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Probabilistic Multiple Poly Bernoulli Polynomials of the Second Kind

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Abstract. The purpose of this paper is to introduce the probabilistic multiple poly Bernoulli polynomials of the second kind under the condition that Y is a random variable. This means that we will consider the probabilistic extension of the multiple poly Bernoulli polynomials of the second kind and study to obtain some new results. Furthermore, we investigate their interesting properties.

2020 Mathematics Subject Classifications: 11B68, 11B73

Key Words and Phrases: Poly Bernoulli polynomials of the second kind, Stirling numbers, Probabilistic poly-Bernoulli polynomials, Probabilistic Bernoulli polynomials.

1. Introduction

Many years ago Qi-Kim-Kim-Dolgy considered poly Bernoulli polynomials of the second kind and multiple poly Bernoulli polynomials of the secon kind in [28]. Recently, researchers considered probabilistic Stirling numbers, bell numbers, Bernoulli polynomials and Euler polynomials. The aim of this paper is to study probabilistic multiple poly Bernoulli polynomials of the second kind. Specifically speaking, we assume that Y is a random variable In section 1, firstly we recall polylogarithm $\mathrm{Li}_k(t)$, multiple polylogarithm $\mathrm{Li}_{k_1,\ldots,k_r}(t)$. we remind poly Bernoulli polynomials of the second kind and Bernoulli polynomials of order α. we recall that probabilistic Stirling numbers of the second kind and Lah numbers. In section 2, we define probabilistic poly Bernoulli polynomials of the second kind $b_{n,Y}^{(k)}$ and probabilistic multiple poly Bernoulli polynomials of the second kind $b_{n,Y}^{(k_1,\ldots,k_r)}$ associated with Y. In Theorem 2.1, we derive an expression for $nb_{n-1,Y}^{(k)}(x)$. When $Y \sim \Gamma(1,1)$ in Theorem 2.2, we get an expression for $b_{n,Y}^{(k)}(x)$ as sum of the products. When Y is the Bernoulli random variable, in Theorem 2.3 we get expression for $b_{n,Y}^{(k)}(x)$. In Theorem 2.4, we obtain an expression for $b_{n,Y}^{(k_1,\ldots,k_r)}(x)$. In Theorem 2.5, we

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find an expression of $\frac{b_{n+1,Y}^{(k_1,\ldots,k_r)}(x+1)-b_{n+1,Y}^{k_1,\ldots,k_r}(x)}{n+1}$. In Theorem 2.6, we derive an expression of $b_{n,Y}^{(k_1,\ldots,k_r)}(x)$ between probabilistic multi-poly-Bernoulli polynomials, Bernoulli polynomials of the second kind of order r and probabilistic the Stirling numbers of the second kind. In Theorem 2.7 we get an expression of $b_{n,Y}^{(k_1,\ldots,k_r)}(x+y)$. Now we recall that

The Bernoulli polynomials of the second kind are defined by

$$\frac{t}{log(1+t)}(1+t)^x = \sum_{n=0}^{\infty} b_n(x)\frac{t^n}{n!}, \quad (\text{see}[12],[13],[25],[29]). \tag{1}$$

The Bernoulli polynomials of the second kind with order r are defined by the generating function

$$\left(\frac{t}{log(1+t)}\right)^r (1+t)^x = \sum_{n=0}^{\infty} b_n^r(x)\frac{t^n}{n!}, (r \in \mathrm{Z}), \quad (\text{see}[17]). \tag{2}$$

It is well known that

$$\frac{t(1+t)^{x-1}}{log(1+t)} = \sum_{n=0}^{\infty} B_n^{(n)}(x)\frac{t^n}{n!}, \quad (\text{see}[9],[22],[5].[7]). \tag{3}$$

where the Bernoulli polynomials of order α which are given by

$$\left(\frac{t}{e^t-1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} B_n^{\alpha}(x)\frac{t^n}{n!}, \quad (\text{see}[9],[22],[5],[29],[8].[7]). \tag{4}$$

From (1) and (3), we note that

$$b_n(x) = B_n^{(n)}(x+1), (n \geq 0), \quad (\text{see}[9],[22]). \tag{5}$$

For $k \in \mathbb{Z}$, the polylogarithm function is defined by

$$\mathrm{Li}_k(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^k}, \quad (\text{see}[4],[13],[9],[22],[11],[16],[21]). \tag{6}$$

The poly Bernoulli polynomials of the second kind are defined by

$$\frac{\mathrm{Li}_k(1-e^{-t})}{log(1+t)}(1+t)^x = \sum_{n=0}^{\infty} b_n^{(k)}(x)\frac{t^n}{n!}, \quad (\text{see}[9],[23]). \tag{7}$$

For $k_1, k_2, \ldots, k_r \in \mathbb{Z}$, the multiple polylogarithm is defined by

$$\mathrm{Li}_{k_1,\ldots,k_r}(z) = \sum_{0<n_1<\cdots<n_r} \frac{z^{n_r}}{n_1^{k_1}\cdots n_r^{k_r}}, (|t|<1), \quad (\text{see}[9],[21],[18],[24],[25],[27],[28],[26]). \tag{8}$$

