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New Generalized Results for Modified Atangana-Baleanu Fractional Derivatives and Integral Operators

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Abstract. In this current study, first we establish the modified power Atangana-Baleanu fractional derivative operators (MPC) in both the Caputo and Riemann-Liouville (MPRL) senses. Using the convolution approach and Laplace transformation, the so-called modified power fractional Caputo and R-L derivative operators with non-singular kernels are introduced. We establish the boundedness of the modified Caputo fractional derivative operator in this study. The fractional differential equations are solved with the generalised Laplace transform (GLT). In addition, the corresponding form of the fractional integral operator is defined. Also, we prove the boundedness and Laplace transform of the fractional integral operator. The composition of power fractional derivative and integral operators is given in the study. Additionally, several examples related to our findings along with their graphical representation are presented.
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1. Introduction

Fractional calculus has a remarkable 325 years history, but there are still many unanswered theoretical and practical questions. Fractional calculus is used by Abel [1] to solve the tautrocrone problem. The use of fractional calculus in differential and integral equations is highlighted in this work [9]. Numerous more articles written by other scholars in [13, 16, 21, 24] provide a variety of concepts and applications related to fractional operators. Many researchers argue that a single Fractional operator cannot correctly represent the complexity of various complex scientific and engineering processes, such as the Caputo ones. Considering that additional experimental proof is needed to verify the fractional models' accuracy [3, 12].

An increasing number of mathematicians and experts have focused on fractional differential and integral equations in recent years [2, 22]. Many phenomena in a variety of disciplines, such as dynamics, physics, biology, and mechanics, have scientific interpretations that align with the fractional order derivatives. The existence theory of solutions is one of the primary study topics for fractional order differential equations, and analysts are closely monitoring. An exact solution to a fractional order differential equation may be difficult to find. It can be difficult to work with fractional calculus's non-singular kernel. By expanding their kernels, the authors [8, 15, 27] has recently defined the generalization of fractional operators. Samraiz *et al.* described the (k, s) form of fractional operators with a non-singular kernel and their physics applications in [27]. They used the (k, s) form of fractional operators to solve the Cauchy problems after proving them. The Hilfer-Prabhakar fractional derivative (k, s) was introduced by Samraiz et al. [28]. The weighted generalized form of fractional operators with a non-singular kernel associated with Mittag-Leffler (M-L) function are presented by [25, 29]. These operators are used to identify Cauchy problems in continuous time random walk theory.By using the multivariate M-L function as a non-singular kernel, the authors in [26] developed the generalized (k, s) fractional operators. To learn more about the applications of fractional operators with non-singular kernels, we refer the interested reader to [4, 5, 10, 14, 23, 30, 31].

2. Preliminaries

Let's recall the following essential definitions.
The following are the definitions of beta and gamma functions found in [32].

Definition 1. *We begin with the well-known gamma function which is defined by*

$$\Gamma(\kappa_1) = \int_0^\infty u^{\kappa_1 - 1} e^{-u} du, \ Re(\kappa_1) > 0.$$

Definition 2. *The well-known beta function* $B(\kappa_1, \zeta_1)$ *can be defined by*

$$B(\kappa_1, \zeta_1) = \int_0^1 \tau^{\kappa_1 - 1}(1-\tau)^{\zeta_1 - 1} d\tau, \ \Re(\kappa_1) > 0, \Re(\zeta_1) > 0$$

and its relation with Γ *function is given by*

$$B(\kappa_1, \zeta_1) = \frac{\Gamma(\kappa_1)\Gamma(\zeta_1)}{\Gamma(\kappa_1 + \zeta_1)}.$$

Definition 3. *The power M-L function is recently introduced by Lotfi et al. [11] as follows:*

$$^{p}\mathbb{E}_{\varrho_1,\kappa_1}(\varrho) = \sum_{n=0}^{\infty} \frac{(\varrho \ln p)^n}{\Gamma(\varrho_1 n + \kappa_1)}, \tag{1}$$

where $\varrho \in \mathbb{C}$, $p > 1$, $\Re(\kappa) > 0$ *and* $\Re(\varrho) > 0$.

Remark 1. *The following are the special cases of Power M-L function:*
i. If we take $\varrho = \kappa = 1$ *and* $p = e$, *we get*

$$^{e}\mathbb{E}_{1,1}(\varrho) = \sum_{n=0}^{\infty} \frac{\varrho^n}{\Gamma(n+1)} = \sum_{n=0}^{\infty} \frac{\varrho^n}{n!} = e^{\varrho}.$$

ii. If we take $\kappa = 1$ *and* $p = e$, *we get*

$$^{e}\mathbb{E}_{\varrho_1,1}(\varrho) = \sum_{n=0}^{\infty} \frac{\varrho^n}{\Gamma(\varrho_1 n + 1)}.$$

iii. If we take $p = e$, *we get*

$$^{e}\mathbb{E}_{\varrho_1,\kappa_1}(\varrho) = \sum_{n=0}^{\infty} \frac{\varrho^n}{\Gamma(\varrho_1 n + \kappa_1)}.$$

Hölder's inequality was first derived by Leonard James Rogers in 1888, then Ludwig Otto Hölder presented it in a different way in 1889. The following definition provides an explanation of Hölder inequality.

Definition 4. *[6] For a given two real numbers,* r_1 *and* s_1, *such that* $r_1, s_1 > 1$ *and* $\frac{1}{r_1} + \frac{1}{s_1} = 1$, *the Hölder integral inequality is stated by*

$$\int_v^u |f_1(\varrho) g_1(\varrho)| d\varrho \leq \left(\int_v^u |f_1(\varrho)|^{r_1} d\varrho \right)^{\frac{1}{r_1}} \left(\int_v^u |g_1(\varrho)|^{s_1} d\varrho \right)^{\frac{1}{s_1}},$$

where $f_1, g_1 \in C^1[u, v]$.

The Lebesgue measurable functions with norm are defined as follows by Kilbas *et al.* in [9].

Definition 5. *Consider a function* g *that is defined on* $[c, d]$. *The Lebesgue measurable functions space* $\chi^q(c, d)$, $1 \leq q \leq \infty$, *for which* $\| \Psi \|_{\chi^q} < \infty$, *i.e.,*

$$\| \Psi \|_{\chi^q} = \left[\int_r^s | \Psi(\zeta) |^q \, d\zeta \right]^{\frac{1}{q}}, \ 1 \leq q < \infty,$$

$$\| \Psi \|_{\chi^\infty} = ess \, sup_{c \leq t \leq d} \, | \Psi(\zeta) | < \infty.$$

Fractional calculus theory still has many remaining challenges since the Riemann and Caputo fractional operators are not enough to solve theoretical and physical problems. The theory still has a lot of challenges, even though researchers have defined their own operators to fill in these gaps. Atangana-Baleanu developed the fractional operators to fill in these gaps. These operators are useful in many theoretical and physical applications, both in the Riemann and Caputo senses. For example, Panda *et al.* [20] investigated the Willis aneurysm system and solved a nonlinear singularity perturbed boundary value problem using the Atangana-Baleanu operator. The investigation of COVID-19 prevalence in France, Italy, and the US by Panda *et al.* in [17] is one of the other uses of the Atangana-Baleanu operators. He introduced new results on the existence and uniqueness of the 2019-nCOV models with respect to fractional and fractal-fractional operator-based solutions. Moreover, the solutions to the Atangana-Baleanu fractional equations, the complex valued Atangana-Baleanu operator, and the L^p-Fredholm integral equations are examined in [18] and [19]. Using this method, fractional operators in the Caputo sense take on a new shape.

Definition 6. *[2] If $g' \in H^1(0,T)$, then the Atangana-Baleanu fractional operator in Caputo sense of order $0 < \delta < 1$ is defined as follows:*

$$ {}_{ABC}\mathfrak{D}_0^{\delta} g(\xi) = \frac{R(\delta)}{1-\delta}\int_0^{\xi} \mathbb{E}_{\delta}\Big(-\omega_{\delta}\big(\xi-\zeta\big)^{\delta}\Big) g'(\zeta)d\zeta, \quad \zeta \geq 0 $$

and its associated integral operator is provided by

$$ {}_{ABC}\mathfrak{I}_0^{\delta} g(\xi) = \frac{1-\delta}{R(\delta)} g(\xi) + \frac{R(\delta)}{1-\delta}\int_0^{\xi} \big(\xi-\zeta\big)^{\delta} g(\zeta)d\zeta, \quad \zeta \geq 0. \tag{2} $$

Numerous theoretical results can be solved using the aforementioned operators. Al-Refai *et al.* in [23] explains how spaces are utilised extensively in operator applications. Consider the differential equation ${}_{ABC}\mathfrak{D}_0^{\delta} g(\xi) = \lambda g(\xi)$, where $g(\xi) \in C^1(0,T)$. The trivial solution, ${}_{ABC}\mathfrak{D}_0^{\delta} g(0) = 0$, is given by the solution $\lambda g(0) + h(0) = 0$ to the fractional equation ${}_{ABC}\mathfrak{D}_0^{\delta} g(\xi) = -\lambda g(\xi) + h(\xi)$. However, the Caputo derivative is constrained by this area. We will consider $\chi(g) = \big\{g : g' \in L^1[0,1]\big\}$ is the space for the following initial value problem

$$ {}_{ABC}\mathfrak{D}_0^{\delta} g(\xi) = \begin{cases} -\lambda g(\xi) + h(\xi), & \xi \in (0,T); \\ g_0, & \xi = 0, \end{cases} $$

we obtain the following solution for $0 < \delta < 1$,

$$ g(\xi) = g_0 \mathbb{E}_{\delta,1}(-\lambda\theta^{\delta}) + \int_0^{\theta} (\theta-\zeta)^{\delta-1} \mathbb{E}_{\delta,\delta}(-\lambda(\theta-\zeta)^{\delta}) h(\zeta)d\zeta. $$

The related homogeneous equation $g(\xi) = g_0\mathbb{E}_{\delta,1}(-\lambda\theta^{\delta})$ likewise yields a nontrivial solution. The definition and discussion of the weighted form of Atangana-Baeanu operators in differential equations is found in [22].

Definition 7. *[22] The weighted Atangana-Baleanu fractional operator of order* $0 < \delta < 1$ *for a given* $g' \in L^1(0,T)$*, is defined by*

$$ {}_{ABC}\mathfrak{D}_0^{\delta} g(\xi) = \frac{R(\delta) w^{-1}(\zeta)}{1-\delta} \int_0^{\xi} \mathbb{E}_{\delta}\Big(-\omega_{\delta}\big(\xi-\zeta\big)^{\delta}\Big) (w(\zeta)g(\zeta))' d\zeta, \quad \zeta \geq 0, $$

and its associated integral operator is given by

$$ {}_{ABC}\mathfrak{I}_0^{\delta} g(\xi) = \frac{1-\delta}{R(\delta)} g(\xi) + \frac{R(\delta)}{1-\delta} w^{-1}(\zeta) \int_0^{\xi} \big(\xi-\zeta\big)^{\delta} w(\zeta) g(\zeta) d\zeta, \quad \zeta \geq 0, $$

where the normalised function $R(\delta)$*, has the property* $R(0) = R(1) = 1$.

