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Super Vertex Cover of a Graph

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Abstract. A set $S \subseteq V(G)$ is a super vertex cover of G if S is a vertex cover and for every $x \in V(G) \setminus S$, there exists $y \in S$ such that $N_G(y) \cap (V(G) \setminus S) = \{x\}$. The super vertex cover number of G, denoted $\beta_s(G)$, is the smallest cardinality of a super vertex cover of G. In this paper, we show that the difference of the super vertex cover number and the vertex cover number can be made arbitrarily large. Graphs of small values of the parameter are characterized. Moreover, we give necessary and sufficient conditions for a super vertex cover in the join and the corona of graphs. Corresponding value of the super vertex cover number of each these graphs is also determined.

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Key Words and Phrases: Vertex cover, super vertex cover, super vertex cover number

1. Introduction

Numerous studies have been made on the vertex covering of a graph (for some studies, see [9], [10], [20], [23]) since the introduction of the concept. As mentioned by Angel and Amutha in [1], the parameter can be used for safety purposes in a network. In particular, the paper pointed out that in a computer network, the minimum value of the parameter offers an optimal solution for designing the network defense strategy. According to Toregas

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et al. [21], problems involving this concept is also utilized to determine emergency facility location in telecommunication networks. However, the vertex cover problem is an NP-hard optimization problem. In fact, using the known result that the clique problem is NP-complete, Karp [11] proved that the vertex cover problem is also NP-complete. It should be noted that the problem remains NP-complete in cubic graphs and in planar graphs (see [6] and [7]).

Studies that dealt with determining bounds and exact values of the vertex cover numbers of some specific graphs can be found in [2] and [22]. Recently, a number of variations of the vertex cover had been introduced and investigated (see [1], [3], [8], [9], [15], [18], and [19]). Motivated by the introduction of these several variants of the parameter, we introduce and initiate the study of super vertex cover of a graph. As the concept suggests, this new parameter combines two existing concepts, namely; vertex cover and super domination. Some studies on super domination can be found in [5], [12],[13], [14], [16], [17].

2. Terminologies and Notations

Let $G = (V(G), E(G))$ be a simple undirected graph. The *open neighborhood* of a vertex v of G is the set $N_G(v) = \{u \in V(G) : uv \in E(G)\}$, while its *closed neighborhood* is the set $N_G[v] = N_G(v) \cup \{v\}$. The *open neighborhood* of a set $S \subseteq V(G)$ is the set $N_G(S) = \cup_{v \in S} N_G(v)$ and its *closed neighborhood* is the set $N_G[S] = S \cup N_G(S)$. Any $v \in V(G)$ with $|N_G(v)| = 0$ is called an *isolated vertex.* Vertex v is a *leaf* or an *endvertex* if $|N_G(v)| = 1$. A vertex w of G is a *support vertex* if $wv \in E(G)$ for some leaf v in G. The sets $I(G)$, $L(G)$, and $S(G)$ will, respectively, denote the sets containing all the isolated vertices, leaves, and support vertices in G.

A subset A of $V(G)$ is an *independent* set if for every pair of distinct vertices in G do not form an edge. The maximum cardinality of an independent set in G, denoted by $\alpha(G)$, is called the *independence number* of G. Any independent set with cardinality equal to $\alpha(G)$ is called an α-set in G.

A set $S \subseteq V(G)$ is a *dominating set* in G if $N_G[S] = V(G)$. It is a *super dominating set* if for every $v \in V(G) \setminus S$ there exists $w \in S$ such that $N_G(w) \cap [V(G) \setminus S] = \{v\}$. The domination number (super domination number) of G, denoted $\gamma(G)$ (resp. $\gamma_{sp}(G)$) is the minimum cardinality of a dominating (resp. super dominating) set in G. Any dominating set (super dominating set) with cardinality $\gamma(G)$ (resp. $\gamma_{sp}(G)$) is called a γ-set (resp. γ_{sp}-set).