By (8), we note that

$$\frac{d}{dt}\mathrm{Li}_{k,1}(t) = \frac{1}{1-t}\mathrm{Li}_k(t), \quad (\text{see}[9],[21],[24],[25],[27],[28]). \tag{9}$$

It is obvious that

$$\mathrm{Li}_{1,1}(t) = \int_0^t \frac{d}{dt}\mathrm{Li}_{1,1}(t)dt = \int_0^t \frac{1}{1-t}(-log(1-t))dt = \frac{1}{2!}(-log(1-t))^2. \tag{10}$$

Continuing this process in (10), we have

$$\mathrm{Li}_{1,\ldots,1}(t) = \frac{1}{r!}(-log(1-t))^r, (r \in \mathbb{N}), \quad (\text{see}[3],[9],[21],[24],[25],[27],[28]). \tag{11}$$

Throught this paper, we assume that Y is a random variable such that the moment generating function of Y given by

$$E[e^{Yt}] = \sum_{n=0}^{\infty} E[Y^n]\frac{t^n}{n!}, (|t| < r), \quad (\text{see}[15],[12],[13],[19],[10],[30]). \tag{12}$$

exists for some $r > 0$. Let $(Y_i)_{i\geq 1}$ be a sequence of mutually independent copies of random variable Y, and let $S_n = Y_1 + \cdots + Y_n$, $(n \geq 1)$, with $S_0 = 0$. A continuous random variable Y whose density function is given by

$$f(y) = \begin{cases} \beta e^{-\beta y}\frac{(\beta y)^{\alpha-1}}{\Gamma(\alpha)}, & \text{if } y > 0, \\ 0, & \text{if } y \leq 0, \end{cases} \quad (\text{see}[13],[16],[20]), \tag{13}$$

for some $\alpha, \beta > 0$ is said to be the gamma random variable with parameters α, β, which is denoted by $Y \sim \Gamma(\alpha, \beta)$.

The Stirling number of the second kind are defined by

$$x^n = \sum_{k=0}^{n} S_2(n,k)(x)_k, \quad (\text{see}[2],[19],[6],[14],[29]). \tag{14}$$

From (14), we also derive the generating function as follows.

$$\frac{1}{k!}(e^t - 1)^k = \sum_{n=k}^{\infty} S_2(n,k)\frac{t^n}{n!}, \quad (\text{see}[2],[19],[6],[14]). \tag{15}$$

It is well known that the Lah numbers are defined by

$$\frac{1}{k!}\left(\frac{t}{1-t}\right)^k = \sum_{n=k}^{\infty} L(n,k)\frac{t^n}{n!}, (r \geq 0), \quad (\text{see}[13],[6]). \tag{16}$$

The probabilistic Stirling number of the second kind associated with Y are defined by

$$\frac{1}{k!}(E[e^{Yt}] - 1)^k = \sum_{n=k}^{\infty} \left\{ {n \atop k} \right\}_Y \frac{t^n}{n!}, \quad (\text{see}[1],[13],[19]). \tag{17}$$

In [13]. Kim also considered the probabilistic multi-poly-Bernoulli polynomials associated with Y by

$$\frac{\mathrm{Li}_{k_1,\ldots,k_r}\left(1-E[e^{-Yt}]\right)}{(1-E[e^{-Yt}])^r}\left(E[e^{-Yt}]\right)^x=\sum_{n=0}^{\infty}B_{n,Y}^{(k_1,\ldots,k_r)}(x)\frac{t^n}{n!},\quad (\text{see}[13]). \tag{18}$$

2. probabilistic poly and multiple poly Bernoulli polynomials of the second kind

In this section, we consider probabilistic poly Bernoulli polynomials of the second kind associated with Y which are given by

$$\frac{\mathrm{Li}_k(1-E[e^{-Yt}])}{log(1+t)}(1+t)^x=\sum_{n=0}^{\infty}b_{n,Y}^{(k)}(x)\frac{t^n}{n!}. \tag{19}$$

When $x=0$, $b_{n,Y}^{(k)}(0)=b_{n,Y}^{(k)}$ are called probabilistic multiple poly Bernoulli numbers of the second kind.