Our main objective is to define the modified power fractional operators and power fractional differential equations and to determine their exact solutions using a variety of methods. we offer the modified power fractional operators as a tool to characterise numerous power differential equations. The work presented in this paper are the generalization of the work done by [2, 7, 11].

3. The Modified Power Fractional Derivative in Caputo Sense

We introduce the modified power fractional derivative operators in this section. We introduce its Laplace transformation and boundedeness. Moreover, a few relevant examples are shown.

Definition 8. *The generalised form of the modified fractional (MPC) derivative operator of order* $0 < \delta < 1$ *and* $p > 1$ *with regard to another function* $\hbar$*, given* g *as a continuous function and* $g' \in L^1(0,T)$*, is defined as follows:*

$$ {}^{MPC}_{\hbar}\mathfrak{D}_{a^+}^{\delta;\kappa;p} g(\xi) = \frac{R(\delta)}{1-\delta} \int_a^{\xi} {}^{p}\mathbb{E}_{\kappa,1}\Big(-\omega_{\delta}\big(\hbar(\xi)-\hbar(\zeta)\big)^{\kappa}\Big) g'(\zeta) d\zeta, \quad \zeta \geq 0. \tag{3} $$

Integrating by parts leads to

$$ \begin{aligned} {}^{MPC}_{\hbar}\mathfrak{D}_{a^+}^{\delta;\kappa;p} g(\xi) = \frac{R(\delta)}{1-\delta} \Bigg[& g(\xi) - {}^{p}\mathbb{E}_{\kappa,1}\Big(-\omega_{\delta}\big(\hbar(\xi)-\hbar(a)\big)^{\kappa}\Big) g(a) \\ & - \omega_{\delta}(\ln p) \int_a^{\xi} (\hbar(\xi)-\hbar(\zeta))^{\kappa-1} \, {}^{p}\mathbb{E}_{\kappa,\kappa}\Big(-\omega_{\delta}\big(\hbar(\xi)-\hbar(\zeta)\big)^{\kappa}\Big) \hbar'(\zeta) g(\zeta) d\zeta \Bigg]. \end{aligned} $$

The functions $\omega_{\delta} = \frac{\delta}{1-\delta}$*,* $\hbar$*, a strictly increasing function, and* $R(\delta)$*, a normalized function with the condition* $R(0) = R(1) = 1$ *are among them.*

Remark 2. *i. Letting* $\hbar(\xi) = \xi$*,* $\kappa = \delta$ *and* $p = e$ *in* (3) *then we get Definition 7.*
ii. Letting we substitute $\kappa = \delta$ *and* $p = e$ *in* (3) *then we get the operator defined by [7].*
iii. Letting $\hbar(\xi) = \xi$ *in* (3) *then we get Definition of power fractional derivative defined by [11].*

We demonstrate the boundedness of the operator given by Definition 8 in the first result of this section.

Theorem 1. *Considering $g \in X^p(0,T)$, and $\hbar$, $\frac{\delta}{1-\delta} = \omega_\delta$ and $R(\delta)$, permit the same characteristics as stated in 8; hence, the inequality*

$$\left\| {}^{MPC}_{\hbar}\mathfrak{D}^{\delta;\kappa;p}_{a^+} g(\xi) \right\|_{X^p} \leq \frac{R(\delta)}{1-\delta} \|g(\xi)\|_{X^p} + \frac{R(\delta)}{1-\delta} \left\| {}^{p}\mathbb{E}_{\kappa,1}\Big(- \omega_\delta\big(\hbar(\xi) - \hbar(a)\big)^\kappa\Big) g(a) \right\|_{X^p}$$
$$+ \frac{R(\delta)}{1-\delta} \sum_{n=0}^{\infty} \frac{|\omega_\delta(\ln p)|^n}{|\Gamma(\kappa n + \kappa)|} \frac{|\big(\hbar(\theta) - \hbar(a)\big)|^{(\kappa n+\kappa-1)}}{(\kappa n + \kappa - 1)} ||f||_{X^p}, \tag{4}$$

holds for $1 \leq r_1 < \infty$.

Proof. Through application of Definition 8, we have

$$\left\| {}^{MPC}_{\hbar}\mathfrak{D}^{\delta;\kappa;p}_{a^+} g(\xi) \right\|_{X^p} \leq \frac{R(\delta)}{1-\delta} \left\| g(\xi) - \mathbb{E}_{\kappa,1}\Big(- \omega_\delta\big(\hbar(\xi) - \hbar(a)\big)^\kappa\Big) g(a) \right\|_{X^p}$$
$$+ \frac{R(\delta)}{1-\delta} \left\| \omega_\delta(\ln p) \int_a^\xi (\hbar(\xi) - \hbar(\zeta))^{\kappa-1}\, {}^{p}\mathbb{E}_{\kappa,\kappa}\Big(- \omega_\delta\big(\hbar(\xi) - \hbar(\zeta)\big)^\kappa\Big) \hbar'(\zeta) g(\zeta) d\zeta \right\|_{X^p}$$
$$\leq \frac{R(\delta)}{1-\delta} \|g(\xi)\|_{X^p} + \frac{R(\delta)}{1-\delta} \left\| \mathbb{E}_{\kappa,\kappa}\Big(- \omega_\delta\big(\hbar(\xi) - \hbar(a)\big)^\kappa\Big) g(a) \right\|_{X^p}$$
$$+ \frac{R(\delta)}{1-\delta} \left\| \omega_\delta(\ln p) \int_a^\xi (\hbar(\xi) - \hbar(\zeta))^{\kappa-1} \mathbb{E}_{\kappa,\kappa}\Big(- \omega_\delta\big(\hbar(\xi) - \hbar(\zeta)\big)^\kappa\Big) \hbar'(\zeta) g(\zeta) d\zeta \right\|_{X^p}. \tag{5}$$

Consider

$$\left\| \int_0^\xi (\hbar(\xi) - \hbar(\zeta))^{\delta-1}\, {}^{p}\mathbb{E}_{\kappa,\kappa}\Big(- \omega_\delta\big(\hbar(\xi) - \hbar(\zeta)\big)^\kappa\Big) \hbar'(\zeta) g(\zeta) d\zeta \right\|_{X^p}$$
$$= \sum_{n=0}^{\infty} \frac{|\omega_\delta(\ln p)|^n}{|\Gamma(\kappa n + \kappa)|} \left\| \int_0^\xi \big(\hbar(\xi) - \hbar(\zeta)\big)^{\kappa n+\kappa-1} \hbar'(\zeta) g(\zeta) d\zeta \right\|_{X^p}$$
$$= \sum_{n=0}^{\infty} \frac{|\omega_\delta(\ln p)|^n}{|\Gamma(\kappa n + \kappa)|} \left(\int_a^\theta \left| \int_a^\xi \big(\hbar(\xi) - \hbar(\zeta)\big)^{\kappa n+\kappa-1} \hbar'(\zeta) g(\zeta) d\zeta \right|^{r_1} \hbar'(\xi) du \right)^{\frac{1}{r_1}}.$$

Substituting $\lambda = \hbar(\zeta)$ and $\rho = \hbar(\xi)$, we obtain

$$= \sum_{n=0}^{\infty} \frac{|\omega_\delta(\ln p)|^n}{|\Gamma(\kappa n + \kappa)|} \left(\int_{\hbar(a)}^{\hbar(\theta)} \left| \int_{\hbar(a)}^{\hbar(\xi)} |\big(\rho - \lambda\big)|^{\kappa n+\kappa-1} g(\hbar^{-1}(\lambda)) d\lambda \right|^{r_1} d\rho \right)^{\frac{1}{r_1}}.$$

The generalised Minkowski's inequality gives us the following

$$\leq \sum_{n=0}^{\infty} \frac{|\omega_\delta(\ln p)|^n}{|\Gamma(\kappa n + \kappa)|} \left(\int_{\hbar(a)}^{\hbar(\theta)} |g(\hbar^{-1}(\lambda))|^{r_1} \int_{\lambda}^{\hbar(\theta)} |\big(\rho - \lambda\big)|^{(\kappa n+\kappa-1)r_1} d\rho \right)^{\frac{1}{r_1}} d\lambda$$

$$= \sum_{n=0}^{\infty} \frac{|\omega_\delta(\ln p)|^n}{|\Gamma(\kappa n+\kappa)|} \left(\int_{\hbar(a)}^{\hbar(\theta)} |g(\hbar^{-1}(\lambda))|^{r_1} \frac{|\big(\hbar(\theta)-\lambda\big)|^{(\kappa n+\kappa-1)r_1+1}}{(\kappa n+\kappa-1)r_1+1} \right)^{\frac{1}{r_1}} d\lambda.$$

With the help of Hölder inequality, we have

$$\leq \sum_{n=0}^{\infty} \frac{|\omega_\delta(\ln p)|^n}{|\Gamma(\kappa n+\kappa)|} \left(\int_{\hbar(a)}^{\hbar(\theta)} |g(\hbar^{-1}(\lambda))|^{r_1} d\lambda \right)^{\frac{1}{r_1}} \left(\int_{\hbar(a)}^{\hbar(\theta)} \left(\frac{|\big(\hbar(\theta)-\lambda\big)|^{(\kappa n+\kappa-1)r_1+1}}{(\delta n-1)r_1+1} \right)^{\frac{s_1}{r_1}} d\lambda \right)^{\frac{1}{s_1}}$$

$$\leq \sum_{n=0}^{\infty} \frac{|\omega_\delta(\ln p)|^n}{|\Gamma(\kappa n+\kappa)|} \left(\int_{\hbar(a)}^{\hbar(\theta)} |g(\hbar^{-1}(\lambda))|^{r_1} d\lambda \right)^{\frac{1}{r_1}} \frac{|\big(\hbar(\theta)-\hbar(a)\big)|^{(\kappa n+\kappa-1)}}{(\kappa n+\kappa-1)},$$

where $\frac{1}{r_1}+\frac{1}{s_1}=1$. By substituting $\hbar^{-1}(\lambda)=t$, we obtain

$$= \sum_{n=0}^{\infty} \frac{|\omega_\delta(\ln p)|^n}{|\Gamma(\kappa n+\kappa)|} \frac{|\big(\hbar(\theta)-\hbar(a)\big)|^{(\kappa n+\kappa-1)}}{(\kappa n+\kappa-1)} \int_{\hbar(a)}^{\hbar(\theta)} |g(\zeta)|^{r_1} \hbar'(\zeta) d\zeta$$

$$= \sum_{n=0}^{\infty} \frac{|\omega_\delta(\ln p)|^n}{|\Gamma(\kappa n+\kappa)|} \frac{|\big(\hbar(\theta)-\hbar(a)\big)|^{(\kappa n+\kappa-1)}}{(\kappa n+\kappa-1)} ||f||_{X^p}.$$

By using this equation in (5), we have the result (4).

