this, Ahmad et al. [1] explored the comparative analysis that confirmed the model's credibility and reliability while thoroughly outlining its strengths and limitations. They also introduced an enhanced Assessment Founded on Distance from Average Solution approach, integrating novel concepts to address decision-making challenges arising from entirely unknown criteria weights [6]. Akram et al. [3] utilized these operators to investigate a range of recurring issues, implementing them in multi-attribute decision-making problems and for wireless location detection. Foundational ideas and important results for the Pythagorean fuzzy Laplace and Fourier transforms were established by Akram et al. [4, 5]. Zadeh proposed fuzzy sets in 1965, in which every element has a membership function given to it. Since then, several extensions have been created, such as Atanassov's intuitionistic fuzzy sets [7]. Aydin [8] developed a new fuzzy entropy metric based on the Euclidean distance between fuzzy integers and their complements using entropy theory. Additionally, Deng and Wang [10] created two innovative distance measurement techniques especially suited for Fermatean fuzzy sets in order to tackle difficulties in medical diagnosis and pattern identification. Triangular cubic fuzzy sets were proposed by Aliya et al. [14]. In [16], Aliya and colleagues proposed Einstein aggregation operators. The vikor approaches were proposed by Aliya et al. [13]. The operational laws of triangular cubic fuzzy sets were suggested by Aliya et al. [12]. The generalized interval-valued bipolar neutrosophic Einstein fuzzy aggregation operator was proposed by Aliya et al. [18]. The fermatean fuzzy sets were proposed by Aliya et al. [17]. The blending regret philosophy DDAS method in Fermatean fuzzy numbers was proposed by Aliya et al. [15]. The TOPSIS approach was proposed by Aliya et al. [11]. The natural gas was suggested by Aliya et al. [19]. The benefits of these suggested operators were thoroughly discussed by Garg et al. [21], who also presented a multi-attribute decision-making approach and showed how to use it in practice when choosing a trustworthy laboratory for COVID-19 testing. In the context of Fermatean fuzzy sets, Hadi et al. [23] created new techniques based on the Hamacher T-conorm and T-norm, highlighting their key features. Motivated by the ideas of FFSs and Hamacher operations, they also presented FFHamacher arithmetic and geometric AOs. Children from the nearby town of Chendian and those from the e-waste recycling village of Guiyu had their lead levels compared by Hoa et al. [24]. The suggested method was used to present a step-by-step algorithm for decision-making as well as a multi-criteria decision-making strategy [25].

Figure 1 is e-waste as below

Figure 1: Different technique of e-waste

A framework called FF-CRITIC was created by Mishra et al. [29] to address multi-attribute decision-making problems. A number of aggregation operators under Fermatean fuzzy sets were introduced by Shahzadi et al. [35]. In order to efficiently aggregate interval-valued data and metadata, Wang and Liu [37] used Einstein AOs. For evaluation, Wang et al. [36] used PF with entropy weights. Wei [38] contributed by proposing PF interaction AOs and exploring their uses within the MADM framework. Wei [40] presented a comprehensive suite of Pythagorean fuzzy Hamacher power aggregation operators. Their application in multi-attribute decision-making was covered by Wu and Wei [42]. The author concentrates on the similarity measures of Fermatean fuzzy sets, providing definitions for both similarity measures and weighted similarity measures in the contexts of discrete and continuous universes [20, 25–27, 33, 43, 46]. Its goal was to assist decision-makers in formulating effective policies to address the challenges of electric power shortages [48]. Innovative concepts for interval-valued fuzzy Einstein hybrid AOs were introduced by Zhao and Wei [51], who also described how they might be used in MADM scenarios. Recently, the bipolar fuzzy set [9, 28, 39, 49, 50] has drawn interest as a practical approach for dealing with uncertainty in MADM situations. A positive membership degree and a negative membership degree are the two values it uses to represent an object. The bipolar fuzzy set permits membership degrees to fluctuate within the range $[-1, 1]$, in contrast to intuitionistic fuzzy sets (IFS), which have membership degrees ranging from 0 to 1.

Problem statement

There are financial of e-waste disposal techniques of the environmental associated. This necessity of creative approaches to e-waste management of recycling. The subject of electronic recycling is still in its infancy, lacks infrastructure, e-waste recycling partners can be challenging, this study presents certain Hamacher aggregation operators in a bipolar fuzzy framework to help with decision-making. Establishing a MADM framework to determine the best recycling partner is the main goal of this study. Porcelain is currently one of the biggest manufacturers of e-waste in addition to being a significant consumer of electronic goods. These a case analysis centered on choosing a recycling partner in Porcelain is included.

Motivation

In the realm of multi-criteria decision-making, handling uncertainty and imprecision is a crucial challenge. Traditional fuzzy set theories and aggregation operators primarily address situations with binary membership, often overlooking the more complex scenarios where both positive and negative membership, as well as hesitation degrees, are equally important. Bipolar Fermatean Fuzzy Sets, which extend classical fuzzy sets to incorporate dual membership and non-membership, offer a promising solution to this problem. However, existing aggregation methods are not sufficiently equipped to handle the complexity and uncertainty inherent in Bipolar Fermatean Fuzzy Sets based decision-making tasks.

The primary issues identified in the current decision-making frameworks are as follows:

Traditional aggregation operators are designed for conventional fuzzy sets and fail to fully integrate the duality present in Bipolar Fermatean Fuzzy Sets. While they capture membership values well, they struggle to incorporate non-membership values and hesitation degrees in a coherent manner. This limitation hinders the ability to accurately evaluate alternatives that involve both positive and negative criteria.

Classical aggregation techniques, such as the Weighted Average or Ordered Weighted Average, are not well-suited for environments where both positive and negative information are crucial. In particular, these methods cannot address the intricacies of interactions between membership and non-membership components in Bipolar Fermatean Fuzzy Sets, leading to potential inaccuracies in decision-making.

A significant gap in the literature is the lack of operational laws that govern the interactions of membership, non-membership, and hesitation degrees in Bipolar Fermatean Fuzzy Sets. The development of these operational laws is essential for building aggregation functions that can process such complex information consistently and meaningfully.

Limitations in Existing Decision-Making Systems: Many existing decision-making systems based on aggregation operators are primarily designed for conventional fuzzy sets, and they encounter difficulties when applied to Bipolar Fermatean Fuzzy Sets. These systems cannot handle the uncertainties and ambiguities inherent in Bipolar Fermatean Fuzzy Sets based data, limiting their effectiveness in fields requiring nuanced decision-making under uncertainty.

Challenges in Real-World Applications (e.g., E-Waste Management): In practical applications such as e-waste management, where decisions involve selecting appropriate partners or alternatives based on multiple criteria (e.g., cost, efficiency, environmental impact), existing methods fail to account for the complex uncertainty of the decision-making environment. The inability to fully model positive and negative contributions, as well as the hesitation or indecision in choosing the best alternative, leads to suboptimal results.

Novelty and contribution

The proposed work offers several unique contributions and essential advancements:

A novel scoring function is introduced to facilitate the comparison of any number of bipolar fermatean fuzzy sets.

New aggregation operators, including the Hamacher averaging operator is developed for effective Bipolar Fermatean Fuzzy Sets.

The beneficial characteristics of these proposed operators are discussed to highlight their utility and effectiveness.

To address MADM challenges involving unknown decision makers and criteria weights, a composite Bipolar Fermatean Fuzzy based framework is proposed. This framework combines the scoring function with Hamacher aggregation operators for enhanced decision-making. A case study on choosing an e-waste salvage partner in Porcelain highlights the approach's stability and reliability within the Bipolar Fermatean Fuzzy framework Comparative study with existing approaches validates the robustness and effectiveness of our approach, underscoring its improved performance and reliability.

Structure of the study

The remainder of this study is structured as follows: Section 2 offerings a comprehensive evaluation of the proposed study. Section 3 explores the fundamental concepts of bipolar fermatean fuzzy sets. In Section 4, we introduce six aggregation operators within the context of bipolar fuzzy sets. Section 5 applies the multi-attribute decision-making technique consuming the proposed Bipolar Fermatean Fuzzy aggregation operators. Section 6 examines a case study fixated on selecting an e-waste recycling cohort in Porcelain, provides a comparison of our suggested approach with existing approaches. Section 7 defined the conclusion.

2. Preliminaries

In this section, basic definitions are defined.

Definition 1. *[47] Let us consider that $\Phi \neq X$ and by a fuzzy set* $\gamma = \left\{ \begin{array}{c} \langle x, \mu_{\gamma(x)} \rangle \\ : x \in X \end{array} \right\}$, $\mu_{\gamma(x)}$ *is a mapping from X to $[0,1]$ represent membership function of an element x in X .*

Definition 2. *[34] The fixed set C and the FFN A is defined in* $A = \left\{ \begin{array}{c} \langle DH_A(p), \\ \chi_A(p)\rangle \\ : p \in C \end{array} \right\}$, *where $DH_A(p)$ and $\chi_A(p)$ exhibit the MED and NOMED, and $DH_A(p) \in [0,1]$,$\chi_A(p) \in [0,1]$ and $0 \leq DH_A(p)^3 + \chi_A(p)^3 \leq 1$. The degree of indeterminacy is defined as*

$$\pi_A(p) = \sqrt[3]{(DH_A(p)^3 + \chi_A(p)^3 - DH_A(p)^3 \chi_A(p)^3)}.$$

The FFN is denoted as $A = \langle DH_A, \chi_A \rangle$.

Definition 3. *[23] Let $\phi_1 = \{\Psi_1, \chi_1\}$ and $\phi_2 = \{\Psi_2, \chi_2\}$ be two FFNs, $\lambda > 0$, then*

$$\phi_1 \oplus \phi_2 = \left[\begin{array}{c} \sqrt[3]{\frac{(\Psi_1)^3+(\Psi_2)^3-(\Psi_1)^3(\Psi_2)^3-(1-\lambda)(\Psi_1)^3(\Psi_2)^3}{1-(1-\lambda)(\Psi_1)^3(\Psi_2)^3}}, \\ \frac{\chi_1\chi_2}{\sqrt[3]{\lambda+(1-\lambda)(\chi_1)^3+(\chi_2)^3-(\chi_1)^3(\chi_2)^3}} \end{array} \right];$$

$$\phi_1 \otimes \phi_2 = \left[\begin{array}{c} \frac{\Psi_1\Psi_2}{\sqrt[3]{\lambda+(1-\lambda)(\Psi_1)^3+(\Psi_2)^3-(\Psi_1)^3(\Psi_2)^3}}, \\ \sqrt[3]{\frac{(\chi_1)^3+(\chi_2)^3-(\chi_1)^3(\chi_2)^3-(1-\lambda)(\chi_1)^3(\chi_2)^3}{1-(1-\lambda)(\chi_1)^3(\Psi_\chi)^3}} \end{array} \right];$$

$$\lambda\phi_1 = \left[\begin{array}{c} \sqrt[3]{\frac{(1+(\lambda-1)(\Psi_1)^3)^\lambda-(1-\Psi_1)^3)^\lambda}{(1+(\lambda-1)(1-\Psi_1^3)^\lambda+(\lambda-1)((\Psi_1)^3)^\lambda}}, \\ \frac{\sqrt[3]{\lambda}(\chi_1)^\lambda}{\sqrt[3]{(1+(\lambda-1)(\chi_1)^3)^\lambda+(\lambda-1)(1-(\chi_1)^3)^\lambda}} \end{array} \right];$$

$$\phi_1^\lambda = \left[\begin{array}{c} \frac{\sqrt[3]{\lambda}(\Psi_1^+)^\lambda}{\sqrt[3]{(1+(\lambda-1)(\Psi_1^+)^3)^\lambda+(\lambda-1)(1-(\Psi_1^+)^3)^\lambda}}, \\ \sqrt[3]{\frac{(1+(\lambda-1)(\chi_1^3)^\lambda-(1-(\chi_1^3)^\lambda}{(1+(\lambda-1)(1-\chi_1^3)^\lambda+(\lambda-1)((\chi_1^3)^\lambda}} \end{array} \right].$$

