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Analysis of Hardy-type Inequalities Involving Green Functions and Taylor's Polynomial Approximation

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Abstract. Herein, we prove the Hardy-type inequalities using the two-point right focal problem's Green functions, which are convex and continuous concerning both variables. Along with the Green functions, Taylor's polynomial and $n-$convex functions are also considered. In addition, we find the bounds on the remainder using Čebyšev functional in the presence of obtained results in the form of Hardy-type inequalities. Then, we discuss Grüss-type inequalities that enable us to find the bounds on remainders and then Ostrowski-type inequalities are discussed. In the later part of this study, we discuss some results related to the mean value theorem and n-exponential convexity.

2020 Mathematics Subject Classifications: 26D10, 26D15, 26D25, 26D30

Key Words and Phrases: Hardy-type Inequalities, Convex functions, n-Exponential Convex functions, Green function, Taylor interpolating polynomial, Čebyšev functional, Grüss-type inequalities, Ostrowski-type inequalities, Bounds on remainders

1. Introduction

A remarkable development in the theory of Green function was introduced by Green [8] that was almost seminal and later on theory related to Green function plays an important role in dealing with differential equations qualitatively and quantitatively [4],[13]. Mathematicians used Green functions in many other directions of mathematics, other than differential equations. In resent past, it has been used diversely in the mathematical inequalities along with the polynomial interpolation. Like in [2] different aspects have been extensively discussed. Then further generalizations have been made by many researchers like K. K .Himmelreich et al. [10] discuss the Hardy-type inequalities by tkaing Green function into account along with Montgomery identity. D. Pokaz [20] studied the Hardy-type inequalities via Green functions, $n-$convex functions and polynomial interpolation of Abel Gontschorf and find the bounds on the remainder obtained from the assumptions under consideration. A. Rasheed et al. [22] investigate the Levinson-type inequalities using

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Green functions of the two-point right focal problems. Muhammad Adeel et al. investigate Levisnson type inequalities in [1] by using the Green function and Lidstone's polynomial and then he gave the application of these inequalities to the estimate the $f-$divergence and Shannon entropy. In [7] some real-world problems are discussed along with the Hermite-Hadamard inequality and Green functions. Although this inequality is vastly studied in many other ways like it's generalizations so-called Levinson's inequality or involvement of fractional calculus to study this inequality but we analyze the aforementioned inequality for upcoming Green functions and Taylor's polynomial.

Due to the importance of the Hardy-type inequalities to find the prior estimation to the solution of partial differential equations, approximation of a function to the basis of functions or the polynomials, it got the attention of mathematicians. Some nominal works and discussion are made in [2], [22], [9], [23] and [12]. Also, there is scope to extend this work in the fractional calculus side using different operators discussed in [6], [7] and [21] in the presence of Čebyšev functional in fractional sense accordingly. For example, while discussing the aforementioned inequality in conformable fractional operator we use Čebyšev functional given in [24], but in the present study we are considering the ordinary case. For further details it is plausibe to mention some important and nominal work that generalize this inequaltiy by using polynomial interpolation and Green function. Kristina Krulić Himmelreich [15] discuss Hardy-type inequalities via Taylor's polynomial, D Pokaz in [20] use Abel-Gontscharoff interpolation and the Green functions as well to generalize aforementioned inequality and Kristina Krulić Himmelreich et al. [14] generalize Hardy-type inequalities with the Hermite interpolating polynomials.

We observe that the Hardy-type inequalities are not yet studied in the presence of the Green functions presented in Lemma 1 of upcoming section. Herein, we are interested in generalizing Hardy-type inequalities via Taylor's polynomial and the two-point right focal Green function, which is discussed in section 3. After that, we find the Grüss-type and Ostowski-type bounds that are given in section 4. The final section 5 involves some results regarding the n-exponential convexity and utility of our previous results, especially the functional obtained from Theorem 5.

2. Preliminaries

Initiating by considering the following Lemma involving the Green functions that are 3-convex. Also, these Green functions are continuous and convex with respect to the involved variables and given in [22] as;

Lemma 1. *Assuming the real valued function $\mathcal{F}$ defined on $\mathcal{T} = [a_1, a_2]$ and $\mathcal{F}$ is thrice differentiable therein. Let $G_\alpha(\alpha = \{1, 2, 3, 4\})$ be the two-point right-focal problem type Green function. Then*

$$\mathcal{F}(\varpi) = \mathcal{F}(a_1) + (\varpi - a_1)\mathcal{F}'(a_2) + (\varpi - a_1)(\varpi - a_2)\mathcal{F}''(a_1) - \frac{(\varpi - a_1)^2}{2}\mathcal{F}''(a_2)$$

$$+\int_{a_1}^{a_2} G_1(\varpi,\tau)\mathcal{F}'''(\tau)d\tau, \tag{1}$$

$$\mathcal{F}(\varpi)=\mathcal{F}(a_2)+(\varpi-a_2)\mathcal{F}'(a_1)+(\varpi-a_1)(\varpi-a_2)\mathcal{F}''(a_2)-\frac{(\varpi-a_2)^2}{2}\mathcal{F}''(a_1)$$
$$-\int_{a_1}^{a_2} G_2(\varpi,\tau)\mathcal{F}'''(\tau)d\tau, \tag{2}$$

$$\mathcal{F}(\varpi)=\mathcal{F}(a_2)+(\varpi-a_1)\mathcal{F}'(a_1)+(a_2-a_1)\mathcal{F}'(a_2)-\left[\frac{(\varpi-a_1)^2}{2}+(\varpi-a_1)(a-b)\right]\mathcal{F}''(a_1)$$
$$+\left[\frac{(a_2-a_1)^2}{2}+(\varpi-a_1)(\varpi-a_2)\right]\mathcal{F}''(a_2)-\int_{a_1}^{a_2} G_3(\varpi,\tau)\mathcal{F}'''(\tau)d\tau \tag{3}$$

and

$$\mathcal{F}(\varpi)=\mathcal{F}(a_2)+(a_2-a_1)\mathcal{F}'(a_1)+(b-\varpi)\mathcal{F}'(a_2)+\left[\frac{(a_2-a_1)^2}{2}+(\varpi-a_2)(\varpi-a_1)\right]\mathcal{F}''(a_1)$$
$$-\left[\frac{(\varpi-a_2)^2}{2}+(\varpi-a_2)(a_2-a_1)\right]\mathcal{F}''(a_2)+\int_{a_1}^{a_2} G_4(\varpi,\tau)\mathcal{F}'''(\tau)d\tau. \tag{4}$$

where the Green functions are given as

$$G_1(\varpi,\tau)=\begin{cases}\frac{(\tau-a_1)^2}{2}+(\varpi-a_1)(\varpi-a_2) & a\le\tau\le\varpi\\ (\varpi-a_1)(\tau-a_2)+\frac{(\tau-a_1)^2}{2} & \varpi\le\tau\le b\end{cases}, \tag{5}$$