From (19), we have

Proof. Theorem 1.

$$\begin{aligned} t\sum_{n=0}^{\infty}b_{n,Y}^{(k)}(x)\frac{t^n}{n!} &= \frac{t}{log(1+t)}(1+t)^x\mathrm{Li}_k(1-E[e^{-Yt}]) \\ &= \sum_{l=0}^{\infty}B_l^{(l)}(x+1)\frac{t^l}{l!}\sum_{m=1}^{\infty}\frac{(1-E[e^{-Yt}])^m}{m^k} \\ &= \sum_{l=0}^{\infty}B_l^{(l)}(x+1)\frac{t^l}{l!}\sum_{m=1}^{\infty}\frac{(-1)^m m!}{m^k}\sum_{j=m}^{\infty}(-1)^j\left\{\begin{matrix} j \\ m\end{matrix}\right\}_Y\frac{t^j}{j!} \\ &= \sum_{l=0}^{\infty}B_l^{(l)}(x+1)\frac{t^l}{l!}\sum_{j=1}^{\infty}\sum_{m=1}^{j}\frac{(-1)^{m+j}m!}{m^k}\left\{\begin{matrix} j \\ m\end{matrix}\right\}_Y\frac{t^j}{j!} \\ &= \sum_{n=1}^{\infty}\sum_{j=1}^{n}\sum_{m=1}^{j}\frac{(-1)^{m+j}m!}{m^k}\binom{n}{j}\left\{\begin{matrix} j \\ m\end{matrix}\right\}_Y B_{n-j}^{(n-j)}(x+1)\frac{t^n}{n!}. \end{aligned} \tag{20}$$

On the other hand, in (20)

$$t\sum_{n=0}^{\infty}b_{n,Y}^{(k)}(x)\frac{t^n}{n!}=\sum_{n=1}^{\infty}nb_{n-1,Y}^{(k)}(x)\frac{t^n}{n!}. \tag{21}$$

Thus, by comparing the coefficients on both sides of (20) and (21), we have the following theorem.

Theorem 1. *For $n \geq 1$, we have*

$$nb_{n-1,Y}^{(k)}(x) = \sum_{j=1}^{n}\sum_{m=1}^{j}\frac{(-1)^{m+j}m!}{m^k}\binom{n}{j}\left\{\begin{matrix} j \\ m \end{matrix}\right\}_Y B_{n-j}^{(n-j)}(x+1).$$

Let $Y \sim \Gamma(1,1)$, then we note that

$$E[e^{Yt}] = \frac{1}{1-t}. \tag{22}$$

From (19) and (22), we have
Proof. Theorem 2.

$$\begin{aligned}\sum_{n=0}^{\infty} b_{n,Y}^{(k)}(x)\frac{t^n}{n!} &= \frac{t(1+t)^x}{tlog(1+t)}\sum_{m=1}\frac{(1-\frac{1}{1-t})^m}{m^k} \\ &= \frac{1}{t}\sum_{i=0}^{\infty} b_i(x)\frac{t^i}{i!}\sum_{m=1}^{\infty}\frac{(-1)^m m!}{m^k}\frac{1}{m!}\left(\frac{t}{1-t}\right)^m \\ &= \frac{1}{t}\sum_{i=0}^{\infty} b_i(x)\frac{t^i}{i!}\sum_{m=1}^{\infty}\frac{(-1)^m m!}{m^k}\sum_{l=m}^{\infty}L(l,m)\frac{t^l}{l!} \\ &= \frac{1}{t}\sum_{i=0}^{\infty} b_i(x)\frac{t^i}{i!}\sum_{l=1}^{\infty}\sum_{m=1}^{l}\frac{(-1)^m m!}{m^k}L(l,m)\frac{t^l}{l!} \\ &= \sum_{n=1}^{\infty}\sum_{l=1}^{n}\sum_{m=1}^{l}\binom{n}{l}\frac{(-1)^m m!}{m^k}L(l,m)b_{n-l}(x)\frac{t^{n-1}}{n!} \\ &= \sum_{n=0}^{\infty}\sum_{l=1}^{n+1}\sum_{m=1}^{l}\binom{n+1}{l}\frac{(-1)^m m!}{m^k}L(l,m)\frac{b_{n-l+1}(x)}{n+1}\frac{t^n}{n!}.\end{aligned} \tag{23}$$

Thus, we have the following theorem.

Theorem 2. *Let $Y \sim \Gamma(1,1)$. For $n \geq 0$, we have*

$$b_{n,Y}^{(k)}(x) = \sum_{l=1}^{n+1}\sum_{m=1}^{l}\binom{n+1}{l}\frac{(-1)^m m!}{m^k}L(l,m)\frac{b_{n-l+1}(x)}{n+1}.$$

Let Y be the Bernoulli random variable with probability of success p. Then we have

$$E[e^{Yt}] = p(e^t-1)+1. \tag{24}$$

By (19) and (24), we have

Proof. Theorem 3.

