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Bipolar-Valued Intuitionistic Fuzzy Positive Implicative Ideals in $\mathcal{BCK}$-Algebras

D. Ramesh[1], Shake Baji[2], Aiyared Iampan[3,*], R. Durga Prasad[4], B. Satyanarayana[5]

[1] *Department of Engineering Mathematics, College of Engineering, Koneru Lakshmaiah Education Foundation, Vaddeswaram, Andhra Pradesh-522302, India*
[2] *Department of Mathematics, Sir C.R. Reddy College of Engineering, Eluru-534007, Andhra Pradesh, India*
[3] *Department of Mathematics, School of Science, University of Phayao, Mae Ka, Mueang, Phayao 56000, Thailand*
[4] *Department of Mathematics, KG Reddy College of Engineering and Technology, Hyderabad, Telangana-501504, India*
[5] *Department of Mathematics, Acharya Nagarjuna University, Nagarjuna Nagar, Guntur-522510, Andhra Pradesh, India*

Abstract. This study develops a novel framework for bipolar-valued intuitionistic fuzzy positive implicative ideals (BPVIFPIIs) in $\mathcal{BCK}$-algebras by integrating bipolar-valued intuitionistic fuzzy set theory with algebraic structures. The primary objective is to define and explore the properties of BPVIFPIIs in $\mathcal{BCK}$-algebras, providing rigorous theoretical foundations supported by illustrative examples. Key conditions under which a bipolar-valued intuitionistic fuzzy set qualifies as a BPVIFPII are established. The findings reveal significant connections between BPVIFPIIs and other fuzzy ideals, highlighting their role in advancing the understanding of uncertainty and algebraic reasoning. This research opens avenues for further exploration of bipolar fuzzy structures in algebra and their practical implications in decision-making processes involving uncertain data.

2020 Mathematics Subject Classifications: 06F35; 03E72

Key Words and Phrases: Bipolar-valued fuzzy set (BPVFS), bipolar-valued intuitionistic fuzzy set (BPVIFS), bipolar-valued intuitionistic fuzzy ideal (BPVIFI), bipolar-valued intuitionistic fuzzy positive implicative ideal (BPVIFPII)

*Corresponding author.
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Email addresses: ram.fuzzy@gmail.com (D. Ramesh), shakebaji6@gmail.com (S. Baji), aiyared.ia@up.ac.th (A. Iampan), durgaprasad.fuzzy@gmail.com (R. D. Prasad), drbsn63@yahoo.co.in (B. Satyanarayana)

1. Introduction

In this article, we will utilize the following list of abbreviations:

- $\mathcal{BCK}$-A: $\mathcal{BCK}$-algebra
- FS: Fuzzy set
- BPVFS: Bipolar-valued fuzzy set
- IFS: Intuitionistic fuzzy set
- BPVIFS: Bipolar-valued intuitionistic fuzzy set
- BPVIFSA: Bipolar-valued intuitionistic fuzzy subalgebra
- BPVIFI: Bipolar-valued intuitionistic fuzzy ideal
- BPVIFPII: Bipolar-valued intuitionistic fuzzy positive implicative ideal

In many practical scenarios, the handling of information and the process of decision-making often encounter situations where data or results lack a clear definition. This intrinsic absence of accuracy and precision is commonly known as uncertainty. Uncertainty can come up for a few reasons, like not having all the data, mistakes in object measurement, errors in collecting data, and the natural differences in complex systems. Dealing with uncertainty and handling it properly is crucial for making good decisions, particularly in areas like business, engineering, artificial intelligence, etc. A very effective way to deal with the challenge of uncertainty is through the application of fuzzy set theory. This idea was introduced in the mid-20th century by Zadeh [31]. This theory is like a mathematical tool that helps us handle uncertainty and imprecision in a neat and structured way.

Research on $\mathcal{BCK}$/BCI algebras was initiated by Imai and Iséki [6, 7] in 1966, as evidenced by their work on set-theoretic difference and propositional logics. Several researchers, including Jun ([11, 21]), Liu [18], and Lee [16], have extensively explored the fuzzy structures inherent in $\mathcal{BCK}$/BCI algebras. Others ([5, 8, 17, 27, 28]) have also made significant contributions to this field from various perspectives on various branches of algebra. BPVFSs (BPVFS), an extension of fuzzy sets ([32, 33]), are designed to address scenarios in which both negative and positive membership degrees hold significance. This extension allows us to consider the negative and positive aspects of membership and identify their respective roles. Unlike a conventional fuzzy set, where an element is either entirely connected (membership value = 1) or partially connected (membership value $= (0, 1)$), BPVFSs offer a more detailed representation. Alternatively, a BPVFS permits the assignment of degrees to members within the range of $[-1, 1]$. These degrees signify the extent to which an element is connected positively or negatively to the set. This extension offers a complete view of uncertain and changing information, particularly valuable in practical scenarios where both negative and positive aspects matter.

The concept of a BPVFS was applied to study different ideas in $\mathcal{BCK}$/BCI-algebras, like a-ideals of BCI-algebras [16], subalgebras and ideals of $\mathcal{BCK}$/BCI-algebras [15], and many others, as explained in [9, 10]. Recent research in [1] explores bipolar-valued fuzzy BCI-implicative ideals of BCI-algebras. Muhiuddin et al. [22] look at positive implicative and closed bipolar-valued fuzzy ideals in $\mathcal{BCK}$-As. Many other scholars have also added to this field, exploring different aspects of algebra in various ways ([12, 13, 23, 25, 29]). After presenting the idea of the fuzzy set concept, many studies have been carried out to investigate the extension of this concept. In 1986, Atanasov introduced the idea of IFSs, representing an advancement in fuzzy set theory. Ezhilmaran and Shankar [29] present the idea of BPVIFSs. This classification of fuzzy sets encompasses not only negative and positive levels of belongingness but also negative and positive levels of non-belongingness for elements within a given set. Recently, in [26], Satyanarayana et al. introduced the concept of BPVIFII in $\mathcal{BCK}$-A.

The proposed theory of bipolar-valued intuitionistic fuzzy positive implicative ideals (BPVIFPIIs) in $\mathcal{BCK}$-algebras is motivated by the need to address limitations in existing frameworks for handling uncertainty in algebraic structures. Traditional fuzzy and intuitionistic fuzzy set theories provide tools for modeling uncertainty, yet they often fail to account for the dual nature of positive and negative membership degrees simultaneously. This duality becomes crucial in applications where both positive and negative aspects of membership and non-membership must be analyzed, such as in decision-making scenarios involving conflicting or imprecise data. By extending these concepts into the domain of $\mathcal{BCK}$-algebras, the study aims to enrich the theoretical foundations of algebraic reasoning under uncertainty, offering a more comprehensive mathematical model.

The existence of previous studies on related bipolar fuzzy structures, such as bipolar complex fuzzy subgroups [30], bipolar complex fuzzy semigroups [24], and T-bipolar soft groups and their fundamental laws [19], provides a solid foundation for advancing this field. These works have successfully demonstrated the versatility of bipolar fuzzy sets in addressing complex algebraic problems, including Γ-semigroups [20] and bipolar complex fuzzy submodules [2]. However, to the best of our knowledge, no existing literature has explored the bipolar-valued intuitionistic fuzzification of positive implicative ideals in $\mathcal{BCK}$/BCI algebras. This absence motivated us to initiate theoretical research on this specific topic, aiming to bridge the gap in the current body of knowledge.

In addition to its theoretical contributions, the proposed framework has significant practical potential. BPVIFPIIs can be applied in decision-making systems where conflicting or dual aspects of data must be considered, such as in medical diagnostics, where symptoms may simultaneously support and contradict potential diagnoses. Similarly, this framework is relevant in machine learning and artificial intelligence, particularly in bipolar sentiment analysis or systems that require evaluating both positive and negative influences on decisions. By enabling the simultaneous analysis of positive and negative membership values, BPVIFPIIs provide a robust tool for applications where traditional fuzzy models are insufficient.

To enhance clarity, we have illustrated the research process in a flowchart (Figure 1), which provides a structured overview of the development of BPVIFPIIs. We believe this graphical representation will facilitate a deeper understanding of the proposed concepts and their significance within the broader context of algebraic and fuzzy set theory.

This article presents a comprehensive exploration of BPVIFPIIs within the framework of $\mathcal{BCK}$-algebras, accompanied by illustrative examples that illuminate the core concepts. The study meticulously establishes the conditions under which a BPVIFS qualifies as a BPVIFPII and when a BPVIFI attains the structure of a BPVIFPII. By combining rigorous theoretical analysis with practical examples, this work not only expands the algebraic understanding of fuzzy structures but also highlights the critical connections between these fuzzy ideals and their broader implications in algebraic reasoning and uncertainty modeling.

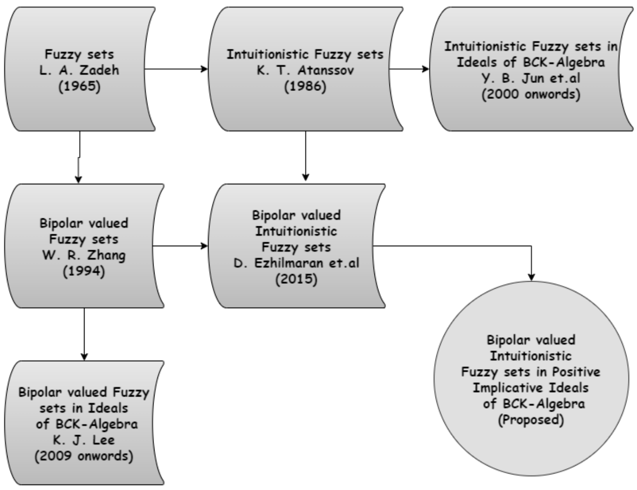

Figure 1: The research process

2. Preliminaries

Definition 1. *[7] A $\mathcal{BCK}$-A $\mathcal{G} = (\mathcal{G}, \diamond, 0)$ is an algebra of type $(2,0)$, where $\mathcal{G}$ is a nonempty set, $\diamond$ is a binary operation on $\mathcal{G}$, and 0 is a fixed element of $\mathcal{G}$ if it satisfies the following axioms: for all g_1, $\hbar_1$, $i_1 \in \mathcal{G}$,*

($\mathcal{BCK}$-1) $((g_1 \diamond \hbar_1) \diamond (g_1 \diamond i_1)) \diamond (i_1 \diamond \hbar_1) = 0$,

($\mathcal{BCK}$-2) $(g_1 \diamond (g_1 \diamond \hbar_1)) \diamond \hbar_1 = 0$,

($\mathcal{BCK}$-3) $g_1 \diamond g_1 = 0$,

($\mathcal{BCK}$-4) $0 \diamond g_1 = 0$,

($\mathcal{BCK}$-5) $g_1 \diamond \hbar_1 = 0$ *and* $\hbar_1 \diamond g_1 = 0 \Rightarrow g_1 = \hbar_1$.

For convenience, we will let $\mathcal{G}$ represent the $\mathcal{BCK}$-A $\mathcal{G} = (\mathcal{G}, \diamond, 0)$ until otherwise specified.

We are able to define a binary operation $\leq$ on $\mathcal{G}$ by assuming $g_1 \leq \hbar_1$ if and only if $g_1 \diamond \hbar_1 = 0$.

In a $\mathcal{BCK}$-A $\mathcal{G}$, the following properties hold.

$$g_1 \diamond 0 = g_1, \tag{1}$$

$$g_1 \diamond \hbar_1 \leq g_1, \tag{2}$$

$$(g_1 \diamond \hbar_1) \diamond i_1 = (g_1 \diamond i_1) \diamond \hbar_1, \tag{3}$$

$$(g_1 \diamond i_1) \diamond (\hbar_1 \diamond i_1) \leq g_1 \diamond \hbar_1, \tag{4}$$

$$g_1 \diamond (g_1 \diamond (g_1 \diamond \hbar_1)) = g_1 \diamond \hbar_1, \tag{5}$$

$$g_1 \leq \hbar_1 \Rightarrow g_1 \diamond i_1 \leq \hbar_1 \diamond i_1 \text{ and } i_1 \diamond \hbar_1 \leq i_1 \diamond g_1, \tag{6}$$

$$g_1 \diamond \hbar_1 \leq i_1 \Rightarrow g_1 \diamond i_1 \leq \hbar_1, \tag{7}$$

for all $g_1, \hbar_1, i_1 \in \mathcal{G}$.

Theorem 1. *[6] In a $\mathcal{BCK}$-A $\mathcal{G}$, the following holds for all $g_1, \hbar_1, i_1 \in \mathcal{G}$,*

(i) $((g_1 \diamond i_1) \diamond i_1) \diamond (\hbar_1 \diamond i_1) \leq (g_1 \diamond \hbar_1) \diamond i_1$,

(ii) $(g_1 \diamond i_1) \diamond (g_1 \diamond (g_1 \diamond i_1)) = (g_1 \diamond i_1) \diamond i_1$,

(iii) $(g_1 \diamond (\hbar_1 \diamond (\hbar_1 \diamond g_1))) \diamond (\hbar_1 \diamond (g_1 \diamond (\hbar_1 \diamond (\hbar_1 \diamond g_1)))) \leq g_1 \diamond \hbar_1$.

Definition 2. *[6] A $\mathcal{BCK}$-A $\mathcal{G}$ is considered to be a positive implicative if the following condition holds*

$$(g_1 \diamond i_1) \diamond (\hbar_1 \diamond i_1) = (g_1 \diamond \hbar_1) \diamond i_1, \tag{8}$$

for all $g_1, \hbar_1, i_1 \in \mathcal{G}$.

Definition 3. *[6] A $\mathcal{BCK}$-A $\mathcal{G}$ is considered to be a commutative if the following condition holds*

$$g_1 \diamond (g_1 \diamond \hbar_1) = \hbar_1 \diamond (\hbar_1 \diamond g_1), \tag{9}$$

for all $g_1, \hbar_1 \in \mathcal{G}$.

Definition 4. *[6] A $\mathcal{BCK}$-A $\mathcal{G}$ is considered to be an implicative if the following condition holds*

$$g_1 \diamond (\hbar_1 \diamond g_1) = g_1, \tag{10}$$

for all $g_1, \hbar_1 \in \mathcal{G}$.