A subset U of vertices of a graph G is called a *vertex cover* of G if for every edge $e = uv \in E(G)$, $u \in U$ or $v \in U$. The minimum cardinality of vertex cover of G is the *vertex cover number* of G and is denoted by $\beta(G)$ and any vertex cover of G with cardinality $\beta(G)$ is called a β-set. Clearly, a vertex cover is also a dominating set in a non-trivial connected graph G. A set $S \subseteq V(G)$ is called *super vertex cover* of G if S is a vertex cover and for every $x \in V(G) \setminus S$, there exists $z \in S$ such that $N_G(z) \cap [V(G) \setminus S] = \{x\}$. In other words, a super vertex cover is any set that is both a vertex cover and super dominating set in G. The *super vertex cover number* of G, denoted by $\beta_s(G)$, is the smallest cardinality of a super vertex cover of G. Any super vertex cover of G with cardinality $\beta_s(G)$ is called a β_s-set.

Let G and H be any two graphs. The *join* $G + H$ is the graph with vertex set $V(G+H) = V(G) \cup V(H)$ and edge set $E(G+H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$. The *corona* $G \circ H$ is the graph obtained by taking one copy of G and $|V(G)|$ copies of H, and then joining the *ith* vertex of G to every vertex of the *ith* copy of H. We denote by H^v the copy of H in $G \circ H$ corresponding to the vertex $v \in G$ and write $v + H^v$ for $\langle \{v\} \rangle + H^v$. Readers are referred to [4] for other basic definitions that are not given here.

3. Results

Remark 1. *Let G be a graph and let S be a super vertex cover of G. Then each of the following holds:*

(*i*) *S is a dominating set in G.*

(*ii*) *$I(G) \subseteq S$ where $I(G)$ is the set containing all the isolated vertices of G.*

(*iii*) *If v is a support vertex in G, then $|(N_G[v] \cap L(G)) \cap (V(G) \setminus S)| \leq 1$, where $L(G)$ is the set of all leaves (end vertices) in G.*

Proposition 1. *Let G be a graph of order n such that $E(G) \neq \varnothing$. Then*

$$\max\{\gamma_{sp}(G), \beta(G)\} \leq \beta_s(G) \leq n - 1.$$

Proof. Since every super vertex cover of G is both a vertex cover and a super dominating set in G, it follows that $\max\{\gamma_{sp}(G), \beta(G)\} \leq \beta_s(G)$. Now let $e = uv \in E(G)$ and set $S = V(G) \setminus \{u\}$. Then clearly, S is a super vertex cover of G. Thus, $\beta_s(G) \leq |S| = n - 1$. □

Remark 2. *The bounds given in Proposition 1 are sharp. Moreover, strict inequality is attainable.*

To see this, consider the graphs $G_1 = P_5 = [a, b, c, d, e]$, $G_2 = C_4 = [x, y, z, w, x]$, and the star graph $G_3 = K_1, 4$. The set $T = \{b, c, d\}$ is a β-set, γ_{sp}-set, and β_s-set in G_1. Hence,

$$\gamma_{sp}(G_1) = \beta(G_1) = \beta_s(G_1) = 3 < 4 = |V(G_1)| - 1.$$

On the other hand, sets $S_1 = \{x, y\}$, $S_2 = \{x, z\}$, and $S_3 = \{x, y, z\}$ are γ_{sp}-set, β-set, and β_s-set in G_2, respectively. Thus,

$$\gamma_{sp}(G_2) = \beta(G_2) = 2 < 3 = \beta_s(G_2) = |V(G_2)| - 1.$$

Finally, one can easily see that

$$\beta(G_3) = 1 < 4 = \beta_s(G_3) = \gamma_{sp}(G_3) = |V(G_3)| - 1.$$

For strict inequality, let G be the graph in Figure 1. Set $\{d, e, c\}$ is a β-set, $\{a, e, f, g\}$ is a γ_{sp}-set, and $\{a, b, e, f, g\}$ is a β_s-set in G. Hence,