$$\begin{aligned}
\sum_{n=0}^{\infty} b_{n,Y}^{(k)} \frac{t^n}{n!} &= \frac{(1+t)^x}{log(1+t)} \sum_{m=1}^{\infty} \frac{(1-E[e^{-Yt}])^m}{m^k} \\
&= \frac{(1+t)^x}{log(1+t)} \sum_{m=1}^{\infty} \frac{(-1)^m p^m (e^t-1)^m}{m^k} \\
&= \frac{t(1+t)^x}{tlog(1+t)} \sum_{m=1}^{\infty} \frac{(-1)^m p^m m!}{m^k} \frac{(e^t-1)^m}{m!} \\
&= \frac{1}{t} \sum_{l=0}^{\infty} b_l(x) \frac{t^l}{l!} \sum_{m=1}^{\infty} \frac{(-1)^m p^m m!}{m^k} \sum_{i=m}^{\infty} S_2(i,m) \frac{t^i}{i!} \\
&= \frac{1}{t} \sum_{l=0}^{\infty} b_l(x) \frac{t^l}{l!} \sum_{i=1}^{\infty} \sum_{m=1}^{i} \frac{(-1)^m p^m m!}{m^k} S_2(i,m) \frac{t^i}{i!} \\
&= \sum_{n=1}^{\infty} \sum_{i=1}^{n} \sum_{m=1}^{i} \binom{n}{i} \frac{(-1)^m p^m m!}{m^k} S_2(i,m) b_{n-i}(x) \frac{t^{n-1}}{n!} \\
&= \sum_{n=0}^{\infty} \sum_{i=1}^{n+1} \sum_{m=1}^{i} \binom{n+1}{i} \frac{(-1)^m p^m m!}{m^k} S_2(i,m) \frac{b_{n-i+1}(x)}{n+1} \frac{t^n}{n!}.
\end{aligned} \tag{25}$$

Thus, we have the following theorem.

Theorem 3. *Let Y be the Bernoulli random variable with probability of success p. For $n \geq 0$, we have*

$$b_{n,Y}^{(k)} = \sum_{i=1}^{n+1} \sum_{m=1}^{i} \binom{n+1}{i} \frac{(-1)^m p^m m!}{m^k} S_2(i,m) \frac{b_{n-i+1}(x)}{n+1}.$$

Now, we consider probabilistic multiple poly Bernoulli polynomials of the second kind which are given by

$$\frac{r!\mathrm{Li}_{k_1,\ldots,k_r}(1-E[e^{-Yt}])}{(log(1+t))^r}(1+t)^x = \sum_{n=0}^{\infty} b_{n,Y}^{(k_1,\ldots,k_r)}(x) \frac{t^n}{n!}. \tag{26}$$

When $k_1 = \cdots = k_r = 1$ and $Y = 1$, we note that

$$\sum_{n=0}^{\infty} b_{n,Y}^{(1,\ldots,1)}(x) = \left(\frac{t}{log(1+t)}\right)^r (1+t)^x = \sum_{n=0}^{\infty} b_n^r(x) \frac{t^n}{n!}. \tag{27}$$

From (26), we have

Proof. Theorem 4.

$$\sum_{n=0}^{\infty} b_{n,Y}^{(k_1,\dots,k_r)}(x)\frac{t^n}{n!} = \frac{r!(1+t)^x}{(log(1+t))^r}\mathrm{Li}_{k_1,\dots,k_r}(1-E[e^{-Yt}]) \tag{28}$$

$$= \frac{r!(1+t)^x}{(log(1+t))^r}\sum_{0<m_1<\cdots<m_r}\frac{\left(1-E[e^{-Yt}]\right)^{m^r}}{m_1^{k_1}\cdots m_r^{k_r}}$$

$$= \frac{r!(1+t)^x}{(log(1+t))^r}\sum_{0<m_1<\cdots<m_{r-1}}\frac{1}{m_1^{k_1}\cdots m_{r-1}^{k_{r-1}}}\sum_{m_r=m_{r-1}+1}^{\infty}\frac{\left(1-E[e^{-Yt}]\right)^{m^r}}{m_r^{k_r}}$$

$$= \frac{r!(1+t)^x}{(log(1+t))^r}\sum_{0<m_1<\cdots<m_{r-1}}\frac{\left(1-E[e^{-Yt}]\right)^{m_{r-1}}}{m_1^{k_1}\cdots m_{r-1}^{k_{r-1}}}$$

$$\times\sum_{m_r=0}^{\infty}\frac{(-1)^{m_r+1}(m_r+1)!}{(m_r+m_{r-1}+1)^{k_r}}\frac{\left(E[e^{-Yt}-1]\right)^{m_r+1}}{(m_r+1)!}$$

$$= \frac{r!(1+t)^x}{(log(1+t))^r}\sum_{0<m_1<\cdots<m_{r-1}}\frac{\left(1-E[e^{-Yt}]\right)^{m_{r-1}}}{m_1^{k_1}\cdots m_{r-1}^{k_{r-1}}}$$

$$\times\sum_{m_r=0}^{\infty}\frac{(-1)^{m_r+1}(m_r+1)!}{(m_r+m_{r-1}+1)^{k_r}}\sum_{l=m_r+1}^{\infty}\left\{\begin{matrix} l \\ m_r+1\end{matrix}\right\}_Y\frac{(-1)^l t^l}{l!}$$

$$= \frac{r!(1+t)^x}{(log(1+t))^r}\sum_{0<m_1<\cdots<m_{r-1}}\frac{\left(1-E[e^{-Yt}]\right)^{m_{r-1}}}{m_1^{k_1}\cdots m_{r-1}^{k_{r-1}}}$$