Definition 5. *[6] A subalgebra of a $\mathcal{BCK}$-A $\mathcal{G}$ is defined as a non-empty subset $\Im$ of $\mathcal{G}$ satisfying the following condition:*

$$g_1 \diamond \hbar_1 \in \Im, \tag{11}$$

for all $g_1, \hbar_1 \in \Im$.

Definition 6. *[6] An ideal of a $\mathcal{BCK}$-A $\mathcal{G}$ is defined as a non-empty subset $\Im$ of $\mathcal{G}$ satisfying the following condition:*

(I1) $0 \in \Im$,

(I2) $g_1 \diamond \hbar_1 \in \Im, \hbar_1 \in \Im \Rightarrow g_1 \in \Im$,

for all $g_1, \hbar_1 \in \mathcal{G}$.

Definition 7. *[6] A commutative ideal of a* $\mathcal{BCK}$*-A* $\mathcal{G}$ *is defined as a non-empty subset* $\mathfrak{I}$ *of* $\mathcal{G}$ *satisfying the condition (I1) and the following*

(CI1) $(g_1 \diamond \hbar_1) \diamond i_1 \in \mathfrak{I}, i_1 \in \mathfrak{I} \Rightarrow g_1 \diamond (\hbar_1 \diamond (\hbar_1 \diamond g_1)) \in \mathfrak{I}$,

for all $g_1, \hbar_1, i_1 \in \mathcal{G}$.

Definition 8. *[6] A positive implicative ideal of a* $\mathcal{BCK}$*-A* $\mathcal{G}$ *is defined as a non-empty subset* $\mathfrak{I}$ *of* $\mathcal{G}$ *satisfying the condition (I1) and the following*

(PII1) $(g_1 \diamond \hbar_1) \diamond i_1 \in \mathfrak{I}, \hbar_1 \diamond i_1 \in \mathfrak{I} \Rightarrow g_1 \diamond i_1 \in \mathfrak{I}$,

for all $g_1, \hbar_1, i_1 \in \mathcal{G}$.

Definition 9. *[6] An implicative ideal of a* $\mathcal{BCK}$*-A* $\mathcal{G}$ *is defined as a non-empty subset* $\mathfrak{I}$ *of* $\mathcal{G}$ *satisfying the condition (I1) and the following*

(II1) $(g_1 \diamond (\hbar_1 \diamond g_1)) \diamond i_1 \in \mathfrak{I}, i_1 \in \mathfrak{I} \Rightarrow g_1 \in \mathfrak{I}$,

for all $g_1, \hbar_1, i_1 \in \mathcal{G}$.

Definition 10. *[31] Let* $\mathcal{G}$ *be a non-empty set. An FS in* $\mathcal{G}$ *is a mapping* $\mathcal{M} : \mathcal{G} \to [0, 1]$.

Definition 11. *[31] The complement of an FS* $\mathcal{M}$ *denoted by* $\overline{\mathcal{M}}$ *is also an FS defined as* $\overline{\mathcal{M}}(g_1) = 1 - \mathcal{M}(g_1)$ *for all* $g_1 \in \mathcal{G}$*. Also* $\overline{(\overline{\mathcal{M}})} = \mathcal{M}$.

Definition 12. *[14] A BPVFS* $\mathscr{B}$ *of* $\mathcal{G}$ *is defined as*

$$\mathscr{B} = \{(g_1, \mathcal{M}^+_{\mathscr{B}}(g_1), \mathcal{M}^-_{\mathscr{B}}(g_1)) | g_1 \in \mathcal{G}\},$$

where $\mathcal{M}^+_{\mathscr{B}} : \mathcal{G} \to [0, 1]$ *and* $\mathcal{M}^-_{\mathscr{B}} : \mathcal{G} \to [-1, 0]$ *are mappings. The degree of positive membership* $\mathcal{M}^+_{\mathscr{B}}$ *indicates how the member of* $\mathcal{G}$ *satisfies the property related to the BPVFS* $\mathscr{B}$*, while the negative degree of membership* $\mathcal{M}^-_{\mathscr{B}}$ *indicates the grade of satisfaction that the member of* $\mathcal{G}$ *has toward some implicit counter property of* $\mathscr{B}$*. We shall use the symbol* $\mathscr{B} = (g_1, \mathcal{M}^+_{\mathscr{B}}, \mathcal{M}^-_{\mathscr{B}})$ *for a BPVFS* $\mathscr{B} = \{(g_1, \mathcal{M}^+_{\mathscr{B}}(g_1), \mathcal{M}^-_{\mathscr{B}}(g_1)) | g_1 \in \mathcal{G}\}$.

Definition 13. *[15] A BPVFS* $\mathscr{B} = (g_1, \mathcal{M}^+_{\mathscr{B}}, \mathcal{M}^-_{\mathscr{B}})$ *in* $\mathcal{G}$ *is said to be a bipolar fuzzy subalgebra if the following conditions are satisfied:*

$$\mathcal{M}^+_{\mathscr{B}}(g_1 \diamond \hbar_1) \geq \min\{\mathcal{M}^+_{\mathscr{B}}(g_1), \mathcal{M}^+_{\mathscr{B}}(\hbar_1)\},$$
$$\mathcal{M}^-_{\mathscr{B}}(g_1 \diamond \hbar_1) \leq \max\{\mathcal{M}^-_{\mathscr{B}}(g_1), \mathcal{M}^-_{\mathscr{B}}(\hbar_1)\},$$

for all $g_1, \hbar_1 \in \mathcal{G}$.

Definition 14. *[15] A BPVFS* $\mathscr{B} = (g_1, \mathcal{M}^+_{\mathscr{B}}, \mathcal{M}^-_{\mathscr{B}})$ *in* $\mathcal{G}$ *is said to be a bipolar fuzzy ideal if the following conditions are satisfied:* $\mathcal{M}^+_{\mathscr{B}}(0) \geq \mathcal{M}^+_{\mathscr{B}}(g_1)$, $\mathcal{M}^-_{\mathscr{B}}(0) \leq \mathcal{M}^-_{\mathscr{B}}(g_1)$ *and*

$$\mathcal{M}^+_{\mathscr{B}}(g_1) \geq \min\{\mathcal{M}^+_{\mathscr{B}}(g_1 \diamond \hbar_1), \mathcal{M}^+_{\mathscr{B}}(\hbar_1)\},$$
$$\mathcal{M}^-_{\mathscr{B}}(g_1) \leq \max\{\mathcal{M}^-_{\mathscr{B}}(g_1 \diamond \hbar_1), \mathcal{M}^-_{\mathscr{B}}(\hbar_1)\},$$

for all $g_1, \hbar_1 \in \mathcal{G}$.

Definition 15. *[22] A BPVFS* $\mathscr{B} = (g_1, \mathcal{M}^+_{\mathscr{B}}, \mathcal{M}^-_{\mathscr{B}})$ *in* $\mathcal{G}$ *is said to be a bipolar fuzzy implicative ideal of* $\mathcal{G}$ *if it satisfies* $\mathcal{M}^+_{\mathscr{B}}(0) \geq \mathcal{M}^+_{\mathscr{B}}(g_1)$, $\mathcal{M}^-_{\mathscr{B}}(0) \leq \mathcal{M}^-_{\mathscr{B}}(g_1)$ *and*

$$\mathcal{M}^+_{\mathscr{B}}(g_1) \geq \min\{\mathcal{M}^+_{\mathscr{B}}((g_1 \diamond (\hbar_1 \diamond g_1)) \diamond i_1), \mathcal{M}^+_{\mathscr{B}}(i_1)\},$$
$$\mathcal{M}^-_{\mathscr{B}}(g_1) \leq \max\{\mathcal{M}^-_{\mathscr{B}}((g_1 \diamond (\hbar_1 \diamond g_1)) \diamond i_1), \mathcal{M}^-_{\mathscr{B}}(i_1)\},$$

for all $g_1, \hbar_1, i_1 \in \mathcal{G}$.

Definition 16. *A BPVFS* $\mathscr{B} = (g_1, \mathcal{M}^+_{\mathscr{B}}, \mathcal{M}^-_{\mathscr{B}})$ *in* $\mathcal{G}$ *is said to be a bipolar fuzzy commutative ideal of* $\mathcal{G}$ *if it satisfies* $\mathcal{M}^+_{\mathscr{B}}(0) \geq \mathcal{M}^+_{\mathscr{B}}(g_1)$, $\mathcal{M}^-_{\mathscr{B}}(0) \leq \mathcal{M}^-_{\mathscr{B}}(g_1)$ *and*

$$\mathcal{M}^+_{\mathscr{B}}(g_1 \diamond (\hbar_1 \diamond (\hbar_1 \diamond g_1))) \geq \min\{\mathcal{M}^+_{\mathscr{B}}((g_1 \diamond \hbar_1) \diamond i_1), \mathcal{M}^+_{\mathscr{B}}(i_1)\},$$
$$\mathcal{M}^-_{\mathscr{B}}(g_1 \diamond (\hbar_1 \diamond (\hbar_1 \diamond g_1))) \leq \max\{\mathcal{M}^-_{\mathscr{B}}((g_1 \diamond \hbar_1) \diamond i_1), \mathcal{M}^-_{\mathscr{B}}(i_1)\},$$

for all $g_1, \hbar_1, i_1 \in \mathcal{G}$.

Definition 17. *[22] A BPVFS* $\mathscr{B} = (g_1, \mathcal{M}^+_{\mathscr{B}}, \mathcal{M}^-_{\mathscr{B}})$ *in* $\mathcal{G}$ *is said to be a bipolar fuzzy positive implicative ideal of* $\mathcal{G}$ *if it satisfies* $\mathcal{M}^+_{\mathscr{B}}(0) \geq \mathcal{M}^+_{\mathscr{B}}(g_1)$, $\mathcal{M}^-_{\mathscr{B}}(0) \leq \mathcal{M}^-_{\mathscr{B}}(g_1)$ *and*

$$\mathcal{M}^+_{\mathscr{B}}(g_1 \diamond i_1) \geq \min\{\mathcal{M}^+_{\mathscr{B}}((g_1 \diamond \hbar_1) \diamond i_1), \mathcal{M}^+_{\mathscr{B}}(\hbar_1 \diamond i_1)\},$$
$$\mathcal{M}^-_{\mathscr{B}}(g_1 \diamond i_1) \leq \max\{\mathcal{M}^-_{\mathscr{B}}((g_1 \diamond \hbar_1) \diamond i_1), \mathcal{M}^-_{\mathscr{B}}(\hbar_1 \diamond i_1)\},$$

for all $g_1, \hbar_1, i_1 \in \mathcal{G}$.

Definition 18. *[3] An IFS* $\mathscr{B}$ *in a non-empty set* $\mathcal{G}$ *is an object having the form*

$$\mathscr{B} = \{(g_1, \mathcal{M}(g_1), \mathcal{N}(g_1)) | g_1 \in \mathcal{G}\}$$

where $\mathcal{M}(g_1), \mathcal{N}(g_1)$ *are level of belongingness and level of non-belongingness of* $g_1 \in \mathcal{G}$ *respectively and* $0 \leq \mathcal{M}(g_1) + \mathcal{N}(g_1) \leq 1$ *for all* $g_1 \in \mathcal{G}$. *We shall use the symbol* $\mathscr{B} = (g_1, \mathcal{M}, \mathcal{N})$ *for an IFS* $\mathscr{B} = \{(g_1, \mathcal{M}(g_1), \mathcal{N}(g_1)) | g_1 \in \mathcal{G}\}$.

Definition 19. *[4] A BPVIFS* $\mathscr{B}$ *in a non-empty set* $\mathcal{G}$ *is an object having the form*

$$\mathscr{B} = \{(g_1, \mathcal{M}^+_{\mathscr{B}}(g_1), \mathcal{M}^-_{\mathscr{B}}(g_1), \mathcal{N}^+_{\mathscr{B}}(g_1), \mathcal{N}^-_{\mathscr{B}}(g_1)) | g_1 \in \mathcal{G}\},$$

where $\mathcal{M}^+_{\mathscr{B}}(g_1) : \mathcal{G} \to [0,1]$, $\mathcal{M}^-_{\mathscr{B}}(g_1) : \mathcal{G} \to [-1,0]$, $\mathcal{N}^+_{\mathscr{B}}(g_1) : \mathcal{G} \to [0,1]$, *and* $\mathcal{N}^-_{\mathscr{B}}(g_1) : \mathcal{G} \to [-1,0]$ *are such that* $0 \leq \mathcal{M}^+_{\mathscr{B}}(g_1) + \mathcal{N}^+_{\mathscr{B}}(g_1) \leq 1$ *and* $-1 \leq \mathcal{M}^-_{\mathscr{B}}(g_1) + \mathcal{N}^-_{\mathscr{B}}(g_1) \leq 0$. *In this,* $\mathcal{M}^+\mathscr{B}$ *is used to indicate the level of positive membership level, showing the extent to which a member of* $\mathcal{G}$ *satisfies a property within a BPVIFS* $\mathscr{B}$. *On the other side,* $\mathcal{M}^-\mathscr{B}$ *indicates the level of negative membership, showing the extent to which a member of* $\mathcal{G}$ *satisfies the implicit counter property associated with the BPVIFS. The terms* $\mathcal{N}^+_{\mathscr{B}}(g_1)$ *and* $\mathcal{N}^-_{\mathscr{B}}(g_1)$ *refer to the level of positive non-membership and level negative non-membership respectively. We calculate* $\mathcal{N}^+_{\mathscr{B}}(g_1)$ *and* $\mathcal{N}^-_{\mathscr{B}}(g_1)$ *as* $\mathcal{N}^+_{\mathscr{B}}(g_1) = 1 - \mathcal{M}^+_{\mathscr{B}}(g_1)$ *and* $\mathcal{N}^-_{\mathscr{B}}(g_1) = -1 - \mathcal{M}^-_{\mathscr{B}}(g_1)$.