$$\beta(G) = 3 < 4 = \gamma_{sp}(G) < 5 = \beta_s(G) < 6 = |V(G)| - 1.$$

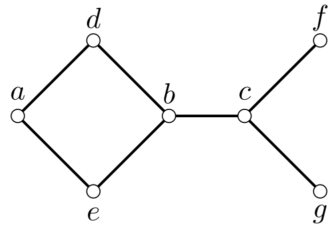

Figure 1: A graph G with $\beta_s(G) = 5$

Theorem 1. *Let G be a graph of order n such that $E(G) \neq \varnothing$. Then $\beta_s(G) = n - 1$ if and only if for every pair of non-adjacent vertices v and w of G, and for any $x \in V(G)$ with $x \in N_G(v) \setminus N_G(w)$ $(x \in N_G(w) \setminus N_G(v))$, it holds that $N_G(w) \setminus N_G(x) = \varnothing$ (resp. $N_G(v) \setminus N_G(x) = \varnothing$).*

Proof. Suppose $\beta_s(G) = n-1$. Suppose further that there exist a pair of non-adjacent vertices v and w and a vertex x such that $x \in N_G(v) \setminus N_G(w)$ but $N_G(w) \setminus N_G(x) \neq \varnothing$. Pick any $z \in N_G(w) \setminus N_G(x)$ and let $S = V(G) \setminus \{x, w\}$. Then S is a super vertex cover of G. Hence, $\beta_s(G) \leq |S| = n - 2$, a contradiction. Therefore, the condition or property holds.

For the converse, suppose that the property holds. Let S_0 be a β_s-set in G. By Proposition 1, $\beta_s = |S_0| \leq n-1$. Suppose $|S_0| < n-1$. Then there exist $p, q \in V(G) \setminus S_0$. Since S_0 is a vertex cover of G, $pq \notin E(G)$. Since S is a super dominating set in G, there exist $v_p, v_q \in S$ such that $N_G(v_p) \cap [V(G) \setminus S_0] = \{p\}$ and $N_G(v_q) \cap [V(G) \setminus S_0] = \{q\}$. This implies that $p \in N_G(v_p) \setminus N_G(q)$. By our assumption that the given property holds, it follows that $N_G(q) \setminus N_G(p) = \varnothing$. This forces $v_q \in N_G(p)$ because $v_q \in N_G(q)$. This contradicts the property of vertex v_q. Thus, $\beta_s = |S_0| = n - 1$. □

The next result is immediate from Theorem 1.

Corollary 1. *Let n be a positive integer. Then each of the following holds:*

(i) $\beta_s(K_n) = n - 1$ *for* $n \geq 2$.

(ii) $\beta_s(K_{1,n-1}) = n - 1$ *for* $n \geq 2$.

It is well known that $\gamma_{sp}(G) \geq \frac{n}{2}$ and $\alpha(G) + \beta(G) = n$ for every graph G of order n. The next remark follows from these facts and Proposition 1.

Remark 3. *If G is a graph of order n, then $\beta_s(G) \geq \max\{\frac{n}{2}, n - \alpha(G)\}$.*

Proposition 2. *Let $G_1, G_2, \dots, G_k$ be the components of G. Then $\beta_s(G) = \sum_{j=1}^{k} \beta_s(G_j)$.*

Proof. Let S be a β_s-set in G. For each $j \in [k] = \{1, 2, \cdots, k\}$, let $S_j = S \cap V(G_j)$. Then $S = \cup_{j=1}^{k} S_j$. Let $j \in [k]$. Since S is a super vertex cover of G, S_j is a vertex cover of G_j for each $j \in [k]$. Thus,

$$\beta_s(G) = |S| = |\cup_{j=1}^{k} S_j| = \sum_{j=1}^{k} |S_j| \geq \sum_{j=1}^{k} \beta_s(G_j).$$

For each $j \in [k]$, let D_j be a β_s-set in G_j. Clearly, $D = \cup_{j=1}^{k} D_j$ is a super vertex cover of G. Hence,

$$\beta_s(G) \leq |D| = |\cup_{j=1}^{k} D_j| = \sum_{j=1}^{k} |D_j| = \sum_{j=1}^{k} \beta_s(G_j).$$

This proves the assertion. □

Theorem 2. *Let G be any graph on $n \geq 1$ vertices. Then each of the following holds:*

(i) $\beta_s(G) = 1$ *if and only if* $G \in \{K_1, K_2\}$.