$$\times\sum_{l=1}^{\infty}\sum_{m_r=0}^{l-1}\frac{(-1)^{m^r+l+1}(m_r+1)!}{(m_r+m_{r-1}+1)^{k_r}}\left\{\begin{matrix} l \\ m_r+1\end{matrix}\right\}_Y\frac{t^l}{l!}$$

$$= \frac{r!(1+t)^x}{(log(1+t))^r}\sum_{0<m_1<\cdots<m_{r-1}}\frac{\left(1-E[e^{-Yt}]\right)^{m_{r-1}}}{m_1^{k_1}\cdots m_{r-1}^{k_{r-1}-j}}$$

$$\times\sum_{l=1}^{\infty}\sum_{m_r=0}^{l-1}(-1)^{m^r+l+j+1}(m_r+1)!\left\{\begin{matrix} l \\ m_r+1\end{matrix}\right\}\frac{t^l}{l!}\sum_{j=0}^{\infty}\binom{k_r+j-1}{j}(m_r+1)^{-k_r-j}\frac{t^j}{j!}$$

$$= \frac{r!(1+t)^x}{(log(1+t))^r}\sum_{0<m_1<\cdots<m_{r-1}}\frac{\left(1-E[e^{-Yt}]\right)^{m_{r-1}}}{m_1^{k_1}\cdots m_{r-1}^{k_{r-1}-j}}$$

$$\times\sum_{l=1}^{\infty}\sum_{m_r=0}^{l-1}\sum_{j=0}^{\infty}(-1)^{m_r+l+j+1}(m_r+1)!(m_r+1)^{-k_r-j}\left\{\begin{matrix} l \\ m_r+1\end{matrix}\right\}_Y\binom{k_r+j-1}{j}\frac{t^l}{l!}$$

$$= \frac{r}{log(1+t)}\sum_{l=1}^{\infty}\sum_{m_r=0}^{l-1}\sum_{j=0}^{\infty}(-1)^{m_r+l+j+1}(m_r+1)!(m_r+1)^{-k_r-j}\left\{\begin{matrix} l \\ m_r+1\end{matrix}\right\}_Y\binom{k_r+j-1}{j}\frac{t^l}{l!}$$

$$\times \sum_{i=0}^{\infty} b_{i,Y}^{(k_1,\ldots,k_{r-1}-j)}(x)\frac{t^i}{i!}$$
$$= \frac{r}{\mathrm{Li}_1(-t)} \sum_{n=1}^{\infty}\sum_{l=0}^{n-1}\sum_{m_r=0}^{l-1}\sum_{j=0}^{\infty}(-1)^{m_r+l+j}\binom{n}{l+1}\binom{k_r+j-1}{j}\left\{\begin{matrix} l+1 \\ m_r+1 \end{matrix}\right\}_Y$$
$$\times (m_r+1)!(m_r+1)^{-k_r-j} b_{n-l-1,Y}^{(k_1,\ldots,k_{r-1}-j)}(x)\frac{t^n}{n!}.$$

Therefore, by (28), we have the following theorem.

Theorem 4. *For $n \geq 0$, $t \neq 0$, we have*

$$b_{n,Y}^{(k_1,\ldots,k_r)}(x) = \frac{r}{\mathrm{Li}_k(-t)} \sum_{l=0}^{n-1}\sum_{m_r=0}^{l-1}\sum_{j=0}^{\infty}(-1)^{m_r+l+j}\binom{n}{l+1}\binom{k_r+j-1}{j}\left\{\begin{matrix} l+1 \\ m_r+1 \end{matrix}\right\}_Y$$
$$\times (m_r+1)!(m_r+1)^{-k_r-j} b_{n-l-1,Y}^{(k_1,\ldots,k_{r-1}-j)}(x).$$

From (26), we obtain
Proof. Theorem 5.

$$\frac{1}{t}\left(\sum_{n=0}^{\infty} b_{n,Y}^{(k_1,\ldots,k_r)}(x+1)\frac{t^n}{n!} - \sum_{n=0}^{\infty} b_{n,Y}^{(k_1,\ldots,k_r)}(x)\frac{t^n}{n!}\right) = \frac{r!\mathrm{Li}_{k_1,\ldots,k_r}\left(1-E[e^{-Yt}]\right)}{(log(1+t))^r}(1+t)^x \tag{29}$$