Now, we present some illustrative examples of new fractional derivative operator.

Example 1. *Assume $0<\delta<1$ and a constant function $g(\zeta)=C$. The Definition 8 allows us to write*

$$\begin{aligned}
({}^{MPC}_{\hbar}\mathfrak{D}^{\delta;\kappa;p}_{a^+}C)(\zeta) &= \frac{R(\delta)}{1-\delta}\Bigg[C - {}^{p}\mathbb{E}_{\kappa,1}\Big(-\omega_\delta\big(\hbar(\xi)-\hbar(a)\big)^{\kappa}\Big)C \\
&- \omega_\delta(\ln p)\int_a^{\xi} (\hbar(\xi)-\hbar(\zeta))^{\kappa-1}\, {}^{p}\mathbb{E}_{\kappa,\kappa}\Big(-\omega_\delta\big(\hbar(\xi)-\hbar(\zeta)\big)^{\kappa}\Big)\hbar'(\zeta)Cd\zeta\Bigg] \\
&= \frac{R(\delta)}{1-\delta}\Bigg[C - {}^{p}\mathbb{E}_{\kappa,1}\Big(-\omega_\delta\big(\hbar(\xi)-\hbar(a)\big)^{\kappa}\Big)C \\
&- \omega_\delta(\ln p)\Big(\frac{-1}{\omega_\delta(\ln p)}\big({}^{p}\mathbb{E}_{\kappa,1}\Big(-\omega_\delta\big(\hbar(\xi)-\hbar(a^+)\big)^{\kappa}\Big)-1\big)\Big)C \\
({}^{MABC}_{\hbar}\mathfrak{D}^{\delta}_{0}C)(\zeta) &= 0.
\end{aligned}$$

This shows that the differentiation of a constant is zero.

Example 2. *If we choose $g(\xi)=\big(\hbar(\xi)-\hbar(\zeta)\big)^{\rho}$, then we have*

$${}^{MPC}_{\hbar}\mathfrak{D}^{\delta;\eta;p}_{a^+}g(\xi) = \frac{R(\delta)}{1-\delta}\int_a^{\xi} {}^{p}\mathcal{E}_{\kappa,1}\Big(-\omega_\delta\big(\hbar(\xi)-\hbar(\zeta)\big)^{\kappa}\Big)\frac{d}{d\zeta}\big(\hbar(\xi)-\hbar(\zeta)\big)^{\rho}d\zeta$$

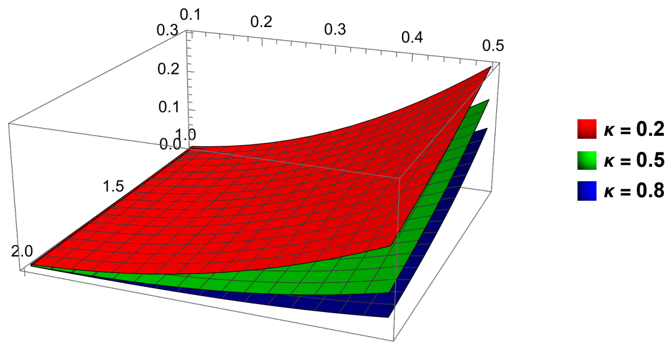

Figure 1: The graphical representation of absolute value of (6) corresponding to choice $0 \leq \xi \leq 1$.

$$= -\frac{\rho R(\delta)}{1-\delta}\sum_{n=0}^{\infty}\frac{(\ln p)^n(-\omega_\delta)^n}{\Gamma(\kappa n+1)}\int_a^\xi \big(\hbar(\xi)-\hbar(\zeta)\big)^{\kappa n+\rho-1}\hbar'(\zeta)d\zeta$$
$$= \frac{\rho R(\delta)}{1-\delta}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}(\ln p)^n(\omega_\delta)^n}{\Gamma(\kappa n+1)}\frac{\big(\hbar(\xi)-\hbar(a)\big)^{\kappa n+\rho}}{\kappa n+\rho}$$

Example 3. *If we choose* $\hbar(\xi) = \xi$, $\rho = 2$, $R(\delta) = 1$, $\delta = \frac{1}{2}$, $a = 0$, $\hbar(0) = 0$ *and* $p = 2$ *in Example 2, then we have*

$$^{MPC}_{\hbar}\mathfrak{D}^{\delta;\eta;p}_{a^+}f(\xi) = 4\sum_{n=0}^{\infty}\frac{(-1)^{n+1}(\ln 2)^n}{\Gamma(\kappa n+1)}\frac{(\xi)^{\kappa n+2}}{\kappa n+2}. \tag{6}$$

The graphical representation of (6) *for the choice of order* $\kappa = 0.2, 0.5, 0.8$, *is given by the following graph.*

Example 4. *If we choose* $\hbar(\xi) = \ln(\xi+1)$, $\rho = 2$, $R(\delta) = 1$, $\delta = \frac{1}{2}$, $a = 0$, $\hbar(0) = 0$ *and* $p = 2$ *in Example 2, then we have*

$$^{MPC}_{\hbar}\mathfrak{D}^{\delta;\eta;p}_{a^+}f(\xi) = 4\sum_{n=0}^{\infty}\frac{(-1)^{n+1}(\ln 2)^n}{\Gamma(\kappa n+1)}\frac{(\ln(\xi+1))^{\kappa n+2}}{\kappa n+2}. \tag{7}$$

The graphical representation of (7) *for the choice of order* $\kappa = 0.2, 0.5, 0.8$, *is given by the following graph*

Example 5. *If we choose* $\hbar(\xi) = \sqrt{\xi+1}$, $\zeta = 2$, $R(\delta) = 1$, $\delta = \frac{1}{2}$, $a = 0$, $\hbar(0) = 0$ *and* $p = 2$ *in Example 2, then we have*

$$^{MPC}_{\hbar}\mathfrak{D}^{\delta;\eta;p}_{a^+}f(\xi) = 4\sum_{n=0}^{\infty}\frac{(-1)^{n+1}(\ln 2)^n}{\Gamma(\kappa n+1)}\frac{(\sqrt{\xi+1})^{\kappa n+2}}{\kappa n+2}. \tag{8}$$

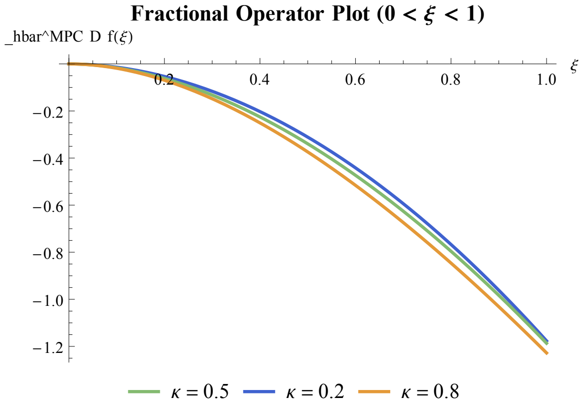

Figure 2: The 2-dimensional graphical representation of (6) corresponding to choice $0 \leq \xi \leq 1$.

The graphical representation of (8) *for the choice of order* $\kappa = 0.2, 0.5, 0.8$, *is given by the following graph.*

Example 6. *Suppose that function g be piecewise continuous*

$$g(\zeta) = \begin{cases} \hbar^{\frac{-1}{2}}(\zeta), & \zeta \neq 0; \\ A, & \zeta = 0. \end{cases} \tag{9}$$

Putting $A \in \mathbb{R} \setminus \{0\}$, *now* $\kappa = \frac{1}{2} = \delta$, $\hbar(0) = 0$, $u_\delta = 1$, $a = 0$ *and* $R(\delta) = 1$ *according to Definition 8, we get*

$$\begin{aligned}
{}_{\hbar}^{MPC}\mathfrak{D}_0^{\frac{1}{2};\frac{1}{2};p} g(\xi) &= 2\Big[g(\xi) - A\,{}^{p}\mathbb{E}_{\frac{1}{2},1}\Big(-\big(\hbar(\xi)\big)^{\frac{1}{2}}\Big) \\
&- \ln p \int_0^{\xi} (\hbar(\xi) - \hbar(\zeta))^{-\frac{1}{2}}\,{}^{p}\mathbb{E}_{\frac{1}{2},\frac{1}{2}}\Big(-\big(\hbar(\xi) - \hbar(\zeta)\big)^{\frac{1}{2}}\Big)\hbar'(\zeta)\hbar^{\frac{-1}{2}}(\zeta)d\zeta\Big] \\
&= 2\Big[g(\xi) - A\mathbb{E}_{\frac{1}{2},1}\Big(-\big(\hbar(\xi)\big)^{\frac{1}{2}}\Big) - \ln p\sqrt{\pi}\,{}^{p}\mathbb{E}_{\frac{1}{2},1}\Big(-\big(\hbar(\xi)\big)^{\frac{1}{2}}\Big) \\
&= 2\Big[g(\xi) - (A + \ln p\sqrt{\pi})\,{}^{p}\mathbb{E}_{\frac{1}{2},1}\Big(-\big(\hbar(\xi)\big)^{\frac{1}{2}}\Big)\Big].
\end{aligned} \tag{10}$$

By using series expansion of
${}^{p}\mathbb{E}_{\frac{1}{2},1}\Big(-\big(\hbar(\xi)\big)^{\frac{1}{2}}\Big) = 1 + \frac{(\hbar(\xi))^{\frac{1}{2}}\ln p}{\Gamma(\frac{3}{2})} + \cdots$, *we have*
${}^{p}\mathbb{E}_{\frac{1}{2},1}\Big(0\Big) = 1$. *Hence, we get*

$${}_{\hbar}^{MABC}\mathfrak{D}_0^{\frac{1}{2};\frac{1}{2};p} g(0) = -2\ln p\sqrt{\pi} \neq 0.$$

Remark 3. *i. If we put* $p = e$, *then we get the result proved by Huang et al. [7].*
ii. If we put $p = e$ *and* $\hbar(\xi) = \xi$, *then it reduce to the example solved by Al-Refai et al. [23].*
iii. If we put $\hbar(\xi) = \xi$, *then we get the solution of the problem for power fractional derivative defined by Lotfi et al. [11].*

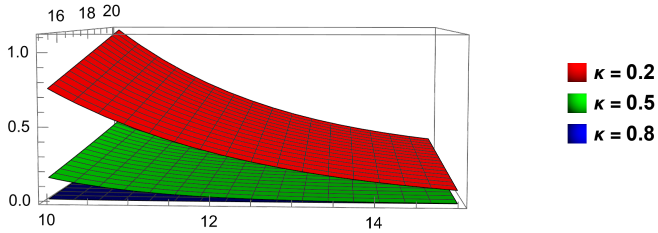

Figure 3: The graphical representation of absolute value of (7) corresponding to choice $0 \leq \xi \leq 1$.