$$G_2(\varpi,\tau)=\begin{cases}\frac{(\tau-a_2)^2}{2}+(\tau-a_1)(\varpi-a_2) & a\le\tau\le\varpi\\ (\varpi-a_1)(\varpi-a_2)+\frac{(\tau-a_2)^2}{2} & \varpi\le\tau\le b\end{cases}, \tag{6}$$

$$G_3(\varpi,\tau)=\begin{cases}\frac{(\tau-a_1)^2}{2}+(\varpi-a_1)(\tau-a_2) & a\le\tau\le\varpi\\ (\varpi-a_1)(\varpi-a_2)+\frac{(\tau-a_1)^2}{2} & \varpi\le\tau\le b\end{cases}, \tag{7}$$

$$G_4(\varpi,\tau)=\begin{cases}\frac{(\tau-a_2)^2}{2}+(\varpi-a_1)(\varpi-a_2) & a\le\tau\le\varpi\\ (\varpi-a_2)(\tau-a_1)+\frac{(\varpi-a_2)^2}{2} & \varpi\le\tau\le b.\end{cases} \tag{8}$$

The Taylor's formula for the function $\mathcal{F}:\mathcal{T}=[a_1,a_2]\to\mathbb{R}$ at $c\in\mathcal{T}$ and $\mathcal{F}^{(n-1)}$ have the property of absolutely continuity, is;

$$\mathcal{F}(\varpi)=\sum_{\kappa=0}^{n-1}\frac{f^{(\kappa)}(\varpi)}{\kappa!}(\varpi-c)^{\kappa}+\frac{1}{n-1}\int_c^{\varpi}\mathcal{F}^{(n)}(\tau)(\varpi-\tau)^{m-1}d\tau. \tag{9}$$

Lemma 2. *Assuming that $\mathcal{F} : \mathcal{T} = [a_1, a_2] \to \mathbb{R}$ is such that $\mathcal{F}^{(n-1)}$ is absolute continuous and $\varpi \in \mathcal{T}$. Then the Taylor's formula at point a_1 and a_2 is given in [2] as;*

$$\mathcal{F}(\varpi) = \sum_{\kappa=0}^{n-1} \frac{\mathcal{F}^{(\kappa)}(a_1)}{\kappa!}(\varpi - a_1)^{\kappa} + \frac{1}{(n-1)!}\int_{a_1}^{a_2} \mathcal{F}^{(n)}(\tau)(\varpi - \tau)_+^{n-1} d\tau. \tag{10}$$

where

$$\int_{a_1}^{a_2} (\varpi - \tau)_+^{n-1} d\tau = \int_{a_1}^{\varpi} (\varpi - \tau)^{n-1} d\tau + \int_{\varpi}^{a_2} 0 d\tau,$$

and

$$\mathcal{F}(\varpi) = \sum_{\kappa=0}^{n-1} \frac{\mathcal{F}^{(\kappa)}(a_2)}{\kappa!}(-1)^{\kappa}(a_2 - \varpi)^{\kappa} - \frac{(-1)^{n-1}}{(n-1)!}\int_{a_1}^{a_2} \mathcal{F}^{(n)}(\tau)(\tau - \varpi)_+^{n-1} d\tau. \tag{11}$$

where

$$\int_{a_1}^{a_2} (\tau - \varpi)_+^{n-1} d\tau = \int_{a_1}^{\varpi} 0 d\tau + \int_{\varpi}^{a_2} (\tau - \varpi)^{n-1} d\tau,$$

$$(\tau - \varpi)_+ = \begin{cases} \tau - \varpi & \varpi \leq \tau \\ 0 & \varpi > \tau \end{cases}. \tag{12}$$

Definition 1. *The n-th order divided difference of a real valued function f defined on $[a_1, a_2]$ at distinct points $\varpi_0, ..., \varpi_n \in [a_1, a_2]$ is defined for $(i = 0, 1, ..., n)$ as*

$$f[\varpi_i] = f(\varpi_i),$$
$$f[\varpi_0, ..., \varpi_n] = \frac{f[\varpi_1, ..., \varpi_n] - f[\varpi_0, ..., \varpi_{n-1}]}{\varpi_n - \varpi_0}.$$

Here the value obtained from $f[\varpi_0, ..., \varpi_n]$ is not depending on the order of points $\varpi_0, ..., \varpi_n$. In case when all points coincides then this definition will be extended to the following form if $f^{(j-1)}(\varpi)$ exists

$$f[\varpi, ..._{j-times}, \varpi] = \frac{f^{(j-1)}(\varpi)}{(j-1)!}. \tag{13}$$

The British mathematician G H Hardy introduce an inequality so-called Hardy inequality in [9] given as;

$$\int_0^{\infty} \Big(\frac{1}{\varpi}\int_0^{\varpi} f(\tau) d\tau\Big)^{\gamma} d\varpi \leq \Big(\frac{\gamma}{\gamma - 1}\Big)^{\gamma} \int_0^{\infty} f^{\gamma}(\varpi) d\varpi, \ \ \gamma > 1, \tag{14}$$

where he took a non-negative function f with $f \in L^{\gamma}(\mathbb{R}_+)$ and $\mathbb{R}_+ = (0, \infty)$ along with sharp constant $\left(\frac{\gamma}{\gamma-1}\right)^{\gamma}$. With the passage of time it has been generalized in many ways.

Like in 1964 N Levinson generalized Hardy inequality given [19], [18]. But [18], [16] and [17] generalized the Hardy-type inequalities and operator. After some generalizations S. Kaisjer et al. [12] gave some nominal concept of Hardy-type inequalities via convexity. Here we throw a glance on some of their important results. For positive $\sigma-$finite measures, the two measure spaces $(\sum_1, \Omega_1, \sigma_1)$ and $(\sum_2, \Omega_2, \sigma_2)$, we have $\mathcal{A}_k$ the operator given in [12] as

$$\mathcal{A}_k f(\varpi) = \frac{1}{\mathcal{K}(\varpi)} \int_{\Omega_2} \mathcal{G}(\varpi, \tau) f(\tau) d\mu_2(\tau), \tag{15}$$

where f is the measurable function and $\mathcal{G} : \Omega_1 \times \Omega_2 \to \mathbb{R}$ is nonnegative measurable kernel and obeys the following inequality

$$0 < \mathcal{K}(\varpi) = \int_{\Omega_2} \mathcal{G}(\varpi, \tau) d\mu_2(\tau), \quad \varpi \in \Omega_1. \tag{16}$$

Furthermore S. Kaisjer et al. gave another useful result in [12] in the form of following theorem.