$$= \frac{r!(1+t)^x}{(log(1+t))^r} \sum_{0<m_1<\cdots<m_r} \frac{\left(1-E[e^{-Yt}]\right)^{m_r}}{m_1^{k_1}\cdots m_r^{k_r}}$$
$$= \frac{r!(1+t)^x}{(log(1+t))^r} \sum_{0<m_1<\cdots<m_{r-1}} \frac{1}{m_1^{k_1}\cdots m_{r-1}^{k_{r-1}}} \sum_{m_r=m_{r-1}+1} \frac{\left(1-E[e^{-Yt}]\right)^{m_r}}{m_r^{k_r}}$$
$$= \frac{r!(1+t)^x}{(log(1+t))^r} \sum_{0<m_1<\cdots<m_{r-1}} \frac{\left(1-E[e^{Yt}]\right)^{m_{r-1}}}{m_1^{k_1}\cdots m_{r-1}^{k_{r-1}}} \sum_{m_r=1}^{\infty} \frac{(-1)^{m_r} m_r!}{(m_r+m_{r-1})^{k_r}} \frac{\left(E[e^{-Yt}]-1\right)^{m_r}}{m_r!}$$
$$= \frac{r!(1+t)^x}{(log(1+t))^r} \sum_{0<m_1<\cdots<m_{r-1}} \frac{\left(1-E[e^{Yt}]\right)^{m_{r-1}}}{m_1^{k_1}\cdots m_{r-1}^{k_{r-1}}} \sum_{m_r=1}^{\infty} \frac{(-1)^{m_r+l} m_r!}{(m_r+m_{r-1})^{k_r}} \sum_{l=m_r}^{\infty}\left\{\begin{matrix} l \\ m_r \end{matrix}\right\}_Y \frac{t^l}{l!}$$
$$= \frac{r!(1+t)^x}{(log(1+t))^r} \sum_{0<m_1<\cdots<m_{r-1}} \frac{\left(1-E[e^{Yt}]\right)^{m_{r-1}}}{m_1^{k_1}\cdots m_{r-1}^{k_{r-1}}} \sum_{l=1}^{\infty}\sum_{m_r=1}^{l} \frac{(-1)^{m_r+l} m_r!}{(m_r+m_{r-1})^{k_r}} \left\{\begin{matrix} l \\ m_r \end{matrix}\right\}_Y \frac{t^l}{l!}$$
$$= \frac{r!(1+t)^x}{(log(1+t))^r} \sum_{0<m_1<\cdots<m_{r-1}} \frac{\left(1-E[e^{Yt}]\right)^{m_{r-1}}}{m_1^{k_1}\cdots m_{r-1}^{k_{r-1}-j}}$$
$$\times \sum_{l=1}^{\infty}\sum_{m_r=1}^{l}\sum_{j=0}^{\infty}\binom{k_r+j-1}{j}(-1)^{m_r+l+j} m_r^{-k_r-j} m_r! \left\{\begin{matrix} l \\ m_r \end{matrix}\right\}_Y \frac{t^l}{l!}$$

$$= \frac{r}{log(1+t)} \sum_{i=0}^{\infty} b_{i,Y}^{(k_1,\ldots,k_{r-1}-j)}(x)\frac{t^i}{i!} \sum_{l=1}^{\infty} \sum_{m_r=1}^{l} \sum_{j=0}^{\infty} \binom{k_r+j-1}{j} (-1)^{m_r+l+j} m_r^{-k_r-j} m_r! \left\{ \begin{matrix} l \\ m_r \end{matrix} \right\}_Y \frac{t^l}{l!}$$

$$= \frac{r}{\mathrm{Li}_1(-t)} \sum_{n=1}^{\infty} \sum_{l=1}^{n} \sum_{m_r=1}^{l} \sum_{j=0}^{\infty} \binom{k_r+j-1}{j} \binom{n}{l} (-1)^{m_r+l+j} m_r^{-k_r-j} m_r! \left\{ \begin{matrix} l \\ m_r \end{matrix} \right\}_Y b_{n-l,Y}^{(k_1,\ldots,k_{r-1}-j)}(x)\frac{t^n}{n!}.$$

On the other hand in (29)

$$\frac{1}{t}\left(\sum_{n=0}^{\infty} b_{n,Y}^{(k_1,\ldots,k_r)}(x+1)\frac{t^n}{n!} - \sum_{n=0}^{\infty} b_{n,Y}^{(k_1,\ldots,k_r)}(x)\frac{t^n}{n!} \right) = \sum_{n=0}^{\infty} \frac{\left(b_{n+1,Y}^{k_1,\ldots,k_r}(x+1) - b_{n+1,Y}^{k_1,\ldots,k_r}(x) \right)}{n+1} \frac{t^n}{n!}. \tag{30}$$

By comparing the coefficients on both sides of (29) and (30), we have the following theorem.