Definition 9. *The modified power fractional derivative operator in the R-L sense of order $0 < \delta < 1$, $\kappa > 0$, and $p > 1$ with respect to another function $\hbar$, is defined as follows, assuming g is a continuous function and $g' \in L^1(0,T)$ by*

$$ {}_{\hbar}^{MPRL}\mathfrak{D}_{a^+}^{\delta;\kappa;p} g(\xi) = \frac{R(\delta)}{1-\delta}\frac{d}{d\zeta}\int_a^{\xi} {}^{p}\mathbb{E}_{\kappa,1}\Big(-\omega_\delta\big(\hbar(\xi)-\hbar(\zeta)\big)^{\kappa}\Big) g(\zeta)d\zeta, \quad \zeta \geq 0, \tag{11} $$

where $R(\delta)$ is a normalised function with the property $R(0) = R(1) = 1$, $\omega_\delta = \frac{\delta}{1-\delta}$, and $\hbar$ is a strictly rising function.

Remark 4. *i. If we substitute $\hbar(\xi) = \xi$, $\kappa = \delta$ and $p = e$ in (11), then we get the well-known definition Atangana-Baleanu fractional derivative operator in the R-L sense.*
ii. If we substitute $\hbar(\xi) = \xi$, in (11), then we get definition of power fractional derivative in R-L sense defined by Lotfi et al. [11].

Remark 5. *The modified power fractional derivative in R-L sense satisfies the following property*

$$ {}_{\hbar}^{MPRL}\mathfrak{D}_{a^+}^{0;\kappa;p} g(\xi) = g(\xi). $$

Theorem 2. *The modified fractional derivative in both Caputo and R-L senses satisfy the property of linearity for all α, λ and $f, g \in L^1(a,b)$.*

Proof. One can easily prove that

$$ {}_{\hbar}^{MPC}\mathfrak{D}_{a^+}^{\delta;\kappa;p}\big(\alpha g(\xi) + \lambda g(\xi)\big) = \alpha\, {}_{\hbar}^{MPC}\mathfrak{D}_{a^+}^{\delta;\kappa;p} g(\xi) + \lambda\, {}_{\hbar}^{MPC}\mathfrak{D}_{a^+}^{\delta;\kappa;p} g(\xi), $$

and

$$ {}_{\hbar}^{MPRL}\mathfrak{D}_{a^+}^{\delta;\kappa;p}\big(\alpha g(\xi) + \lambda g(\xi)\big) = \alpha\, {}_{\hbar}^{MPRL}\mathfrak{D}_{a^+}^{\delta;\kappa;p} g(\xi) + \lambda\, {}_{\hbar}^{MPRL}\mathfrak{D}_{a^+}^{\delta;\kappa;p} g(\xi). $$

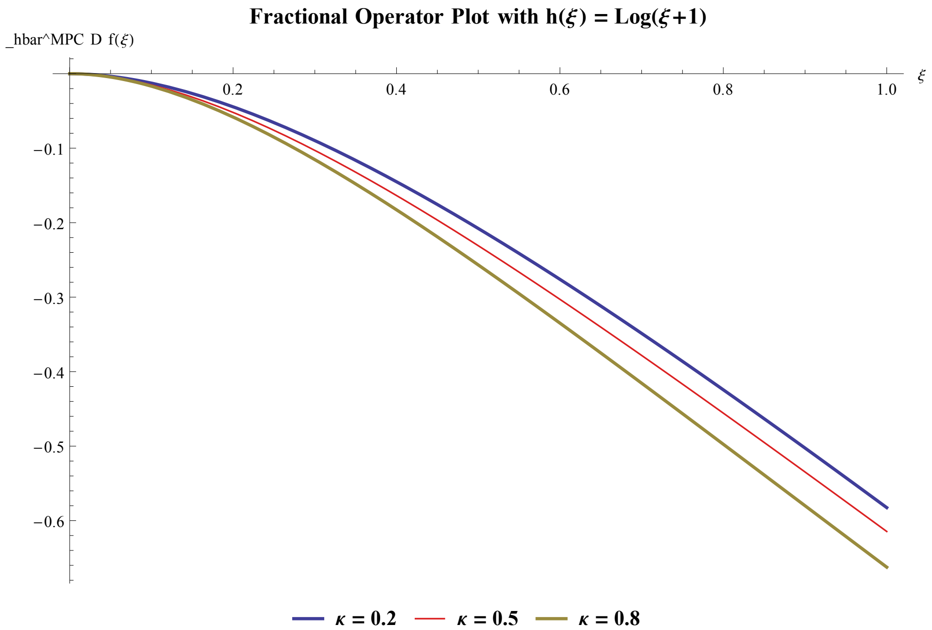

Figure 4: The 2-dimensional graphical representation of (7) corresponding to choice $0 \leq \xi \leq 1$.

Definition 10. *The Laplace transform of ψ is produced by letting $\hbar$ (where $\hbar$ is a monotonically increasing) and ψ be defined on $[a, \infty)$ by*

$$L_{\hbar}(\psi)(s) = \int_{a}^{\infty} e^{-s(\hbar(\xi)-\hbar(a))}\hbar'(\xi)\psi(\xi)d\zeta$$

such that the equation (10) *holds for all values of s.*

Definition 11. *The convolution of the function Φ and ω associated with ϕ is provided by*

$$(\Phi *_{\phi} \omega)(\xi) = \int_{a}^{\theta} \Phi\left(\phi^{-1}\left(\phi(\theta) + \phi(a) - \phi(\zeta)\right)\right)\omega(\zeta)\phi'(\zeta)d\zeta.$$

For FO (fractional operator) in Definition 8, the convolution form is provided by

$$\begin{aligned}
{}^{MPC}_{\hbar}\mathfrak{D}^{\delta;\kappa;p}_{a^+}g(\xi) &= \frac{R(\delta)}{1-\delta}\Bigg[g(\xi) - {}^{p}\mathbb{E}_{\delta,1}\Big(-\omega_{\delta}\big(\hbar(\xi)-\hbar(a)\big)^{\kappa}\Big)g(a) \\
&- \omega_{\delta}(\ln p)\int_{a}^{\xi}(\hbar(\xi)-\hbar(\zeta))^{\kappa-1}\,{}^{p}\mathbb{E}_{\kappa,\kappa}\Big(-\omega_{\delta}\big(\hbar(\xi)-\hbar(\zeta)\big)^{\kappa}\Big)\hbar'(\zeta)g(\zeta)d\zeta\Bigg] \\
&= \frac{R(\delta)}{1-\delta}\Bigg[g(\xi) - {}^{p}\mathbb{E}_{\delta,1}\Big(-\omega_{\delta}\big(\hbar(\xi)-\hbar(a)\big)^{\kappa}\Big)g(a) \\
&- \omega_{\delta}(\ln p)(\hbar(\xi)-\hbar(a))^{\kappa-1}\,{}^{p}\mathbb{E}_{\kappa,\kappa}\Big(-\omega_{\delta}\big(\hbar(\xi)-\hbar(a)\big)^{\kappa}\Big) * g(\zeta)\Bigg]. \qquad (12)
\end{aligned}$$

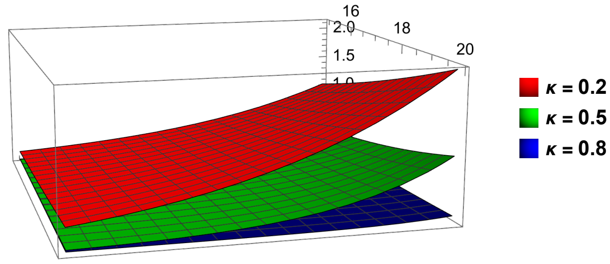

Figure 5: The graphical representation of absolute value of (8) corresponding to choice $0 \leq \xi \leq 1$.

The power M-L function involved in the newly established operator specified by Definition 8 is then evaluated using the Laplace transform in the subsequent lemma.

Lemma 1. *Given an increasing function* $\hbar$ *and a range* $0 < \delta < 1$, *we have*

$$L_{\hbar}\Bigg({}^{p}\mathbb{E}_{\kappa,1}\Big(-\omega_{\delta}\big(\hbar(\xi)-\hbar(a)\big)^{\kappa}\Big)\Bigg)(s)=\frac{s^{\kappa-1}}{s^{\kappa}+\omega_{\delta}\ln p},\Big|\frac{\omega_{\delta}\ln p}{s^{\kappa}}\Big|<1. \tag{13}$$

Proof. By employing Definition 10, we have

$$L_{\hbar}\Bigg({}^{p}\mathbb{E}_{\kappa,1}\Big(-\omega_{\delta}\big(\hbar(\xi)-\hbar(a)\big)^{\kappa}\Big)\Bigg)(s)=\int_{a}^{\infty}e^{-s(\hbar(\xi)-\hbar(a))}\hbar'(\xi)$$
$$\times\ {}^{p}\mathbb{E}_{\kappa,1}\Big(-\omega_{\delta}\big(\hbar(\xi)-\hbar(a)\big)^{\kappa}\Big)d\zeta$$
$$=\sum_{n=0}^{\infty}\frac{(-\omega_{\delta}\ln p)^{n}}{\Gamma(\kappa n+1)}\int_{a}^{\infty}e^{-s(\hbar(\xi)-\hbar(a))}\hbar'(\xi)\big(\hbar(\xi)-\hbar(a)\big)^{\kappa n}d\zeta.$$

Substituting $(\hbar(\xi)-\hbar(a))=\zeta$, we obtain

$$L_{\hbar}\Bigg({}^{p}\mathbb{E}_{\kappa,1}\Big(-\omega_{\delta}\big(\hbar(\xi)-\hbar(a)\big)^{\kappa}\Big)\Bigg)(s)=\sum_{n=0}^{\infty}\frac{(-\omega_{\delta}\ln p)^{n}}{\Gamma(\delta n+1)}\int_{a}^{\infty}e^{-s\zeta}t^{\delta n}d\zeta=\sum_{n=0}^{\infty}\frac{(-\omega_{\delta}\ln p)^{n}}{\Gamma(\kappa n+1)}L\{t^{\kappa n}\}$$
$$=\sum_{n=0}^{\infty}\frac{(-\omega_{\delta}\ln p)^{n}}{s^{\kappa n+1}}=\frac{s^{\kappa-1}}{s^{\kappa}+\omega_{\delta}\ln p},$$

which gives the desired result.