Definition 20. *A BPVIFS* $\mathscr{B} = (\mathcal{M}^+_{\mathscr{B}}, \mathcal{M}^-_{\mathscr{B}}, \mathcal{N}^+_{\mathscr{B}}, \mathcal{N}^-_{\mathscr{B}})$ *in* $\mathcal{G}$ *is a BPVIFI of* $\mathcal{G}$ *if it satisfies* $\mathcal{M}^+_{\mathscr{B}}(0) \geq \mathcal{M}^+_{\mathscr{B}}(g_1)$, $\mathcal{M}^-_{\mathscr{B}}(0) \leq \mathcal{M}^-_{\mathscr{B}}(g_1)$, $\mathcal{N}^+_{\mathscr{B}}(0) \leq \mathcal{N}^+_{\mathscr{B}}(g_1)$, $\mathcal{N}^-_{\mathscr{B}}(0) \geq \mathcal{N}^-_{\mathscr{B}}(g_1)$, *and*

$$\begin{aligned}
\mathcal{M}^+_{\mathscr{B}}(g_1) &\geq \min\{\mathcal{M}^+_{\mathscr{B}}(g_1 \diamond \hbar_1), \mathcal{M}^+_{\mathscr{B}}(\hbar_1)\},\\
\mathcal{M}^-_{\mathscr{B}}(g_1) &\leq \max\{\mathcal{M}^-_{\mathscr{B}}(g_1 \diamond \hbar_1), \mathcal{M}^-_{\mathscr{B}}(\hbar_1)\},\\
\mathcal{N}^+_{\mathscr{B}}(g_1) &\leq \max\{\mathcal{N}^+_{\mathscr{B}}(g_1 \diamond \hbar_1), \mathcal{N}^+_{\mathscr{B}}(\hbar_1)\},\\
\mathcal{N}^-_{\mathscr{B}}(g_1) &\geq \min\{\mathcal{N}^-_{\mathscr{B}}(g_1 \diamond \hbar_1), \mathcal{N}^-_{\mathscr{B}}(\hbar_1)\},
\end{aligned}$$

for all $g_1, \hbar_1 \in \mathcal{G}$.

Theorem 2. *A BPVIFS* $\mathscr{B} = (\mathcal{M}^+_{\mathscr{B}}, \mathcal{M}^-_{\mathscr{B}}, \mathcal{N}^+_{\mathscr{B}}, \mathcal{N}^-_{\mathscr{B}})$ *in* $\mathcal{G}$ *is a BPVIFI of* $\mathcal{G}$ *if and only if it satisfies the condition if* $g_1 \diamond \hbar_1 \leq i_1$ *for all* $g_1, \hbar_1, i_1 \in \mathcal{G}$, *then*

$$\left(\begin{array}{c}
\mathcal{M}^+_{\mathscr{B}}(g_1) \geq \min\{\mathcal{M}^+_{\mathscr{B}}(\hbar_1), \mathcal{M}^+_{\mathscr{B}}(i_1)\},\\
\mathcal{M}^-_{\mathscr{B}}(g_1) \leq \max\{\mathcal{M}^-_{\mathscr{B}}(\hbar_1), \mathcal{M}^-_{\mathscr{B}}(i_1)\},\\
\mathcal{N}^+_{\mathscr{B}}(g_1) \leq \max\{\mathcal{N}^+_{\mathscr{B}}(\hbar_1), \mathcal{N}^+_{\mathscr{B}}(i_1)\},\\
\mathcal{N}^-_{\mathscr{B}}(g_1) \geq \min\{\mathcal{N}^-_{\mathscr{B}}(\hbar_1), \mathcal{N}^-_{\mathscr{B}}(i_1)\},
\end{array}\right) \tag{12}$$

Proof. Assume that $\mathscr{B} = (\mathcal{M}^+_{\mathscr{B}}, \mathcal{M}^-_{\mathscr{B}}, \mathcal{N}^+_{\mathscr{B}}, \mathcal{N}^-_{\mathscr{B}})$ is a BPIFI of $\mathcal{G}$. Let $g_1, \hbar_1, i_1 \in \mathcal{G}$ be such that $g_1 \diamond \hbar_1 \leq i_1$. Then $(g_1 \diamond \hbar_1) \diamond i_1 = 0$. Thus,

$$\begin{aligned}
\mathcal{M}^+_{\mathscr{B}}(g_1) &\geq \min\{\mathcal{M}^+_{\mathscr{B}}(g_1 \diamond \hbar_1), \mathcal{M}^+_{\mathscr{B}}(\hbar_1)\}\\
&\geq \min\{\min\{\mathcal{M}^+_{\mathscr{B}}((g_1 \diamond \hbar_1) \diamond i_1), \mathcal{M}^+_{\mathscr{B}}(i_1)\}, \mathcal{M}^+_{\mathscr{B}}(\hbar_1)\}\\
&\geq \min\{\min\{\mathcal{M}^+_{\mathscr{B}}(0), \mathcal{M}^+_{\mathscr{B}}(i_1)\}, \mathcal{M}^+_{\mathscr{B}}(\hbar_1)\}\\
&= \min\{\mathcal{M}^+_{\mathscr{B}}(\hbar_1), \mathcal{M}^+_{\mathscr{B}}(i_1)\},\\
\mathcal{M}^-_{\mathscr{B}}(g_1) &\leq \max\{\mathcal{M}^-_{\mathscr{B}}(g_1 \diamond \hbar_1), \mathcal{M}^-_{\mathscr{B}}(\hbar_1)\}\\
&\leq \max\{\max\{\mathcal{M}^-_{\mathscr{B}}((g_1 \diamond \hbar_1) \diamond i_1), \mathcal{M}^-_{\mathscr{B}}(i_1)\}, \mathcal{M}^-_{\mathscr{B}}(\hbar_1)\}\\
&\leq \max\{\max\{\mathcal{M}^-_{\mathscr{B}}(0), \mathcal{M}^-_{\mathscr{B}}(i_1)\}, \mathcal{M}^-_{\mathscr{B}}(\hbar_1)\}\\
&= \max\{\mathcal{M}^-_{\mathscr{B}}(\hbar_1), \mathcal{M}^-_{\mathscr{B}}(i_1)\},\\
\mathcal{N}^+_{\mathscr{B}}(g_1) &\leq \max\{\mathcal{N}^+_{\mathscr{B}}(g_1 \diamond \hbar_1), \mathcal{N}^+_{\mathscr{B}}(\hbar_1)\}\\
&\leq \max\{\max\{\mathcal{N}^+_{\mathscr{B}}((g_1 \diamond \hbar_1) \diamond i_1), \mathcal{N}^+_{\mathscr{B}}(i_1)\}, \mathcal{N}^+_{\mathscr{B}}(\hbar_1)\}\\
&\leq \max\{\max\{\mathcal{N}^+_{\mathscr{B}}(0), \mathcal{N}^+_{\mathscr{B}}(i_1)\}, \mathcal{N}^+_{\mathscr{B}}(\hbar_1)\}\\
&= \max\{\mathcal{N}^+_{\mathscr{B}}(\hbar_1), \mathcal{N}^+_{\mathscr{B}}(i_1)\},\\
\mathcal{N}^-_{\mathscr{B}}(g_1) &\geq \min\{\mathcal{N}^-_{\mathscr{B}}(g_1 \diamond \hbar_1), \mathcal{N}^-_{\mathscr{B}}(\hbar_1)\}\\
&\geq \min\{\min\{\mathcal{N}^-_{\mathscr{B}}((g_1 \diamond \hbar_1) \diamond i_1), \mathcal{N}^-_{\mathscr{B}}(i_1)\}, \mathcal{N}^-_{\mathscr{B}}(\hbar_1)\}\\
&\geq \min\{\min\{\mathcal{N}^-_{\mathscr{B}}(0), \mathcal{N}^-_{\mathscr{B}}(i_1)\}, \mathcal{N}^-_{\mathscr{B}}(\hbar_1)\}\\
&= \min\{\mathcal{N}^-_{\mathscr{B}}(\hbar_1), \mathcal{N}^-_{\mathscr{B}}(i_1)\}.
\end{aligned}$$

Hence, (12) is valid.

Conversely, let $\mathscr{B} = (\mathcal{M}^+_{\mathscr{B}}, \mathcal{M}^-_{\mathscr{B}}, \mathcal{N}^+_{\mathscr{B}}, \mathcal{N}^-_{\mathscr{B}})$ be a BPIFS in $\mathcal{G}$ that satisfies (12). Since $0 \diamond g_1 \leq g_1$ for all $g_1 \in \mathcal{G}$, we have $\mathcal{M}^+_{\mathscr{B}}(0) \geq \mathcal{M}^+_{\mathscr{B}}(g_1)$, $\mathcal{M}^-_{\mathscr{B}}(0) \leq \mathcal{M}^-_{\mathscr{B}}(g_1)$, $\mathcal{N}^+_{\mathscr{B}}(0) \leq$

$\mathcal{N}^{+}_{\mathscr{B}}(g_1)$, and $\mathcal{N}^{-}_{\mathscr{B}}(0) \geq \mathcal{N}^{-}_{\mathscr{B}}(g_1)$. Also, since $g_1 \diamond (g_1 \diamond \hbar_1) \leq \hbar_1$ for all $g_1, \hbar_1 \in \mathcal{G}$, we have

$$\begin{aligned}
\mathcal{M}^{+}_{\mathscr{B}}(g_1) &\geq \min\{\mathcal{M}^{+}_{\mathscr{B}}(g_1 \diamond \hbar_1), \mathcal{M}^{+}_{\mathscr{B}}(\hbar_1)\},\\
\mathcal{M}^{-}_{\mathscr{B}}(g_1) &\leq \max\{\mathcal{M}^{-}_{\mathscr{B}}(g_1 \diamond \hbar_1), \mathcal{M}^{-}_{\mathscr{B}}(\hbar_1)\},\\
\mathcal{N}^{+}_{\mathscr{B}}(g_1) &\leq \max\{\mathcal{N}^{+}_{\mathscr{B}}(g_1 \diamond \hbar_1), \mathcal{N}^{+}_{\mathscr{B}}(\hbar_1)\},\\
\mathcal{N}^{-}_{\mathscr{B}}(g_1) &\geq \min\{\mathcal{N}^{-}_{\mathscr{B}}(g_1 \diamond \hbar_1), \mathcal{N}^{-}_{\mathscr{B}}(\hbar_1)\}.
\end{aligned}$$

Therefore, $\mathscr{B} = (\mathcal{M}^{+}_{\mathscr{B}}, \mathcal{M}^{-}_{\mathscr{B}}, \mathcal{N}^{+}_{\mathscr{B}}, \mathcal{N}^{-}_{\mathscr{B}})$ is a BPIFI of $\mathcal{G}$.

3. Bipolar Valued Intuitionistic Fuzzy Positive Implicative Ideals

Definition 21. *A BPVIFS $\mathscr{B} = (\mathcal{M}^{+}_{\mathscr{B}}, \mathcal{M}^{-}_{\mathscr{B}}, \mathcal{N}^{+}_{\mathscr{B}}, \mathcal{N}^{-}_{\mathscr{B}})$ in $\mathcal{G}$ is a BPVIFPII of $\mathcal{G}$ if it satisfies for all $g_1, \hbar_1, i_1 \in \mathcal{G}$, $\mathcal{M}^{+}_{\mathscr{B}}(0) \geq \mathcal{M}^{+}_{\mathscr{B}}(g_1)$, $\mathcal{M}^{-}_{\mathscr{B}}(0) \leq \mathcal{M}^{-}_{\mathscr{B}}(g_1)$, $\mathcal{N}^{+}_{\mathscr{B}}(0) \leq \mathcal{N}^{+}_{\mathscr{B}}(g_1)$, $\mathcal{N}^{-}_{\mathscr{B}}(0) \geq \mathcal{N}^{-}_{\mathscr{B}}(g_1)$, and*

$$\begin{aligned}
\mathcal{M}^{+}_{\mathscr{B}}(g_1 \diamond i_1) &\geq \min\{\mathcal{M}^{+}_{\mathscr{B}}((g_1 \diamond \hbar_1) \diamond i_1), \mathcal{M}^{+}_{\mathscr{B}}(\hbar_1 \diamond i_1)\},\\
\mathcal{M}^{-}_{\mathscr{B}}(g_1 \diamond i_1) &\leq \max\{\mathcal{M}^{-}_{\mathscr{B}}((g_1 \diamond \hbar_1) \diamond i_1), \mathcal{M}^{-}_{\mathscr{B}}(\hbar_1 \diamond i_1)\},\\
\mathcal{N}^{+}_{\mathscr{B}}(g_1 \diamond i_1) &\leq \max\{\mathcal{N}^{+}_{\mathscr{B}}((g_1 \diamond \hbar_1) \diamond i_1), \mathcal{N}^{+}_{\mathscr{B}}(\hbar_1 \diamond i_1)\},\\
\mathcal{N}^{-}_{\mathscr{B}}(g_1 \diamond i_1) &\geq \min\{\mathcal{N}^{-}_{\mathscr{B}}((g_1 \diamond \hbar_1) \diamond i_1), \mathcal{N}^{-}_{\mathscr{B}}(\hbar_1 \diamond i_1)\}.
\end{aligned}$$

Example 1. *Consider $\mathcal{G} = \{0, 1, 2, 3\}$ be a set in which the binary operation $\diamond$ is defined as follows:*

$$0 \diamond g_1 = 0 \ \forall g_1 \in \mathcal{G}$$

$$1 \diamond g_1 = \begin{cases} 0, & \text{if } g_1 \in \{1, 3\} \\ 1, & \text{if } g_1 \in \{0, 2\} \end{cases}$$

$$2 \diamond g_1 = \begin{cases} 0, & \text{if } g_1 \in \{2, 3\} \\ 2, & \text{if } g_1 \in \{0, 1\} \end{cases}$$

$$3 \diamond g_1 = \begin{cases} 0, & \text{if } g_1 = 3 \\ 1, & \text{if } g_1 = 2 \\ 2, & \text{if } g_1 = 1 \\ 3, & \text{if } g_1 = 0 \end{cases}$$

Then $\mathcal{G}$ is a $\mathcal{BCK}$-A. Let us define a BPVIFS $\mathscr{B} = (\mathcal{M}^{+}_{\mathscr{B}}, \mathcal{M}^{-}_{\mathscr{B}}, \mathcal{N}^{+}_{\mathscr{B}}, \mathcal{N}^{-}_{\mathscr{B}})$ in $\mathcal{G}$ as shown in Table 1.