(ii) $\beta_s(G) = 2$ *if and only if* $G \in \{\overline{K}_2, K_3, P_3, K_1 \cup K_2, P_4, K_2 \cup K_2\}$.

(iii) $\beta_s(G) = n$ *if and only if* $G = \overline{K}_n$

Proof. (i) Suppose $\beta_s(G) = 1$. By Remark 3, $n \leq 2$. If $n = 1$, then $G = K_1$. If $n = 2$, then $G = K_2$. Hence, $G \in \{K_1, K_2\}$.

The converse is clear.

(ii) Suppose $\beta_s(G) = 2$. By Remark 3 and part (i), $2 \leq n \leq 4$. If $n = 2$, then $G = \overline{K}_2$ by part (i) and Proposition 2. If $n = 3$ and G is connected, then $G \in \{K_3, P_3\}$. If G is disconnected, then $G = K_1 \cup K_2$ by part (i) and Proposition 2. Suppose $n = 4$. Let $S = \{a, b\}$ be a β_s-set in G and let $x, y \in V(G) \setminus S$. Since S is a super dominating set in G, we may assume that $N_G(a) \cap (V(G) \setminus S) = \{x\}$ and $N_G(b) \cap (V(G) \setminus S) = \{y\}$. Since S is a vertex cover and $x, y \notin S$, it follows that $xy \notin E(G)$. If $ab \in E(G)$, then $G = P_4$. If $ab \notin E(G)$, then $G = \langle\{a, x\}\rangle \cup \langle\{b, y\}\rangle = K_2 \cup K_2$. Accordingly, $G \in \{\overline{K}_2, K_3, P_3, K_1 \cup K_2, P_4, K_2 \cup K_2\}$.

The converse is clear.

(iii) Suppose $\beta_s(G) = n$. Suppose $G \neq \overline{K}_n$. Then $E(G) \neq \varnothing$. By Proposition 1, we have $\beta_s(G) \leq n - 1$, a contradiction. Thus, $G = \overline{K}_n$.

For the converse, suppose $G = \overline{K}_n$. By (i) and Proposition 2, $\beta_s(G) = n$. □

Proposition 3. *Let n be any positive integer. Then*

$$\beta_s(P_n) = \begin{cases} 1 & \text{if } n = 1, 2 \\ 2 & \text{if } n = 3, 4 \\ 3r & \text{if } n = 5r \\ 3r+1, & \text{if } n = 5r+1 \text{ or } n = 5r+2 \\ 3r+2, & \text{if } n = 5r+3 \text{ or } n = 5r+4. \end{cases}$$

Proof. Clearly, $\beta_s(P_1) = \beta_s(P_2) = 1$ and $\beta_s(P_3) = \beta_s(P_4) = 2$. Next, let $n = 5r$ and let $P_{5r} = [v_1, v_2, v_3, v_4, v_5, \cdots, v_{5r-3}, v_{5r-2}, v_{5r-1}, v_{5r}]$. Then $S_1 = \{v_2, v_7, \cdots, v_{5r-3}\} \cup \{v_3, v_8, \cdots, v_{5r-2}\} \cup \{v_5, v_{10}, \cdots, v_{5r}\}$ is a β_s-set in P_{5r}. Hence, $\beta_s(P_{5r}) = |S_1| = 3r$. If $n = 5r+1$ and if $P_{5r+1} = [v_1, v_2, v_3, v_4, v_5, \cdots, v_{5r-3}, v_{5r-2}, v_{5r-1}, v_{5r}, v_{5r+1}]$, then $S_2 = \{v_2, v_7, \cdots, v_{5r-3}\} \cup \{v_3, v_8, \cdots, v_{5r-2}\} \cup \{v_5, v_{10}, \cdots, v_{5r}\} \cup \{v_{5r+1}\}$ is a β_s-set in P_{5r+1}. This implies that $\beta_s(P_{5r+1}) = |S_2| = 3r+1$. If $n = 5r+2$ and if $P_{5r+2} = [v_1, v_2, v_3, v_4, v_5, \cdots, v_{5r-3}, v_{5r-2}, v_{5r-1}, v_{5r}, v_{5r+1}, v_{5r+2}]$, then $S_3 = \{v_2, v_7, \cdots, v_{5r-3}\} \cup \{v_3, v_8, \cdots, v_{5r-2}\} \cup \{v_5, v_{10}, \cdots, v_{5r}\} \cup \{v_{5r+2}\}$ is a β_s-set in P_{5r+2}. It follows that $\beta_s(P_{5r+2}) = |S_3| = 3r+1$. It is routine to show that $\beta_s(P_{5r+3}) = \beta_s(P_{5r+4}) = 3r+2$. □