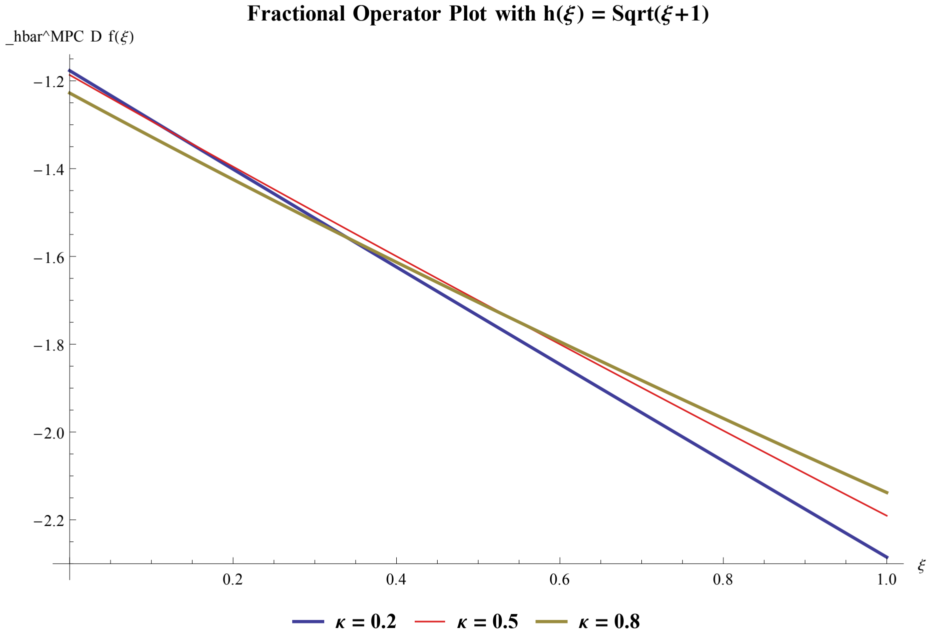

Figure 6: The 2-dimensional graphical representation of (8) corresponding to choice $0 \leq \xi \leq 1$.

Theorem 3. *For a given continuous function g and $g' \in L^1(0,T)$, the GLT of the modified derivative operator* (3) *of order $0 < \delta < 1$ can be defined as follows:*

$$L_\hbar\Big({}^{MPC}_{\hbar}\mathfrak{D}^{\delta;\kappa;p}_{a^+} g(\xi)\Big)(s) = \frac{R(\delta)}{1-\delta}\frac{s^\kappa L_\hbar\{g(\xi)\} - s^{\kappa-1}g(a)}{s^\kappa + \omega_\delta \ln p},$$

where $|\frac{\omega_\delta \ln p}{s^\kappa}| < 1$.

Proof. By using the equation (12), we have

$$L_\hbar\{{}^{MPC}_{\hbar}\mathfrak{D}^{\delta;\kappa;p}_{a^+} g(\xi)\}(s) = \frac{R(\delta)}{1-\delta}\Bigg(L_\hbar\{g(\xi)\} - g(a)L_\hbar\Big({}^{p}\mathbb{E}_{\kappa,1}\big(-\omega_\delta\big(\hbar(\xi)-\hbar(a)\big)^\kappa\big)\Big)$$

$$-\,\omega_\delta \ln p L_\hbar\Big(\big(\hbar(\xi)-\hbar(a)\big)^{\kappa-1}\,{}^{p}\mathbb{E}_{\kappa,1}\big(-\omega_\delta\big(\hbar(\xi)-\hbar(a)\big)^\kappa\big) * g(\xi)\Big)\Bigg)$$

$$= \frac{R(\delta)}{1-\delta}\Bigg(L_\hbar\{g(\xi)\} - g(a)\frac{s^{\kappa-1}}{s^\kappa+\omega_\delta\ln p} - \omega_\delta \ln p\sum_{n=0}^{\infty}\frac{(-\omega_\delta\ln p)^n}{\Gamma(\kappa n+\kappa)}$$

$$\times L_\hbar\Big(\big(\hbar(\xi)-\hbar(a)\big)^\kappa\big(\hbar(\xi)-\hbar(a)\big)^{\kappa n-1}\Big)L_\hbar\{g(\xi)\}$$

$$= \frac{R(\delta)}{1-\delta}\Bigg(L_\hbar\{g(\xi)\} - g(a)\frac{s^{\kappa-1}}{s^\kappa+\omega_\delta\ln p} - \omega_\delta \ln p\sum_{n=0}^{\infty}\frac{(-\omega_\delta\ln p)^n}{\Gamma(\kappa n+\kappa)}$$

$$\times L_\hbar\Big(\big(\hbar(\xi)-\hbar(a)\big)^{\kappa(n+1)-1}\Big)L_\hbar\{g(\xi)\}\Bigg)$$

$$= \frac{R(\delta)}{1-\delta}\left(L_\hbar\{g(\xi)\} - g(a)\frac{s^{\kappa-1}}{s^\kappa + \omega_\delta \ln p} - \omega_\delta \ln p \sum_{n=0}^{\infty} \frac{(-\omega_\delta \ln p)^n}{s^{\kappa(n+1)}} L_\hbar\{g(\xi)\}\right)$$

$$= \frac{R(\delta)}{1-\delta}\left(L_\hbar\{g(\xi)\} - g(a)\frac{s^{\kappa-1}}{s^\kappa + \omega_\delta \ln p} - \frac{\omega_\delta \ln p}{s^\kappa + \omega_\delta \ln p} L_\hbar\{g(\xi)\}\right)$$

$$= \frac{R(\delta)}{1-\delta}\frac{s^\kappa L_\hbar\{g(\xi)\} - s^{\kappa-1}g(a)}{s^\kappa + \omega_\delta \ln p}.$$

Hence the result is proved.

Example 7. *When $0 < \delta < 1$ and $\kappa > 0$ are chosen as the parameters, and $|\frac{\omega_\delta \ln p}{s^\kappa}| < 1$, the equation's solution*

$$ {}_{\hbar}^{MPC}\mathfrak{D}_0^\delta g(\xi) = \mathcal{C}$$

is provided by

$$g(\zeta) = \begin{cases} \frac{\mathcal{C}(1-\delta)}{R(\delta)}\left(1 + \omega_\delta \ln p \frac{\left(\hbar(\zeta)\right)^\kappa}{\Gamma(\kappa+1)}\right), & \zeta \neq 0; \\ 0, & \zeta = 0. \end{cases}$$

Proof. By given hypothesis $g(0) = 0$, and for $\zeta > 0$, we have

$$L_\hbar\{{}_{\hbar}^{MPC}\mathfrak{D}_0^{\delta;\kappa;p} g(s)\} == \frac{R(\delta)}{1-\delta}\frac{s^\kappa}{s^\kappa + \omega_\delta \ln p}\left(L_\hbar\{g(\zeta)\}(s)\right)$$

$$= \mathcal{C}\frac{s^\kappa}{s^\kappa + \omega_\delta \ln p} L_\hbar\big(1 + \omega_\delta \ln p \frac{\left(\hbar(\zeta)\right)^\kappa}{\Gamma(\kappa+1)}\big)(s)$$

$$= \mathcal{C}\frac{s^\kappa}{s^\kappa + \omega_\delta \ln p}\big(\frac{1}{s} + \frac{\omega_\delta \ln p}{s^{\kappa+1}}\big)$$

$$= \frac{\mathcal{C}}{s}.$$

This implies that

$$L_\hbar\{{}_{\hbar}^{MPC}\mathfrak{D}_0^{\delta;\kappa;p} g(s)\} = L_\hbar(\mathcal{C}).$$

The action of inverse Laplace will give the desired result.

Theorem 4. *Suppose that g be a continuous function and $g' \in L^1(0,T)$, then the GLT of the modified fractional derivative (11) of order $0 < \delta < 1$, $p > 1$, and $\kappa > 0$ is given by*

$$L_\hbar\Big({}_{\hbar}^{MPRL}\mathfrak{D}_{a^+}^{\delta;\kappa;p} g(\xi)\Big)(s) = \frac{R(\delta)}{1-\delta}\frac{s^\kappa L_\hbar\{g(\xi)\}}{s^\kappa + \omega_\delta \ln p},$$

where $|\frac{\omega_\delta \ln p}{s^\kappa}| < 1$.

Proof. Applying Laplace transformation on both sides of (11), we have

$$\begin{aligned} L_\hbar\Big({}^{MPRL}_{\hbar}\mathfrak{D}^{\delta;\kappa;p}_{a^+}g(\xi)\Big)(s) &= \frac{R(\delta)}{1-\delta}L_\hbar\Big[\frac{d}{d\zeta}\,{}^{p}\mathbb{E}_{\kappa,1}\big(-\omega_\delta\big(\hbar(\xi)-\hbar(a)\big)^\kappa\big)*g(\xi)\Big](s)\\ &= \frac{sR(\delta)}{1-\delta}L_\hbar\big[{}^{p}\mathbb{E}_{\kappa,1}\big(-\omega_\delta\big(\hbar(\xi)-\hbar(a)\big)^\kappa\big)*g(\xi)\big](s)\\ &= \frac{sR(\delta)}{1-\delta}L_\hbar\big[{}^{p}\mathbb{E}_{\kappa,1}\big(-\omega_\delta\big(\hbar(\xi)-\hbar(a)\big)^\kappa\big)\big](s)L_\hbar\big[g(\zeta)\big](s)\\ &= \frac{R(\delta)}{1-\delta}\frac{s^\kappa L_\hbar\big[g(\zeta)\big](s)}{s^\kappa+\omega_\delta\ln p}. \end{aligned}$$

which proves the required result.