By using standard computation, it is clear that $\mathscr{B} = (\mathcal{M}^{+}_{\mathscr{B}}, \mathcal{M}^{-}_{\mathscr{B}}, \mathcal{N}^{+}_{\mathscr{B}}, \mathcal{N}^{-}_{\mathscr{B}})$ is a BPV-IFPII of $\mathcal{G}$.

Example 2. *Consider $\mathcal{G} = \{0, 1, 2, 3, 4\}$ be a set in which the binary operation $\diamond$ is defined as follows:*

$$0 \diamond g_1 = 0 \ \forall g_1 \in \mathcal{G}$$

Table 1: BPVIFPII

$\mathfrak{g}_1$	$\mathcal{M}^+_{\mathscr{B}}(\mathfrak{g}_1)$	$\mathcal{M}^-_{\mathscr{B}}(\mathfrak{g}_1)$	$\mathcal{N}^+_{\mathscr{B}}(\mathfrak{g}_1)$	$\mathcal{N}^-_{\mathscr{B}}(\mathfrak{g}_1)$
0	0.65	−0.85	0.35	−0.15
1	0.53	−0.53	0.47	−0.47
2	0.31	−0.23	0.69	−0.77
3	0.31	−0.23	0.69	−0.77

$$1 \diamond g_1 = \begin{cases} 0, & if\ g_1 \in \{1,2,3,4\} \\ 1, & if\ g_1 = 0 \end{cases}$$

$$2 \diamond g_1 = \begin{cases} 0, & if\ g_1 \in \{2,3\} \\ 2, & if\ g_1 \in \{0,1,4\} \end{cases}$$

$$3 \diamond g_1 = \begin{cases} 0, & if\ g_1 = 3 \\ 3, & if\ g_1 \in \{0,1,2,4\} \end{cases}$$

$$4 \diamond g_1 = \begin{cases} 0, & if\ g_1 = 4 \\ 4, & if\ g_1 \in \{0,1,2,3\} \end{cases}$$

Then $\mathcal{G}$ is a $\mathcal{BCK}$-A. Let us define a BPVIFS $\mathscr{B} = (\mathcal{M}^+_{\mathscr{B}}, \mathcal{M}^-_{\mathscr{B}}, \mathcal{N}^+_{\mathscr{B}}, \mathcal{N}^-_{\mathscr{B}})$ in $\mathcal{G}$ as shown in Table 2.

Table 2: BPVIFPII

$\mathfrak{g}_1$	$\mathcal{M}^+_{\mathscr{B}}(\mathfrak{g}_1)$	$\mathcal{M}^-_{\mathscr{B}}(\mathfrak{g}_1)$	$\mathcal{N}^+_{\mathscr{B}}(\mathfrak{g}_1)$	$\mathcal{N}^-_{\mathscr{B}}(\mathfrak{g}_1)$
0	0.91	−0.79	0.09	−0.21
1	0.75	−0.59	0.25	−0.41
2	0.53	−0.38	0.47	−0.62
3	0.32	−0.29	0.68	−0.71
4	0.11	−0.11	0.89	−0.89

By using standard computation, it is clear that $\mathscr{B} = (\mathcal{M}^+_{\mathscr{B}}, \mathcal{M}^-_{\mathscr{B}}, \mathcal{N}^+_{\mathscr{B}}, \mathcal{N}^-_{\mathscr{B}})$ is a BPVIFPII of $\mathcal{G}$.

Theorem 3. *Every BPVIFPII of $\mathcal{G}$ is also a BPVIFI of $\mathcal{G}$.*

Proof. Let $\mathscr{B} = (\mathcal{M}^+_{\mathscr{B}}, \mathcal{M}^-_{\mathscr{B}}, \mathcal{N}^+_{\mathscr{B}}, \mathcal{N}^-_{\mathscr{B}})$ be a BPVIFPII of $\mathcal{G}$, and put $\mathfrak{i}_1 = 0$ in Definition 21. Then, using (1), we obtain

$$\begin{aligned}
&\mathcal{M}^+_{\mathscr{B}}(\mathfrak{g}_1 \diamond 0) \geq \min\{\mathcal{M}^+_{\mathscr{B}}((\mathfrak{g}_1 \diamond \hbar_1) \diamond 0), \mathcal{M}^+_{\mathscr{B}}(\hbar_1 \diamond 0)\} \\
&\Rightarrow \mathcal{M}^+_{\mathscr{B}}(\mathfrak{g}_1) \geq \min\{\mathcal{M}^+_{\mathscr{B}}(\mathfrak{g}_1 \diamond \hbar_1), \mathcal{M}^+_{\mathscr{B}}(\hbar_1)\}, \\
&\mathcal{M}^-_{\mathscr{B}}(\mathfrak{g}_1 \diamond 0) \leq \max\{\mathcal{M}^-_{\mathscr{B}}((\mathfrak{g}_1 \diamond \hbar_1) \diamond 0), \mathcal{M}^-_{\mathscr{B}}(\hbar_1 \diamond 0)\} \\
&\Rightarrow \mathcal{M}^-_{\mathscr{B}}(\mathfrak{g}_1) \leq \max\{\mathcal{M}^-_{\mathscr{B}}(\mathfrak{g}_1 \diamond \hbar_1), \mathcal{M}^-_{\mathscr{B}}(\hbar_1)\}, \\
&\mathcal{N}^+_{\mathscr{B}}(\mathfrak{g}_1 \diamond 0) \leq \max\{\mathcal{N}^+_{\mathscr{B}}((\mathfrak{g}_1 \diamond \hbar_1) \diamond 0), \mathcal{N}^+_{\mathscr{B}}(\hbar_1 \diamond 0)\} \\
&\Rightarrow \mathcal{N}^+_{\mathscr{B}}(\mathfrak{g}_1) \leq \max\{\mathcal{N}^+_{\mathscr{B}}(\mathfrak{g}_1 \diamond \hbar_1), \mathcal{N}^+_{\mathscr{B}}(\hbar_1)\},
\end{aligned}$$

$$\mathcal{N}^-_{\mathscr{B}}(g_1 \diamond 0) \geq \min\{\mathcal{N}^-_{\mathscr{B}}((g_1 \diamond \hbar_1) \diamond 0), \mathcal{N}^-_{\mathscr{B}}(\hbar_1 \diamond 0)\}$$
$$\Rightarrow \mathcal{N}^-(g_1) \geq \min\{\mathcal{N}^-_{\mathscr{B}}(g_1 \diamond \hbar_1), \mathcal{N}^-_{\mathscr{B}}(\hbar_1)\},$$

for all $g_1, \hbar_1, i_1 \in \mathcal{G}$.

This shows that $\mathscr{B} = (\mathcal{M}^+_{\mathscr{B}}, \mathcal{M}^-_{\mathscr{B}}, \mathcal{N}^+_{\mathscr{B}}, \mathcal{N}^-_{\mathscr{B}})$ is a BPVIFPII of $\mathcal{G}$.

The following example shows that the converse of Theorem 3 may not be true.

Example 3. *Let $g_1 = \{0, 1, 2, 3\}$ be a set in which the binary operation $\diamond$ is defined as given below*

$$0 \diamond g_1 = 0 \ \forall g_1 \in \mathcal{G}$$
$$1 \diamond g_1 = \begin{cases} 0, & \text{if } g_1 \in \{1, 2\} \\ 1, & \text{if } g_1 \in \{0, 3\} \end{cases}$$
$$2 \diamond g_1 = \begin{cases} 0, & \text{if } g_1 = 2 \\ 1, & \text{if } g_1 = 1 \\ 2, & \text{if } g_1 \in \{0, 3\} \end{cases}$$
$$3 \diamond g_1 = \begin{cases} 0, & \text{if } g_1 = 3 \\ 3, & \text{if } g_1 \in \{0, 1, 2\} \end{cases}$$

Then $\mathcal{G}$ is a $\mathcal{BCK}$-A. Let $\mathscr{B} = (\mathcal{M}^+_{\mathscr{B}}, \mathcal{M}^-_{\mathscr{B}}, \mathcal{N}^+_{\mathscr{B}}, \mathcal{N}^-_{\mathscr{B}})$ be a BPVIFS in $\mathcal{G}$ defined as shown in Table 3.

Table 3: BPVIFI

g_1	$\mathcal{M}^+_{\mathscr{B}}(g_1)$	$\mathcal{M}^-_{\mathscr{B}}(g_1)$	$\mathcal{N}^+_{\mathscr{B}}(g_1)$	$\mathcal{N}^-_{\mathscr{B}}(g_1)$
0	0.95	−0.87	0.05	−0.13
1	0.73	−0.43	0.27	−0.57
2	0.73	−0.43	0.27	−0.57
3	0.33	−0.15	0.67	−0.85

It is easy to check that $\mathscr{B} = (\mathcal{M}^+_{\mathscr{B}}, \mathcal{M}^-_{\mathscr{B}}, \mathcal{N}^+_{\mathscr{B}}, \mathcal{N}^-_{\mathscr{B}})$ is a BPVIFI of $\mathcal{G}$. However, it is not a BPVIFPII of $\mathcal{G}$ because

$\mathcal{M}^+_{\mathscr{B}}(2 \diamond 1) = \mathcal{M}^+_{\mathscr{B}}(1) = 0.73 < 0.95 = \mathcal{M}^+_{\mathscr{B}}(0) = \min\{\mathcal{M}^+_{\mathscr{B}}((2 \diamond 1) \diamond 1), \mathcal{M}^+_{\mathscr{B}}(1 \diamond 1)\}$,
$\mathcal{M}^-_{\mathscr{B}}(2 \diamond 1) = \mathcal{M}^-_{\mathscr{B}}(1) = -0.43 > -0.87 = \mathcal{M}^-_{\mathscr{B}}(0) = \max\{\mathcal{M}^-_{\mathscr{B}}((2 \diamond 1) \diamond 1), \mathcal{M}^-_{\mathscr{B}}(1 \diamond 1)\}$,
$\mathcal{N}^+_{\mathscr{B}}(2 \diamond 1) = \mathcal{N}^+_{\mathscr{B}}(1) = 0.27 > 0.05 = \mathcal{N}^+_{\mathscr{B}}(0) = \max\{\mathcal{N}^+_{\mathscr{B}}((2 \diamond 1) \diamond 1), \mathcal{N}^+_{\mathscr{B}}(1 \diamond 1)\}$,
$\mathcal{N}^-_{\mathscr{B}}(2 \diamond 1) = \mathcal{N}^-_{\mathscr{B}}(1) = -0.57 < -0.13 = \mathcal{N}^-_{\mathscr{B}}(0) = \min\{\mathcal{N}^-_{\mathscr{B}}((2 \diamond 1) \diamond 1), \mathcal{N}^-_{\mathscr{B}}(1 \diamond 1)\}$.

Corollary 1. *Let $\mathscr{B} = (\mathcal{M}^+_{\mathscr{B}}, \mathcal{M}^-_{\mathscr{B}}, \mathcal{N}^+_{\mathscr{B}}, \mathcal{N}^-_{\mathscr{B}})$ be a BPVIFPII of $\mathcal{G}$. If $g_1 \leq \hbar_1$ in $\mathcal{G}$, then $\mathcal{M}^+_{\mathscr{B}}(g_1) \geq \mathcal{M}^+_{\mathscr{B}}(\hbar_1)$, $\mathcal{M}^-_{\mathscr{B}}(g_1) \leq \mathcal{M}^-_{\mathscr{B}}(\hbar_1)$, $\mathcal{N}^+_{\mathscr{B}}(g_1) \leq \mathcal{N}^+_{\mathscr{B}}(\hbar_1)$, and $\mathcal{N}^-_{\mathscr{B}}(g_1) \geq \mathcal{N}^-_{\mathscr{B}}(\hbar_1)$. i.e., $\mathcal{M}^+_{\mathscr{B}}, \mathcal{N}^-_{\mathscr{B}}$ are order-reversing and $\mathcal{M}^-_{\mathscr{B}}, \mathcal{N}^+_{\mathscr{B}}$ are order-preserving.*

Corollary 2. *In $\mathcal{G}$, every BPVIFPII of $\mathcal{G}$ is a BPVIFSA.*

Theorem 4. *If $\mathscr{B} = (\mathcal{M}^+_{\mathscr{B}}, \mathcal{M}^-_{\mathscr{B}}, \mathcal{N}^+_{\mathscr{B}}, \mathcal{N}^-_{\mathscr{B}})$ is a BPVIFPII in $\mathcal{G}$, and*
$\mathcal{J}(0) = \{g_1 \in \mathcal{G} | \mathcal{M}^+_{\mathscr{B}}(g_1) = \mathcal{M}^+_{\mathscr{B}}(0), \mathcal{M}^-_{\mathscr{B}}(g_1) = \mathcal{M}^-_{\mathscr{B}}(0), \mathcal{N}^+_{\mathscr{B}}(g_1) = \mathcal{N}^+_{\mathscr{B}}(0), \mathcal{N}^-_{\mathscr{B}}(g_1) = \mathcal{N}^-(0)\}$, *then $\mathcal{J}(0)$ is a positive implicative ideal of $\mathcal{G}$.*