Definition 4. *[23] Let $a = \{\varsigma, \chi\}$ be the FFNs, the score function is given as $a = \varsigma_\alpha^3 - \chi_\alpha^3$.*

Definition 5. *[23] Let $a = \{\varsigma, \chi\}$ be the FFNs, the accuracy function is given as $a = \varsigma_\alpha^3 + \chi_\alpha^3$.*

Definition 6. *[49, 50] Let X be a fix set. A BFS is an object having the form $A = \{\langle E_A^+(x), \chi_A^-(x)\rangle, : x \in C\}$. The fixed set C and the BN A is defined in , where the positive membership degree function $E_A^+(x) : X \longmapsto [0,1]$ denotes the satisfaction degree of an element x to the property corresponding to a BFS A and the negative membership degree function $\chi_A^-(x) : X \longmapsto [0,1]$, denotes satisfaction degree of an element x to some implicit counter-property corresponding to a BFS A, respectively, and, for every $x \in X$. The BN is denoted as $A = \{\langle E_A^+(x), \chi_A^-(x)\rangle, : x \in C\}$.*

Definition 7. *[39] Let $a_1 = [\kappa_1^+, \varsigma_1^-]$ and $a_2 = [\kappa_2^+, \varsigma_2^-]$ be two BFHFNs and $\lambda > 0$, then*

$$a_1 \oplus a_2 = \left[\frac{(\kappa_1^+ + (\kappa_2^+ - \kappa_1^+\kappa_2^+ - (1-\lambda)\kappa_1^+\kappa_2^+}{1-(1-\lambda)\kappa_1^+\kappa_2^+}, \frac{-\varsigma_1^-\varsigma_2^-}{\lambda+(1-\lambda)(\varsigma_1^- + \varsigma_2^- - \varsigma_1^-\varsigma_2^-} \right];$$

$$a_1 \otimes a_2 = \left[\frac{\kappa_1^+\kappa_2^+}{\lambda+(1-\lambda)(\kappa_1^+ + \kappa_1^+ - \kappa_1^+\kappa_1^+}, \frac{-(\varsigma_1^- + (\varsigma_2^- - \varsigma_1^-\varsigma_2^- - (1-\lambda)\varsigma_1^-\varsigma_2^-}{1-(1-\lambda)\varsigma_1^-\varsigma_2^-} \right];$$

$$\lambda a_1 = \left[\frac{(1+(\lambda-1)(\kappa_1^+)^\lambda-(1-(\kappa_1^+)^\lambda}{(1+(\lambda-1)(1-\kappa_1^+)^\lambda+(\lambda-1)(\kappa_1^+)^\lambda}, \frac{-\lambda(\varsigma_1^-)^\lambda}{(1+(\lambda-1)(\varsigma_1^-)^\lambda+(\lambda-1)(1-\varsigma_1^-)^\lambda}\right];$$
$$a_1^\lambda = \left[\frac{\lambda(\kappa_1^+)^\lambda}{(1+(\lambda-1)(\kappa_1^+)^\lambda+(\lambda-1)(1-\kappa_1^+)^\lambda}, \frac{(1+(\lambda-1)(\varsigma_1^-)^\lambda-(1-(\varsigma_1^-)^\lambda}{(1+(\lambda-1)(\varsigma_1^-)^\lambda+(\lambda-1)(1-\varsigma_1^-)^\lambda}\right].$$

Definition 8. *[39] The BFNs are* $a = [\kappa^+, \varsigma^-]$*, then the score function* $\breve{H}$ *is define as:*$\breve{H} = \frac{1+\kappa^+-\varsigma^-}{2}$.

Definition 9. *[39] The BFNs are* $a = [\kappa^+, \varsigma^-]$*, then the accuracy function* $\breve{H}$ *is define as:*$\breve{H} = \frac{1+\kappa^++\varsigma^-}{2}$.

3. Bipolar Fermatean fuzzy Number and operational laws on hamacher

In this section, we define the definition and operational laws of BFFNs.

Definition 10. *The fixed set* C *and the BFFN* A *is defined in* $A = \left\{ \begin{array}{c} \langle E_A^+(x), \chi_A^+(x)\rangle, \\ \langle \varsigma_A^-(x), \chi_A^-(x)\rangle \\ : x \in C \end{array} \right\}$, *where* $E_A^+(x), \chi_A^+(x)$ *and* $\varsigma_A^-(x), \chi_A^-(x)$*represent the MED and NOMED, and* $E_A^+(x) \in [0,1], \chi_A^+(x) \in [0,1], \varsigma_A^-(x) \in [0,1], \chi_A^-(x) \in [0,1]$ *and* $0 \le E_A^+(x)^3\chi_A^+(x)^3 + \varsigma_A^-(x)^3\chi_A^-(x)^3 \le 1$.*The degree of indeterminacy is defined as*

$$\pi_A(x) = \sqrt[3]{\begin{array}{c}(E_A^+(x)^3\chi_A^+(x)^3 + \varsigma_A^-(x)^3\chi_A^-(x)^3 - \\ (E_A^+(x)^3\chi_A^+(x)^3)(\varsigma_A^-(x)^3\chi_A^-(x)^3))\end{array}}.$$

The BFFN is denoted as $A = \left\{\langle E_A^+(x), \chi_A^+(x)\rangle, \langle \varsigma_A^-(x), \chi_A^-(x)\rangle : x \in C\right\}$.

Definition 11. *Let* $a_1 = \left\{ \begin{array}{c} [\kappa_1^+, \varsigma_1^+], \\ [\Upsilon_1^-, \vartheta_1^-] \end{array} \right\}$ *and* $a_2 = \left\{ \begin{array}{c} [\kappa_2^+, \varsigma_2^+], \\ [\Upsilon_2^-, \vartheta_2^-] \end{array} \right\}$ *be two BFHFNs and* $\lambda > 0$*, then*

$$a_1 \oplus a_2 = \left[\begin{array}{c} \left(\begin{array}{c} \sqrt[3]{\frac{(\kappa_1^+)^3+(\kappa_2^+)^3-(\kappa_1^+)^3(\kappa_2^+)^3-(1-\lambda)(\kappa_1^+)^3(\kappa_2^+)^3}{1-(1-\lambda)(\kappa_1^+)^3(\kappa_2^+)^3}}, \\ \sqrt[3]{\frac{(\varsigma_1^+)^3+(\varsigma_2^+)^3-(\varsigma_1^+)^3(\varsigma_2^+)^3-(1-\lambda)(\varsigma_1^+)^3(\varsigma_2^+)^3}{1-(1-\lambda)(\varsigma_1^+)^3(\varsigma_2^+)^3}} \end{array}\right), \\ \left(\begin{array}{c} \frac{\Upsilon_1^-\Upsilon_2^-}{\sqrt[3]{\lambda+(1-\lambda)(\Upsilon_1^-)^3+(\Upsilon_2^-)^3-(\Upsilon_1^-)^3(\Upsilon_2^-)^3}}, \\ \frac{\vartheta_1^-\vartheta_2^-}{\sqrt[3]{\lambda+(1-\lambda)(\vartheta_1^-)^3+(\vartheta_2^-)^3-(\vartheta_1^-)^3(\vartheta_2^-)^3}} \end{array}\right) \end{array}\right];$$

$$a_1 \otimes a_2 = \left[\begin{array}{c} \left(\begin{array}{c} \frac{\kappa_1^+\kappa_2^+}{\sqrt[3]{\lambda+(1-\lambda)(\kappa_1^+)^3+(\kappa_2^+)^3-(\kappa_1^+)^3(\kappa_2^+)^3}}, \\ \frac{\varsigma_1^+\varsigma_2^+}{\sqrt[3]{\lambda+(1-\lambda)(\varsigma_1^+)^3+(\varsigma_2^+)^3-(\varsigma_1^+)^3(\varsigma_2^+)^3}} \end{array}\right) \\ \left(\begin{array}{c} \sqrt[3]{\frac{(\Upsilon_1^-)^3+(\Upsilon_2^-)^3-(\Upsilon_1^-)^3(\Upsilon_2^-)^3-(1-\lambda)(\Upsilon_1^-)^3(\Upsilon_2^-)^3}{1-(1-\lambda)(\Upsilon_1^-)^3(\Upsilon_2^-)^3}}, \\ \sqrt[3]{\frac{(\vartheta_1^-)^3+(\vartheta_2^-)^3-(\vartheta_1^-)^3(\vartheta_2^-)^3-(1-\lambda)(\vartheta_1^-)^3(\vartheta_2^-)^3}{1-(1-\lambda)(\vartheta_1^-)^3(\vartheta_2^-)^3}} \end{array}\right), \end{array}\right];$$

$$\lambda a_1 = \left[\begin{array}{c} \left(\begin{array}{c} \sqrt[3]{\frac{(1+(\lambda-1)(\kappa_1^+)^3)^\lambda-(1-(\kappa_1^+)^3)^\lambda}{(1+(\lambda-1)(1-\kappa_1^+)^3)^\lambda+(\lambda-1)((\kappa_1^+)^3)^\lambda}}, \\ \sqrt[3]{\frac{(1+(\lambda-1)(\varsigma_1^+)^3)^\lambda-(1-(\varsigma_1^+)^3)^\lambda}{(1+(\lambda-1)(\varsigma_1^+)^3)^\lambda+(\lambda-1)(1-(\varsigma_1^+)^3)^\lambda}} \end{array} \right), \\ \left(\begin{array}{c} \frac{\sqrt[3]{\lambda}(\Upsilon_1^-)^\lambda}{\sqrt[3]{(1+(\lambda-1)(\Upsilon_1^-)^3)^\lambda+(\lambda-1)(1-(\Upsilon_1^-)^3)^\lambda}}, \\ \frac{\sqrt[3]{\lambda}(\vartheta_1^-)^\lambda}{\sqrt[3]{(1+(\lambda-1)(\vartheta_1^-)^3)^\lambda+(\lambda-1)(1-(\vartheta_1^-)^3)^\lambda}} \end{array} \right) \end{array} \right];$$

$$a_1^\lambda = \left[\begin{array}{c} \left(\begin{array}{c} \frac{\sqrt[3]{\lambda}(\kappa_1^+)^\lambda}{\sqrt[3]{(1+(\lambda-1)(\kappa_1^+)^3)^\lambda+(\lambda-1)(1-(\kappa_1^+)^3)^\lambda}}, \\ \frac{\sqrt[3]{\lambda}(\varsigma_1^+)^\lambda}{\sqrt[3]{(1+(\lambda-1)(\varsigma_1^+)^3)^\lambda+(\lambda-1)(1-(\varsigma_1^+)^3)^\lambda}} \end{array} \right), \\ \left(\begin{array}{c} \sqrt[3]{\frac{(1+(\lambda-1)(\Upsilon_1^-)^3)^\lambda-(1-(\Upsilon_1^-)^3)^\lambda}{(1+(\lambda-1)(1-\Upsilon_1^-)^3)^\lambda+(\lambda-1)((\Upsilon_1^-)^3)^\lambda}}, \\ \sqrt[3]{\frac{(1+(\lambda-1)(\vartheta_1^-)^3)^\lambda-(1-(\vartheta_1^-)^3)^\lambda}{(1+(\lambda-1)(\vartheta_1^-)^3)^\lambda+(\lambda-1)(1-(\vartheta_1^-)^3)^\lambda}} \end{array} \right) \end{array} \right].$$

Definition 12. *The BFFNs are* $a = \left\{ \begin{array}{c} [\kappa^+, \varsigma^+], \\ [\Upsilon^-, \vartheta^-] \end{array} \right\}$*, then the score function* $\breve{H}$ *is define as:*$\breve{H} = \frac{\{[(\kappa^+)^3+(\varsigma^+)^3]-[(\Upsilon^-)^3+(\vartheta^-)^3]\}}{4}$.