Theorem 1. *Assuming that* $\frac{\mathcal{G}(\varpi,s)}{\mathcal{K}(\varpi)} u(\varpi)$ *is integrable locally on* Ω_1 *for each fixed* $s \in \Omega_2$*, where* u *be the weight function,* $\mathcal{G}(\varpi, s) \geq 0$*. Define* v *as*

$$\nu(s) = \int_{\Omega_1} \frac{\mathcal{G}(\varpi, s)}{\mathcal{K}(\varpi)} u(\varpi) d\mu_1(\varpi) < \infty. \tag{17}$$

If $\mathcal{F}$ *is supposed to be the convex function on* $I \subset \mathbb{R}$ *which is open, then following relation holds*

$$\int_{\Omega_1} \mathcal{F}(\mathcal{A}_k f(\varpi)) u(\varpi) d\mu_1(\varpi) \leq \int_{\Omega_2} \mathcal{F}(f(s)) \nu(s) d\mu_2(s). \tag{18}$$

for measurable function $f : \Omega_1 \to \mathbb{R}$*, with image of* f *is subset of* I*, and* $\mathcal{A}_k$ *is given in* (15)*.*

Now we are discussing some of the existing results from literature which help us to find some important bounds using Ostrowski and Grüss-type inequalities presented in section 4. Considering the following functional for two real valued Lebesgue integrable functions f, g over an interval $\mathcal{T}$ so-called Čebyšev functional given by P. Cerone et al. in [5] as

$$\mathfrak{F}(f, g) = \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} f(\eta) g(\eta) d\eta - \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} f(\eta) d\eta . \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} g(\eta) d\eta. \tag{19}$$

Next result is given in the same article [5] which is used to find the bounds of the remainder of Hardy-type inequalities using Taylor's polynimial and Green function.

Theorem 2. *Let the functions f, g are as taken in* (19) *and uniformly continuous on* $[a_1, a_2]$ *as well. Additionally assuming* $(. - a), (b - .)(g')^2 \in L[a_1, a_2]$*, then following inequality holds*

$$\Big|\mathfrak{F}(f,g)\Big| \leq \frac{1}{\sqrt{2}}\Big(\mathfrak{F}(f,f)\Big)^{\frac{1}{2}} \frac{1}{\sqrt{a_2 - a_1}}\left(\int_{a_1}^{a_2} (\eta - a_1)(a_2 - \eta)(g'(\eta))^2 d\eta\right)^2, \tag{20}$$

where $\frac{1}{\sqrt{2}}$ *is the best possible approximation.*

Also, the Grüss-type inequality was given in [5], which is given as follows;

Theorem 3. *Let f is absolutely continuous on* $[a_1, a_2]$ *and* $f' \in L_\infty[a_1, a_2]$ *and* $g : [a_1, a_2] \to \mathbb{R}$ *is monotonic nondecreasing then following result holds*

$$\Big|\mathfrak{F}(f,g)\Big| \leq \frac{1}{2(a_2 - a_1)} ||f'||_\infty \int_{a_1}^{a_2} (\eta - a_1)(a_2 - \eta)(g'(\eta))^2 dg(\eta), \tag{21}$$

where $\frac{1}{2}$ *is the best possible approximation.*

The following definitions of n-exponential convexity from [11] is used in section 5 and related results of exponential convexity are under consideration there.

Definition 2. *A function* $\mathcal{F} : \mathcal{T} \to \mathbb{R}$ *is n-exponentially convex in Jensen sense on* $\mathcal{T}$ *if*

$$\sum_{i,j}^{n} c_i c_j \mathcal{F}\Big(\frac{\zeta_i + \zeta_j}{2}\Big) \geq 0, \tag{22}$$

holds for all $c_1, c_2, ..., c_n \in \mathbb{R}$ *and all choices of* $\zeta_1, ..., \zeta_n \in \mathcal{T}$.

A function $\mathcal{F} : \mathcal{T} \to \mathbb{R}$ will be n-exponentially convex if it meets the criteria for n-exponential convexity in Jensen sense and continuous on $\mathcal{T}$.

Remark 1. *The aforementioned definition ensures that the function which is 1-exponentially convex in Jensen sense is indeed non-negative. Furthermore, an n-exponentially convex function in Jensen sense is also l-exponentially convex in Jensen sense for each* $l \in \mathbb{N}$*, where* $l \leq n$.

Employing the definition of semi-definite matrices and some fundamental results from linear algebra, J.Pečarić [11] propose the following result:

Proposition 1. *If* $\mathcal{F}$ *is n-exponentially convex in Jensen sense, then the matrix*

$$\Big[\mathcal{F}\Big(\frac{\zeta_i + \zeta_j}{2}\Big)\Big]_{i,j=1}^{l}, \quad \forall\, l \in \mathbb{N}, \;\; l \leq n,$$

is positive semi definite. In particular following result holds for all such l

$$\det\Big[\mathcal{F}\Big(\frac{\zeta_i + \zeta_j}{2}\Big)\Big]_{i,j=1}^{l} \geq 0.$$

Definition 3. *If a function $\mathcal{F}:\mathcal{T}\to\mathbb{R}$ is n-exponentially convex in Jensen sense for all $n\in\mathbb{N}$ then $\mathcal{F}$ is eponentially convex in Jensen sense. A continuous function $\mathcal{F}:\mathcal{T}\to\mathbb{R}$ that is eponentially convex in Jensen sense is exponentially convex.*

Remark 2. *It is easy to show that $\mathcal{F}:\mathcal{T}\to\mathbb{R}$ is a log-convex in the Jensen sense if and only if*

$$a_1^2\mathcal{F}(\zeta_1)+2a_1a_2\mathcal{F}\Big(\frac{\zeta_1+\zeta_2}{2}\Big)+a_2^2\mathcal{F}(\zeta_2)\geq 0, \tag{23}$$

holds for all $a_1,a_2\in\mathbb{R}$ and $\zeta_1,\zeta_2\in\mathcal{T}$. Consequently, the function is 2-exponentially convex in the Jensen sense if and only if it is log-convex in the Jensen sense.

A positive function is log-convex if and only if it is 2-exponentially convex.