Proof. Let $g_1, \hbar_1 \in \mathcal{G}$ be such that $(g_1 \diamond \hbar_1) \diamond i_1, \hbar_1 \diamond i_1 \in \mathcal{J}(0)$. Since $\mathscr{B} = (g_1, \mathcal{M}^+_{\mathscr{B}}, \mathcal{M}^-_{\mathscr{B}}, \mathcal{N}^+_{\mathscr{B}}, \mathcal{N}^-_{\mathscr{B}})$ is a BPVIFPII in $\mathcal{G}$, we have
$\mathcal{M}^+_{\mathscr{B}}(g_1 \diamond i_1) \geq \min\{\mathcal{M}^+_{\mathscr{B}}((g_1 \diamond \hbar_1) \diamond i_1), \mathcal{M}^+_{\mathscr{B}}(\hbar_1 \diamond i_1)\} = \min\{\mathcal{M}^+_{\mathscr{B}}(0), \mathcal{M}^+_{\mathscr{B}}(0)\} = \mathcal{M}^+_{\mathscr{B}}(0)$,
$\mathcal{M}^-_{\mathscr{B}}(g_1 \diamond i_1) \leq \max\{\mathcal{M}^-_{\mathscr{B}}((g_1 \diamond \hbar_1) \diamond i_1), \mathcal{M}^-_{\mathscr{B}}(\hbar_1 \diamond i_1)\} = \max\{\mathcal{M}^-_{\mathscr{B}}(0), \mathcal{M}^-_{\mathscr{B}}(0)\} = \mathcal{M}^-_{\mathscr{B}}(0)$,
$\mathcal{N}^+_{\mathscr{B}}(g_1 \diamond i_1) \leq \max\{\mathcal{N}^+_{\mathscr{B}}((g_1 \diamond \hbar_1) \diamond i_1), \mathcal{N}^+_{\mathscr{B}}(\hbar_1 \diamond i_1)\} = \max\{\mathcal{N}^+_{\mathscr{B}}(0), \mathcal{N}^+_{\mathscr{B}}(0)\} = \mathcal{N}^+_{\mathscr{B}}(0)$,
$\mathcal{N}^-_{\mathscr{B}}(g_1 \diamond i_1) \geq \min\{\mathcal{N}^-_{\mathscr{B}}((g_1 \diamond \hbar_1) \diamond i_1), \mathcal{N}^-_{\mathscr{B}}(\hbar_1 \diamond i_1)\} = \min\{\mathcal{N}^-_{\mathscr{B}}(0), \mathcal{N}^-_{\mathscr{B}}(0)\} = \mathcal{N}^-_{\mathscr{B}}(0)$.
On the other hand, we know from (BPVIFPII-1) that $\mathcal{M}^+_{\mathscr{B}}(0) \geq \mathcal{M}^+_{\mathscr{B}}(g_1)$, $\mathcal{M}^-_{\mathscr{B}}(0) \leq \mathcal{M}^-_{\mathscr{B}}(g_1)$, $\mathcal{N}^+_{\mathscr{B}}(0) \leq \mathcal{N}^+_{\mathscr{B}}(g_1)$, and $\mathcal{N}^-_{\mathscr{B}}(0) \geq \mathcal{N}^-_{\mathscr{B}}(g_1)$ for all $g_1 \in \mathcal{G}$. Thus, $\mathcal{M}^+_{\mathscr{B}}(g_1 \diamond i_1) = \mathcal{M}^+_{\mathscr{B}}(0)$, $\mathcal{M}^-_{\mathscr{B}}(g_1 \diamond i_1) = \mathcal{M}^-_{\mathscr{B}}(0)$, $\mathcal{N}^+_{\mathscr{B}}(g_1 \diamond i_1) = \mathcal{N}^+_{\mathscr{B}}(0)$, and $\mathcal{N}^-_{\mathscr{B}}(g_1 \diamond i_1) = \mathcal{N}^-_{\mathscr{B}}(0)$. This implies $g_1 \diamond i_1 \in \mathcal{J}(0)$. Obviously, $0 \in \mathcal{J}(0)$. Therefore, $\mathcal{J}(0)$ is a positive implicative ideal of $\mathcal{G}$.

Theorem 5. *In a positive implicative $\mathcal{BCK}$-A $\mathcal{G}$, every BPVIFI of $\mathcal{G}$ is a BPVIFPII of $\mathcal{G}$.*

Proof. Let $\mathcal{G}$ be a positive implicative $\mathcal{BCK}$-A, and $\mathscr{B} = (\mathcal{M}^+_{\mathscr{B}}, \mathcal{M}^-_{\mathscr{B}}, \mathcal{N}^+_{\mathscr{B}}, \mathcal{N}^-_{\mathscr{B}})$ be a BPVIFI of $\mathcal{G}$. If we replace g_1 with $g_1 \diamond i_1$ and $\hbar_1$ with $\hbar_1 \diamond i_1$ in BPVIFI-2, 3, 4, 5, then

$$\begin{aligned}
\mathcal{M}^+_{\mathscr{B}}(g_1 \diamond i_1) &\geq \min\{\mathcal{M}^+_{\mathscr{B}}((g_1 \diamond i_1) \diamond (\hbar_1 \diamond i_1)), \mathcal{M}^+_{\mathscr{B}}(\hbar_1 \diamond i_1)\} \\
&= \min\{\mathcal{M}^+_{\mathscr{B}}((g_1 \diamond \hbar_1) \diamond i_1), \mathcal{M}^+_{\mathscr{B}}(\hbar_1 \diamond i_1)\}, \\
\mathcal{M}^-_{\mathscr{B}}(g_1 \diamond i_1) &\leq \max\{\mathcal{M}^-_{\mathscr{B}}((g_1 \diamond i_1) \diamond (\hbar_1 \diamond i_1)), \mathcal{M}^-_{\mathscr{B}}(\hbar_1 \diamond i_1)\} \\
&= \max\{\mathcal{M}^-_{\mathscr{B}}((g_1 \diamond \hbar_1) \diamond i_1), \mathcal{M}^-_{\mathscr{B}}(\hbar_1 \diamond i_1)\}, \\
\mathcal{N}^+_{\mathscr{B}}(g_1 \diamond i_1) &\leq \max\{\mathcal{N}^+_{\mathscr{B}}((g_1 \diamond i_1) \diamond (\hbar_1 \diamond i_1)), \mathcal{N}^+_{\mathscr{B}}(\hbar_1 \diamond i_1)\} \\
&= \max\{\mathcal{N}^+_{\mathscr{B}}((g_1 \diamond \hbar_1) \diamond i_1), \mathcal{N}^+_{\mathscr{B}}(\hbar_1 \diamond i_1)\}, \\
\mathcal{N}^-_{\mathscr{B}}(g_1 \diamond i_1) &\geq \min\{\mathcal{N}^-_{\mathscr{B}}((g_1 \diamond i_1) \diamond (\hbar_1 \diamond i_1)), \mathcal{N}^-_{\mathscr{B}}(\hbar_1 \diamond i_1)\} \\
&= \min\{\mathcal{N}^-_{\mathscr{B}}((g_1 \diamond \hbar_1) \diamond i_1), \mathcal{N}^-_{\mathscr{B}}(\hbar_1 \diamond i_1)\},
\end{aligned}$$

for all $g_1, \hbar_1, i_1 \in \mathcal{G}$. Obviously, $\mathcal{M}^+_{\mathscr{B}}(0) \geq \mathcal{M}^+_{\mathscr{B}}(g_1)$, $\mathcal{M}^-_{\mathscr{B}}(0) \leq \mathcal{M}^-_{\mathscr{B}}(g_1)$, $\mathcal{N}^+_{\mathscr{B}}(0) \leq \mathcal{N}^+_{\mathscr{B}}(g_1)$, and $\mathcal{N}^-_{\mathscr{B}}(0) \geq \mathcal{N}^-_{\mathscr{B}}(g_1)$ for all $g_1 \in \mathcal{G}$. Therefore, $\mathscr{B} = (\mathcal{M}^+_{\mathscr{B}}, \mathcal{M}^-_{\mathscr{B}}, \mathcal{N}^+_{\mathscr{B}}, \mathcal{N}^-_{\mathscr{B}})$ is a BPVIFPII of $\mathcal{G}$.

Theorem 6. *A BPVIFS $\mathscr{B} = (\mathcal{M}^+_{\mathscr{B}}, \mathcal{M}^-_{\mathscr{B}}, \mathcal{N}^+_{\mathscr{B}}, \mathcal{N}^-_{\mathscr{B}})$ in $\mathcal{G}$ is a BPVIFPII of $\mathcal{G}$ if and only if it is a BPVIFI satisfying the following condition:*

$$\begin{pmatrix}
\mathcal{M}^+_{\mathscr{B}}(g_1 \diamond \hbar_1) \geq \mathcal{M}^+_{\mathscr{B}}((g_1 \diamond \hbar_1) \diamond \hbar_1), \\
\mathcal{M}^-_{\mathscr{B}}(g_1 \diamond \hbar_1) \leq \mathcal{M}^-_{\mathscr{B}}((g_1 \diamond \hbar_1) \diamond \hbar_1), \\
\mathcal{N}^+_{\mathscr{B}}(g_1 \diamond \hbar_1) \leq \mathcal{N}^+_{\mathscr{B}}((g_1 \diamond \hbar_1) \diamond \hbar_1), \\
\mathcal{N}^-_{\mathscr{B}}(g_1 \diamond \hbar_1) \geq \mathcal{N}^-_{\mathscr{B}}((g_1 \diamond \hbar_1) \diamond \hbar_1),
\end{pmatrix} \tag{13}$$

for all $g_1, \hbar_1 \in \mathcal{G}$.

Proof. Assume that $\mathscr{B} = (\mathcal{M}^+_{\mathscr{B}}, \mathcal{M}^-_{\mathscr{B}}, \mathcal{N}^+_{\mathscr{B}}, \mathcal{N}^-_{\mathscr{B}})$ is a BPVIFPII of $\mathcal{G}$. Write $\mathfrak{i}_1 = \hbar_1$ in Definition 21, we obtain

$$\begin{aligned}
\mathcal{M}^+_{\mathscr{B}}(\mathfrak{g}_1 \diamond \hbar_1) &\geq \min\{\mathcal{M}^+_{\mathscr{B}}((\mathfrak{g}_1 \diamond \hbar_1) \diamond \hbar_1), \mathcal{M}^+_{\mathscr{B}}(\hbar_1 \diamond \hbar_1)\} \\
&= \min\{\mathcal{M}^+_{\mathscr{B}}((\mathfrak{g}_1 \diamond \hbar_1) \diamond \hbar_1), \mathcal{M}^+_{\mathscr{B}}(0)\} \\
&= \mathcal{M}^+_{\mathscr{B}}((\mathfrak{g}_1 \diamond \hbar_1) \diamond \hbar_1), \\
\mathcal{M}^-_{\mathscr{B}}(\mathfrak{g}_1 \diamond \hbar_1) &\leq \max\{\mathcal{M}^-_{\mathscr{B}}((\mathfrak{g}_1 \diamond \hbar_1) \diamond \hbar_1), \mathcal{M}^-_{\mathscr{B}}(\hbar_1 \diamond \hbar_1)\} \\
&= \max\{\mathcal{M}^-_{\mathscr{B}}((\mathfrak{g}_1 \diamond \hbar_1) \diamond \hbar_1), \mathcal{M}^-_{\mathscr{B}}(0)\} \\
&= \mathcal{M}^-_{\mathscr{B}}((\mathfrak{g}_1 \diamond \hbar_1) \diamond \hbar_1), \\
\mathcal{N}^+_{\mathscr{B}}(\mathfrak{g}_1 \diamond \hbar_1) &\leq \max\{\mathcal{N}^+_{\mathscr{B}}((\mathfrak{g}_1 \diamond \hbar_1) \diamond \hbar_1), \mathcal{N}^+_{\mathscr{B}}(\hbar_1 \diamond \hbar_1)\} \\
&= \max\{\mathcal{N}^+_{\mathscr{B}}((\mathfrak{g}_1 \diamond \hbar_1) \diamond \hbar_1), \mathcal{N}^+_{\mathscr{B}}(0)\} \\
&= \mathcal{N}^+_{\mathscr{B}}((\mathfrak{g}_1 \diamond \hbar_1) \diamond \hbar_1), \\
\mathcal{N}^-_{\mathscr{B}}(\mathfrak{g}_1 \diamond \hbar_1) &\geq \min\{\mathcal{N}^-_{\mathscr{B}}((\mathfrak{g}_1 \diamond \hbar_1) \diamond \hbar_1), \mathcal{N}^-_{\mathscr{B}}(\hbar_1 \diamond \hbar_1)\} \\
&= \min\{\mathcal{N}^-_{\mathscr{B}}((\mathfrak{g}_1 \diamond \hbar_1) \diamond \hbar_1), \mathcal{N}^-_{\mathscr{B}}(0)\} \\
&= \mathcal{N}^-_{\mathscr{B}}((\mathfrak{g}_1 \diamond \hbar_1) \diamond \hbar_1),
\end{aligned}$$

for all $\mathfrak{g}_1, \hbar_1 \in \mathcal{G}$. Thus, condition (13) holds.