Definition 13. *The BFFNs are* $a = \left\{ \begin{array}{c} [\kappa^+, \varsigma^+], \\ [\Upsilon^-, \vartheta^-] \end{array} \right\}$*, then the accuracy function* $\breve{H}$ *is define as:*$\breve{H} = \frac{\{[(\kappa^+)^3+(\varsigma^+)^3]+[(\Upsilon^-)^3+(\vartheta^-)^3]\}}{4}$.

4. Bipolar Fermatean Fuzzy aggregation operator based on Hamacher

This section defines the BFHFWA, BFHFOWA and BFHFHWA operators.

4.1. Bipolar Fermatean Fuzzy Hamacher weighted average operator

Definition 14. *The set of BFFNs can be represented as* $h_j = \left\{ \begin{array}{c} [p^+, r^+], \\ [q^-, r^-] \end{array} \right\}$ *and the weight vector is* $g = (g_1, g_2, ..., g_n)^T$ *with* $g_j \in [0, 1]$ *and* $\sum_{j=1}^{n} g_j = 1$. *Then* $BFHFWA(h_1, h_2, ..., h_n) = \bigoplus_{j=1}^{n} g_j h_j$ *is said BFHFWA operator.*

Theorem 1. *The gathering of BFFNs are* $a_j = \left\{ \begin{array}{c} [\kappa^+, \varsigma^+], \\ [\Upsilon^-, \vartheta^-] \end{array} \right\}$ *and the weight vector is* $g = (g_1, g_2, ..., g_L)^T$ *with* $g_j \in [0, 1]$ *and* $\sum_{j=1}^{L} g_j = 1$. *Then it is said BFHFWA operator and* $BFHFWA(a_1, a_2, ..., a_L) =$

$$\left[\begin{array}{c}\left(\sqrt[3]{\frac{\prod_{j=1}^{L}(1+(g-1)(\kappa_j^+)^3)^g-\prod_{j=1}^{L}(1-(\kappa_j^+)^3)^g}{\prod_{j=1}^{L}(1+(g-1)(1-\kappa_j^+)^3)^g+(g-1)\prod_{j=1}^{L}((1-\kappa_j^+)^3)^g}}, \sqrt[3]{\frac{\prod_{j=1}^{L}(1+(g-1)(\varsigma_j^+)^3)^g-\prod_{j=1}^{L}(1-(\varsigma_j^+)^3)^g}{\prod_{j=1}^{L}(1+(g-1)(\varsigma_j^+)^3)^g+(g-1)\prod_{j=1}^{L}(1-(\varsigma_j^+)^3)^g}}\right),\\ \left(\frac{\sqrt[3]{g}\prod_{j=1}^{L}(\Upsilon_j^-)^g}{\sqrt[3]{\prod_{j=1}^{L}(1+(g-1)(\Upsilon_j^-)^3)^g+(g-1)\prod_{j=1}^{L}(1-(\Upsilon_j^-)^3)^g}}, \frac{\sqrt[3]{g}\prod_{j=1}^{L}(\vartheta_j^-)^g}{\sqrt[3]{\prod_{j=1}^{L}(1+(g-1)(\vartheta_j^-)^3)^g+(g-1)\prod_{j=1}^{L}(1-(\vartheta_j^-)^3)^g}}\right)\end{array}\right].$$

Proof. Since L is true and $L=1$

$$g_1a_1=\left[\begin{array}{c}\left(\sqrt[3]{\frac{\prod_{j=1}^{L}(1+(g-1)(\kappa_1^+)^3)^{g_1}-\prod_{j=1}^{L}(1-(\kappa_1^+)^3)^{g_1}}{\prod_{j=1}^{L}(1+(g-1)(1-\kappa_1^+)^3)^{g_1}+(g-1)\prod_{j=1}^{L}((1-\kappa_1^+)^3)^{g_1}}}, \sqrt[3]{\frac{\prod_{j=1}^{L}(1+(g-1)(\varsigma_1^+)^3)^{g_1}-\prod_{j=1}^{L}(1-(\varsigma_1^+)^3)^{g_1}}{\prod_{j=1}^{L}(1+(g-1)(\varsigma_1^+)^3)^{g_1}+(g-1)\prod_{j=1}^{L}(1-(\varsigma_1^+)^3)^{g_1}}}\right),\\ \left(\frac{\sqrt[3]{g}\prod_{j=1}^{L}(\Upsilon_1^-)^{g_1}}{\sqrt[3]{\prod_{j=1}^{L}(1+(g-1)(\Upsilon_1^-)^3)^{g_1}+(g-1)\prod_{j=1}^{L}(1-(\Upsilon_1^-)^3)^{g_1}}}, \frac{\sqrt[3]{g}\prod_{j=1}^{L}(\vartheta_1^-)^{g_1}}{\sqrt[3]{\prod_{j=1}^{L}(1+(g-1)(\vartheta_1^-)^3)^{g_1}+(g-1)\prod_{j=1}^{L}(1-(\vartheta_1^-)^3)^{g_1}}}\right)\end{array}\right]$$

$$g_1 a_1 = \left[\begin{array}{c} \left(\begin{array}{c} \sqrt[3]{\frac{(1+(g-1)(\kappa_1^+)^3)^{g_1}-(1-(\kappa_1^+)^3)^{g_1}}{(1+(g-1)(1-\kappa_1^+)^3)^{g_1}+(g-1)((1-\kappa_1^+)^3)^{g_1}}}, \\ \sqrt[3]{\frac{(1+(g-1)(\varsigma_1^+)^3)^{g_1}-(1-(\varsigma_1^+)^3)^{g_1}}{(1+(g-1)(\varsigma_1^+)^3)^{g_1}+(g-1)(1-(\varsigma_1^+)^3)^{g_1}}} \end{array} \right), \\ \left(\begin{array}{c} \frac{\sqrt[3]{g}(\Upsilon_1^-)^{g_1}}{\sqrt[3]{(1+(g-1)(\Upsilon_1^-)^3)^{g_1}+(g-1)(1-(\Upsilon_1^-)^3)^{g_1}}}, \\ \frac{\sqrt[3]{g}(\vartheta_1^-)^{g_1}}{\sqrt[3]{(1+(g-1)(\vartheta_1^-)^3)^{g_1}+(g-1)(1-(\vartheta_1^-)^3)^{g_1}}} \end{array} \right) \end{array} \right]$$

$$g_2 a_2 = \left[\begin{array}{c} \left(\begin{array}{c} \sqrt[3]{\frac{(1+(g-1)(\kappa_2^+)^3)^{g_2}-(1-(\kappa_2^+)^3)^{g_2}}{(1+(g-1)(1-\kappa_2^+)^3)^{g_2}+(g-1)((1-\kappa_2^+)^3)^{g_2}}}, \\ \sqrt[3]{\frac{(1+(g-1)(\varsigma_2^+)^3)^{g_2}-(1-(\varsigma_2^+)^3)^{g_2}}{(1+(g-1)(\varsigma_2^+)^3)^{g_2}+(g-1)(1-(\varsigma_2^+)^3)^{g_2}}} \end{array} \right), \\ \left(\begin{array}{c} \frac{\sqrt[3]{g}(\Upsilon_2^-)^{g_2}}{\sqrt[3]{(1+(g-1)(\Upsilon_2^-)^3)^{g_2}+(g-1)(1-(\Upsilon_2^-)^3)^{g_2}}}, \\ \frac{\sqrt[3]{g}(\vartheta_1^-)^{g_2}}{\sqrt[3]{(1+(g-1)(\vartheta_2^-)^3)^{g_2}+(g-1)(1-(\vartheta_2^-)^3)^{g_2}}} \end{array} \right) \end{array} \right]$$

$$g_1 a_1 \oplus g_2 a_2 = \left[\begin{array}{c} \left(\begin{array}{c} \sqrt[3]{\frac{(1+(g-1)(\kappa_1^+)^3)^{g_1}-(1-(\kappa_1^+)^3)^{g_1}}{(1+(g-1)(1-\kappa_1^+)^3)^{g_1}+(g-1)((1-\kappa_1^+)^3)^{g_1}}}, \\ \sqrt[3]{\frac{(1+(g-1)(\varsigma_1^+)^3)^{g_1}-(1-(\varsigma_1^+)^3)^{g_1}}{(1+(g-1)(\varsigma_1^+)^3)^{g_1}+(g-1)(1-(\varsigma_1^+)^3)^{g_1}}} \end{array} \right), \\ \left(\begin{array}{c} \frac{\sqrt[3]{g}(\Upsilon_1^-)^{g_1}}{\sqrt[3]{(1+(g-1)(\Upsilon_1^-)^3)^{g_1}+(g-1)(1-(\Upsilon_1^-)^3)^{g_1}}}, \\ \frac{\sqrt[3]{g}(\vartheta_1^-)^{g_1}}{\sqrt[3]{(1+(g-1)(\vartheta_1^-)^3)^{g_1}+(g-1)(1-(\vartheta_1^-)^3)^{g_1}}} \end{array} \right) \end{array} \right] \oplus$$

$$\left[\begin{array}{c} \left(\begin{array}{c} \sqrt[3]{\frac{(1+(g-1)(\kappa_2^+)^3)^{g_2}-(1-(\kappa_2^+)^3)^{g_2}}{(1+(g-1)(1-\kappa_2^+)^3)^{g_2}+(g-1)((1-\kappa_2^+)^3)^{g_2}}}, \\ \sqrt[3]{\frac{(1+(g-1)(\varsigma_2^+)^3)^{g_2}-(1-(\varsigma_2^+)^3)^{g_2}}{(1+(g-1)(\varsigma_2^+)^3)^{g_2}+(g-1)(1-(\varsigma_2^+)^3)^{g_2}}} \end{array} \right), \\ \left(\begin{array}{c} \frac{\sqrt[3]{g}(\Upsilon_2^-)^{g_2}}{\sqrt[3]{(1+(g-1)(\Upsilon_2^-)^3)^{g_2}+(g-1)(1-(\Upsilon_2^-)^3)^{g_2}}}, \\ \frac{\sqrt[3]{g}(\vartheta_1^-)^{g_2}}{\sqrt[3]{(1+(g-1)(\vartheta_2^-)^3)^{g_2}+(g-1)(1-(\vartheta_2^-)^3)^{g_2}}} \end{array} \right) \end{array} \right]$$