Theorem 5. *For the following FDE (fractional differential equation)*

$${}^{MPRL}_{\hbar}\mathfrak{D}^{\delta;\kappa;p}_{a^+}x(\xi) = g(\xi), \tag{14}$$

there exist a unique solution as follows:

$$x(\xi) = \frac{R(\delta)}{1-\delta}g(\xi) + \frac{\delta\ln p}{R(\delta)}\,{}^{RL}_{\hbar}\mathfrak{I}^{\kappa}_{0}g(\xi), \tag{15}$$

where ${}^{RL}\mathfrak{I}^{\kappa}_{0}$ *is the generalized R-L fractional integral.*

Proof. By applying Laplace on (14), we have

$$L_\hbar\big\{{}^{MPRL}_{\hbar}\mathfrak{D}^{\delta;\kappa;p}_{a^+}x(\xi)\big\}(s) = L_\hbar\big\{g(\xi)\big\}(s).$$

By using Theorem 4, we obtain

$$\begin{aligned} L_\hbar\big\{{}^{MPRL}_{\hbar}\mathfrak{D}^{\delta;\kappa;p}_{a^+}x(\xi)\big\}(s) &= \frac{R(\delta)}{1-\delta}L_\hbar\big\{g(\xi)\big\}(s) + \frac{\delta\ln p}{R(\delta)s^\kappa}L_\hbar\big\{g(\xi)\big\}(s)\\ &= \frac{R(\delta)}{1-\delta}L_\hbar\big\{g(\xi)\big\}(s) + \frac{\delta\ln p}{R(\delta)\Gamma(\kappa)}L_\hbar\big\{\big(\hbar(\zeta)\big)^{\lambda-1}*g(\xi)\big\}(s)\\ &= L_\hbar\Big\{\frac{R(\delta)}{1-\delta}g(\xi) + \frac{\delta\ln p}{R(\delta)\Gamma(\kappa)}\big(\hbar(\zeta)\big)^{\lambda-1}*g(\xi)\Big\}(s). \end{aligned}$$

By taking inverse Laplace transformation, we get

$$x(\xi) = \frac{R(\delta)}{1-\delta}g(\xi) + \frac{\delta\ln p}{R(\delta)}\,{}^{RL}_{\hbar}\mathfrak{I}^{\kappa}_{0}g(\xi),$$

which gives the desired result.

Example 8. *Let us take the following power fractional differential equation on* $[0, 100]$ $(\xi \in [0, 100])$ *into consideration:*

$$ {}^{MPRL}_{\hbar}\mathfrak{D}^{\delta;\kappa;p}_{a^+} x(\xi) = \hbar^2(\xi). \tag{16} $$

By employing Theorem 5, we get

$$ \begin{aligned} x(\xi) &= \frac{R(\delta)}{1-\delta}\hbar^2(\xi) + \frac{\delta \ln p}{R(\delta)} {}^{RL}_{\hbar}\mathfrak{I}^{\kappa}_{0}\hbar^2(\xi) \\ &= \frac{R(\delta)}{1-\delta}\hbar^2(\xi) + 2\frac{\delta \ln p}{R(\delta)} \frac{(\hbar(\xi))^{\kappa+2}}{\Gamma(\kappa+3)}. \end{aligned} \tag{17} $$

Example 9. *Letting* $\hbar(\xi) = \xi$, *we get*

$$ {}^{MPRL}_{\hbar}\mathfrak{D}^{\delta;\kappa;p}_{a^+} x(\xi) = \xi^2. \tag{18} $$

By employing Theorem 5, we get

$$ \begin{aligned} x(\xi) &= \frac{R(\delta)}{1-\delta}\xi^2 + \frac{\delta \ln p}{R(\delta)} {}^{RL}_{\hbar}\mathfrak{I}^{\kappa}_{0}\xi^2 \\ &= \frac{R(\delta)}{1-\delta}\xi^2 + 2\frac{\delta \ln p}{R(\delta)} \frac{\xi^{\kappa+2}}{\Gamma(\kappa+3)}. \end{aligned} \tag{19} $$

Example 10. *If we fixed* $R(\delta) = 1$, $p = 2, 3, 4, 5$, $0 < \kappa < 3$ *and* $\delta = \frac{1}{2}$ *in Example 9, then we have*

$$ x(\xi) = 2\xi^2 + \ln p \frac{\xi^{\kappa+2}}{\Gamma(\kappa+3)}. \tag{20} $$

Corresponding to the fixed values of $\kappa = 0.2, 0.5, 0.8$, *we have the following 2-dimensional*

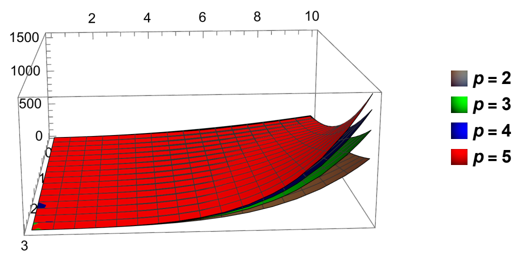

Figure 7: The graphical representation of (20) corresponding to choice $0 \leq \xi \leq 100$.

graphical representation.

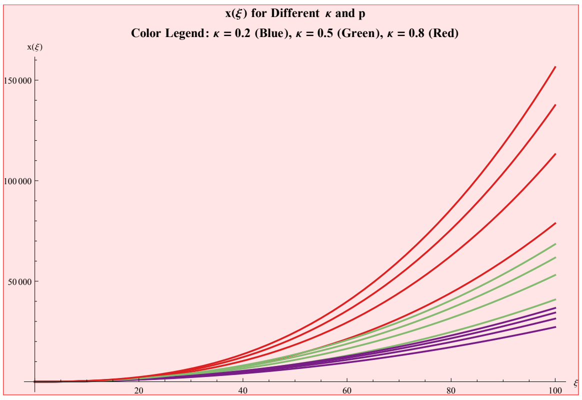

Figure 8: The 2-dimensional graphical representation of (20) corresponding to choice $0 \leq \xi \leq 100$.

Example 11. *If we fixed* $R(\delta) = 1$, $p = 2, 3, 4, 5$, $0 < \kappa < 4$ *and* $\delta = \frac{1}{2}$ *in Example 9, then we have*

$$x(\xi) = 2\xi^2 + \ln p \frac{\xi^{\kappa+2}}{\Gamma(\kappa+3)}. \tag{21}$$

Corresponding to the fixed values of $\kappa = 3.2, 3.5, 3.8$, *we have the following* 2*-dimensional*

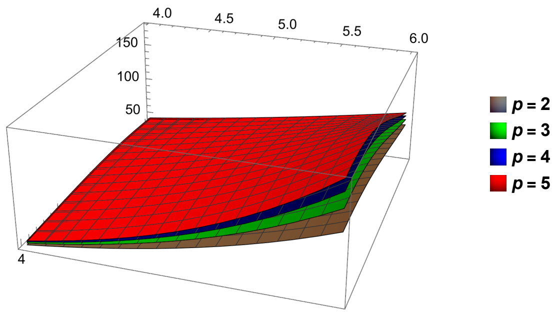

Figure 9: The graphical representation of (21) corresponding to choice $0 \leq \xi \leq 100$.

graphical representation

Example 12. *If we fixed* $R(\delta) = 1$, $p = 2, 3, 4, 5$, $0 < \kappa < 7$ *and* $\delta = \frac{1}{2}$ *in Example 9, then we have*

$$x(\xi) = 2\xi^2 + \ln p \frac{\xi^{\kappa+2}}{\Gamma(\kappa+3)}. \tag{22}$$

Corresponding to the fixed values of $\kappa = 6.2, 6.5, 6.8$, *we have the following* 2*-dimensional graphical representation.*

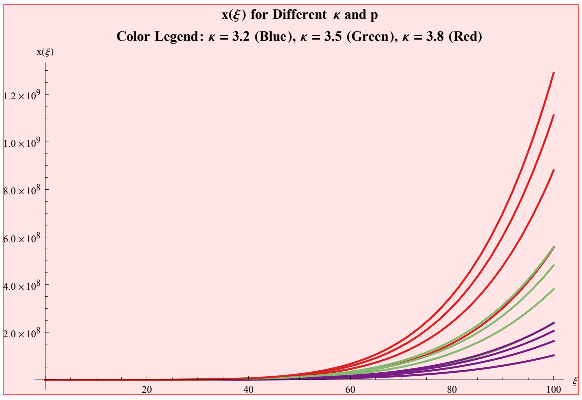

Figure 10: The 2-dimensional graphical representation of (21) corresponding to choice $0 \leq \xi \leq 100$.

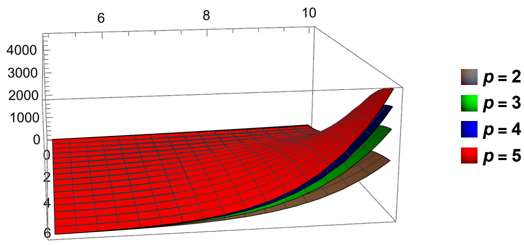

Figure 11: The graphical representation of (22) corresponding to choice $0 \leq \xi \leq 100$.

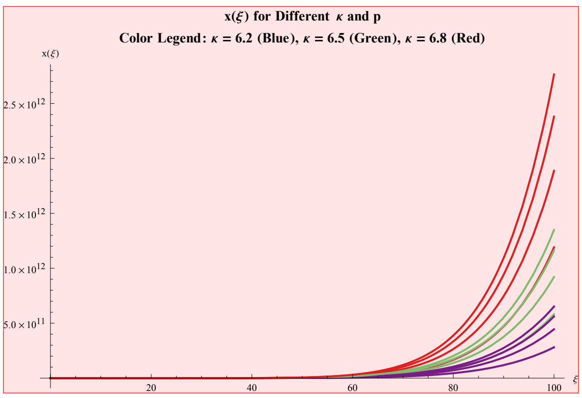

Figure 12: The 2-dimensional graphical representation of (22) corresponding to choice $0 \leq \xi \leq 100$.

Example 13. *If we fixed* $R(\delta) = 1$, $p = 2, 3, 4, 5$, $0 < \kappa < 8$ *and* $\delta = \frac{1}{2}$ *in Example 9, then we have*

$$x(\xi) = 2\xi^2 + \ln p \frac{\xi^{\kappa+2}}{\Gamma(\kappa + 3)}. \tag{23}$$

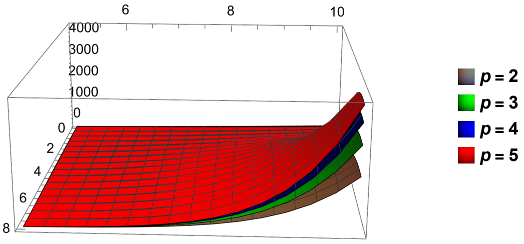

Figure 13: The graphical representation of (23) corresponding to choice $0 \le \xi \le 100$.

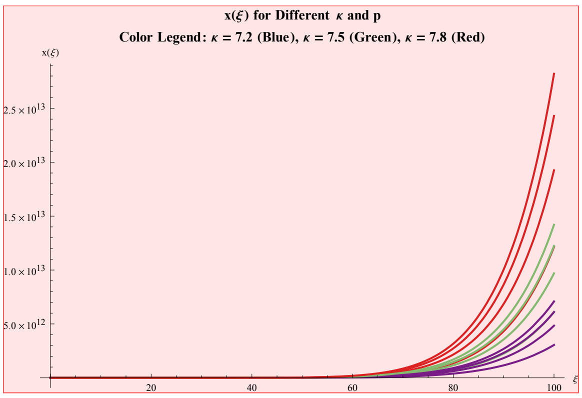

Figure 14: The 2-dimensional graphical representation of (23) corresponding to choice $0 \le \xi \le 100$.