Conversely, let $\mathscr{B} = (\mathcal{M}^+_{\mathscr{B}}, \mathcal{M}^-_{\mathscr{B}}, \mathcal{N}^+_{\mathscr{B}}, \mathcal{N}^-_{\mathscr{B}})$ be a BPVIFI of $\mathcal{G}$ satisfying the condition (13). By using $\mathcal{BCK}$-1, (3), and (6), we obtain for all $\mathfrak{g}_1, \hbar_1, \mathfrak{i}_1 \in \mathcal{G}$,

$$\begin{aligned}
((\mathfrak{g}_1 \diamond \mathfrak{i}_1) \diamond (\mathfrak{g}_1 \diamond \hbar_1)) \leq (\hbar_1 \diamond \mathfrak{i}_1) &\Rightarrow ((\mathfrak{g}_1 \diamond \mathfrak{i}_1) \diamond (\hbar_1 \diamond \mathfrak{i}_1)) \leq (\mathfrak{g}_1 \diamond \hbar_1) \\
&\Rightarrow ((\mathfrak{g}_1 \diamond \mathfrak{i}_1) \diamond (\hbar_1 \diamond \mathfrak{i}_1)) \diamond \mathfrak{i}_1 \leq (\mathfrak{g}_1 \diamond \hbar_1) \diamond \mathfrak{i}_1 \\
&\Rightarrow ((\mathfrak{g}_1 \diamond \mathfrak{i}_1) \diamond \mathfrak{i}_1) \diamond (\hbar_1 \diamond \mathfrak{i}_1) \leq (\mathfrak{g}_1 \diamond \hbar_1) \diamond \mathfrak{i}_1.
\end{aligned}$$

It follows from Corollary 1 that

$$\left(\begin{array}{c}
\mathcal{M}^+_{\mathscr{B}}(((\mathfrak{g}_1 \diamond \mathfrak{i}_1) \diamond \mathfrak{i}_1) \diamond (\hbar_1 \diamond \mathfrak{i}_1)) \geq \mathcal{M}^+_{\mathscr{B}}((\mathfrak{g}_1 \diamond \hbar_1) \diamond \mathfrak{i}_1), \\
\mathcal{M}^-_{\mathscr{B}}(((\mathfrak{g}_1 \diamond \mathfrak{i}_1) \diamond \mathfrak{i}_1) \diamond (\hbar_1 \diamond \mathfrak{i}_1)) \leq \mathcal{M}^-_{\mathscr{B}}((\mathfrak{g}_1 \diamond \hbar_1) \diamond \mathfrak{i}_1), \\
\mathcal{N}^+_{\mathscr{B}}(((\mathfrak{g}_1 \diamond \mathfrak{i}_1) \diamond \mathfrak{i}_1) \diamond (\hbar_1 \diamond \mathfrak{i}_1)) \leq \mathcal{N}^+_{\mathscr{B}}((\mathfrak{g}_1 \diamond \hbar_1) \diamond \mathfrak{i}_1), \\
\mathcal{N}^-_{\mathscr{B}}(((\mathfrak{g}_1 \diamond \mathfrak{i}_1) \diamond \mathfrak{i}_1) \diamond (\hbar_1 \diamond \mathfrak{i}_1)) \geq \mathcal{N}^-_{\mathscr{B}}((\mathfrak{g}_1 \diamond \hbar_1) \diamond \mathfrak{i}_1),
\end{array}\right) \tag{14}$$

for all $\mathfrak{g}_1, \hbar_1, \mathfrak{i}_1 \in \mathcal{G}$. Now, by using (13), Definition 20, and (14), we obtain

$$\begin{aligned}
\mathcal{M}^+_{\mathscr{B}}(\mathfrak{g}_1 \diamond \mathfrak{i}_1) &\geq \mathcal{M}^+_{\mathscr{B}}((\mathfrak{g}_1 \diamond \mathfrak{i}_1) \diamond \mathfrak{i}_1) \\
&\geq \min\{\mathcal{M}^+_{\mathscr{B}}(((\mathfrak{g}_1 \diamond \mathfrak{i}_1) \diamond \mathfrak{i}_1) \diamond (\hbar_1 \diamond \mathfrak{i}_1)), \mathcal{M}^+_{\mathscr{B}}(\hbar_1 \diamond \mathfrak{i}_1)\} \\
&\geq \min\{\mathcal{M}^+_{\mathscr{B}}((\mathfrak{g}_1 \diamond \hbar_1) \diamond \mathfrak{i}_1), \mathcal{M}^+_{\mathscr{B}}(\hbar_1 \diamond \mathfrak{i}_1)\}, \\
\mathcal{M}^-_{\mathscr{B}}(\mathfrak{g}_1 \diamond \mathfrak{i}_1) &\leq \mathcal{M}^-_{\mathscr{B}}((\mathfrak{g}_1 \diamond \mathfrak{i}_1) \diamond \mathfrak{i}_1) \\
&\leq \max\{\mathcal{M}^-_{\mathscr{B}}(((\mathfrak{g}_1 \diamond \mathfrak{i}_1) \diamond \mathfrak{i}_1) \diamond (\hbar_1 \diamond \mathfrak{i}_1)), \mathcal{M}^-_{\mathscr{B}}(\hbar_1 \diamond \mathfrak{i}_1)\} \\
&\leq \max\{\mathcal{M}^-_{\mathscr{B}}((\mathfrak{g}_1 \diamond \hbar_1) \diamond \mathfrak{i}_1), \mathcal{M}^-_{\mathscr{B}}(\hbar_1 \diamond \mathfrak{i}_1)\}, \\
\mathcal{N}^+_{\mathscr{B}}(\mathfrak{g}_1 \diamond \mathfrak{i}_1) &\leq \mathcal{N}^+_{\mathscr{B}}((\mathfrak{g}_1 \diamond \mathfrak{i}_1) \diamond \mathfrak{i}_1) \\
&\leq \max\{\mathcal{N}^+_{\mathscr{B}}(((\mathfrak{g}_1 \diamond \mathfrak{i}_1) \diamond \mathfrak{i}_1) \diamond (\hbar_1 \diamond \mathfrak{i}_1)), \mathcal{N}^+_{\mathscr{B}}(\hbar_1 \diamond \mathfrak{i}_1)\}
\end{aligned}$$

$$
\begin{aligned}
&\le \max\{\mathcal{N}^+_{\mathscr{B}}((g_1 \diamond \hbar_1) \diamond i_1), \mathcal{N}^+_{\mathscr{B}}(\hbar_1 \diamond i_1)\},\\
\mathcal{N}^-_{\mathscr{B}}(g_1 \diamond i_1) &\ge \mathcal{N}^-_{\mathscr{B}}((g_1 \diamond i_1) \diamond i_1)\\
&\ge \min\{\mathcal{N}^-_{\mathscr{B}}(((g_1 \diamond i_1) \diamond i_1) \diamond (\hbar_1 \diamond i_1)), \mathcal{N}^-_{\mathscr{B}}(\hbar_1 \diamond i_1)\}\\
&\ge \min\{\mathcal{N}^-_{\mathscr{B}}((g_1 \diamond \hbar_1) \diamond i_1), \mathcal{N}^-_{\mathscr{B}}(\hbar_1 \diamond i_1)\}.
\end{aligned}
$$

Therefore, $\mathscr{B} = (\mathcal{M}^+_{\mathscr{B}}, \mathcal{M}^-_{\mathscr{B}}, \mathcal{N}^+_{\mathscr{B}}, \mathcal{N}^-_{\mathscr{B}})$ is a BPVIFPII of $\mathcal{G}$.

Theorem 7. *A BPVIFS $\mathscr{B} = (\mathcal{M}^+_{\mathscr{B}}, \mathcal{M}^-_{\mathscr{B}}, \mathcal{N}^+_{\mathscr{B}}, \mathcal{N}^-_{\mathscr{B}})$ in $\mathcal{G}$ is a BPVIFPII of $\mathcal{G}$ if and only if it satisfies (BPVIFI-1) and the following condition:*

$$
\begin{pmatrix}
\mathcal{M}^+_{\mathscr{B}}(g_1 \diamond \hbar_1) \ge \min\{\mathcal{M}^+_{\mathscr{B}}(((g_1 \diamond \hbar_1) \diamond \hbar_1) \diamond i_1), \mathcal{M}^+_{\mathscr{B}}(i_1)\},\\
\mathcal{M}^-_{\mathscr{B}}(g_1 \diamond \hbar_1) \le \max\{\mathcal{M}^-_{\mathscr{B}}(((g_1 \diamond \hbar_1) \diamond \hbar_1) \diamond i_1), \mathcal{M}^-_{\mathscr{B}}(i_1)\},\\
\mathcal{N}^+_{\mathscr{B}}(g_1 \diamond \hbar_1) \le \max\{\mathcal{N}^+_{\mathscr{B}}(((g_1 \diamond \hbar_1) \diamond \hbar_1) \diamond i_1), \mathcal{N}^+_{\mathscr{B}}(i_1)\},\\
\mathcal{N}^-_{\mathscr{B}}(g_1 \diamond \hbar_1) \ge \min\{\mathcal{N}^-_{\mathscr{B}}(((g_1 \diamond \hbar_1) \diamond \hbar_1) \diamond i_1), \mathcal{N}^-_{\mathscr{B}}(i_1)\},
\end{pmatrix} \tag{15}
$$

for all $g_1, \hbar_1, i_1 \in \mathcal{G}$.

Proof. Assume that $\mathscr{B} = (\mathcal{M}^+_{\mathscr{B}}, \mathcal{M}^-_{\mathscr{B}}, \mathcal{N}^+_{\mathscr{B}}, \mathcal{N}^-_{\mathscr{B}})$ is a BPVIFPII of $\mathcal{G}$. Then, $\mathscr{B} = (\mathcal{M}^+_{\mathscr{B}}, \mathcal{M}^-_{\mathscr{B}}, \mathcal{N}^+_{\mathscr{B}}, \mathcal{N}^-_{\mathscr{B}})$ is a BPVIFI of $\mathcal{G}$ by Theorem 3, and thus, it satisfies (BPVIFI-1). Now,

$$
\begin{aligned}
\mathcal{M}^+_{\mathscr{B}}(g_1 \diamond \hbar_1) &\ge \min\{\mathcal{M}^+_{\mathscr{B}}((g_1 \diamond \hbar_1) \diamond i_1), \mathcal{M}^+_{\mathscr{B}}(i_1)\}\\
&= \min\{\mathcal{M}^+_{\mathscr{B}}(((g_1 \diamond \hbar_1) \diamond i_1) \diamond (\hbar_1 \diamond \hbar_1)), \mathcal{M}^+_{\mathscr{B}}(i_1)\}\\
&= \min\{\mathcal{M}^+_{\mathscr{B}}(((g_1 \diamond i_1) \diamond \hbar_1) \diamond (\hbar_1 \diamond \hbar_1)), \mathcal{M}^+_{\mathscr{B}}(i_1)\}\\
&\ge \min\{\mathcal{M}^+_{\mathscr{B}}((g_1 \diamond i_1) \diamond \hbar_1), \mathcal{M}^+_{\mathscr{B}}(i_1)\}\\
&\ge \min\{\mathcal{M}^+_{\mathscr{B}}(((g_1 \diamond i_1) \diamond \hbar_1) \diamond \hbar_1), \mathcal{M}^+_{\mathscr{B}}(i_1)\}\\
&= \min\{\mathcal{M}^+_{\mathscr{B}}(((g_1 \diamond \hbar_1) \diamond \hbar_1) \diamond i_1), \mathcal{M}^+_{\mathscr{B}}(i_1)\},
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{M}^-_{\mathscr{B}}(g_1 \diamond \hbar_1) &\le \max\{\mathcal{M}^-_{\mathscr{B}}((g_1 \diamond \hbar_1) \diamond i_1), \mathcal{M}^-_{\mathscr{B}}(i_1)\}\\
&= \max\{\mathcal{M}^-_{\mathscr{B}}(((g_1 \diamond \hbar_1) \diamond i_1) \diamond (\hbar_1 \diamond \hbar_1)), \mathcal{M}^-_{\mathscr{B}}(i_1)\}\\
&= \max\{\mathcal{M}^-_{\mathscr{B}}(((g_1 \diamond i_1) \diamond \hbar_1) \diamond (\hbar_1 \diamond \hbar_1)), \mathcal{M}^-_{\mathscr{B}}(i_1)\}\\
&\le \max\{\mathcal{M}^-_{\mathscr{B}}((g_1 \diamond i_1) \diamond \hbar_1), \mathcal{M}^-_{\mathscr{B}}(i_1)\}\\
&\le \max\{\mathcal{M}^-_{\mathscr{B}}(((g_1 \diamond i_1) \diamond \hbar_1) \diamond \hbar_1), \mathcal{M}^-_{\mathscr{B}}(i_1)\}\\
&= \max\{\mathcal{M}^-_{\mathscr{B}}(((g_1 \diamond \hbar_1) \diamond \hbar_1) \diamond i_1), \mathcal{M}^-_{\mathscr{B}}(i_1)\},
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{N}^+_{\mathscr{B}}(g_1 \diamond \hbar_1) &\le \max\{\mathcal{N}^+_{\mathscr{B}}((g_1 \diamond \hbar_1) \diamond i_1), \mathcal{N}^+_{\mathscr{B}}(i_1)\}\\
&= \max\{\mathcal{N}^+_{\mathscr{B}}(((g_1 \diamond \hbar_1) \diamond i_1) \diamond (\hbar_1 \diamond \hbar_1)), \mathcal{N}^+_{\mathscr{B}}(i_1)\}\\
&= \max\{\mathcal{N}^+_{\mathscr{B}}(((g_1 \diamond i_1) \diamond \hbar_1) \diamond (\hbar_1 \diamond \hbar_1)), \mathcal{N}^+_{\mathscr{B}}(i_1)\}\\
&\le \max\{\mathcal{N}^+_{\mathscr{B}}((g_1 \diamond i_1) \diamond \hbar_1), \mathcal{N}^+_{\mathscr{B}}(i_1)\}\\
&\le \max\{\mathcal{N}^+_{\mathscr{B}}(((g_1 \diamond i_1) \diamond \hbar_1) \diamond \hbar_1), \mathcal{N}^+_{\mathscr{B}}(i_1)\}
\end{aligned}
$$

$$= \max\{\mathcal{N}^+_{\mathscr{B}}(((\mathrm{g}_1 \diamond \hbar_1) \diamond \hbar_1) \diamond \mathrm{i}_1), \mathcal{N}^+_{\mathscr{B}}(\mathrm{i}_1)\},$$

$$\begin{aligned}\mathcal{N}^-_{\mathscr{B}}(\mathrm{g}_1 \diamond \hbar_1) &\geq \min\{\mathcal{N}^-_{\mathscr{B}}((\mathrm{g}_1 \diamond \hbar_1) \diamond \mathrm{i}_1), \mathcal{N}^-_{\mathscr{B}}(\mathrm{i}_1)\}\\ &= \min\{\mathcal{N}^-_{\mathscr{B}}(((\mathrm{g}_1 \diamond \hbar_1) \diamond \mathrm{i}_1) \diamond (\hbar_1 \diamond \hbar_1)), \mathcal{N}^-_{\mathscr{B}}(\mathrm{i}_1)\}\\ &= \min\{\mathcal{N}^-_{\mathscr{B}}(((\mathrm{g}_1 \diamond \mathrm{i}_1) \diamond \hbar_1) \diamond (\hbar_1 \diamond \hbar_1)), \mathcal{N}^-_{\mathscr{B}}(\mathrm{i}_1)\}\\ &\geq \min\{\mathcal{N}^-_{\mathscr{B}}((\mathrm{g}_1 \diamond \mathrm{i}_1) \diamond \hbar_1), \mathcal{N}^-_{\mathscr{B}}(\mathrm{i}_1)\}\\ &\geq \min\{\mathcal{N}^-_{\mathscr{B}}(((\mathrm{g}_1 \diamond \mathrm{i}_1) \diamond \hbar_1) \diamond \hbar_1), \mathcal{N}^-_{\mathscr{B}}(\mathrm{i}_1)\}\\ &= \min\{\mathcal{N}^-_{\mathscr{B}}(((\mathrm{g}_1 \diamond \hbar_1) \diamond \hbar_1) \diamond \mathrm{i}_1), \mathcal{N}^-_{\mathscr{B}}(\mathrm{i}_1)\},\end{aligned}$$

for all $\mathrm{g}_1, \hbar_1, \mathrm{i}_1 \in \mathcal{G}$. Hence, (15) holds.