Theorem 5. *For $n \geq 1$, $t \neq 0$, we have*

$$\frac{\left(b_{n+1,Y}^{(k_1,\ldots,k_r)}(x+1) - b_{n+1,Y}^{(k_1,\ldots,k_r)}(x) \right)}{n+1}$$
$$= \frac{r}{\mathrm{Li}_1(-t)} \sum_{l=1}^{n} \sum_{m_r=1}^{l} \sum_{j=0}^{\infty} \binom{k_r+j-1}{j} \binom{n}{l} (-1)^{m_r+l+j} m_r^{-k_r-j} m_r! \left\{ \begin{matrix} l \\ m_r \end{matrix} \right\}_Y b_{n-l,Y}^{(k_1,\ldots,k_{r-1}-j)}(x).$$

From (26), we have

Proof. Theorem 6.

$$\begin{aligned} \sum_{n=0}^{\infty} b_{n,Y}^{(k_1,\ldots,k_r)}(x)\frac{t^n}{n!} &= \frac{r!\mathrm{Li}_{k_1,\ldots,k_r}\left(1-E[e^{-Yt}]\right)}{(log(1+t))^r}(1+t)^x \qquad (31) \\ &= \frac{t^r r!(1+t)^x}{t^r((log(1+t))^r} \frac{\mathrm{Li}_{k_1,\ldots,k_r}\left(1-E[e^{-Yt}]\right)}{(1-E[e^{-Yt}])^r} \left(1-E[e^{-Yt}]\right)^r \\ &= \frac{(r!)^2}{t^r} \sum_{m=0}^{\infty} B_{m,Y}^{(k_1,\ldots,k_r)} \frac{t^m}{m!} \sum_{l=r}^{\infty} \left\{ \begin{matrix} l \\ r \end{matrix} \right\}_Y (-1)^l \frac{t^l}{l!} \sum_{i=0}^{\infty} b_i^r(x)\frac{t^i}{i!} \\ &= \frac{(r!)^2}{t^r} \sum_{j=0}^{\infty} \sum_{m=0}^{j} \binom{j}{m} B_{m,Y}^{(k_1,\ldots,k_r)} b_{j-m}^r(x) \frac{t^j}{j!} \sum_{l=r}^{\infty} \left\{ \begin{matrix} l \\ r \end{matrix} \right\}_Y (-1)^l \frac{t^l}{l!} \\ &= \frac{(r!)^2}{t^r} \sum_{n=r}^{\infty} \sum_{l=0}^{n} \sum_{m=0}^{n-l} \binom{n-l}{m} \binom{n}{l} (-1)^l B_{m,Y}^{(k_1,\ldots,k_r)} b_{n-l-m}^{(r)}(x) \left\{ \begin{matrix} l \\ r \end{matrix} \right\}_Y \frac{t^n}{n!} \\ &= (r!)^2 \sum_{n=0}^{\infty} \sum_{l=0}^{n+r} \sum_{m=0}^{n+r-l} \binom{j}{m} \binom{n+r-l}{l} (-1)^l \frac{B_{m,Y}^{(k_1,\ldots,k_r)} b_{n+r-l-m}^{(r)}(x)}{(n+r)_r} \left\{ \begin{matrix} l \\ r \end{matrix} \right\}_Y \frac{t^n}{n!}. \end{aligned}$$

Therefore, by (31), we have the following theorem.

Theorem 6. *For $n \geq 0$, we have*

$$b_{n,Y}^{(k_1,\ldots,k_r)}(x) = (r!)^2 \sum_{l=0}^{n+r} \sum_{m=0}^{r-l} \binom{j}{m} \binom{n+r}{j} (-1)^l \frac{B_{m,Y}^{(k_1,\ldots,k_r)} b_{n+r-l-m}^{(r)}(x)}{(n)_r} \left\{ \begin{matrix} l \\ r \end{matrix} \right\}_Y .$$

From (26), we have
Proof. Theorem 7.

$$\sum_{n=0}^{\infty} b_{n,Y}^{(k_1,\ldots,k_r)}(x+y) \frac{t^n}{n!} = \frac{r!\mathrm{Li}_{k_1,\ldots,k_r}(1-E[e^{-Yt}])}{(log(1+t))^r}(1+t)^{x+y} \tag{32}$$

$$= \frac{r!(1+t)^x}{(log(1+t))^r} \sum_{l=0}^{\infty} (y)_l \frac{t^l}{l!} \sum_{0<m_1<\cdots<m_{r-1}} \frac{1}{m_1^{k_1}\cdots m_{r-1}^{k_{r-1}}} \sum_{m_r=m_{r-1}+1}^{\infty} \frac{\left(1-E[e^{-Yt}]\right)^{m_r}}{m_r^{k_r}}$$

$$= \frac{r!(1+t)^x}{(log(1+t))^r} \sum_{l=0}^{\infty} (y)_l \frac{t^l}{l!} \sum_{0<m_1<\cdots<m_{r-1}} \frac{\left(1-E[e^{-Yt}]\right)^{m_{r-1}}}{m_1^{k_1}\cdots m_{r-1}^{k_{r-1}}}$$

$$\times \sum_{m_r=1}^{\infty} \frac{(-1)^{m_r} m_r!}{(m_r+m_{r-1})^k} \frac{\left(E[e^{-Yt}]-1\right)^{m_r}}{m_r!}$$

$$= \frac{r!(1+t)^x}{(log(1+t))^r} \sum_{l=0}^{\infty} (y)_l \frac{t^l}{l!} \sum_{0<m_1<\cdots<m_{r-1}} \frac{\left(1-E[e^{-Yt}]\right)^{m_{r-1}}}{m_1^{k_1}\cdots m_{r-1}^{k_{r-1}}}$$