3. Main Results

In this section, we present Hardy-type inequalities as a fundamental result, employing Green functions obtained from the two-point right focal problem in addition to this, Taylor's polynomial is also taken into account, after that, using these fundamental inequalities we analyze the Grüss-type and Ostrowski-type bounds. Starting with our first result which is given as;

Theorem 4. *Let $\mathcal{F}$ be defined on $\mathcal{T}$ such that $\mathcal{F}'''$ exists and $\mathcal{F}^{(n-1)}$ is absolutely continuous therein, for $n\in\mathbb{N}$. Considering two measure spaces $(\sum_1,\Omega_1,\sigma_1)$ and $(\sum_2,\Omega_2,\sigma_2)$ with positive $\sigma-$finite measures and $\mathcal{A}_k$ and $\mathcal{K}$ are mentioned in* (15) *and* (16) *respectively. Let G_α, $\{\alpha=1,2,3,4\}$ be defined in* (5), (6), (7) *and* (8) *respectively and $u:\Omega_1\to\mathbb{R}$ be the weight function and v is given in* (17)*, then:*

- $$\begin{aligned}&\int_{\Omega_2}\mathcal{F}(f(s))\nu(s)d\mu_2(s)-\int_{\Omega_1}\mathcal{F}(\mathcal{A}_kf(\varpi))u(\varpi)d\mu_1(\varpi)\\&=\int_{a_1}^{a_2}\mathcal{J}G_\alpha(f,\tau)\Big[\sum_{\kappa}\frac{\mathcal{F}^{(\kappa)}(a_1)}{\kappa!}(\tau-a_1)^\kappa\\&+\frac{1}{(n-4)!}\int_{a_1}^{a_2}\mathcal{F}^{(n)}(\eta)(\tau-\eta)_+^{n-4}d\eta\Big]d\tau,\end{aligned}\tag{24}$$

- $$\begin{aligned}&\int_{\Omega_2}\mathcal{F}(f(s))\nu(s)d\mu_2(s)-\int_{\Omega_1}\mathcal{F}(\mathcal{A}_kf(\varpi))u(\varpi)d\mu_1(\varpi)\\&=\int_{a_1}^{a_2}\mathcal{J}G_\alpha(f,\tau)\Big[\sum_{\kappa}\frac{(-1)^\kappa\mathcal{F}^{(\kappa)}(a_2)}{\kappa!}(a_2-\tau)^\kappa\\&-\frac{(-1)^{n-4}}{(n-4)!}\int_{a_1}^{a_2}\mathcal{F}^{(n)}(\eta)(\tau-\eta)_+^{n-4}d\eta\Big]d\tau,\end{aligned}\tag{25}$$

where

$$\mathcal{J}G_\alpha(f,\tau) = \int_{\Omega_2} G_\alpha(f(s),\tau)\nu(s)d\mu_2(s) - \int_{\Omega_1} G_\alpha(\mathcal{A}_k f(\varpi),\tau)u(\varpi)d\mu_1(\varpi) \tag{26}$$

and notation $\sum_\kappa$ *is used for* $\sum_{\kappa=3}^{n-1}$ *throughout this paper.*

Proof.

- First we consider $\alpha = 1$ and from (1) and we can write

$$\int_{\Omega_2} \mathcal{F}(f(s))\nu(s)d\mu_2(s) = \int_{\Omega_2} \Big[\mathcal{F}(a_1) + (f(s)-a)\mathcal{F}'(a_2) + (f(s)-a)(f(s)-b)\mathcal{F}''(a_1) - \frac{(f(s)-a)^2}{2}\mathcal{F}''(a_2) + \int_{a_1}^{a_2} G_1(f(s),\tau)\mathcal{F}'''(\tau)d\tau\Big]\nu(s)d\mu_2(s). \tag{27}$$

Similarly, we can write

$$\begin{aligned}&\int_{\Omega_1} \mathcal{F}(\mathcal{A}_k f(\varpi))u(\varpi)d\mu_1(\varpi)\\ &= \int_{\Omega_2} \Big[\mathcal{F}(a_1) + (\mathcal{A}_k f(\varpi)-a)\mathcal{F}'(a_2) + (\mathcal{A}_k f(\varpi)-a)(\mathcal{A}_k f(\varpi)-b)\mathcal{F}''(a_1)\\ &- \frac{(\mathcal{A}_k f(\varpi)-a)^2}{2}\mathcal{F}''(a_2) + \int_{a_1}^{a_2} G_1(\mathcal{A}_k f(\varpi),\tau)\mathcal{F}'''(\tau)d\tau\Big]u(\varpi)d\mu_1(\varpi).\end{aligned} \tag{28}$$

Now subtracting (28) from (27) we obtain

$$\begin{aligned}&\int_{\Omega_2} \mathcal{F}(f(s))\nu(s)d\mu_2(s) - \int_{\Omega_1} \mathcal{F}(\mathcal{A}_k f(\varpi))u(\varpi)d\mu_1(\varpi)\\ &= \int_{\Omega_2} \Big[\int_{a_1}^{a_2} G_1(f(s),\tau)\mathcal{F}'''(\tau)d\tau\Big]\nu(s)d\mu_2(s)\\ &- \int_{\Omega_1} \Big[\int_{a_1}^{a_2} G_1(\mathcal{A}_k f(\varpi),\tau)\mathcal{F}'''(\tau)d\tau\Big]u(\varpi)d\mu_1(\varpi).\end{aligned} \tag{29}$$

The fact used in obtaining the equation (29) is as follows;

$$\begin{aligned}&\int_{\Omega_2} \nu(s)d\mu_2(s) = \int_{\Omega_2}\int_{\Omega_1} \frac{k(\varpi,s)}{\mathcal{K}(\varpi)}u(\varpi)d\mu_1(\varpi)d\mu_2(s)\\ &= \int_{\Omega_1} \frac{u(\varpi)}{\mathcal{K}(\varpi)}\int_{\Omega_2} \mathcal{G}(\varpi,s)d\mu_2(s)d\mu_1(\varpi)\\ &= \int_{\Omega_1} \frac{u(\varpi)}{\mathcal{K}(\varpi)}\mathcal{K}(\varpi)d\mu_1(\varpi) = \int_{\Omega_1} u(\varpi)d\mu_1(\varpi)\end{aligned}$$

and

$$\int_{\Omega_2} \mathcal{A}_k f(\varpi)u(\varpi)d\mu_1(\varpi) = \int_{\Omega_1} \Big[\frac{1}{\mathcal{K}(\varpi)} \int_{\Omega_2} \mathcal{G}(\varpi, s)f(s)d\mu_2(s)\Big]u(\varpi)d\mu_1(\varpi)$$
$$= \int_{\Omega_2} f(s) \int_{\Omega_1} \frac{\mathcal{G}(\varpi, s)}{\mathcal{K}(\varpi)} u(\varpi)d\mu_1(\varpi)d\mu_2(s)$$
$$= \int_{\Omega_2} f(s)\nu(s)d\mu_2(s).$$

Now applying (10) in $\mathcal{F}'''$ at point a we obtain

$$\mathcal{F}'''(\tau) = \sum_{\kappa} \frac{\mathcal{F}^{(\kappa)}(a_1)}{\kappa!}(\tau - a_1)^{j-3} + \frac{1}{(n-4)!} \int_{a_1}^{a_2} \mathcal{F}^{(n)}(\eta)(\tau - \eta)_+^{n-4} d\eta. \tag{30}$$