Conversely, let $\mathscr{B} = (\mathcal{M}^+_{\mathscr{B}}, \mathcal{M}^-_{\mathscr{B}}, \mathcal{N}^+_{\mathscr{B}}, \mathcal{N}^-_{\mathscr{B}})$ be a BPVIFS in $\mathcal{G}$ which satisfies (BPVIFI-1) and (15). Then,

$$\begin{aligned}\mathcal{M}^+_{\mathscr{B}}(\mathrm{g}_1) &= \mathcal{M}^+_{\mathscr{B}}(\mathrm{g}_1 \diamond 0)\\ &\geq \min\{\mathcal{M}^+_{\mathscr{B}}(((\mathrm{g}_1 \diamond 0) \diamond 0) \diamond \mathrm{i}_1), \mathcal{M}^+_{\mathscr{B}}(\mathrm{i}_1)\}\\ &= \min\{\mathcal{M}^+_{\mathscr{B}}(\mathrm{g}_1 \diamond \mathrm{i}_1), \mathcal{M}^+_{\mathscr{B}}(\mathrm{i}_1)\},\\ \mathcal{M}^-_{\mathscr{B}}(\mathrm{g}_1) &= \mathcal{M}^-_{\mathscr{B}}(\mathrm{g}_1 \diamond 0)\\ &\leq \max\{\mathcal{M}^-_{\mathscr{B}}(((\mathrm{g}_1 \diamond 0) \diamond 0) \diamond \mathrm{i}_1), \mathcal{M}^-_{\mathscr{B}}(\mathrm{i}_1)\}\\ &= \max\{\mathcal{M}^-_{\mathscr{B}}(\mathrm{g}_1 \diamond \mathrm{i}_1), \mathcal{M}^-_{\mathscr{B}}(\mathrm{i}_1)\},\\ \mathcal{N}^+_{\mathscr{B}}(\mathrm{g}_1) &= \mathcal{N}^+_{\mathscr{B}}(\mathrm{g}_1 \diamond 0)\\ &\leq \max\{\mathcal{N}^+_{\mathscr{B}}(((\mathrm{g}_1 \diamond 0) \diamond 0) \diamond \mathrm{i}_1), \mathcal{N}^+_{\mathscr{B}}(\mathrm{i}_1)\}\\ &= \max\{\mathcal{N}^+_{\mathscr{B}}(\mathrm{g}_1 \diamond \mathrm{i}_1), \mathcal{N}^+_{\mathscr{B}}(\mathrm{i}_1)\},\\ \mathcal{N}^-_{\mathscr{B}}(\mathrm{g}_1) &= \mathcal{N}^-_{\mathscr{B}}(\mathrm{g}_1 \diamond 0)\\ &\geq \min\{\mathcal{N}^-_{\mathscr{B}}(((\mathrm{g}_1 \diamond 0) \diamond 0) \diamond \mathrm{i}_1), \mathcal{N}^-_{\mathscr{B}}(\mathrm{i}_1)\}\\ &= \min\{\mathcal{N}^-_{\mathscr{B}}(\mathrm{g}_1 \diamond \mathrm{i}_1), \mathcal{N}^-_{\mathscr{B}}(\mathrm{i}_1)\},\end{aligned}$$

for all $\mathrm{g}_1, \mathrm{i}_1 \in \mathcal{G}$. Thus, $\mathscr{B} = (\mathcal{M}^+_{\mathscr{B}}, \mathcal{M}^-_{\mathscr{B}}, \mathcal{N}^+_{\mathscr{B}}, \mathcal{N}^-_{\mathscr{B}})$ is a BPVIFI of $\mathcal{G}$. Taking $\mathrm{i}_1 = 0$ in (15), we obtain

$$\begin{aligned}\mathcal{M}^+_{\mathscr{B}}(\mathrm{g}_1 \diamond \hbar_1) &\geq \min\{\mathcal{M}^+_{\mathscr{B}}(((\mathrm{g}_1 \diamond \hbar_1) \diamond \hbar_1) \diamond 0), \mathcal{M}^+_{\mathscr{B}}(0)\}\\ &= \mathcal{M}^+_{\mathscr{B}}((\mathrm{g}_1 \diamond \hbar_1) \diamond \hbar_1),\\ \mathcal{M}^-_{\mathscr{B}}(\mathrm{g}_1 \diamond \hbar_1) &\leq \max\{\mathcal{M}^-_{\mathscr{B}}(((\mathrm{g}_1 \diamond \hbar_1) \diamond \hbar_1) \diamond 0), \mathcal{M}^-_{\mathscr{B}}(0)\}\\ &= \mathcal{M}^-_{\mathscr{B}}((\mathrm{g}_1 \diamond \hbar_1) \diamond \hbar_1),\\ \mathcal{N}^+_{\mathscr{B}}(\mathrm{g}_1 \diamond \hbar_1) &\leq \max\{\mathcal{N}^+_{\mathscr{B}}(((\mathrm{g}_1 \diamond \hbar_1) \diamond \hbar_1) \diamond 0), \mathcal{N}^+_{\mathscr{B}}(0)\}\\ &= \mathcal{N}^+_{\mathscr{B}}((\mathrm{g}_1 \diamond \hbar_1) \diamond \hbar_1),\\ \mathcal{N}^-_{\mathscr{B}}(\mathrm{g}_1 \diamond \hbar_1) &\geq \min\{\mathcal{N}^-_{\mathscr{B}}(((\mathrm{g}_1 \diamond \hbar_1) \diamond \hbar_1) \diamond 0), \mathcal{N}^-_{\mathscr{B}}(0)\}\\ &= \mathcal{N}^-_{\mathscr{B}}((\mathrm{g}_1 \diamond \hbar_1) \diamond \hbar_1),\end{aligned}$$

for all $\mathrm{g}_1, \hbar_1 \in \mathcal{G}$. It follows from Theorem 6 that $\mathscr{B} = (\mathcal{M}^+_{\mathscr{B}}, \mathcal{M}^-_{\mathscr{B}}, \mathcal{N}^+_{\mathscr{B}}, \mathcal{N}^-_{\mathscr{B}})$ is a BPVIFPII of $\mathcal{G}$.

Theorem 8. *Let* $\mathscr{B} = (\mathcal{M}^+_{\mathscr{B}}, \mathcal{M}^-_{\mathscr{B}}, \mathcal{N}^+_{\mathscr{B}}, \mathcal{N}^-_{\mathscr{B}})$ *be a BPVIFI of* $\mathcal{G}$*. Then* $\mathscr{B} = (\mathcal{M}^+_{\mathscr{B}}, \mathcal{M}^-_{\mathscr{B}}, \mathcal{N}^+_{\mathscr{B}}, \mathcal{N}^-_{\mathscr{B}})$ *is a BPVIFPII of* $\mathcal{G}$ *if and only if it satisfies the condition*

$$\begin{pmatrix} \mathcal{M}^+_{\mathscr{B}}((g_1 \diamond i_1) \diamond (\hbar_1 \diamond i_1)) \geq \mathcal{M}^+_{\mathscr{B}}((g_1 \diamond \hbar_1) \diamond i_1), \\ \mathcal{M}^-_{\mathscr{B}}((g_1 \diamond i_1) \diamond (\hbar_1 \diamond i_1)) \leq \mathcal{M}^-_{\mathscr{B}}((g_1 \diamond \hbar_1) \diamond i_1), \\ \mathcal{N}^+_{\mathscr{B}}((g_1 \diamond i_1) \diamond (\hbar_1 \diamond i_1)) \leq \mathcal{N}^+_{\mathscr{B}}((g_1 \diamond \hbar_1) \diamond i_1), \\ \mathcal{N}^-_{\mathscr{B}}((g_1 \diamond i_1) \diamond (\hbar_1 \diamond i_1)) \geq \mathcal{N}^-_{\mathscr{B}}((g_1 \diamond \hbar_1) \diamond i_1), \end{pmatrix} \tag{16}$$

for all $g_1, \hbar_1, i_1 \in \mathcal{G}$*.*

Proof. Assume that $\mathscr{B} = (\mathcal{M}^+_{\mathscr{B}}, \mathcal{M}^-_{\mathscr{B}}, \mathcal{N}^+_{\mathscr{B}}, \mathcal{N}^-_{\mathscr{B}})$ is a BPVIFPII of $\mathcal{G}$. Then, by Theorem 3, $\mathscr{B} = (\mathcal{M}^+_{\mathscr{B}}, \mathcal{M}^-_{\mathscr{B}}, \mathcal{N}^+_{\mathscr{B}}, \mathcal{N}^-_{\mathscr{B}})$ is a BPVIFI of $\mathcal{G}$ and satisfies (13). Since $((g_1 \diamond (\hbar_1 \diamond i_1)) \diamond i_1) \diamond i_1 = ((g_1 \diamond i_1) \diamond (\hbar_1 \diamond i_1)) \diamond i_1 \leq (g_1 \diamond \hbar_1) \diamond i_1$ for all $g_1, \hbar_1, i_1 \in \mathcal{G}$, it follows from Corollary 1 that

$$\begin{pmatrix} \mathcal{M}^+_{\mathscr{B}}(((g_1 \diamond (\hbar_1 \diamond i_1)) \diamond i_1) \diamond i_1) \geq \mathcal{M}^+_{\mathscr{B}}((g_1 \diamond \hbar_1) \diamond i_1), \\ \mathcal{M}^-_{\mathscr{B}}(((g_1 \diamond (\hbar_1 \diamond i_1)) \diamond i_1) \diamond i_1) \leq \mathcal{M}^-_{\mathscr{B}}((g_1 \diamond \hbar_1) \diamond i_1), \\ \mathcal{N}^+_{\mathscr{B}}(((g_1 \diamond (\hbar_1 \diamond i_1)) \diamond i_1) \diamond i_1) \leq \mathcal{N}^+_{\mathscr{B}}((g_1 \diamond \hbar_1) \diamond i_1), \\ \mathcal{N}^-_{\mathscr{B}}(((g_1 \diamond (\hbar_1 \diamond i_1)) \diamond i_1) \diamond i_1) \geq \mathcal{N}^-_{\mathscr{B}}((g_1 \diamond \hbar_1) \diamond i_1), \end{pmatrix} \tag{17}$$

for all $g_1, \hbar_1, i_1 \in \mathcal{G}$. Now, by using (3), (13), and (17), we obtain

$$\begin{aligned} \mathcal{M}^+_{\mathscr{B}}((g_1 \diamond i_1) \diamond (\hbar_1 \diamond i_1)) &= \mathcal{M}^+_{\mathscr{B}}((g_1 \diamond (\hbar_1 \diamond i_1)) \diamond i_1) \\ &\geq \mathcal{M}^+_{\mathscr{B}}(((g_1 \diamond (\hbar_1 \diamond i_1)) \diamond i_1) \diamond i_1) \\ &\geq \mathcal{M}^+_{\mathscr{B}}((g_1 \diamond \hbar_1) \diamond i_1), \\ \mathcal{M}^-_{\mathscr{B}}((g_1 \diamond i_1) \diamond (\hbar_1 \diamond i_1)) &= \mathcal{M}^-_{\mathscr{B}}((g_1 \diamond (\hbar_1 \diamond i_1)) \diamond i_1) \\ &\leq \mathcal{M}^-_{\mathscr{B}}(((g_1 \diamond (\hbar_1 \diamond i_1)) \diamond i_1) \diamond i_1) \\ &\leq \mathcal{M}^-_{\mathscr{B}}((g_1 \diamond \hbar_1) \diamond i_1), \\ \mathcal{N}^+_{\mathscr{B}}((g_1 \diamond i_1) \diamond (\hbar_1 \diamond i_1)) &= \mathcal{N}^+_{\mathscr{B}}((g_1 \diamond (\hbar_1 \diamond i_1)) \diamond i_1) \\ &\leq \mathcal{N}^+_{\mathscr{B}}(((g_1 \diamond (\hbar_1 \diamond i_1)) \diamond i_1) \diamond i_1) \\ &\leq \mathcal{N}^+_{\mathscr{B}}((g_1 \diamond \hbar_1) \diamond i_1), \\ \mathcal{N}^-_{\mathscr{B}}((g_1 \diamond i_1) \diamond (\hbar_1 \diamond i_1)) &= \mathcal{N}^-_{\mathscr{B}}((g_1 \diamond (\hbar_1 \diamond i_1)) \diamond i_1) \\ &\geq \mathcal{N}^-_{\mathscr{B}}(((g_1 \diamond (\hbar_1 \diamond i_1)) \diamond i_1) \diamond i_1) \\ &\geq \mathcal{N}^-_{\mathscr{B}}((g_1 \diamond \hbar_1) \diamond i_1), \end{aligned}$$

for all $g_1, \hbar_1, i_1 \in \mathcal{G}$. Hence, (16) is valid.