Since $L = k$

$\text{BFHFWA}(a_1, a_2, ..., a_L) =$

$$\left[\begin{array}{c}\left(\begin{array}{c}\sqrt[3]{\dfrac{\prod_{j=1}^{k}(1+(g-1)(\kappa_j^{+})^3)^g-\prod_{j=1}^{k}(1-(\kappa_j^{+})^3)^g}{\prod_{j=1}^{k}(1+(g-1)(1-\kappa_j^{+})^3)^g+(g-1)\prod_{j=1}^{k}((1-\kappa_j^{+})^3)^g}},\\ \sqrt[3]{\dfrac{\prod_{j=1}^{k}(1+(g-1)(\varsigma_j^{+})^3)^g-\prod_{j=1}^{k}(1-(\varsigma_j^{+})^3)^g}{\prod_{j=1}^{k}(1+(g-1)(\varsigma_j^{+})^3)^g+(g-1)\prod_{j=1}^{k}(1-(\varsigma_j^{+})^3)^g}}\end{array}\right),\\ \left(\begin{array}{c}\dfrac{\sqrt[3]{g}\prod_{j=1}^{k}(\Upsilon_j^{-})^g}{\sqrt[3]{\prod_{j=1}^{k}(1+(g-1)(\Upsilon_j^{-})^3)^g+(g-1)\prod_{j=1}^{k}(1-(\Upsilon_j^{-})^3)^g}},\\ \dfrac{\sqrt[3]{g}\prod_{j=1}^{k}(\vartheta_j^{-})^g}{\sqrt[3]{\prod_{j=1}^{k}(1+(g-1)(\vartheta_j^{-})^3)^g+(g-1)\prod_{j=1}^{k}(1-(\vartheta_j^{-})^3)^g}}\end{array}\right)\end{array}\right]$$

Since $L = k+1$

$\text{BFHFWA}(a_1, a_2, ..., a_L) =$

$$\left[\begin{array}{c}\left(\begin{array}{c}\sqrt[3]{\dfrac{\prod_{j=1}^{k+1}(1+(g-1)(\kappa_j^{+})^3)^g-\prod_{j=1}^{k+1}(1-(\kappa_j^{+})^3)^g}{\prod_{j=1}^{k+1}(1+(g-1)(1-\kappa_j^{+})^3)^g+(g-1)\prod_{j=1}^{k+1}((1-\kappa_j^{+})^3)^g}},\\ \sqrt[3]{\dfrac{\prod_{j=1}^{k+1}(1+(g-1)(\varsigma_j^{+})^3)^g-\prod_{j=1}^{k+1}(1-(\varsigma_j^{+})^3)^g}{\prod_{j=1}^{k+1}(1+(g-1)(\varsigma_j^{+})^3)^g+(g-1)\prod_{j=1}^{k+1}(1-(\varsigma_j^{+})^3)^g}}\end{array}\right),\\ \left(\begin{array}{c}\dfrac{\sqrt[3]{g}\prod_{j=1}^{k+1}(\Upsilon_j^{-})^g}{\sqrt[3]{\prod_{j=1}^{k+1}(1+(g-1)(\Upsilon_j^{-})^3)^g+(g-1)\prod_{j=1}^{k+1}(1-(\Upsilon_j^{-})^3)^g}},\\ \dfrac{\sqrt[3]{g}\prod_{j=1}^{k+1}(\vartheta_j^{-})^g}{\sqrt[3]{\prod_{j=1}^{k+1}(1+(g-1)(\vartheta_j^{-})^3)^g+(g-1)\prod_{j=1}^{k+1}(1-(\vartheta_j^{-})^3)^g}}\end{array}\right)\end{array}\right]$$

Theorem 2. ***(Idempotency):****If* $\widetilde{V} = \left\{\begin{array}{c}[q^{+}, \varsigma^{+}],\\ [q^{-}, \vartheta^{-}]\end{array}\right\}$ *for all* $P = 1, 2, 3, ..., m,$ *then*

$BFHFWA(V, V, V, ..., V) = V.$

Proof. Since $\widetilde{V_P} = \widetilde{V}$ are equal to $\left\{ \begin{array}{c} [q^+, \varsigma^+], \\ [q^-, \vartheta^-] \end{array} \right\}$ for $P = 1, 2, 3, ..., m$, then

$$\text{BFHFWA}(V,V,V,...,V) = \left[\begin{array}{c} \left(\begin{array}{c} \sqrt[3]{\frac{\prod_{j=1}^{m}(1+(g-1)(q_j^+)^3)^g - \prod_{j=1}^{m}(1-(q_j^+)^3)^g}{\prod_{j=1}^{m}(1+(g-1)(1-q_j^+)^3)^g + (g-1)\prod_{j=1}^{m}((1-q_j^+)^3)^g}}, \\ \sqrt[3]{\frac{\prod_{j=1}^{m}(1+(g-1)(\varsigma_j^+)^3)^g - \prod_{j=1}^{m}(1-(\varsigma_j^+)^3)^g}{\prod_{j=1}^{m}(1+(g-1)(\varsigma_j^+)^3)^g + (g-1)\prod_{j=1}^{m}(1-(\varsigma_j^+)^3)^g}} \end{array} \right), \\ \left(\begin{array}{c} \frac{\sqrt[3]{g}\prod_{j=1}^{m}(q_j^-)^g}{\sqrt[3]{\prod_{j=1}^{m}(1+(g-1)(q_j^-)^3)^g + (g-1)\prod_{j=1}^{m}(1-(q_j^-)^3)^g}}, \\ \frac{\sqrt[3]{g}\prod_{j=1}^{m}(\vartheta_j^-)^g}{\sqrt[3]{\prod_{j=1}^{m}(1+(g-1)(\vartheta_j^-)^3)^g + (g-1)\prod_{j=1}^{m}(1-(\vartheta_j^-)^3)^g}} \end{array} \right) \end{array} \right]$$

$$= \left[\begin{array}{c} \left(\begin{array}{c} \sqrt[3]{\frac{(1+(g-1)(q_j^+)^3)^g - (1-(q_j^+)^3)^g}{(1+(g-1)(1-q_j^+)^3)^g + (g-1)((1-q_j^+)^3)^g}}, \\ \sqrt[3]{\frac{(1+(g-1)(\varsigma_j^+)^3)^g - (1-(\varsigma_j^+)^3)^g}{(1+(g-1)(\varsigma_j^+)^3)^g + (g-1)(1-(\varsigma_j^+)^3)^g}} \end{array} \right), \\ \left(\begin{array}{c} \frac{\sqrt[3]{g}(q_j^-)^g}{\sqrt[3]{(1+(g-1)(q_j^-)^3)^g + (g-1)(1-(q_j^-)^3)^g}}, \\ \frac{\sqrt[3]{g}(\vartheta_j^-)^g}{\sqrt[3]{(1+(g-1)(\vartheta_j^-)^3)^g + (g-1)(1-(\vartheta_j^-)^3)^g}} \end{array} \right) \end{array} \right] =$$

$$\left\{ \begin{array}{c} [q^+, \varsigma^+], \\ [q^-, \vartheta^-] \end{array} \right\} = \widetilde{V}$$

$$\text{BFHFWA}(V,V,V,...,V) = \widetilde{V}$$

Theorem 3. ***(Boundedness):*** *If* $Y^- = min(y_1, y_2, ..., y_n)$, $Y^+ = \max(y_1, y_2, ..., y_n)$, *then* $Y^- \leq$*BFHFWA*$(y_1, y_2, ..., y_n) \leq Y^+$.

Proof. Let $Y = \left\{ \begin{array}{c} [q^+, \varsigma^+], \\ [q^-, \vartheta^-] \end{array} \right\}$ be the accumulation of BFFNs

$Y^- = min(y_1, y_2, ..., y_n) = \left\{ \begin{array}{c} [q^+, \varsigma^+], \\ [q^-, \vartheta^-] \end{array} \right\}$,

$Y^+ = \max(y_1, y_2, ..., y_n) = \left\{ \begin{array}{c} [q^+, \varsigma^+], \\ [q^-, \vartheta^-] \end{array} \right\}$

Since $\min_j(q^+) \leq \max_j(q^+)$,$\min_j(\varsigma^+) \leq \max_j(\varsigma^+)$,

$\min_j(q^-) \leq \max_j(q^-)$,$\min_j(\vartheta^-) \leq \max_j(\vartheta^-)$

Which implies

$$\sqrt[3]{\frac{(1+(g-1)(q_j^+)^3)^g-(1-(q_j^+)^3)^g}{(1+(g-1)(1-q_j^+)^3)^g+(g-1)((1-q_j^+)^3)^g}}$$

$$\geq \sqrt[3]{\frac{(1+(g-1)(min_j(q_j^+)^3)^g-(1-(min_j(q_j^+)^3)^g}{(1+(g-1)(1-(min_j(q_j^+)^3)^g+(g-1)((1-(min_j(q_j^+)^3)^g}}$$

$=\min_j(q^+)$;

$$\sqrt[3]{\frac{(1+(g-1)(\varsigma_j^+)^3)^g-(1-(\varsigma_j^+)^3)^g}{(1+(g-1)(1-\varsigma_j^+)^3)^g+(g-1)((1-\varsigma_j^+)^3)^g}}$$

$$\geq \sqrt[3]{\frac{(1+(g-1)(min_j(\varsigma_j^+)^3)^g-(1-(min_j(\varsigma_j^+)^3)^g}{(1+(g-1)(1-(min_j(\varsigma_j^+)^3)^g+(g-1)((1-(min_j(\varsigma_j^+)^3)^g}}$$

$=\min_j(\varsigma^+)$;

$$\frac{\sqrt[3]{g}(q_j^-)^g}{\sqrt[3]{(1+(g-1)(q_j^-)^3)^g+(g-1)(1-(q_j^-)^3)^g}}$$

$$\geq \frac{\sqrt[3]{g}(min_j(q_j^-)^g)}{\sqrt[3]{(1+(g-1)(min_j(q_j^-)^3)^g)+(g-1)(1-(min_j(q_j^-)^3)^g)}}$$

$= min_j(q_j^-)$

$$\frac{\sqrt[3]{g}(\vartheta_j^-)^g}{\sqrt[3]{(1+(g-1)(\vartheta_j^-)^3)^g+(g-1)(1-(\vartheta_j^-)^3)^g}}$$

$$\geq \frac{\sqrt[3]{g}(\max_j(\vartheta_j^-)^g)}{\sqrt[3]{(1+(g-1)(\max_j(\vartheta_j^-)^3)^g)+(g-1)(1-(\max_j(\vartheta_j^-)^3)^g)}}$$

$= \max_j(\vartheta_j^-)$

$\text{BFHFWA}(y_1, y_2, ..., y_n) = \{[q^+, \varsigma^+], [q^-, \vartheta^-]\}$

$S(Y) = \frac{\{[(q^+)^3+(\varsigma^+)^3]-[(q^-)^3+(\vartheta^-)^3]\}}{4}$

$\leq \frac{\{[\max_j(q^+)^3+\max_j(\varsigma^+)^3]-[min_j(q^-)^3+min_j(\vartheta^-)^3]\}}{4}$

$= S(Y^-)$

$S(Y) = \frac{\{[(q^+)^3+(\varsigma^+)^3]-[(q^-)^3+(\vartheta^-)^3]\}}{4}$

$\geq \frac{\{[\max_j(q^+)^3+\max_j(\varsigma^+)^3]-[min_j(q^-)^3+min_j(\vartheta^-)^3]\}}{4}$