Proposition 4. *Let n be any positive integer where $n \geq 3$. Then*

$$\beta_s(C_n) = \begin{cases} 2 & \text{if } n = 3 \\ 3 & \text{if } n = 4 \\ 3r & \text{if } n = 5r \\ 3r+1, & \text{if } n = 5r+1 \text{ or } n = 5r+2 \\ 3r+2, & \text{if } n = 5r+3 \text{ or } n = 5r+4. \end{cases}$$

Proof. It can be verified easily that $\beta_s(C_3) = 2$ and $\beta_s(C_4) = 3$. Let $n = 5r$ and let $C_{5r} = [v_1, v_2, v_3, v_4, v_5, \cdots, v_{5r-3}, v_{5r-2}, v_{5r-1}, v_{5r}, v_1]$. Then $D_1 = \{v_1, v_6, \cdots, v_{5r-4}\} \cup \{v_2, v_7, \cdots, v_{5r-3}\} \cup \{v_4, v_9, \cdots, v_{5r-1}\}$ is a β_s-set in C_{5r}. Hence, $\beta_s(C_{5r}) = |D_1| = 3r$. If $n = 5r+1$ and if $C_{5r+1} = [v_1, v_2, v_3, v_4, v_5, \cdots, v_{5r-3}, v_{5r-2}, v_{5r-1}, v_{5r}, v_{5r+1}]$, then $D_2 = \{v_1, v_6, \cdots, v_{5r-4}\} \cup \{v_2, v_7, \cdots, v_{5r-3}\} \cup \{v_4, v_9, \cdots, v_{5r-6}, v_{5r-1}\}$ is a β_s-set in C_{5r+1}. This implies that $\beta_s(C_{5r+1}) = |D_2| = 3r+1$. If $n = 5r+2$ and if $C_{5r+2} = [v_1, v_2, v_3, v_4, v_5, \cdots, v_{5r-4}, v_{5r-3}, v_{5r-2}, v_{5r-1}, v_{5r}, v_{5r+1}, v_{5r+2}]$, then $D_3 = \{v_1, v_6, \cdots, v_{5r-4}\} \cup \{v_2, v_7, \cdots, v_{5r-3}\} \cup \{v_4, v_9, \cdots, v_{5r-1}\} \cup \{v_{5r}\}$ is a β_s-set in C_{5r+2}. It follows that $\beta_s(C_{5r+2}) = |D_3| = 3r+1$. That $\beta_s(C_{5r+3}) = \beta_s(C_{5r+4}) = 3r+2$ can be shown easily. □

Theorem 3. *Let a and b be positive integers such that $1 \leq a \leq b$. Then there exists a connected graph G such that $\beta(G) = a$ and $\beta_s(G) = b$.*