Corresponding to the fixed values of $\kappa = 7.2, 7.5, 7.8$, we have the following 2-dimensional and 3-dimensional graphical representation.

The three-dimensional and two-dimensional graphical representations illustrate the convergence of the solution to the differential equation. This behavior demonstrates the boundedness and convergence of the solution.

4. The Modified Power Fractional Integral

In this section, we present an modified version of the power fractional integral operator. Also, we discuss its boundedness, the Laplace transform, and a few other related features.

Definition 12. *For any $g \in L^1(0,T]$, the modified power fractional operator with $0 < \delta < 1$ associated $\hbar$ is stated by as*

$$ {}_{\hbar}^{MPI}\Im_{a^+}^{\delta;\kappa;p} g(\xi) = \frac{1-\delta}{R(\delta)} g(\xi) + \frac{\delta \ln p}{R(\delta)\Gamma(\kappa)} \int_a^{\xi} (\hbar(\xi) - \hbar(\zeta))^{\kappa-1} \hbar'(\zeta) g(\zeta) d\zeta. \tag{24} $$

Remark 6. *i. If we consider $p = e$ in (24), we get the fractional integral defined by [7].*
ii. If we consider $p = e$ and $\hbar(\xi) = \xi$ in (24), we get the fractional integral defined by [2].
iii. If we consider $\hbar(\xi) = \xi$ in (24), we get the fractional integral defined by [11].

First, we establish the boundedness of this operator.

Theorem 6. *Given a strictly increasing function $g \in X^p(0,T)$, $\hbar$, we get $\omega_\delta = \frac{\delta}{1-\delta}$.If $1 \le r_1 < \infty$ and $R(\delta)$, is a normalised function with the property $R(0) = R(1) = 1$, then the following inequality is true.*

$$ \left\| {}_{\hbar}^{MPI}\Im_{a^+}^{\delta;\kappa;p} g(\xi) \right\|_{X^p} \le \frac{1-\delta}{R(\delta)} \|g(\xi)\|_{X^p} + \frac{\delta \ln p}{R(\delta)\Gamma(\kappa)} \|f\|_{X^p} \frac{(\hbar(\theta) - \hbar(a))^{\kappa-1}}{\kappa - 1}. \tag{25} $$

Proof. By employing the Definition 24, we have

$$ \begin{aligned} &\left\| {}_{\hbar}^{MPI}\Im_{a^+}^{\delta;\kappa;p} g(\xi) \right\|_{X^p} \le \frac{1-\delta}{R(\delta)} \|g(\xi)\|_{X^p} + \frac{\delta \ln p}{R(\delta)\Gamma(\kappa)} \\ &\left\| \int_a^{\xi} (\hbar(\xi) - \hbar(\zeta))^{\kappa-1} \hbar'(\zeta) g(\zeta) d\zeta \right\|_{X^p} \\ &= \frac{1-\delta}{R(\delta)} \|g(\xi)\|_{X^p} + \frac{\delta \ln p}{R(\delta)\Gamma(\kappa)} \Big(\int_0^{\theta} \Big| \int_a^{\xi} (\hbar(\xi) - \hbar(\zeta))^{\kappa-1} \hbar'(\zeta) g(\zeta) d\zeta \Big|^p \hbar'(\xi) du \Big)^{\frac{1}{r_1}}. \end{aligned} $$

Substituting $\varrho = \hbar(\xi)$ and $\lambda = \hbar(\zeta)$, we have

$$ \begin{aligned} &\left\| {}_{\hbar}^{MPI}\Im_{a^+}^{\delta;\kappa;p} g(\xi) \right\|_{X^p} \le \frac{1-\delta}{R(\delta)} \|g(\xi)\|_{X^p} + \frac{\delta \ln p}{R(\delta)\Gamma(\kappa)} \\ &\times \Big(\int_{\hbar(a)}^{\hbar(\theta)} \Big| \int_{\hbar(0)}^{\varrho} (\varrho - \lambda)^{\kappa-1} g(\hbar^{-1}(\lambda)) d\lambda \Big|^{r_1} d\varrho \Big)^{\frac{1}{r_1}} \\ &= \frac{1-\delta}{R(\delta)} \|g(\xi)\|_{X^p} + \frac{\delta \ln p}{R(\delta)\Gamma(\kappa)} \\ &\times \int_{\hbar(a)}^{\hbar(\theta)} \Big(|g(\hbar^{-1}(\lambda))|^p \Big| \int_{\lambda}^{\hbar(\theta)} (\rho - \lambda)^{\kappa-1} \Big|^{r_1} d\varrho \Big)^{\frac{1}{r_1}} d\lambda \end{aligned} $$

$$= \frac{1-\delta}{R(\delta)}\|g(\xi)\|_{X^p} + \frac{\delta \ln p}{R(\delta)\Gamma(\kappa)}$$
$$\times \int_{\hbar(a)}^{\hbar(\theta)} |g(\hbar^{-1}(\lambda))|\Big(\int_{\lambda}^{\hbar(\theta)} (\rho-\lambda)^{r_1(\kappa-1)} d\varrho\Big)^{\frac{1}{r_1}} d\lambda$$
$$= \frac{1-\delta}{R(\delta)}\|g(\xi)\|_{X^p} + \frac{\delta \ln p}{R(\delta)\Gamma(\kappa)}$$
$$\times \int_{\hbar(a)}^{\hbar(\theta)} |g(\hbar^{-1}(\lambda))|\Big(\frac{(\hbar(\theta)-\lambda)^{r(\kappa-1)+1}}{r_1(\kappa-1)+1}\Big)^{\frac{1}{r_1}} d\lambda.$$

With the help of Hölder, inequality, we have

$$\Big\|{}_{\hbar}^{MABC}\Im_{a^+}^{\delta;\kappa;p} g(\xi)\Big\|_{X^p} \le \frac{1-\delta}{R(\delta)}\|g(\xi)\|_{X^p} + \frac{\delta \ln p}{R(\delta)\Gamma(\kappa)}\int_{\hbar(a)}^{\hbar(\theta)}\Big(|g(\hbar^{-1}(\lambda))|^{r_1} d\lambda\Big)^{\frac{1}{r_1}}$$
$$\times \left(\Big(\int_{\hbar(a)}^{\hbar(\theta)} \frac{(\hbar(\theta)-\lambda)^{r_1(\kappa-1)+1}}{p(\kappa-1)+1}\Big)^{\frac{s_1}{r_1}} d\lambda\right)^{\frac{1}{s_1}},$$

where $\frac{1}{r_1}+\frac{1}{s_1}=1$, Now substituting $\hbar^{-1}(\lambda)=t$, we have

$$\Big\|{}_{\hbar}^{MPI}\Im_{a^+}^{\delta;\kappa;p} g(\xi)\Big\|_{X^p} \le \frac{1-\delta}{R(\delta)}\|g(\xi)\|_{X^p} + \frac{\delta \ln p}{R(\delta)\Gamma(\kappa)}\int_{a}^{\theta}\Big(|g(\zeta)|^{r_1}\hbar'(\zeta)d\zeta\Big)^{\frac{1}{r_1}}$$
$$\times \left(\Big(\int_{\hbar(a)}^{\hbar(\theta)} \frac{(\hbar(\theta)-\hbar(\zeta))^{r_1(\kappa-1)+1}}{r_1(\kappa-1)+1}\Big)^{\frac{s_1}{r_1}} \hbar'(\zeta)d\zeta\right)^{\frac{1}{s_1}}$$
$$= \frac{1-\delta}{R(\delta)}\|g(\xi)\|_{X^p} + \frac{\delta \ln p}{R(\delta)\Gamma(\kappa)}\int_{\hbar(a)}^{\hbar(\theta)}\Big(|g(\zeta)|^{r_1}\hbar'(\zeta)d\zeta\Big)^{\frac{1}{r_1}}$$
$$\times \left(\Big(\int_{\hbar(a)}^{\hbar(\theta)} \frac{(\hbar(\theta)-\hbar(\zeta))^{r_1(\kappa-1)+1}}{r_1(\kappa-1)+1}\Big)^{\frac{s_1}{r_1}} \hbar'(\zeta)d\zeta\right)^{\frac{1}{s_1}}$$
$$\le \frac{1-\delta}{R(\delta)}\|g(\xi)\|_{X^p} + \frac{\delta \ln p}{R(\delta)\Gamma(\kappa)}\frac{(\hbar(\theta)-\hbar(a))^{\kappa-1}}{\kappa-1}\|f\|_{X^p}.$$

Next, we determine our generalised fractional integral operator's Laplace transform.

Theorem 7. *The modified fractional integral of order* $0<\delta<1$, $\kappa>0$ *associated with* $\hbar$ *is defined for* $g \in L^1(0,T]$ *as follows:*

$$L_{\hbar}({}_{\hbar}^{MPI}\Im_{a^+}^{\delta;\kappa;p} g(\xi)) = \frac{1-\delta}{R(\delta)}\frac{s^{\kappa}+\omega_{\delta}\ln p}{s^{\kappa}}L_{\hbar}(g(\xi)).$$

Proof. Through application of Definition 24, we have

$$L_{\hbar}({}_{\hbar}^{MPI}\Im_{a}^{\delta;\kappa;p} g(\xi)) = \frac{1-\delta}{R(\delta)}L_{\hbar}(g(\xi)) + \frac{\delta \ln p}{R(\delta)\Gamma(\kappa)}L_{\hbar}\Big((\hbar(\xi)-\hbar(a))^{\kappa-1} * g(\xi)\Big)$$

$$= \frac{1-\delta}{R(\delta)} L_\hbar(g(\xi)) + \frac{\delta \ln p}{R(\delta)\Gamma(\kappa)} L_\hbar\Big((\hbar(\xi) - \hbar(a))^{\kappa-1}\Big) L_\hbar(g(\xi))$$
$$= \frac{1-\delta}{R(\delta)} L_\hbar(g(\xi)) + \frac{\delta \ln p}{R(\delta)s^\kappa} L_\hbar(g(\xi))$$
$$= \frac{1-\delta}{R(\delta)} \frac{s^\kappa + \omega_\delta \ln p}{s^\kappa} L_\hbar(g(\xi)),$$

which gives the desired result.