$= S(Y^+)$

$S(Y^-) \geq S(Y^+)$

$\text{BFHFWA}(y_1, y_2, ..., y_n) = Y^-$ and

$\text{BFHFWA}(y_1, y_2, ..., y_n) = Y^+$

4.2. Bipolar Fermatean Fuzzy Hamacher Ordered Weighted Average operator

Definition 15. *The gathering of BFFNs are* $f_j = \left\{ \begin{array}{c} [p^+, r^+], \\ [q^-, r^-] \end{array} \right\}$ *and the weight vector is* $g = (g_1, g_2, ..., g_n)^T$ *with* $g_j \in [0,1]$ *and* $\sum_{j=1}^{n} g_j = 1$. *Then* $BFHFOWA(f_1, f_2, ..., f_n) =$

$\bigoplus_{j=1}^{n} g_j f_j$ *is expressed BFHFOWA operator.*

Theorem 4. *The gathering of BFFNs are* $a_j = \left\{ \begin{array}{c} [\kappa^+, \varsigma^+], \\ [\Upsilon^-, \vartheta^-] \end{array} \right\}$ *and the weight vector is* $g = (g_1, g_2, ..., g_n)^T$ *with* $g_j \in [0, 1]$ *and* $\sum_{j=1}^{n} g_j = 1$*. Then it is said BFHFOWA operator and* $BFHFOWA(a_1, a_2, ..., a_n) =$

$$\left[\begin{array}{c} \left(\begin{array}{c} \sqrt[3]{\dfrac{\prod_{j=1}^{n}(1+(g-1)(\kappa_j^+)^3)^g - \prod_{j=1}^{n}(1-(\kappa_j^+)^3)^g}{\prod_{j=1}^{n}(1+(g-1)(1-\kappa_j^+)^3)^g + (g-1)\prod_{j=1}^{n}((1-\kappa_j^+)^3)^g}}, \\ \sqrt[3]{\dfrac{\prod_{j=1}^{n}(1+(g-1)(\varsigma_j^+)^3)^g - \prod_{j=1}^{n}(1-(\varsigma_j^+)^3)^g}{\prod_{j=1}^{n}(1+(g-1)(\varsigma_j^+)^3)^g + (g-1)\prod_{j=1}^{n}(1-(\varsigma_j^+)^3)^g}} \end{array} \right), \\ \left(\begin{array}{c} \dfrac{\sqrt[3]{g}\prod_{j=1}^{n}(\Upsilon_j^-)^g}{\sqrt[3]{\prod_{j=1}^{n}(1+(g-1)(\Upsilon_j^-)^3)^g + (g-1)\prod_{j=1}^{n}(1-(\Upsilon_j^-)^3)^g}}, \\ \dfrac{\sqrt[3]{g}\prod_{j=1}^{n}(\vartheta_j^-)^g}{\sqrt[3]{\prod_{j=1}^{n}(1+(g-1)(\vartheta_j^-)^3)^g + (g-1)\prod_{j=1}^{n}(1-(\vartheta_j^-)^3)^g}} \end{array} \right) \end{array} \right].$$

Proof:This proof is straightforward

Theorem 5. ***(Idempotency):*** *If* $\widetilde{XD} = \left\{ \begin{array}{c} [r^+, s^+], \\ [r^-, s^-] \end{array} \right\}$ *for all* $k = 1, 2, 3, ..., n$*, then* $BFHFOWA(XD, XD, XD, ..., XD) = XD$*.*

Proof:This proof is straightforward

4.3. Bipolar Fermatean Fuzzy-Hamacher hybrid Weighted Average operator

Definition 16. *The gathering of BFFNs are* $f_k = \left\{ \begin{array}{c} [p^+, r^+], \\ [q^-, r^-] \end{array} \right\}$ *and the weight vector is* $e = (e_1, e_2, ..., e_n)^T$ *with* $e_k \in [0, 1]$ *and* $\sum_{k=1}^{n} e_k = 1$*. Then* $BFHFHWA\,(f_1, f_2, ..., f_n) = \bigoplus_{k=1}^{n} e_k f_k$ *is expressed BFHFHWA operator.*

Theorem 6. *The gathering of BFFNs are* $h_j = \left\{ \begin{array}{c} [\kappa^+, \varsigma^+], \\ [\Upsilon^-, \vartheta^-] \end{array} \right\}$ *and the weight vector is* $f = (f_1, f_2, ..., f_n)^T$ *with* $f_j \in [0,1]$ *and* $\sum_{j=1}^{n} f_j = 1$*. Then it is said BFHFHWA operator and* $BFHFHWA(h_1, h_2, ..., h_n) =$

$$\left[\begin{array}{c} \left(\begin{array}{c} \sqrt[3]{\dfrac{\prod_{j=1}^{n}(1+(f-1)(\kappa_j^+)^3)^f - \prod_{j=1}^{n}(1-(\kappa_j^+)^3)^f}{\prod_{j=1}^{n}(1+(f-1)(1-\kappa_j^+)^3)^f + (f-1)\prod_{j=1}^{n}((1-\kappa_j^+)^3)^f}}, \\ \sqrt[3]{\dfrac{\prod_{j=1}^{n}(1+(f-1)(\varsigma_j^+)^3)^f - \prod_{j=1}^{n}(1-(\varsigma_j^+)^3)^f}{\prod_{j=1}^{n}(1+(f-1)(\varsigma_j^+)^3)^f + (f-1)\prod_{j=1}^{n}(1-(\varsigma_j^+)^3)^f}} \end{array} \right), \\ \left(\begin{array}{c} \dfrac{\sqrt[3]{f}\prod_{j=1}^{n}(\Upsilon_j^-)^f}{\sqrt[3]{\prod_{j=1}^{n}(1+(f-1)(\Upsilon_j^-)^3)^f + (f-1)\prod_{j=1}^{n}(1-(\Upsilon_j^-)^3)^f}}, \\ \dfrac{\sqrt[3]{f}\prod_{j=1}^{n}(\vartheta_j^-)^f}{\sqrt[3]{\prod_{j=1}^{n}(1+(f-1)(\vartheta_j^-)^3)^f + (f-1)\prod_{j=1}^{n}(1-(\vartheta_j^-)^3)^f}} \end{array} \right) \end{array} \right].$$

Proof:This proof is straightforward

Theorem 7. ***(Idempotency):****If* $\widetilde{RY} = \left\{ \begin{array}{c} [\kappa^+, \varsigma^+], \\ [\Upsilon^-, \vartheta^-] \end{array} \right\}$ *for all* $L = 1, 2, 3, ..., m$*, then* $BFHFHWA(RY, RY, RY, ..., RY) = RY$*.*

Proof:This proof is straightforward

Theorem 8. ***(Boundedness):****If* $Y^- = min(b_1, b_2, ..., b_n)$*,* $Y^+ = \max(b_1, b_2, ..., b_n)$*, then* $Y^- \leq BFHFHWA(b_1, b_2, ..., b_n) \leq Y^+$*.*

Proof:This proof is straightforward

5. Multiple Criteria Decision Making technique on Bipolar fermatean fuzzy idea

Step 1:Explain the BFF decision matrix
Step 2:Explain the BFHFWA operator and $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$.
BFHFWA$(a_1, a_2, ..., a_n)$ =

$$
\left[\begin{array}{c}
\left(\begin{array}{c}
\sqrt[3]{\dfrac{\prod\limits_{j=1}^{n}(1+(\lambda-1)(\kappa_j^{+})^3)^{\lambda}-\prod\limits_{j=1}^{n}(1-(\kappa_j^{+})^3)^{\lambda}}{\prod\limits_{j=1}^{n}(1+(\lambda-1)(1-\kappa_j^{+})^3)^{\lambda}+(\lambda-1)\prod\limits_{j=1}^{n}((1-\kappa_j^{+})^3)^{\lambda}}},\\
\sqrt[3]{\dfrac{\prod\limits_{j=1}^{n}(1+(\lambda-1)(\varsigma_j^{+})^3)^{\lambda}-\prod\limits_{j=1}^{n}(1-(\varsigma_j^{+})^3)^{\lambda}}{\prod\limits_{j=1}^{n}(1+(\lambda-1)(\varsigma_j^{+})^3)^{\lambda}+(\lambda-1)\prod\limits_{j=1}^{n}(1-(\varsigma_j^{+})^3)^{\lambda}}}
\end{array}\right),\\
\left(\begin{array}{c}
\dfrac{\sqrt[3]{\lambda}\prod\limits_{j=1}^{n}(\Upsilon_j^{-})^{\lambda}}{\sqrt[3]{\prod\limits_{j=1}^{n}(1+(\lambda-1)(\Upsilon_j^{-})^3)^{\lambda}+(\lambda-1)\prod\limits_{j=1}^{n}(1-(\Upsilon_j^{-})^3)^{\lambda}}},\\
\dfrac{\sqrt[3]{\lambda}\prod\limits_{j=1}^{n}(\vartheta_j^{-})^{\lambda}}{\sqrt[3]{\prod\limits_{j=1}^{n}(1+(\lambda-1)(\vartheta_j^{-})^3)^{\lambda}+(\lambda-1)\prod\limits_{j=1}^{n}(1-(\vartheta_j^{-})^3)^{\lambda}}}
\end{array}\right)
\end{array}\right]
$$

Step 3:Describe the BFHFWA operator and $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$.

$\mathrm{BFHFWA}(a_1, a_2, ..., a_n) =$

$$
\left[\begin{array}{c}
\left(\begin{array}{c}
\sqrt[3]{\dfrac{\prod\limits_{j=1}^{n}(1+(\lambda-1)(\kappa_j^{+})^3)^{\lambda}-\prod\limits_{j=1}^{n}(1-(\kappa_j^{+})^3)^{\lambda}}{\prod\limits_{j=1}^{n}(1+(\lambda-1)(1-\kappa_j^{+})^3)^{\lambda}+(\lambda-1)\prod\limits_{j=1}^{n}((1-\kappa_j^{+})^3)^{\lambda}}},\\
\sqrt[3]{\dfrac{\prod\limits_{j=1}^{n}(1+(\lambda-1)(\varsigma_j^{+})^3)^{\lambda}-\prod\limits_{j=1}^{n}(1-(\varsigma_j^{+})^3)^{\lambda}}{\prod\limits_{j=1}^{n}(1+(\lambda-1)(\varsigma_j^{+})^3)^{\lambda}+(\lambda-1)\prod\limits_{j=1}^{n}(1-(\varsigma_j^{+})^3)^{\lambda}}}
\end{array}\right),\\
\left(\begin{array}{c}
\dfrac{\sqrt[3]{\lambda}\prod\limits_{j=1}^{n}(\Upsilon_j^{-})^{\lambda}}{\sqrt[3]{\prod\limits_{j=1}^{n}(1+(\lambda-1)(\Upsilon_j^{-})^3)^{\lambda}+(\lambda-1)\prod\limits_{j=1}^{n}(1-(\Upsilon_j^{-})^3)^{\lambda}}},\\
\dfrac{\sqrt[3]{\lambda}\prod\limits_{j=1}^{n}(\vartheta_j^{-})^{\lambda}}{\sqrt[3]{\prod\limits_{j=1}^{n}(1+(\lambda-1)(\vartheta_j^{-})^3)^{\lambda}+(\lambda-1)\prod\limits_{j=1}^{n}(1-(\vartheta_j^{-})^3)^{\lambda}}}
\end{array}\right)
\end{array}\right]
$$

Step 4:Find the score function

Step 5:Find the ranking.