Using (30) in (29) we obtain for $\alpha = 1$ as

$$\int_{\Omega_2} \mathcal{F}(f(s))\nu(s)d\mu_2(s) - \int_{\Omega_1} \mathcal{F}(\mathcal{A}_k f(\varpi))u(\varpi)d\mu_1(\varpi)$$
$$= \int_{a_1}^{a_2} \left[\int_{\Omega_2} G_\alpha(f(s), \tau)\nu(s)d\mu_2(s) - \int_{\Omega_1} G_\alpha(\mathcal{A}_k f(\varpi), \tau)u(\varpi)d\mu_1(\varpi) \right]$$
$$\times \sum_{\kappa} \frac{\mathcal{F}^{(\kappa)}(a_1)}{\kappa!}(\tau - a_1)^{\kappa-3} d\tau$$
$$+ \int_{a_1}^{a_2} \Bigg(\left[\int_{\Omega_2} G_\alpha(f(s), \tau)\nu(s)d\mu_2(s) - \int_{\Omega_1} G_\alpha(\mathcal{A}_k f(\varpi), \tau)u(\varpi)d\mu_1(\varpi) \right]$$
$$\times \frac{1}{(n-4)!} \int_{a_1}^{a_2} \mathcal{F}^{(n)}(\eta)(\tau - \eta)_+^{n-4} d\eta \Bigg) d\tau.$$

After giving the notation mentioned in (26) we arrived at the required result.

For $\alpha = 2, 3, 4$ consider the left hand sides of (27) and (28), then make use of (2), (3) and (4), after that subtracting the obtained results to get (29) then follow the same steps as for $\alpha = 1$ to obtain the required result.

- Proof is analogous to the first part, use the Taylor polynomial (11) and simplify we get required.

Remark 3. *As G_α are convex with respect to both variables, replacing the convex function $\mathcal{F}(.)$ by $G_\alpha(., \tau)$ ensures that relation in* (18) *remains valid, that is;*

$$\int_{\Omega_1} G_\alpha(\mathcal{A}_k f(\varpi), \tau)u(\varpi)d\mu_1(\varpi) \leq \int_{\Omega_2} G_\alpha(f(s), \tau)\nu(s)d\mu_2(s). \tag{31}$$

Theorem 5. *Let all conditions of Theorem 4 holds and if $\mathcal{F}$ is $n-$ convex on $\mathcal{T}$, then;*

- *(i)*

$$\int_{\Omega_2} \mathcal{F}(f(s))\nu(s)d\mu_2(s) - \int_{\Omega_1} \mathcal{F}(\mathcal{A}_k f(\varpi))u(\varpi)d\mu_1(\varpi)$$
$$\geq \int_{a_1}^{a_2} \mathcal{J}G_\alpha(f,\tau) \times \sum_{\kappa} \frac{\mathcal{F}^{(\kappa)}(a_1)}{\kappa!}(\tau - a_1)^\kappa d\tau. \tag{32}$$

- *(ii) For any odd $n > 4$;*

$$\int_{\Omega_2} \mathcal{F}(f(s))\nu(s)d\mu_2(s) - \int_{\Omega_1} \mathcal{F}(\mathcal{A}_k f(\varpi))u(\varpi)d\mu_1(\varpi)$$
$$\geq \int_{a_1}^{a_2} \mathcal{J}G_\alpha(f,\tau) \times \sum_{\kappa}(-1)^\kappa \frac{\mathcal{F}^{(\kappa)}(a_2)}{\kappa!}(a_2 - \tau)^\kappa d\tau. \tag{33}$$

Proof.

- As $\mathcal{F}$ is $n-$convex for all $n \in \mathbb{N}$ implies $\mathcal{F}^{(n)} \geq 0$ for $\tau \in \mathcal{T}$. Also, $(\tau - \eta)_+^{n-4}$ is non-negative for all $n \in \mathbb{N}$ because $\tau \geq \eta$. By using aforementioned reasons in equation (24) we obtain the required result.
- As $\mathcal{F}$ is $n-$convex this implies $\mathcal{F}^{(n)} \geq 0$ on $\mathcal{T}$. Also,

$$(-1)^{n-4}(\tau - \eta)_+^{n-4} = \begin{cases} 0 & \tau \leq \eta \\ (\eta - \tau)^{n-4} & \tau \geq \eta \end{cases}, \tag{34}$$

Since for $\tau \geq \eta$, so for any odd $n \geq 4$, $(\eta - \tau)^{n-4}$ is negative and $-(\eta - \tau)^{n-4}$ will be nonnegative. Thus last term in (25) is nonnegative. Hence we can write (33).

Theorem 6. *Let the conditions of Theorem 4 hold entirely. Additionally, if $\mathcal{F}$ is $n-$convex and the function;*

(i)

$$\mathcal{L}_1(.) = \sum_{\kappa} \frac{\mathcal{F}^{(\kappa)}(a_1)}{\kappa!} \int_{a_1}^{a_2} G_\alpha(.,\tau)(\tau - a_1)^\kappa d\tau, \tag{35}$$

is convex on $[a_1, a_2]$, then

$$\int_{\Omega_1} \mathcal{F}(\mathcal{A}_k f(\varpi))u(\varpi)d\mu_1(\varpi) \leq \int_{\Omega_2} \mathcal{F}(f(s))\nu(s)d\mu_2(s). \tag{36}$$

(ii)

$$\mathcal{L}_2(.) = \sum_{\kappa} \frac{(-1)^{\kappa}\mathcal{F}^{(\kappa)}(a_1)}{\kappa!} \int_{a_1}^{a_2} G_\alpha(.,\tau)(\tau - a_1)^{\kappa} d\tau, \tag{37}$$

is convex on $[a_1, a_2]$, *then*

$$\int_{\Omega_1} \mathcal{F}(\mathcal{A}_k f(\varpi))u(\varpi)d\mu_1(\varpi) \leq \int_{\Omega_2} \mathcal{F}(f(s))\nu(s)d\mu_2(s).$$

Proof.

- From right hand side of (32) we can write

$$\int_{\Omega_2} \mathcal{L}_1(f(s))\nu(s)d\mu_2(s) - \int_{\Omega_1} \mathcal{L}_1(\mathcal{A}_k f(\varpi))u(\varpi)d\mu_1(\varpi).$$

  Since $\mathcal{L}_1$ is convex, using Theorem 1 we find that the last relation is nonnegative. Consequently, our stated inequality results from (32).

- Analogously, the assertion for the functional $\mathcal{L}_2$ can be derived from (33).