$$\times \sum_{m_r=1}^{\infty} \frac{(-1)^{m_r} m_r!}{(m_r+m_{r-1})^k} \sum_{i=m_r}^{\infty} \left\{ \begin{matrix} i \\ m_r \end{matrix} \right\}_Y (-1)^i \frac{t^i}{i!}$$

$$= \frac{r!(1+t)^x}{(log(1+t))^r} \sum_{l=0}^{\infty} (y)_l \frac{t^l}{l!} \sum_{0<m_1<\cdots<m_{r-1}} \frac{\left(1-E[e^{-Yt}]\right)^{m_{r-1}}}{m_1^{k_1}\cdots m_{r-1}^{k_{r-1}}} \sum_{i=1}^{\infty} \sum_{m_r=1}^{i} \frac{(-1)^{m_r+i} m_r! \left\{ \begin{smallmatrix} i \\ m_r \end{smallmatrix} \right\}}{(m_r+m_{r-1})^k} \frac{t^i}{i!}$$

$$= \sum_{i=1}^{\infty} \sum_{m_r=1}^{i} \sum_{j=0}^{\infty} \binom{m_r+j-1}{j} (-1)^{m_r+i-j} m_r^{-k_r-j} m_r! \left\{ \begin{matrix} i \\ m_r \end{matrix} \right\}_Y \frac{t^i}{i!}$$

$$\times \frac{r!(1+t)^x}{(log(1+t))^r} \mathrm{Li}_{k_1,\ldots,k_{r-1}-j}\left(1-E[e^{-Yt}]\right) \sum_{l=0}^{\infty} (y)_l \frac{t^l}{l!}$$

$$= \frac{r}{log(1+t)} \sum_{i=1}^{\infty} \sum_{m_r=1}^{i} \sum_{j=0}^{\infty} \binom{m_r+j-1}{j} (-1)^{m_r+i-j} m_r^{-k_r-j} m_r! \left\{ \begin{matrix} i \\ m_r \end{matrix} \right\}_Y \frac{t^i}{i!}$$

$$\times \sum_{s=0}^{\infty} b_{s,Y}^{(k_1,\ldots,k_{r-1}-j)}(x) \frac{t^s}{s!} \sum_{l=0}^{\infty} (y)_l \frac{t^l}{l!}$$

$$
\begin{aligned}
&= \frac{r}{log(1+t)} \sum_{u=0}^{\infty} \sum_{l=0}^{u} \binom{u}{l} b_{u-l,Y}^{(k_1,\ldots,k_{r-1}-j)}(x) \frac{t^u}{u!} \\
&\times \sum_{i=1}^{\infty} \sum_{m_r=1}^{i} \sum_{j=0}^{\infty} \binom{m_r+j-1}{j} (-1)^{m_r+i+j} m_r^{-k_r-j} m_r! \left\{ \begin{matrix} i \\ m_r \end{matrix} \right\}_Y \frac{t^i}{i!} \\
&= \frac{r}{\mathrm{Li}_1(-t)} \sum_{n=1}^{\infty} \sum_{i=1}^{n} \sum_{m_r=1}^{i} \sum_{j=0}^{\infty} \binom{n-i}{l} \binom{n}{i} \binom{m_r+j-1}{j} (-1)^{m_r+i+j} m_r^{-k_r-j} m_r! \\
&\times \left\{ \begin{matrix} i \\ m_r \end{matrix} \right\}_Y b_{n-i-l,Y}^{(k_1,\ldots,k_{r-1}-j)}(x) \frac{t^n}{n!}.
\end{aligned}
$$

By comparing the coefficients on both sides of (32), we have following theorem.

Theorem 7. *For $n \geq 1$, we have*

$$
\begin{aligned}
& b_{n,Y}^{(k_1,\ldots,k_r)}(x+y) \\
&= \frac{r}{\mathrm{Li}_1(-t)} \sum_{i=1}^{n} \sum_{m_r=1}^{i} \sum_{j=0}^{\infty} \binom{n-i}{l} \binom{n}{i} \binom{m_r+j-1}{j} (-1)^{m_r+i+j} m_r^{-k_r-j} m_r! \\
&\times \left\{ \begin{matrix} i \\ m_r \end{matrix} \right\}_Y b_{n-i-l,Y}^{(k_1,\ldots,k_{r-1}-j)}(x).
\end{aligned}
$$

3. Conclusion

An increasing number of scholars have studied many special polynomials and numbers and have applied the properties of these polynomials in probability theory. In view of this, we explored the properties of probabilistic multiple poly Bernoulli polynomials of the second kind. That implies that we have conducted an in-depth study of these polynomials and numbers. We obtained several interesting results and formula when Y is the Bernoulli random variable and gamma random variable. In the future, we will consider more valuable polynomials and numbers and get some significant results.

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