6. Case study

This section examines the potential environmental of the health impacts of e-waste, focusing on the complex challenges and issues related with e-waste organization in Porcelain. Additionally, it aims to provide valuable insights for improving the country's e-waste recycling framework. Presently, China stands as a significant consumer of electronic products and a major importer of e-waste, driven by the rapid advancement of electrical and electronic systems alongside its economic growth.

Domestic e-waste flows in China

China currently has three main types of sites for e-waste disposal. First, used electronics and appliances are commonly sold in second-hand markets. Instead of discarding old household items, many consumers prefer to keep them in their homes or offices, and they are willing to sell e-waste if offered a reasonable price. Outdated appliances are frequently recycled by private companies that focus on extracting raw materials. These recyclers typically acquire waste electrical and electronic equipment from households at low prices but often lack the necessary facilities for safe disposal. As a result, this method of e-waste disposal can lead to significant environmental pollution [46].

Casual recycling area of Porcelain

In Porcelain, most domestic e-waste is funneled into an informal recycling sector that also handles imported waste electrical and electronic equipment. By 2007, this industry employed more than 700,000 people, with 98% working in unregulated recycling operations (Ongondo et al.[30]), acknowledged as the largest e-waste recycling site in both Porcelain and globally, has a population of approximately 150,000, including nearly 100,000 migrant workers engaged in e-waste recycling activities. These facilities usually consist of numerous small workshops focused on recycling waste electrical and electronic equipment. However, the recycling techniques used are often outdated and basic. These methods include: (1) dismantling electronic devices; (2) heating and manually removing components from printed circuit boards; (3) burning cables and wires to retrieve valuable metals; (4) melting and shredding plastics; (5) collecting toner; and (6) performing open acid leaching on e-waste to extract valuable metals.

Figure 2 is given as below

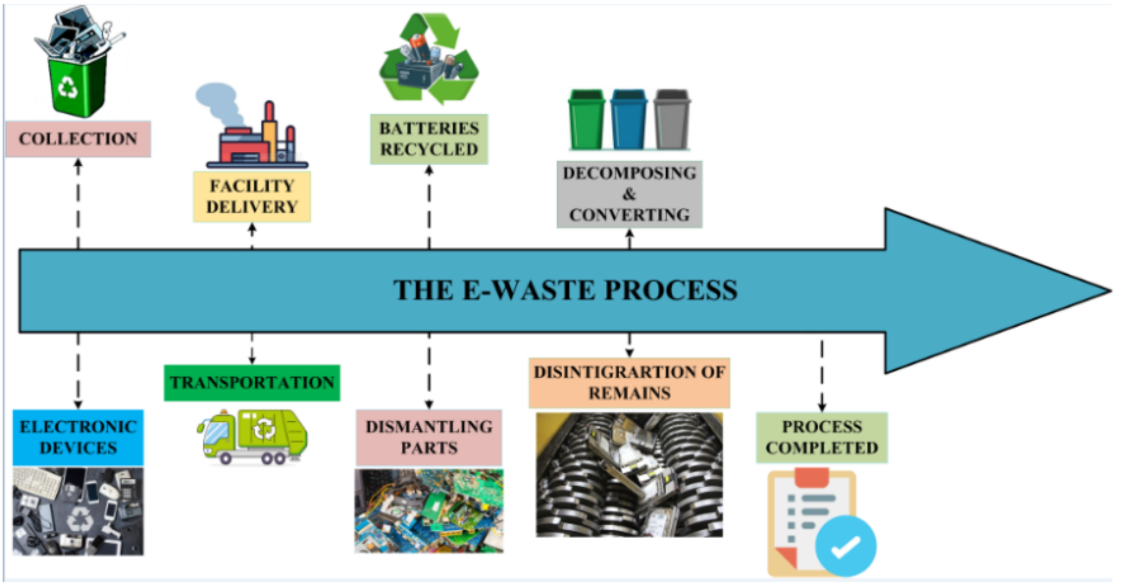

Figure 2: The e-waste process

To establish a comprehensive excess electrical and electronic apparatus recycling scheme, the National Growth and Improvement Commission designated Qingdao Haier, Hangzhou Dadi, Beijing Huaxing, and Tianjin Datong as national pilot projects in 2004. However, progress has been limited since then. For example, Haier, the fourth-largest white goods manufacturer in China, was selected to develop a producer-responsibility recycling model to improve the collection of used household appliances. In collaboration with Tsinghua University, Haier sought to enhance recycling technology. Despite these efforts, by May 2007, the company had disposed of only 8,000 domestic appliances, equating to an annual collection rate of approximately 600,000 devices.

In parallel, isolated ecological organizations have also taken initiatives to increase e-waste recovery rates. Shenzhen Green Eco-Manufacture Tech Co., Ltd. opened its first e-waste recycling supermarket in Wuhan [22]. GEM established specific pricing for used or obsolete electrical and electronic equipment based on their condition. In addition, the company has formed strategic partnerships with retailers such as Wuhan Zhongbai, Gome, and Suning, aiming to reduce electronic waste and promote low-carbon feasting through market-driven plans. Figure 3 is material of the case study is given as

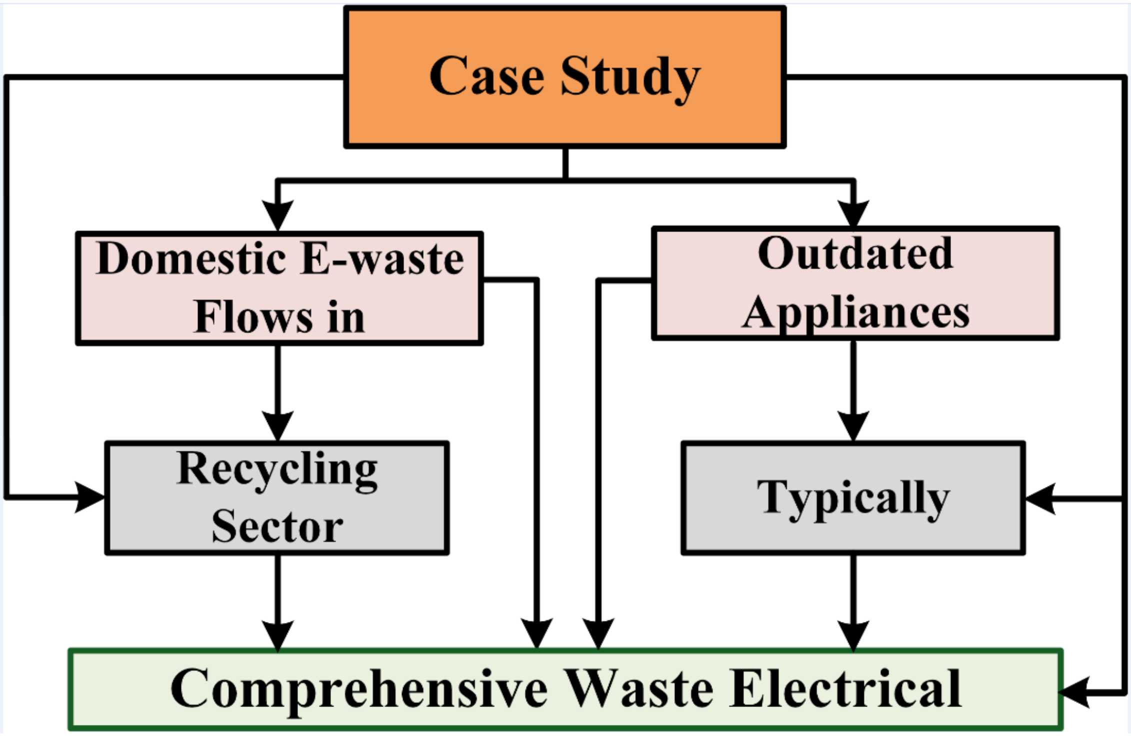

Figure 3: Material of the case study

6.1. Numerical example

In this subsection, the ecological assessment standards and their meanings. Standards meaning the proposed cohesive model is practical to this instance as follows:

LHR :Ecological contamination

This standard is connected to the projected equal of production of airborne contaminants, injurious materials, solid wildernesses, production of air impurities waste marine, which statements by a provider in its making development.

KHI :Reserve feeding

This criterion is correlated to the appraised equal of rare factual eating, liveliness feeding and river eating throughout the progression of construction.

FSD :Biological revolution This criterion is interrelated to the advance of progressions and harvests that can help to maintainable growth using the profitable request of information to spread straight or unintended environmental developments.

ISB :Organic organization

This criterion is connected to preparation of capitals for emerging, physical construction and applying rules for ecological defense. ISO 14000 and ISO14001 are the greatest extensively secondhand values in an ecological organization method.

Step 1:Describe the BFF decision matrix provided in Tables 1 and 2.

BFF decision matrix table 1

	LHR_1	KHI_2	FSD_3	ISB_4
PS_1	$\begin{bmatrix} [0.11, 0.13], [-0.12, -0.16] \end{bmatrix}$	$\begin{bmatrix} [0.12, 0.14], [-0.13, -0.15] \end{bmatrix}$	$\begin{bmatrix} [0.21, 0.22], [-0.23, -0.26] \end{bmatrix}$	$\begin{bmatrix} [0.3, 0.5], [-0.4, -0.6] \end{bmatrix}$
PS_2	$\begin{bmatrix} [0.3, 0.5], [-0.4, -0.6] \end{bmatrix}$	$\begin{bmatrix} [0.11, 0.13], [-0.12, -0.16] \end{bmatrix}$	$\begin{bmatrix} [0.12, 0.14], [-0.13, -0.15] \end{bmatrix}$	$\begin{bmatrix} [0.21, 0.22], [-0.23, -0.26] \end{bmatrix}$
PS_3	$\begin{bmatrix} [0.12, 0.14], [-0.13, -0.15] \end{bmatrix}$	$\begin{bmatrix} [0.3, 0.5], [-0.4, -0.6] \end{bmatrix}$	$\begin{bmatrix} [0.21, 0.22], [-0.23, -0.26] \end{bmatrix}$	$\begin{bmatrix} [0.11, 0.13], [-0.12, -0.16] \end{bmatrix}$
PS_4	$\begin{bmatrix} [0.21, 0.22], [-0.23, -0.26] \end{bmatrix}$	$\begin{bmatrix} [0.11, 0.13], [-0.12, -0.16] \end{bmatrix}$	$\begin{bmatrix} [0.3, 0.5], [-0.4, -0.6] \end{bmatrix}$	$\begin{bmatrix} [0.12, 0.14], [-0.13, -0.15] \end{bmatrix}$

BFF decision matrix table 2

	LHR_1	KHI_2	FSD_3	ISB_4
PS_1	$\begin{bmatrix} [0.11, 0.34], [-0.40, -0.54] \end{bmatrix}$	$\begin{bmatrix} [0.102, 0.104], [-0.103, -0.105] \end{bmatrix}$	$\begin{bmatrix} [0.09, 0.011], [-0.01, -0.012] \end{bmatrix}$	$\begin{bmatrix} [0.01, 0.03], [-0.02, -0.04] \end{bmatrix}$
PS_2	$\begin{bmatrix} [0.102, 0.104], [-0.103, -0.105] \end{bmatrix}$	$\begin{bmatrix} [0.11, 0.34], [-0.40, -0.54] \end{bmatrix}$	$\begin{bmatrix} [0.01, 0.03], [-0.02, -0.04] \end{bmatrix}$	$\begin{bmatrix} [0.09, 0.011], [-0.01, -0.012] \end{bmatrix}$
PS_3	$\begin{bmatrix} [0.09, 0.011], [-0.01, -0.012] \end{bmatrix}$	$\begin{bmatrix} [0.01, 0.03], [-0.02, -0.04] \end{bmatrix}$	$\begin{bmatrix} [0.11, 0.34], [-0.40, -0.54] \end{bmatrix}$	$\begin{bmatrix} [0.102, 0.104], [-0.103, -0.105] \end{bmatrix}$
PS_4	$\begin{bmatrix} [0.01, 0.03], [-0.02, -0.04] \end{bmatrix}$	$\begin{bmatrix} [0.102, 0.104], [-0.103, -0.105] \end{bmatrix}$	$\begin{bmatrix} [0.09, 0.011], [-0.01, -0.012] \end{bmatrix}$	$\begin{bmatrix} [0.11, 0.34], [-0.40, -0.54] \end{bmatrix}$

Step 2:Explain the BHFFWA operator presented in table 3 and $\xi = (0.26, 0.21, 0.25, 0.28)$.
BFHFWA operator table 3.