Proof. If $a = b$, then consider $G = K_{a+1}$. Clearly, $\beta(G) = a$. By Theorem 1, $\beta_s(G) = a$. Next, suppose $a < b$ and let $m = b - a$. Let G be the graph obtained from K_{a+1} by adding m pendant edges $v_1x_1, v_1x_2, \ldots, v_1x_m$, where $V(K_{a+1}) = \{v_1, v_2, \cdots, v_a, v_{a+1}\}$ (see Figure 2). Clearly, $S_1 = \{v_1, v_2, \cdots, v_a\}$ is a vertex cover of G. Hence, $\beta(G) \leq |S_1| = a$. Let S be a β-set in G. If $v_1 \notin S$, then $\{v_2, v_3, \cdots, v_{a+1}\} \subseteq S$ since S is a vertex cover of G. Again, since S is a vertex cover of G, it follows that $\{x_1, x_2, \cdots, x_m\} \subseteq S$. Thus, $S = \{x_1, x_2, \cdots, x_m, v_2, v_3, \cdots, v_{a+1}\}$. Consequently, $\beta(G) = |S| = m + a = b - a + a = b$, which is not possible. Thus, $v_1 \in S$. Suppose $|(V(K_{a+1}) \setminus \{v_1\}) \cap S| < a - 1$. Then there exist $r, t \in \{2, 3, \cdots, a + 1\}$ such that $v_r, v_t \notin S$. This, however, is not possible because $v_rv_t \in E(G)$ and S is a vertex cover. Therefore, $|(V(K_{a+1}) \setminus \{v_1\}) \cap S| = a - 1$. Therefore, since S is a β-set in G, $\beta(G) = |S| = a$. Now if $m = 1$, then clearly, $S_2 = \{v_1, v_2, \cdots, v_{a+1}\}$ is a β_s-set in G. Thus, $\beta_s(G) = a + 1 = b$. Suppose $m \geq 2$. Note that the set $S_3 = \{v_1, v_2, \cdots, v_{a+1}, x_2, \cdots, x_m\}$ is a super vertex cover of G. It follows that $\beta(G) \leq |S_3| = a + 1 + m - 1 = a + m = b$. Let S_0 be a β_s-set in G. If $v_1 \notin S_0$, then $S_0 = \{v_2, \cdots, v_{a+1}, x_1, x_2, \cdots, x_m\}$ because S_0 is a vertex cover of G. Thus, $\beta(G) = |S| = m + a = b - a + a = b$. Suppose $v_1 \in S$. Since S is a vertex cover, $|V(K_{a+1}) \setminus S| \leq 1$. If $|V(K_{a+1}) \setminus S| = 0$, then $|S \cap \{x_1, x_2, \cdots, x_m\}| = m - 1$ because S is a super dominating set in G. Again, since S is a super dominating set, $|S \cap \{x_1, x_2, \cdots, x_m\}| = m$ whenever $|V(K_{a+1}) \setminus S| = 1$. In both cases, we have $\beta_s(G) = |S_0| = b$.

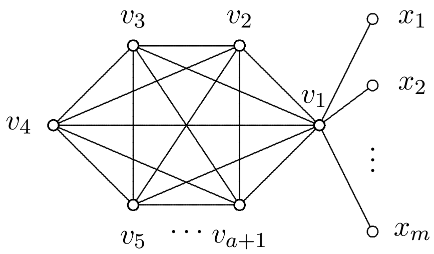

Figure 2

Therefore, the assertion holds. □

The next result is a consequence of Theorem 3.

Corollary 2. *Let n be a positive integer. Then there exists a connected graph G such that $\beta_s(G) - \beta(G) = n$. In other words, the difference $\beta_s(G) - \beta(G)$ can be made arbitrarily large.*

Theorem 4. *Let G and H be any graphs. A set $C \subseteq V(G + H)$ is a super vertex cover of $G + H$ if and only if $C = C_G \cup C_H$ and satisfies one of the following conditions:*

(i) $C_G = V(G)$ *and* C_H *is a super vertex cover of* H.

(ii) $C_G = V(G)$ *and* $C_H = V(H) \setminus \{q\}$ *where* q *is an isolated vertex in* H.

(iii) $C_H = V(H)$ *and* C_G *is a super vertex cover of* G.

(iv) $C_H = V(H)$ *and* $C_G = V(G) \setminus \{x\}$ *where* x *is an isolated vertex in* G.