4. Bounds on Remainders

In this section, we make use of the Theorems 2 and 3 to discuss the Grüss and Ostrowski-type inequalities and bounds on the remainders. Consider the following notations for the sake of brevity.

$$\mathcal{B}_1(\tau) = \int_{a_1}^{a_2} \mathcal{J}G_\alpha(f,\tau) \times (\tau - \eta)_+^{n-4} d\tau \tag{38}$$

and

$$\mathcal{B}_2(\tau) = (-1)^{n-4} \int_{a_1}^{a_2} \mathcal{J}G_\alpha(f,\tau) \times (\tau - \eta)_+^{n-4} d\tau. \tag{39}$$

Theorem 7. *Let $\mathcal{F}$ be defined on $\mathcal{T}$, for $n \in \mathbb{N}$, $\mathcal{F}^{(n-1)}$ has the property that it is absolutely continuous therein and $(\tau - a_1)(a_2 - \tau)[\mathcal{F}^{(n+1)}]^2 \in L_1([a_1, a_2])$. Let $(\sum_1, \Omega_1, \sigma_1)$ and $(\sum_2, \Omega_2, \sigma_2)$ with positive $\sigma-$finite measures, $\mathcal{A}_k$ and $\mathcal{K}$ are given in* (15) *and* (16) *respectively. Let G_α, $\{\alpha = 1, 2, 3, 4\}$ be defined in* (5), (6), (7) *and* (8) *respectively, $\mathcal{J}G_\alpha$ is defined in* (26) *$u : \Omega_1 \to \mathbb{R}$ be the weight function and v is given in* (17) *and $\mathcal{B}_1, \mathcal{B}_2$ are given in* (38) *and* (39) *respectively and f is measurable, then;*

(i) the remainder $\mathcal{R}_1$ is

$$\mathcal{R}_1(\mathcal{F}; a_1, a_2)$$

$$= \int_{\Omega_2} \mathcal{F}(f(s))\nu(s)d\mu_2(s) - \int_{\Omega_1} \mathcal{F}(\mathcal{A}_k f(\varpi))u(\varpi)d\mu_1(\varpi)$$
$$- \sum_{\kappa} \frac{\mathcal{F}^{(\kappa)}(a_1)}{\kappa!} \int_{a_1}^{a_2} \mathcal{J}G_\alpha(f,\tau)(\tau-a_1)^\kappa d\tau$$
$$- \frac{\mathcal{F}^{(n-1)}(a_2) - \mathcal{F}^{(n-1)}(a_1)}{(a_2-a_1)(n-4)!} \int_{a_1}^{a_2} \mathcal{B}_1(\tau)d\tau, \tag{40}$$

bounded by

$$|\mathcal{R}_1(\mathcal{F};a_1,a_2)| \leq \frac{\sqrt{a_2-a_1}}{\sqrt{2}(n-4)!}\big(\mathcal{F}(\mathcal{B}_1,\mathcal{B}_1)\big)^{\frac{1}{2}}\Big(\int_{a_1}^{a_2}(\tau-a_1)(a_2-\tau)[\mathcal{F}^{(n+1)}]^2 d\tau\Big)^{\frac{1}{2}}. \tag{41}$$

(ii) and the remainder $\mathcal{R}_2$ is

$$\mathcal{R}_2(\mathcal{F};a_1,a_2) = \int_{\Omega_2} \mathcal{F}(f(s))\nu(s)d\mu_2(s) - \int_{\Omega_1} \mathcal{F}(\mathcal{A}_k f(\varpi))u(\varpi)d\mu_1(\varpi)$$
$$- \sum_{\kappa} \frac{(-1)^\kappa \mathcal{F}^{(\kappa)}(a_2)}{\kappa!} \int_{a_1}^{a_2} \mathcal{J}G_\alpha(f,\tau)(a_2-\tau)^\kappa d\tau$$
$$- \frac{\mathcal{F}^{(n-1)}(a_2) - \mathcal{F}^{(n-1)}(a_1)}{(a_2-a_1)(n-4)!} \int_{a_1}^{a_2} \mathcal{B}_2(\tau)d\tau, \tag{42}$$

bounded by

$$|\mathcal{R}_2(\mathcal{F};a_1,a_2)| \leq \frac{\sqrt{a_2-a_1}}{\sqrt{2}(n-4)!}\big(\mathcal{F}(\mathcal{B}_2,\mathcal{B}_2)\big)^{\frac{1}{2}}\Big(\int_{a_1}^{a_2}(\tau-a_1)(a_2-\tau)[\mathcal{F}^{(n+1)}]^2 d\tau\Big)^{\frac{1}{2}}. \tag{43}$$

Proof.

- From (24) and (40) we have

$$\mathcal{R}_1(\mathcal{F};a_1,a_2) = \frac{1}{(n-4)!}\Bigg(\int_{a_1}^{a_2} \mathcal{B}_1(\tau)\mathcal{F}^{(n)}(\tau)d\tau - \frac{\mathcal{F}^{(n-1)}(a_2) - \mathcal{F}^{(n-1)}(a_1)}{(a_2-a_1)} \int_{a_1}^{a_2} \mathcal{B}_1(\tau)d\tau\Bigg). \tag{44}$$

Taking $f = \mathcal{B}_1$ and $g = \mathcal{F}^{(n)}$ along with (44), then using Theorem 2 we obtain

$$\frac{1}{a_2-a_1}\Big|\mathcal{R}_1(\mathcal{F};a_1,a_2)\Big|$$
$$= \frac{1}{(n-4)!}\left|\frac{1}{a_2-a_1}\int_{a_1}^{a_2} \mathcal{B}_1(\tau)\mathcal{F}^{(n)}(\tau)d\tau - \frac{1}{(a_2-a_1)}\int_{a_1}^{a_2} \mathcal{B}_1(\tau)d\tau \frac{1}{(a_2-a_1)}\int_{a_1}^{a_2} \mathcal{F}^{(n)}(\tau)d\tau\right|$$
$$\leq \frac{1}{\sqrt{2}(n-4)!}\frac{1}{\sqrt{a_2-a_1}}\big(\mathcal{F}(\mathcal{B}_1,\mathcal{B}_1)\big)^{\frac{1}{2}}\Big(\int_{a_1}^{a_2}(\tau-a_1)(a_2-\tau)[\mathcal{F}^{(n+1)}]^2 d\tau\Big)^{\frac{1}{2}}. \tag{45}$$

After simplification of involved integral we get the required result.

- Following similar steps of first part we obtain the required result.

Now we use Theorem 3 to discuss the Grüss-type inequalities that enable us to find the bounds of remainder.