Theorem 8. *Assuming $g' \in L^1(0,T)$, and $0 < \delta < 1$, the following outcome can be obtained.*

$$ {}^{MPC}_{\hbar}\mathfrak{D}^{\delta;\kappa;p}_{a^+}\big({}^{MPI}_{\hbar}\mathfrak{I}^{\delta;\kappa;p}_{a^+}\big)g(\xi) = g(\xi) - g(0)\,{}^{p}\mathbb{E}_{\kappa,1}\Big(-\omega_\delta\big(\hbar(\xi) - \hbar(a)\big)^\kappa\Big). \tag{26}$$

Proof. Using the Laplace transform definition, we have

$$L_\hbar\Big({}^{MPC}_{\hbar}\mathfrak{D}^{\delta;\kappa;p}_{a^+}\big({}^{MPC}_{\hbar}\mathfrak{I}^{\delta;\kappa;p}_{a}\big)g(\xi)\Big)$$
$$= \frac{R(\delta)}{1-\delta} \frac{s^\kappa}{s^\kappa + \omega_\delta \ln p} L_\hbar\Big({}^{MPI}_{\hbar}\mathfrak{I}^{\delta;\kappa;p}_{a} g(\xi)\Big) - \frac{R(\delta)}{1-\delta} \frac{s^{\kappa-1}}{s^\kappa + \omega_\delta \ln p}\Big({}^{MPI}_{\hbar}\mathfrak{I}^{\delta;\kappa;p}_{a} g(0)\Big)$$
$$= \frac{s^\kappa}{s^\kappa + \omega_\delta \ln p}\Big(\frac{s^\kappa + \omega_\delta \ln p}{s^\kappa}\Big) L_\hbar(g(\xi)) - \frac{s^{\kappa-1}}{s^\kappa + \omega_\delta \ln p} g(0)$$
$$= L_\hbar(g(\xi)) - \frac{s^{\kappa-1}}{s^\kappa + \omega_\delta \ln p} g(0).$$

Utilising the inverse Laplace transform, we arrive at

$$\Big({}^{MPC}_{\hbar}\mathfrak{D}^{\delta;\kappa;p}_{a}\big({}^{MPI}_{\hbar}\mathfrak{I}^{\delta;\kappa;p}_{a}\big)g(\xi)\Big) = g(\xi) - g(0)\,{}^{p}\mathbb{E}_{\kappa,1}\Big(-\omega_\delta\big(\hbar(\xi) - \hbar(a)\big)^\kappa\Big),$$

which completes the required result.

Theorem 9. *Assuming that $g' \in L^1(0,T)$, $a = 0$, and $0 < \delta < 1$, the following outcome can be obtained.*

$${}^{MPI}_{\hbar}\mathfrak{I}^{\delta;\kappa;p}_{0}\big({}^{MPC}_{\hbar}\mathfrak{D}^{\delta;\kappa;p}_{0}\big)g(\xi) = g(\xi) - g(0).$$

Proof. By using the Definition 8 and 24, we have

$${}^{MPI}_{\hbar}\mathfrak{I}^{\delta;\kappa;p}_{0}\big({}^{MPC}_{\hbar}\mathfrak{D}^{\delta;\kappa;p}_{0}\big)g(\xi) = \tfrac{1-\delta}{R(\delta)}\big({}^{MPC}_{\hbar}\mathfrak{D}^{\delta;\kappa;p}_{0}\big)g(\xi) + \tfrac{\delta \ln p}{R(\delta)\Gamma(\kappa)} \int_0^t (\hbar(\xi) - \hbar(\zeta))^{\kappa-1}$$
$$\times \hbar'(\zeta)\big({}^{MPC}_{\hbar}\mathfrak{D}^{\delta;\kappa;p}_{0}\big)g(\zeta)d\zeta$$
$$= \int_0^\xi {}^{p}\mathbb{E}_{\kappa,1}\Big(-\omega_\delta\big(\hbar(\xi) - \hbar(\zeta)\big)^\kappa\Big)g'(\zeta)d\zeta + \tfrac{R(\delta)}{1-\delta}\tfrac{\delta \ln p}{R(\delta)\Gamma(\kappa)} \int_0^\xi (\hbar(\xi) - \hbar(\zeta))^{\kappa-1}$$
$$\times \hbar'(\zeta) \int_0^t {}^{p}\mathbb{E}_{\kappa,1}\Big(-\omega_{\kappa,1}\big(\hbar(\zeta) - \hbar(\vartheta)\big)^\kappa\Big)g'(\vartheta)d\vartheta$$

$$
\begin{aligned}
&= \int_0^\xi {}^p\mathbb{E}_\kappa\Big(-\omega_\delta\big(\hbar(\xi)-\hbar(\zeta)\big)^\kappa\Big)g'(\zeta)d\zeta + \tfrac{\delta\ln p}{(1-\delta)\Gamma(\kappa)} \\
&\times \int_0^\xi g'(\vartheta)\int_z^\xi (\hbar(\xi)-\hbar(\zeta))^{\kappa-1}\hbar'(\zeta)\,{}^p\mathbb{E}_{\kappa,1}\Big(-\omega_\delta\big(\hbar(\zeta)-\hbar(\vartheta)\big)^\kappa\Big)d\zeta d\vartheta \\
&= \int_0^\xi {}^p\mathbb{E}_{\kappa,1}\Big(-\omega_\delta\big(\hbar(\xi)-\hbar(\zeta)\big)^\kappa\Big)g'(\zeta)d\zeta + \tfrac{\delta\ln p}{(1-\delta)\Gamma(\kappa)}\sum_{n=0}^\infty \tfrac{(-\omega_\delta\ln p)^n}{\Gamma(\kappa n+1)} \\
&\times \int_0^\xi g'(\vartheta)\int_z^\xi (\hbar(\xi)-\hbar(\zeta))^{\kappa-1}\hbar'(\zeta)\big(\hbar(\zeta)-\hbar(\vartheta)\big)^{\kappa n}d\zeta d\vartheta \\
&= \int_0^\xi {}^p\mathbb{E}_{\kappa,1}\Big(-\omega_\delta\big(\hbar(\xi)-\hbar(\zeta)\big)^\kappa\Big)g'(\zeta)d\zeta + \tfrac{\delta\ln p}{(1-\delta)\Gamma(\kappa)}\sum_{n=0}^\infty \tfrac{(-\omega_\delta\ln p)^n}{\Gamma(\kappa n+1)} \\
&\times \int_0^\xi g'(\vartheta)\Big(\hbar(\xi)-\hbar(\vartheta)\Big)^{\kappa(n+1)-1}\int_z^\xi \left(\tfrac{\hbar(\xi)-\hbar(\zeta)}{(\hbar(\xi)-\hbar(\vartheta))}\right)^{\kappa-1}\left(\tfrac{\hbar(\zeta)-\hbar(\vartheta)}{(\hbar(\xi)-\hbar(\vartheta))}\right)^{\kappa n}\hbar'(\zeta)d\zeta d\vartheta \\
&= \int_0^\xi {}^p\mathbb{E}_{\kappa,1}\Big(-\omega_\delta\big(\hbar(\xi)-\hbar(\zeta)\big)^\kappa\Big)g'(\zeta)d\zeta + \tfrac{\delta\ln p}{(1-\delta)\Gamma(\kappa)}\sum_{n=0}^\infty \tfrac{(-\omega_\delta)^n}{\Gamma(\kappa n+1)} \\
&\times B(\kappa n+1,\kappa)\int_0^\xi g'(\vartheta)\Big(\hbar(\xi)-\hbar(\vartheta)\Big)^{\kappa(n+1)}d\vartheta \\
&= \int_0^\xi {}^p\mathbb{E}_{\kappa,1}\Big(-\omega_\delta\big(\hbar(\xi)-\hbar(\zeta)\big)^\kappa\Big)g'(\zeta)d\zeta + \tfrac{\omega_\delta\ln p}{\Gamma(\kappa)}\sum_{n=0}^\infty \tfrac{(-\omega_\delta\ln p)^n}{\Gamma(\kappa n+1)} \\
&\times \tfrac{\Gamma(\kappa n+1)\Gamma(\kappa)}{\Gamma(\kappa(n+1)+1)}\int_0^\xi g'(\vartheta)\Big(\hbar(\xi)-\hbar(\vartheta)\Big)^{\kappa(n+1)}d\vartheta \\
&= \int_0^\xi {}^p\mathbb{E}_{\kappa,1}\Big(-\omega_\delta\big(\hbar(\xi)-\hbar(\zeta)\big)^\kappa\Big)g'(\zeta)d\zeta - \sum_{n=0}^\infty \tfrac{(-\omega_\delta\ln p)^{n+1}}{\Gamma(\kappa(n+1)+1)} \\
&\times \int_0^\xi g'(\vartheta)\Big(\hbar(\xi)-\hbar(\vartheta)\Big)^{\kappa(n+1)}d\vartheta \\
&= \sum_{n=0}^\infty \tfrac{(-\omega_\delta\ln p)^n}{\Gamma(\kappa n+1)}\int_0^\xi g'(\zeta)\Big(\hbar(\xi)-\hbar(\zeta)\Big)^{\kappa n}d\zeta \\
&- \sum_{n=0}^\infty \tfrac{(-\omega_\delta\ln p)^{n+1}}{\Gamma(\kappa(n+1)+1)}\int_0^\xi g'(\zeta)\Big(\hbar(\xi)-\hbar(\zeta)\Big)^{\kappa(n+1)}d\zeta \\
&= \sum_{n=0}^\infty \tfrac{(-\omega_\delta\ln p)^n}{\Gamma(\kappa n+1)}\int_0^\xi g'(\zeta)\Big(\hbar(\xi)-\hbar(\zeta)\Big)^{\kappa n}d\zeta \\
&- \sum_{n=1}^\infty \tfrac{(-\omega_\delta\ln p)^n}{\Gamma(\kappa n+1)}\int_0^\xi g'(\zeta)\Big(\hbar(\xi)-\hbar(\zeta)\Big)^{\kappa n}d\zeta \\
&= \int_0^\xi g'(\zeta)d\zeta + \sum_{n=1}^\infty \tfrac{(-\omega_\delta\ln p)^n}{\Gamma(\kappa n+1)}\int_0^\xi g'(\zeta)\Big(\hbar(\xi)-\hbar(\zeta)\Big)^{\kappa n}d\zeta \\
&- \sum_{n=1}^\infty \tfrac{(-\omega_\delta\ln p)^n}{\Gamma(\kappa(n)+1)}\int_0^\xi g'(\zeta)\Big(\hbar(\xi)-\hbar(\zeta)\Big)^{\kappa n}d\zeta \\
&= \int_0^\xi g'(\zeta)d\zeta = g(\xi)-g(0).
\end{aligned}
$$

Hence the result is completed.

5. Conclusion

In this paper, the generalization of power fractional integral and derivative operators in both the Caputo and R-L sense are presented. These new operators allowed us to solve differential equations, and we investigated their solutions using the generalized Laplace transform. It is shown that the defined operators in X^p have norm and are bounded. We evaluate the Laplace transform of both FOs. The inverse property of the operators is presented under certain condition $g(0) = 0$. Additionally, we presented some examples both analytically and graphically. This behavior demonstrates the boundedness and convergence of the solution. The graphical representations are given in both two and three dimensions. The operators presented in this paper are more general than the existing operators cited in literature.

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Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Competing interests

The authors declare that they have no competing interests.

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