Proof. Suppose C is super vertex cover of $G + H$. Let $C_G = C \cap V(G)$ and $C_H = C \cap V(H)$. Suppose $C_G \neq V(G)$ and $C_H \neq V(H)$. Pick any $v \in V(G) \setminus C_G$ and $p \in V(H) \setminus C_H$. Then $e = vp \in E(G)$ and none of v and p is in C. This implies that C is not a vertex cover in $G + H$, a contradiction. Thus, $C_G = V(G)$ or $C_H = V(H)$. Suppose $C_G = V(G)$ and let $ab \in E(H)$. Since C is a vertex cover of $G + H$, it follows that $a \in C_H$ or $b \in C_H$. This implies that C_H is a vertex cover of H. If C_H is a super dominating set in H, then (i) holds. So suppose that C_H is not a super dominating set in H. Then there exists a vertex $q \in V(H) \setminus C_H$ such that for all $z \in C_H$, we have $N_H(z) \cap [V(H) \setminus C_H] \neq \{q\}$. Suppose q is not an isolated vertex in H. Let $z_0 \in C_H \cap N_H(q)$ (z_0 exists because C_H is a vertex cover of H). Then $N_H(z_0) \cap [V(H) \setminus C_H] \neq \{q\}$. It follows that there exists $t \in N_H(z_0) \cap [V(H) \setminus C_H]$ such that $q \neq t$. This implies that $N_{G+H}(w) \cap [V(G + H) \setminus C] \neq \{q\}$ for all $w \in C$, contrary to the assumption that C is a super dominating set in $G + H$. Thus, q is an isolated vertex in H. Since C is a super dominating set in $G + H$, and $q \notin N_H[C_H]$, it follows that $C_H = V(H) \setminus \{q\}$. This shows that (ii) holds. Similarly, (iii) or (iv) holds.

Conversely, suppose that $C = C_G \cup C_H$. Suppose that (i) holds. Then clearly, C is a super vertex cover of $G + H$. Suppose (ii) holds. Since $V(G) \subset C$, every edge of the form vw or vq, where $v, w \in V(G)$, is incident to a vertex in C. Now let $ab \in E(H)$. Since $C_H = V(H) \setminus \{q\}$, $a, b \in C_H \subset C$. Hence, C is a vertex cover of $G+H$. Since $V(G+H) \setminus C = \{q\}$, pick any $y \in C_G = V(G)$. Then $yq \in E(G + H)$ and $N_{G+H}(y) \cap [V(G + H) \setminus C] = \{q\}$. Thus, C is a super dominating set in $G + H$. Therefore, C is a super vertex cover in $G + H$. The same conclusion holds for C if (iii) or (iv) holds. □

Corollary 3. *Let* G *and* H *be any two graphs of orders* m *and* n*, respectively. Then* $\beta_s(G + H) = m + n - 1$ *if and only if one of the following holds:*

(i) $\beta_s(G) = m - 1$ *and* $\beta_s(H) = n - 1$.

(ii) $\beta_s(G) = m - 1$ *and* $H = \overline{K}_n$ *(or* $\beta_s(H) = n - 1$ *and* $G = \overline{K}_m$*).*

(iii) $G = \overline{K}_m$ *and* $H = \overline{K}_n$.

In particular, $\beta_s(K_{m,n}) = m + n - 1$ *for* $m, n \geq 1$.

Proof. Suppose $\beta_s(G + H) = m + n - 1$ and let C be a β_s-set in $G + H$. Suppose $\beta_s(G) < m-1$ or $\beta_s(H) < n-1$, say $\beta_s(G) < m-1$. Let S_G be a β_s-set in G. By Theorem 4, $C = S_G \cup V(H)$ is a super vertex cover of $G+H$. Hence, $\beta_s(G+H) \leq |C| = n+\beta_s(G) < |C|$, a contradiction. It follows that $\beta_s(G) \geq m-1$ and $\beta_s(H) \geq n-1$. Suppose $\beta_s(G) = m-1$. If $\beta_s(H) = n - 1$, then (i) holds. Suppose $\beta_s(H) = n$. Then $H = \overline{K}_n$ by Theorem 2(iii). Hence, (ii) holds. If $\beta_s(G) = m$, then (ii) or (iii) holds.