Conversely, let $\mathscr{B} = (\mathcal{M}^+_{\mathscr{B}}, \mathcal{M}^-_{\mathscr{B}}, \mathcal{N}^+_{\mathscr{B}}, \mathcal{N}^-_{\mathscr{B}})$ be a BPVIFI of $\mathcal{G}$ which satisfies (16). Write $i_1 = \hbar_1$ in (16), we obtain

$$\begin{aligned} \mathcal{M}^+_{\mathscr{B}}((g_1 \diamond \hbar_1) \diamond (\hbar_1 \diamond \hbar_1)) &= \mathcal{M}^+_{\mathscr{B}}((g_1 \diamond \hbar_1) \diamond 0) \\ &= \mathcal{M}^+_{\mathscr{B}}(g_1 \diamond \hbar_1) \\ &\geq \mathcal{M}^+_{\mathscr{B}}((g_1 \diamond \hbar_1) \diamond \hbar_1), \end{aligned}$$

$$\begin{aligned}
\mathcal{M}^-_{\mathscr{B}}((\mathrm{g}_1 \diamond \hbar_1) \diamond (\hbar_1 \diamond \hbar_1)) &= \mathcal{M}^-_{\mathscr{B}}((\mathrm{g}_1 \diamond \hbar_1) \diamond 0) \\
&= \mathcal{M}^-_{\mathscr{B}}(\mathrm{g}_1 \diamond \hbar_1) \\
&\leq \mathcal{M}^-_{\mathscr{B}}((\mathrm{g}_1 \diamond \hbar_1) \diamond \hbar_1), \\
\mathcal{N}^+_{\mathscr{B}}((\mathrm{g}_1 \diamond \hbar_1) \diamond (\hbar_1 \diamond \hbar_1)) &= \mathcal{N}^+_{\mathscr{B}}((\mathrm{g}_1 \diamond \hbar_1) \diamond 0) \\
&= \mathcal{N}^+_{\mathscr{B}}(\mathrm{g}_1 \diamond \hbar_1) \\
&\leq \mathcal{N}^+_{\mathscr{B}}((\mathrm{g}_1 \diamond \hbar_1) \diamond \hbar_1), \\
\mathcal{N}^-_{\mathscr{B}}((\mathrm{g}_1 \diamond \hbar_1) \diamond (\hbar_1 \diamond \hbar_1)) &= \mathcal{N}^-_{\mathscr{B}}((\mathrm{g}_1 \diamond \hbar_1) \diamond 0) \\
&= \mathcal{N}^-_{\mathscr{B}}(\mathrm{g}_1 \diamond \hbar_1) \\
&\geq \mathcal{N}^-_{\mathscr{B}}((\mathrm{g}_1 \diamond \hbar_1) \diamond \hbar_1).
\end{aligned}$$

Therefore, $\mathscr{B} = (\mathcal{M}^+_{\mathscr{B}}, \mathcal{M}^-_{\mathscr{B}}, \mathcal{N}^+_{\mathscr{B}}, \mathcal{N}^-_{\mathscr{B}})$ is a BPVIFPII of $\mathcal{G}$ by Theorem 6.

Theorem 9. *Let $\mathscr{B} = (\mathcal{M}^+_{\mathscr{B}}, \mathcal{M}^-_{\mathscr{B}}, \mathcal{N}^+_{\mathscr{B}}, \mathcal{N}^-_{\mathscr{B}})$ be a BPVIFS in $\mathcal{G}$. Then $\mathscr{B} = (\mathcal{M}^+_{\mathscr{B}}, \mathcal{M}^-_{\mathscr{B}}, \mathcal{N}^+_{\mathscr{B}}, \mathcal{N}^-_{\mathscr{B}})$ is a BPVIFPII of $\mathcal{G}$ if and only if it satisfies the condition*

$$(((g_1 \diamond \hbar_1) \diamond \hbar_1) \diamond \mathfrak{u}) \leq \mathfrak{v} \Rightarrow \begin{pmatrix} \mathcal{M}^+_{\mathscr{B}}(g_1 \diamond \hbar_1) \geq \min\{\mathcal{M}^+_{\mathscr{B}}(\mathfrak{u}), \mathcal{M}^+_{\mathscr{B}}(\mathfrak{v})\}, \\ \mathcal{M}^-_{\mathscr{B}}(g_1 \diamond \hbar_1) \leq \max\{\mathcal{M}^-_{\mathscr{B}}(\mathfrak{u}), \mathcal{M}^-_{\mathscr{B}}(\mathfrak{v})\}, \\ \mathcal{N}^+_{\mathscr{B}}(g_1 \diamond \hbar_1) \leq \max\{\mathcal{N}^+_{\mathscr{B}}(\mathfrak{u}), \mathcal{N}^+_{\mathscr{B}}(\mathfrak{v})\}, \\ \mathcal{N}^-_{\mathscr{B}}(g_1 \diamond \hbar_1) \geq \min\{\mathcal{N}^-_{\mathscr{B}}(\mathfrak{u}), \mathcal{N}^-_{\mathscr{B}}(\mathfrak{v})\}, \end{pmatrix} \tag{18}$$

for all $g_1, \hbar_1, \mathfrak{u}, \mathfrak{v} \in \mathcal{G}$.

Proof. Assume that $\mathscr{B} = (\mathcal{M}^+_{\mathscr{B}}, \mathcal{M}^-_{\mathscr{B}}, \mathcal{N}^+_{\mathscr{B}}, \mathcal{N}^-_{\mathscr{B}})$ is a BPVIFPII of $\mathcal{G}$. Then, by Theorem 3, $\mathscr{B} = (\mathcal{M}^+_{\mathscr{B}}, \mathcal{M}^-_{\mathscr{B}}, \mathcal{N}^+_{\mathscr{B}}, \mathcal{N}^-_{\mathscr{B}})$ is a BPVIFI of $\mathcal{G}$. Let $\mathrm{g}_1, \hbar_1, \mathfrak{u}, \mathfrak{v} \in \mathcal{G}$ be such that $(((\mathrm{g}_1 \diamond \hbar_1) \diamond \hbar_1) \diamond \mathfrak{u}) \leq \mathfrak{v}$. Now, by applying (13) and Theorem 2, we obtain

$$\begin{aligned}
\mathcal{M}^+_{\mathscr{B}}(\mathrm{g}_1 \diamond \hbar_1) &\geq \mathcal{M}^+_{\mathscr{B}}((\mathrm{g}_1 \diamond \hbar_1) \diamond \hbar_1) \\
&\geq \min\{\mathcal{M}^+_{\mathscr{B}}(\mathfrak{u}), \mathcal{M}^+_{\mathscr{B}}(\mathfrak{v})\}, \\
\mathcal{M}^-_{\mathscr{B}}(\mathrm{g}_1 \diamond \hbar_1) &\leq \mathcal{M}^-_{\mathscr{B}}((\mathrm{g}_1 \diamond \hbar_1) \diamond \hbar_1) \\
&\leq \max\{\mathcal{M}^-_{\mathscr{B}}(\mathfrak{u}), \mathcal{M}^-_{\mathscr{B}}(\mathfrak{v})\}, \\
\mathcal{N}^+_{\mathscr{B}}(\mathrm{g}_1 \diamond \hbar_1) &\leq \mathcal{N}^+_{\mathscr{B}}((\mathrm{g}_1 \diamond \hbar_1) \diamond \hbar_1) \\
&\leq \max\{\mathcal{N}^+_{\mathscr{B}}(\mathfrak{u}), \mathcal{N}^+_{\mathscr{B}}(\mathfrak{v})\}, \\
\mathcal{N}^-_{\mathscr{B}}(\mathrm{g}_1 \diamond \hbar_1) &\geq \mathcal{N}^-_{\mathscr{B}}((\mathrm{g}_1 \diamond \hbar_1) \diamond \hbar_1) \\
&\geq \min\{\mathcal{N}^-_{\mathscr{B}}(\mathfrak{u}), \mathcal{N}^-_{\mathscr{B}}(\mathfrak{v})\}.
\end{aligned}$$

Therefore, (18) is valid.

Conversely, let $\mathscr{B} = (\mathcal{M}^+_{\mathscr{B}}, \mathcal{M}^-_{\mathscr{B}}, \mathcal{N}^+_{\mathscr{B}}, \mathcal{N}^-_{\mathscr{B}})$ be a BPVIFS in $\mathcal{G}$ that satisfies (18). Let $\mathrm{g}_1, \mathfrak{u}, \mathfrak{v} \in \mathcal{G}$ be such that $\mathrm{g}_1 \diamond \mathfrak{u} \leq \mathfrak{v}$. Then, $(((\mathrm{g}_1 \diamond 0) \diamond 0) \diamond \mathfrak{u}) \diamond \mathfrak{v} = 0$, and so

$$\begin{aligned}
\mathcal{M}^+_{\mathscr{B}}(\mathrm{g}_1) = \mathcal{M}^+_{\mathscr{B}}(\mathrm{g}_1 \diamond 0) &\geq \min\{\mathcal{M}^+_{\mathscr{B}}(\mathfrak{u}), \mathcal{M}^+_{\mathscr{B}}(\mathfrak{v})\}, \\
\mathcal{M}^-_{\mathscr{B}}(\mathrm{g}_1) = \mathcal{M}^-_{\mathscr{B}}(\mathrm{g}_1 \diamond 0) &\leq \max\{\mathcal{M}^-_{\mathscr{B}}(\mathfrak{u}), \mathcal{M}^-_{\mathscr{B}}(\mathfrak{v})\}, \\
\mathcal{N}^+_{\mathscr{B}}(\mathrm{g}_1) = \mathcal{N}^+_{\mathscr{B}}(\mathrm{g}_1 \diamond 0) &\leq \max\{\mathcal{N}^+_{\mathscr{B}}(\mathfrak{u}), \mathcal{N}^+_{\mathscr{B}}(\mathfrak{v})\}, \\
\mathcal{N}^-_{\mathscr{B}}(\mathrm{g}_1) = \mathcal{N}^-_{\mathscr{B}}(\mathrm{g}_1 \diamond 0) &\geq \min\{\mathcal{N}^-_{\mathscr{B}}(\mathfrak{u}), \mathcal{N}^-_{\mathscr{B}}(\mathfrak{v})\}.
\end{aligned}$$

Thus, by Theorem 2, $\mathscr{B} = (\mathcal{M}^+_{\mathscr{B}}, \mathcal{M}^-_{\mathscr{B}}, \mathcal{N}^+_{\mathscr{B}}, \mathcal{N}^-_{\mathscr{B}})$ is a BPVIFI of $\mathcal{G}$. Since $(((\mathrm{g}_1 \diamond \hbar_1) \diamond \hbar_1) \diamond ((\mathrm{g}_1 \diamond \hbar_1) \diamond \hbar_1)) \diamond 0 = 0$ for all $\mathrm{g}_1, \hbar_1 \in \mathcal{G}$, it follows from (18) that

$$\begin{aligned}
\mathcal{M}^+_{\mathscr{B}}(\mathrm{g}_1 \diamond \hbar_1) &\geq \min\{\mathcal{M}^+_{\mathscr{B}}((\mathrm{g}_1 \diamond \hbar_1) \diamond \hbar_1), \mathcal{M}^+_{\mathscr{B}}(0)\} \\
&= \mathcal{M}^+_{\mathscr{B}}((\mathrm{g}_1 \diamond \hbar_1) \diamond \hbar_1), \\
\mathcal{M}^-_{\mathscr{B}}(\mathrm{g}_1 \diamond \hbar_1) &\leq \max\{\mathcal{M}^-_{\mathscr{B}}((\mathrm{g}_1 \diamond \hbar_1) \diamond \hbar_1), \mathcal{M}^-_{\mathscr{B}}(0)\} \\
&= \mathcal{M}^-_{\mathscr{B}}((\mathrm{g}_1 \diamond \hbar_1) \diamond \hbar_1), \\
\mathcal{N}^+_{\mathscr{B}}(\mathrm{g}_1 \diamond \hbar_1) &\leq \max\{\mathcal{N}^+_{\mathscr{B}}((\mathrm{g}_1 \diamond \hbar_1) \diamond \hbar_1), \mathcal{N}^+_{\mathscr{B}}(0)\} \\
&= \mathcal{N}^+_{\mathscr{B}}((\mathrm{g}_1 \diamond \hbar_1) \diamond \hbar_1), \\
\mathcal{N}^-_{\mathscr{B}}(\mathrm{g}_1 \diamond \hbar_1) &\geq \min\{\mathcal{N}^-_{\mathscr{B}}((\mathrm{g}_1 \diamond \hbar_1) \diamond \hbar_1), \mathcal{N}^-_{\mathscr{B}}(0)\} \\
&= \mathcal{N}^-_{\mathscr{B}}((\mathrm{g}_1 \diamond \hbar_1) \diamond \hbar_1).
\end{aligned}$$

Therefore, by Theorem 6, $\mathscr{B} = (\mathcal{M}^+_{\mathscr{B}}, \mathcal{M}^-_{\mathscr{B}}, \mathcal{N}^+_{\mathscr{B}}, \mathcal{N}^-_{\mathscr{B}})$ is a BPVIFPII of $\mathcal{G}$.

4. Conclusion

This study introduces the concept of bipolar-valued intuitionistic fuzzy positive implicative ideals (BPVIFPIIs) within $\mathcal{BCK}$-algebras, offering a robust theoretical framework that expands the algebraic treatment of fuzzy structures. The research thoroughly examines the conditions under which a bipolar-valued intuitionistic fuzzy set qualifies as a BPVIFPII and explores its connections with BPVIFIs, supported by illustrative examples and rigorous proofs.

To provide a clearer understanding of the research process, Figure 1 presents a detailed flowchart outlining the logical progression of this study. This diagram highlights key stages, starting from the foundational definitions of fuzzy and intuitionistic fuzzy sets, extending through the development of bipolar-valued intuitionistic fuzzy structures, and culminating in the formalization of BPVIFPIIs in $\mathcal{BCK}$-algebras. The flowchart serves as a visual guide, summarizing the theoretical steps and linking them to their practical implications, thereby facilitating a comprehensive grasp of the study's contributions.

This work not only establishes new theoretical foundations but also underscores the practical potential of BPVIFPIIs in decision-making systems, sentiment analysis, and artificial intelligence. Future research will focus on exploring these applications and extending the proposed framework to related structures, such as bipolar-valued intuitionistic fuzzy soft ideals and $(\in, \in \vee q)$-BPVIFIs, further enriching the field of algebraic reasoning under uncertainty.

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