	LHR_1	KHI_2	FSD_3	ISB_4
PS_1	$\begin{bmatrix} [0.1341, \\ 0.3081], \\ [-0.2051, \\ -0.6341] \end{bmatrix}$	$\begin{bmatrix} [0.1022, \\ 0.1012], \\ [-0.3323, \\ -0.3325] \end{bmatrix}$	$\begin{bmatrix} [0.0329, \\ 0.4211], \\ [-0.5021, \\ -0.7213] \end{bmatrix}$	$\begin{bmatrix} [0.4513, \\ 0.4315], \\ [-0.2314, \\ -0.1216] \end{bmatrix}$
PS_2	$\begin{bmatrix} [0.9101, \\ 0.1112], \\ [-0.1132, \\ -0.1223] \end{bmatrix}$	$\begin{bmatrix} [0.0911, \\ 0.5612], \\ [-0.8721, \\ 0.9812] \end{bmatrix}$	$\begin{bmatrix} [0.1301, \\ 0.3501], \\ [-0.2601, \\ -0.4801] \end{bmatrix}$	$\begin{bmatrix} [0.3112, \\ 0.4313], \\ [-0.4512, \\ -0.5411] \end{bmatrix}$
PS_3	$\begin{bmatrix} [0.3213, \\ 0.5987], \\ [-0.2321, \\ -0.4101] \end{bmatrix}$	$\begin{bmatrix} [0.1081, \\ 0.3091], \\ [-0.2081, \\ -0.4051] \end{bmatrix}$	$\begin{bmatrix} [0.1222, \\ 0.3322], \\ [-0.2122, \\ -0.4222] \end{bmatrix}$	$\begin{bmatrix} [0.1209, \\ 0.3011], \\ [-0.5201, \\ 0.6012] \end{bmatrix}$
PS_4	$\begin{bmatrix} [0.0111, \\ 0.5121], \\ [-0.5011, \\ 0.9121] \end{bmatrix}$	$\begin{bmatrix} [0.3331, \\ 0.3333], \\ [-0.3332, \\ 0.3334] \end{bmatrix}$	$\begin{bmatrix} [0.1239, \\ 0.1411], \\ [-0.6511, \\ -0.6312] \end{bmatrix}$	$\begin{bmatrix} [0.1871, \\ 0.3561], \\ [-0.2741, \\ -0.4571] \end{bmatrix}$

Step 3:Explain the BHFFWA operator as presented in table 4 and $\xi = (0.26, 0.21, 0.25, 0.28)$.

	BHFFWA operator table 4
PS_1	$[[0.1309, 0.3776], [-0.2961, -0.4521]]$
PS_2	$[[0.0901, 0.4512], [-0.1522, -0.6423]]$
PS_3	$[[0.6713, 0.9737], [-0.2321, -0.3001]]$
PS_4	$[[0.3417, 0.6721], [-0.1012, -0.2098]]$

Step 4: Determine score function
$\Psi_1 = 0.0011, \Psi_2 = 0.5672, \Psi_3 = 0.2998, \Psi_4 = 0.3406.$
Step 5: Establish the ranking

Figure 4 is given as

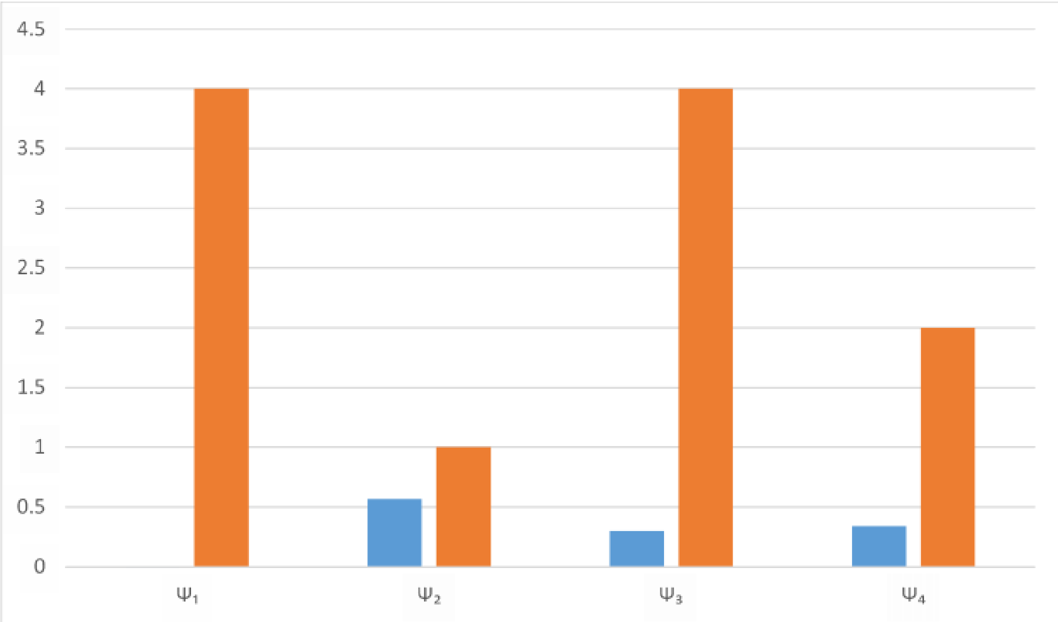

Figure 4: Proposed Method

6.2. Comparsion technique with existing method

To validate the effectiveness of the proposed Bipolar Fermatean Fuzzy Hamacher Aggregation Operators (BFHFWA, BFHFOWA, BFHFHWA), a comparative study was conducted against established aggregation methods in fuzzy decision-making literature. The evaluation criteria include decision accuracy, uncertainty management, computational efficiency, robustness and well-established methods such as those by Senapati et al. [34], Akram et al. [51], Aliya et al. [18, 19], Zhang [50] and Wei et al. [39]. The practical relevance is demonstrated through an e-waste management scenario, emphasizing the selection of optimal recycling partners based on environmental, economic, and operational criteria. Different existing techniques Tables 5 and 6 are given as.

Different existing techniques Table 5			
Methods	Score function	Ranking	Final ranking
BFHFWA operators	$\left\{\begin{array}{c}\Psi_1 = 0.0011,\\ \Psi_2 = 0.5672,\\ \Psi_3 = 0.2998,\\ \Psi_4 = 0.3406\end{array}\right\}$	$\left\{\begin{array}{c}\Psi_2 >\\ \Psi_4 >\\ \Psi_3 >\\ \Psi_1\end{array}\right\}$	$\left\{\begin{array}{c}\Psi_2 >\\ \Psi_4 >\\ \Psi_3 >\\ \Psi_1\end{array}\right\}$
BFHFOWA operators	$\left\{\begin{array}{c}\Psi_1 = 0.0125,\\ \Psi_2 = 0.6263,\\ \Psi_3 = 0.3981,\\ \Psi_4 = 0.4212\end{array}\right\}$	$\left\{\begin{array}{c}\Psi_2 >\\ \Psi_4 >\\ \Psi_3 >\\ \Psi_1\end{array}\right\}$	$\left\{\begin{array}{c}\Psi_2 >\\ \Psi_4 >\\ \Psi_3 >\\ \Psi_1\end{array}\right\}$
BFHFHWA operators	$\left\{\begin{array}{c}\Psi_1 = 0.0234,\\ \Psi_2 = 0.8923,\\ \Psi_3 = 0.4527,\\ \Psi_4 = 0.5028\end{array}\right\}$	$\left\{\begin{array}{c}\Psi_2 >\\ \Psi_4 >\\ \Psi_3 >\\ \Psi_1\end{array}\right\}$	$\left\{\begin{array}{c}\Psi_2 >\\ \Psi_4 >\\ \Psi_3 >\\ \Psi_1\end{array}\right\}$
FFSs [34]	$\left\{\begin{array}{c}\Psi_1 = 0.0107,\\ \Psi_2 = 0.4413,\\ \Psi_3 = 0.1998,\\ \Psi_4 = 0.2451.\end{array}\right\}$	$\left\{\begin{array}{c}\Psi_2 >\\ \Psi_4 >\\ \Psi_3 >\\ \Psi_1\end{array}\right\}$	$\left\{\begin{array}{c}\Psi_2 >\\ \Psi_4 >\\ \Psi_3 >\\ \Psi_1\end{array}\right\}$
MCDM [5]	$\left\{\begin{array}{c}\Psi_1 = 0.0317,\\ \Psi_2 = 0.6001,\\ \Psi_3 = 0.1276,\\ \Psi_4 = 0.1654\end{array}\right\}$	$\left\{\begin{array}{c}\Psi_2 >\\ \Psi_4 >\\ \Psi_3 >\\ \Psi_1\end{array}\right\}$	$\left\{\begin{array}{c}\Psi_2 >\\ \Psi_4 >\\ \Psi_3 >\\ \Psi_1\end{array}\right\}$