Theorem 8. *Let* $(\sum_1, \Omega_1, \sigma_1)$ *and* $(\sum_2, \Omega_2, \sigma_2)$ *with positive* $\sigma-$*finite measures and* $\mathcal{A}_k$ *and* $\mathcal{K}$ *be given in* (15) *and* (16) *respectively. Let* $\mathcal{F}$ *be defined on* $\mathcal{T}$ *such that for* $n \in \mathbb{N}$, $\mathcal{F}^{(n)}$ *is absolutely continuous and* $\mathcal{F}^{(n+1)} \geq 0$ *therein. Let* G_α, $\{\alpha = 1, 2, 3, 4\}$ *be defined in* (5), (6), (7) *and* (8) *respectively and* $u : \Omega_1 \to \mathbb{R}$ *be the weight function and* v *is given in* (17), $\mathcal{B}_1, \mathcal{B}_2$ *are given in* (38) *and* (39) *respectively such that* $\mathcal{B}_1^{'}$ *and* $\mathcal{B}_2^{'} \in L^\infty([a_1, a_2])$, *then;*

$$|\mathcal{R}_1(\mathcal{F}; a_1, a_2)| \leq \frac{(a_2 - a_1)||\mathcal{B}_1^{'}||_\infty}{(n-4)!} \left[\frac{\mathcal{F}^{(n-1)}(a_2) + \mathcal{F}^{(n-1)}(a_1)}{2} - \frac{\mathcal{F}^{(n-2)}(a_2) - \mathcal{F}^{(n-2)}(a_1)}{a_2 - a_1} \right], \tag{46}$$

$$|\mathcal{R}_2(\mathcal{F}; a_1, a_2)| \leq \frac{(a_2 - a_1)||\mathcal{B}_2^{'}||_\infty}{(n-4)!} \left[\frac{\mathcal{F}^{(n-1)}(a_2) + \mathcal{F}^{(n-1)}(a_1)}{2} - \frac{\mathcal{F}^{(n-2)}(a_2) - \mathcal{F}^{(n-2)}(a_1)}{a_2 - a_1} \right]. \tag{47}$$

where $\mathcal{R}_1(\mathcal{F}; a_1, a_2)$ *and* $\mathcal{R}_2(\mathcal{F}; a_1, a_2)$ *are given in* (40) *and* (42) *respectively.*

Proof.

- As all the conditions stated in Theorem 3 are obayed if we take $f = \mathcal{B}_1$ and $g = \mathcal{F}^{(n)}$. So considering (44) we can have

$$\begin{aligned}
&\frac{1}{a_2 - a_1}\Big|\mathcal{R}_1(\mathcal{F}; a_1, a_2)\Big| = \frac{1}{(n-4)!}\Bigg|\frac{1}{a_2 - a_1}\int_{a_1}^{a_2} \mathcal{B}_1(\tau)\mathcal{F}^{(n)}(\tau)d\tau \\
&-\frac{1}{(a_2 - a_1)}\int_{a_1}^{a_2} \mathcal{B}_1(\tau)d\tau \frac{1}{(a_2 - a_1)}\int_{a_1}^{a_2} \mathcal{F}^{(n)}(\tau)d\tau\Bigg| \\
&\leq \frac{1}{2(n-4)!}\frac{||\mathcal{B}_1^{'}||_\infty}{a_2 - a_1}\Big(\int_{a_1}^{a_2} (\tau - a_1)(a_2 - \tau)\mathcal{F}^{(n+1)}(\tau)d\tau\Big).
\end{aligned} \tag{48}$$

  After simplifying integral in last step and taking (44) into account we get the required result.

- Following smiliar steps of first part we obtain the required result.

Now moving forward for the analysis of Ostrowski-type inequalities related to the case under consideration i.e generalized Hardy-type inequalities of convex functions via Taylor polynomial and Green function

Theorem 9. *Let all the conditions of Theorem 4 holds with $n \in \mathbb{N}$ and $n \geq 4$. Let $\mathcal{J}G_\alpha$, $\mathcal{B}_1$ and $\mathcal{B}_2$ be given in (26), (38) and (39) respectively. Assuming that p, q be the conjugate exponents with $1 \leq p,\ q \leq \infty$ such that $\frac{1}{q} + \frac{1}{p} = 1$ and $\mathcal{F} : \mathcal{T} \to \mathbb{R}$ be such that $||\mathcal{F}^{(n)}||_p < \infty$. Then*

$$\Bigg| \int_{\Omega_2} \mathcal{F}(f(s))\nu(s)d\mu_2(s) - \int_{\Omega_1} \mathcal{F}(\mathcal{A}_k f(\varpi))u(\varpi)d\mu_1(\varpi) \\ - \sum_{\kappa} \frac{\mathcal{F}^{(\kappa)}(a_1)}{\kappa!} \int_{a_1}^{a_2} \mathcal{J}G_\alpha(f,\tau)\nu(s)d\mu_2(s)(\tau - a_1)^{\kappa} \Bigg| \leq \frac{1}{(n-4)!} ||\mathcal{F}^{(n)}||_p ||\mathcal{B}_1||_q \tag{49}$$

and

$$\Bigg| \int_{\Omega_2} \mathcal{F}(f(s))\nu(s)d\mu_2(s) - \int_{\Omega_1} \mathcal{F}(\mathcal{A}_k f(\varpi))u(\varpi)d\mu_1(\varpi) \\ - \sum_{\kappa} \frac{\mathcal{F}^{(\kappa)}(a_2)}{\kappa!} \int_{a_1}^{a_2} \mathcal{J}G_\alpha(f,\tau)(\tau - a_2)^{\kappa} d\tau \Bigg| \leq \frac{1}{(n-4)!} ||\mathcal{F}^{(n)}||_p ||\mathcal{B}_1||_q. \tag{50}$$

Proof.

- Using Hölder inequality in (24) we arrive at

$$\Bigg| \int_{\Omega_2} \mathcal{F}(f(s))\nu(s)d\mu_2(s) - \int_{\Omega_1} \mathcal{F}(\mathcal{A}_k f(\varpi))u(\varpi)d\mu_1(\varpi) \\ - \sum_{\kappa} \frac{\mathcal{F}^{(\kappa)}(a_1)}{\kappa!} \int_{a_1}^{a_2} \mathcal{J}G_\alpha(f,\tau)(\tau - a_1)^{\kappa} d\tau \Bigg| = \frac{1}{(n-4)!} \Big| \int_{a_1}^{a_2} \mathcal{B}_1(\tau)\mathcal{F}^{(n)}(\tau)d\tau \Big| \\ \leq \frac{1}{(n-4)!} ||\mathcal{F}^{(n)}||_p \Big(\int_{a_1}^{a_2} |\mathcal{B}_1(\tau)|^q d\tau \Big)^{\frac{1}{q}}.$$

  From this we can write (49).

- Following similar steps of part one after using Hölder inequality in (25) we get the required result.

5. Mean Value Theorem (MVT) and n-Exponential Convexity

It is clear that inequalities (32) and (33) are linear in $\mathcal{F}$. Keeping in mind all the assumptions stated in Theorem 5, two linear functionals can be defined as following;

$$\mathcal{H}_1 = \int_{\Omega_2} \mathcal{F}(f(s))\nu(s)d\mu_2(s) - \int_{\Omega_1} \mathcal{F}(\mathcal{A}_k f(\varpi))u(\varpi)d\mu_1(\varpi)$$

$$-\int_{a_1}^{a_2} \mathcal{J}G_\alpha(f,\tau) \times \sum_\kappa \frac{\mathcal{F}^{(\kappa)}(a_1)}{\kappa!}(\tau - a_1)^\kappa d\tau \tag{51}$$

and

$$\mathcal{H}_2 = \int_{\Omega_2} \mathcal{F}(f(s))\nu(s)d\mu_2(s) - \int_{\Omega_1} \mathcal{F}(\mathcal{A}_k f(\varpi))u(\varpi)d\mu_1(\varpi)$$
$$-\int_{a_1}^{a_2} \mathcal{J}G_\alpha(f,\tau) \times \sum_\kappa (-1)^\kappa \frac{\mathcal{F}^{(\kappa)}(a_2)}{\kappa!}(a_2 - \tau)^\kappa d\tau. \tag{52}$$

For any n-convex function $\mathcal{F} \in C^n([a_1, a_2])$, we have $\mathcal{H}_\gamma(\mathcal{F}) \geq 0$, for $\gamma = 1, 2$. Employing the linearity and non-negativity of the functionals given above, we can obtain the corresponding MVT.

Theorem 10. *Consider the aforementioned functionals $\mathcal{H}_\gamma$, $\gamma = 1, 2$ defined in* (51)*and* (52) *and $\mathcal{F} \in C^n([a_1, a_2])$. Then there exist $a_1 \leq c_\gamma \leq a_2$ such that*

$$\mathcal{H}_\gamma(\mathcal{F}) = \mathcal{F}^{(n)}(c_\gamma)\mathcal{H}_\gamma(\mathcal{F}_0) \;\; \gamma = 1, 2, \tag{53}$$

where $\mathcal{F}_0(x) = \frac{x^n}{n!}$

Proof. Denote $m = \min \mathcal{F}^{(n)}$ and $M = \max \mathcal{F}^{(n)}$. Firstly, we consider $\mathcal{H}(\varpi) = \frac{M\varpi^n}{n!} - \mathcal{F}(\varpi)$. Then $\mathcal{H}_1^{(n)}(\varpi) = M - \mathcal{F}^n \geq 0$, $\varpi \in \mathcal{T}$. So we can say $\mathcal{H}_1$ is n-convex function. Similarly, $\mathcal{H}_2(\varpi) = \mathcal{F}(\varpi) - \frac{m\varpi^n}{n!}$ is n-convex. Following similar steps of [3, Theorem 7] for convex functions $\mathcal{H}_1$ and $\mathcal{H}_2$ we have that there exist c_γ for which (53) holds.

Theorem 11. *Let $\mathcal{F}_1, \mathcal{F}_2 \in C^n([a_1, a_2])$ and $\mathcal{H}_\gamma$, $\gamma = 1, 2$ be defined in* (51) *and* (52). *Then there exist c_γ, $\gamma = 1, 2$ such that*

$$\frac{\mathcal{F}_1^{(n)}(c_\gamma)}{\mathcal{F}_2^{(n)}(c_\gamma)} = \frac{\mathcal{H}_\gamma(\mathcal{F}_1)}{\mathcal{H}_\gamma(\mathcal{F}_2)}, \;\; \gamma = 1, 2, \tag{54}$$

for denominators that are not equal to zero.

Proof. The proof follows from [3, Corollary 12]

The seminal concept of obtaining n-exponentially convex and exponentially convex functions was given in [11]. We apply aforementioned functionals on specific family given in the upcoming theorem which is enough to get criteria for the exponential convexity of the family of functions on selection of any distinct points.

Theorem 12. *Consider $H_\gamma, \gamma = 1, 2$ be the functionals defined in* (51) *and* (52). *Let $\mathcal{S} = \{f_\mu : [a_1, a_2] \to \mathbb{R}\}$ represents the class of functions having property that on choosing any of $r + 1$ distinct points $s_0, ..., s_r \in [a_1, a_2]$, the mapping $\mu \to [s_0, ..., s_r; f_\mu]$ is n-exponentially convex in Jensen sense on $\mathcal{T}$ so does the function $\mu \to \mathcal{H}_\gamma(f_\mu)$ on $\mathcal{T}$. Furthermore, if $\mu \to \mathcal{H}_\gamma(f_\mu)$ is continuous $\mathcal{T}$, then it is n-exponentially convex on $\mathcal{T}$.*

Proof. Define a function for $c_j \in \mathbb{R}$ and $\mu_j \in \mathcal{T}$, $j = 1, ..., n$ and $\mu_j^k = \frac{\mu_i + \mu_j}{2}$, $1 \leq j, k \leq n$ as;

$$\xi(x) = \sum_{j,k=1}^{n} c_j c_k f_{\mu_j^k}(x), \tag{55}$$

where $f_{\mu_j^k}(x) \in \mathcal{S}$. Imploying the assumption that $\mu \to [s_0, ..., s_n; f_\mu]$ is n-exponentially convex in Jensen sense, we have

$$[s_0, ..., s_n; \xi] = \sum_{j,k=1}^{n} c_j c_k [s_0, ..., s_n; f_{\mu_j^k}] \geq 0. \tag{56}$$

Consequently, we have

$$\mathcal{H}_\gamma(\xi) = \sum_{j,k=1}^{n} c_j c_k \mathcal{H}_\gamma(f_{\mu_j^k}) \geq 0, \tag{57}$$

for each $\gamma = 1, 2$. From this relation, it is evident that $\mathcal{H}_\gamma(f_\mu)$ is n-exponentially convex in Jensen sense on $\mathcal{T}$. Also if we take continuity of $\mu \to \mathcal{H}_\gamma(f_\mu)$ under consideration then it is n-exponentially convex on $\mathcal{T}$.

Some of the consequences of Theorem 12 can be obtained in the form of results given in Corollary 4.4.1 and 4.4.2 in [15]

6. Conclusion

The study under consideration is the advancement and analysis of Hardy-type inequalities using the two-point right focal problem-type Green functions and Taylor's polynomial that can be seen in Theorem 4, Theorem 5 and Theorem 6. Also, bounds on the remainders are found in Theorem 7 by using Čebyšev functional. In the same section 4 we discussed the Grüss-type inequalities in Theorem 8 to find the bound and Ostrowski-type inequalities in Theorem 9 related to the Hardy-type inequalities via Taylor's polynomial and Green function. Next, we discuss the Mean value theorem for the functionals (51) and (52) obtained from our main results and then make use of them to get results in the form of MVT given in Theorem 10 and Theorem 11. Finally, using obtained functionals (51) and (52) the n-exponential convexity is discussed as in Theorem 12.

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