Figure 5 is given as

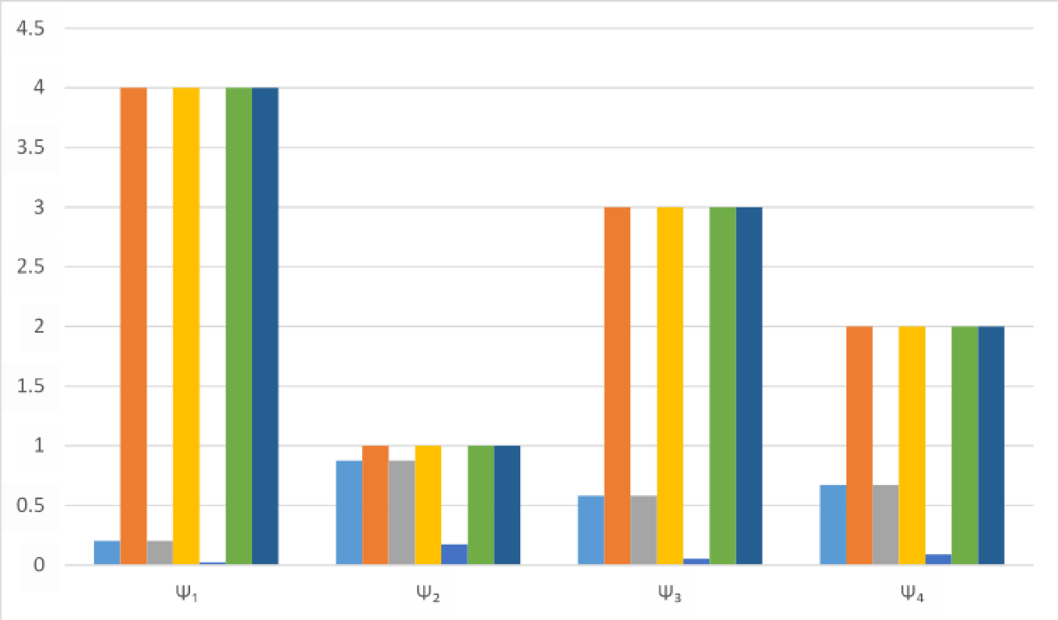

Figure 5: Different existing techniques

Different existing techniques Table 6			
Methods	Score function	Ranking	Final ranking
Cubic einstein [16]	$\left\{\begin{array}{c}\Psi_1 = 0.2014,\\ \Psi_2 = 0.8755,\\ \Psi_3 = 0.5811,\\ \Psi_4 = 0.6711\end{array}\right\}$	$\left\{\begin{array}{c}\Psi_2 >\\ \Psi_4 >\\ \Psi_3 >\\ \Psi_1\end{array}\right\}$	$\left\{\begin{array}{c}\Psi_2 >\\ \Psi_4 >\\ \Psi_3 >\\ \Psi_1\end{array}\right\}$
Natural gas [19]	$\left\{\begin{array}{c}\Psi_1 = 0.2014,\\ \Psi_2 = 0.8755,\\ \Psi_3 = 0.5811,\\ \Psi_4 = 0.6711\end{array}\right\}$	$\left\{\begin{array}{c}\Psi_2 >\\ \Psi_4 >\\ \Psi_3 >\\ \Psi_1\end{array}\right\}$	$\left\{\begin{array}{c}\Psi_2 >\\ \Psi_4 >\\ \Psi_3 >\\ \Psi_1\end{array}\right\}$
Bipolar fuzzy sets [50]	$\left\{\begin{array}{c}\Psi_1 = 0.2176,\\ \Psi_2 = 0.8021,\\ \Psi_3 = 0.6017,\\ \Psi_4 = 0.7081\end{array}\right\}$	$\left\{\begin{array}{c}\Psi_2 >\\ \Psi_4 >\\ \Psi_3 >\\ \Psi_1\end{array}\right\}$	$\left\{\begin{array}{c}\Psi_2 >\\ \Psi_4 >\\ \Psi_3 >\\ \Psi_1\end{array}\right\}$
Hamacher [39]	$\left\{\begin{array}{c}\Psi_1 = 0.0213,\\ \Psi_2 = 0.1745,\\ \Psi_3 = 0.0545,\\ \Psi_4 = 0.0893\end{array}\right\}$	$\left\{\begin{array}{c}\Psi_2 >\\ \Psi_4 >\\ \Psi_3 >\\ \Psi_1\end{array}\right\}$	$\left\{\begin{array}{c}\Psi_2 >\\ \Psi_4 >\\ \Psi_3 >\\ \Psi_1\end{array}\right\}$

The comparison has been made using the following criteria:

Decision Accuracy: Evaluating the ability of methods to rank alternatives consistently and correctly.

Uncertainty Management: Assess the ability to handle imprecise and conflicting data effectively.

Computational Efficiency: Comparison of execution time and resource utilization of the methods.

Robustness: Analyzing sensitivity to variations in input data.

Enhanced Demonstration:

We provide a step-by-step demonstration of the proposed methods, showing how they perform better in real-world decision-making scenarios, particularly the e-waste management case study. These examples illustrate the advantages of our approach in terms of ranking accuracy, reduced computational effort, and better handling of conflicting criteria.

6.3. Experimental of the study

In this subsection, we define the experimental of the study as table 7.

Experimental study table 7

BFF Operators	Uncertainty	Efficiency	Precision	Decision Analysis
BFHFWA operator	High	High	High	Very Low
[48]	Low	Very Low	Very Low	Moderate
[44]	Moderate	High	Moderate	High
[43]	Very Low	Very Low	Low	Very Low
[42]	Very high	High	High	Moderate

6.4. Advantages of the proposed method

BFFS allow for the representation of both positive and negative membership degrees. This flexibility allows for a more thorough representation of uncertainty and ambiguity in real-world situations.

By considering both positive and negative factors, BFFS enhance decision-making processes, particularly in complex environments where various conflicting criteria must be evaluated. BFFS reflect the way humans naturally assess situations, recognizing both favorable and unfavorable attributes. This alignment can lead to more intuitive and relatable decision models.

The bipolar nature of BFFS is particularly advantageous in multi-criteria decision-making scenarios, where various attributes have both positive and negative impacts. This allows for a more holistic view of options and outcomes.

BFFS can be seamlessly integrated with traditional fuzzy sets and other mathematical frameworks, facilitating hybrid approaches that leverage diverse methodologies for improved problem-solving. The use of distinct bipolar membership functions allows for clearer representation of the affirmative and negative aspects of elements. This clarity can enhance data visualization and interpretation.

BFFS excel in environments characterized by contradictory or conflicting information, making them valuable in artificial intelligence, decision support systems, and knowledge representation.

Built on solid mathematical foundations, BFFS provide a rigorous framework for analysis and the development of algorithms, ensuring reliability in computations and conclusions. BFFS can effectively model dynamic systems where relationships between variables may change over time. This adaptability is crucial for applications in fields like economics, environmental science, and engineering.

The versatility of BFFS allows for their application across various disciplines, including social sciences, economics, engineering, and medical diagnostics. This broad applicability highlights their potential to address diverse challenges.

6.5. Results and discussion

This subsection presents a comprehensive analysis of the results derived from applying the proposed aggregation operators based on Hamacher operational laws for Bipolar Fermatean Fuzzy sets. The results are analyzed in terms of the score functions calculated for each alternative and the corresponding rankings, followed by a discussion on the theoretical implications of the findings.

The results and disscussion below in table 8.

Results and disscussion below table 8.

Methods	Score function	Ranking	Final ranking
FFS [8]	$\left\{\begin{array}{c}\Psi_1 = 0.1823,\\ \Psi_2 = 0.7645,\\ \Psi_3 = 0.6649,\\ \Psi_4 = 0.7123\end{array}\right\}$	$\left\{\begin{array}{c}\Psi_2 >\\ \Psi_4 >\\ \Psi_3 >\\ \Psi_1\end{array}\right\}$	$\left\{\begin{array}{c}\Psi_2 >\\ \Psi_4 >\\ \Psi_3 >\\ \Psi_1\end{array}\right\}$
FFYA [19]	$\left\{\begin{array}{c}\Psi_1 = 0.1756,\\ \Psi_2 = 0.7312,\\ \Psi_3 = 0.5511,\\ \Psi_4 = 0.6087\end{array}\right\}$	$\left\{\begin{array}{c}\Psi_2 >\\ \Psi_4 >\\ \Psi_3 >\\ \Psi_1\end{array}\right\}$	$\left\{\begin{array}{c}\Psi_2 >\\ \Psi_4 >\\ \Psi_3 >\\ \Psi_1\end{array}\right\}$
PFIH [36]	$\left\{\begin{array}{c}\Psi_1 = 0.0019,\\ \Psi_2 = 0.1949,\\ \Psi_3 = 0.0156,\\ \Psi_4 = 0.0938\end{array}\right\}$	$\left\{\begin{array}{c}\Psi_2 >\\ \Psi_4 >\\ \Psi_3 >\\ \Psi_1\end{array}\right\}$	$\left\{\begin{array}{c}\Psi_2 >\\ \Psi_4 >\\ \Psi_3 >\\ \Psi_1\end{array}\right\}$
IFE [51]	$\left\{\begin{array}{c}\Psi_1 = 0.0212,\\ \Psi_2 = 0.2876,\\ \Psi_3 = 0.1093,\\ \Psi_4 = 0.2121\end{array}\right\}$	$\left\{\begin{array}{c}\Psi_2 >\\ \Psi_4 >\\ \Psi_3 >\\ \Psi_1\end{array}\right\}$	$\left\{\begin{array}{c}\Psi_2 >\\ \Psi_4 >\\ \Psi_3 >\\ \Psi_1\end{array}\right\}$

The score functions for each method are computed, and the rankings are established by ordering the values in descending order. In all four methods, Ψ_2 consistently ranks the highest, followed by Ψ_4, Ψ_3 , and Ψ_1. This pattern indicates the reliability of Ψ_2 as the most favorable alternative across the different approaches. This analysis shows that the suggested rankings and score functions work well for assessing and contrasting the performance of options, offering a thorough framework for making decisions in the face of uncertainty. The theoretical underpinnings of the methodology, which are based on Hamacher operational principles and fuzzy set theory, enable it to integrate various sources of uncertainty in order to address complicated decision issues. This is particularly beneficial in contexts where alternatives are characterized by inaccurate, partial, or contradictory information, as typically found in domains such as waste management, resource distribution, and risk assessment.

Moreover, the ranking procedure based on aggregated score functions streamlines the decision-making process by offering precise, measurable performance metrics, which is especially helpful in multi-criteria decision analysis, where decision-makers must rank and assess different options according to multiple competing characteristics.

7. Conclusion

This paper introduces advanced aggregation operators based on Hamacher operational laws for Bipolar Fermatean Fuzzy sets, providing an innovative solution for complex decision-making problems that involve multiple uncertainties. A key strength of the proposed methodology is the incorporation of a score function, which allows for the systematic ranking of alternatives based on the aggregated results. This score function enhances decision-making by offering a clear and quantifiable measure of the performance of each alternative, simplifying the selection process, and improving overall decision accuracy. The proposed operators Bipolar Fermatean Fuzzy Hamacher Weighted Average, Bipolar Fermatean Fuzzy Hamacher Ordered Weighted Average, and Bipolar Fermatean Fuzzy Hamacher Hybrid Weighted Average effectively integrate membership, non-membership, and hesitation degrees, making them highly suitable for handling real-world decision challenges. The practical relevance of the methodology is demonstrated through a real-world application in managing electronic waste, where the results show the superiority of the proposed approach over existing methods. The BFHFWA, BFHFOWA, and BFHFHWA operators outperform traditional aggregation techniques in terms of flexibility, efficiency, and accuracy, making them highly effective for handling uncertainty and improving decision-making precision in complex environments.

In comparison with the most recent studies, such as those by Akram et al. [5] and Deng et al. [10], our approach advances the state-of-the-art by providing a more generalized, flexible, and comprehensive aggregation method that handles the dual uncertainty of membership and non-membership, offering substantial improvements in decision-making performance. Overall, this research contributes to the field of fuzzy decision-making by offering a robust, adaptable, and efficient framework for multi-criteria decision analysis. Future work could expand on these operators, exploring their application in other domains and refining their ability to address emerging challenges in uncertain and dynamic decision-making environments.

Compliance with Ethical Standards

The authors declare that there is no conflict of interest regarding the publication of this paper. Ethical approval: This article does not contain any studies with human participants or animals performed by any of the authors.

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