For the converse, suppose (i) holds. By Theorem 4, it follows that $\beta_s(G + H) = m + n - 1$. Next, suppose (ii) holds, i.e., $\beta_s(G) = m - 1$ and $H = \overline{K}_n$. If C is a β_s-set in $G + H$, then C satisfies (ii) or (iii) or (iv) (in case G has an isolated vertex) of Theorem

4. That is, $C = V(H) \cup C_G$, where C_G is β_s-set in G, or $C = V(G) \cup [V(\overline{K}_n) \setminus \{q\}$ for a $q \in V(\overline{K}_n)$ or $C = V(H) \cup [V(G) \setminus \{x\}$ if x is an isolated vertex in G. This implies that $\beta_s(G + H) = m + n - 1$. Lastly, suppose (iii) holds. Since $\beta_s(G) = m$ and $\beta_s(H) = n$ and C is a β_s-set in $G + H$, C satisfies (ii) or (iv) of Theorem 4. Therefore, $\beta_s(G + H) = m + n - 1$. □

Corollary 4. *Let G and H be any two graphs of orders m and n, respectively, such that $\beta_s(G + H) \neq m + n - 1$. Then*

$$\beta_s(G + H) = min\{m + \beta_s(H), n + \beta_s(G)\}.$$

Proof. Let D_1 and D_2 be β_s-sets in G and H, respectively. Then $C_1 = D_2 \cup V(G)$ and $C_2 = D_1 \cup V(H)$ are super vertex covers of $G + H$ by Theorem 4. Therefore,

$$\beta_s(G + H) \leq \min\{|C_1|, |C_2|\} = \min\{m + \beta_s(H), n + \beta_s(G)\}.$$

Next, let C be a β_s-set in $G + H$. Since $\beta_s(G + H) \neq m + n - 1$, $\beta_s(G) < m - 1$ or $\beta_s(H) < n - 1$ by Corollary 3. This implies that C satisfies (i) or (iii) of Theorem 4, i.e., $C = V(G) \cup C_H$ or $C = V(H) \cup C_G$, where C_G and C_H are super vertex covers of G and H, respectively. It follows that $\beta_s(G + H) = |C| \geq \min\{m + \beta_s(H), n + \beta_s(G)\}$. This establishes the desired equality. □

The next result is immediate from Theorem 4 and Corollary 1(i).

Corollary 5. *Let G be a graph of order m and let n be any positive integer. Then $\beta_s(K_n + G) = min\{n + \beta_s(G), m + n - 1\}$. In particular, $\beta_s(K_1 + G) = min\{1 + \beta_s(G), m\}$. Moreover, each of the following holds:*

- (i) *$\beta_s(F_n) = \beta_s(K_1 + P_n) = 1 + \beta_s(P_n)$ for $n \geq 2$.*
- (ii) *$\beta_s(W_n) = \beta_s(K_1 + C_n) = 1 + \beta_s(C_n)$ for $n \geq 3$.*

Theorem 5. *Let G be a non-trivial connected graph and let H be any graph. Then $S \subseteq V(G \circ H)$ is a super vertex cover in $G \circ H$ if and only if $S = A \cup (\cup_{v \in V(G)} S_v)$ and satisfies the following conditions:*

- (i) *A is a vertex cover in G.*
- (ii) *For each $v \in A \cap N_G(V(G) \setminus A)$, it holds that S_v is a super vertex cover in H^v.*
- (iii) *For each $v \in A \setminus N_G(V(G) \setminus A)$, it holds that S_v is a super vertex cover in H^v or $S_v = V(H^v) \setminus \{q_v\}$ for some isolated vertex q_v in H^v.*
- (iv) *For each $v \notin A$, it holds that $S_v = V(H^v)$.*

Proof. Suppose S is a super vertex cover in $G \circ H$. Let $A = C \cap V(G)$ and let $S_v = C \cap V(H^v)$ for each $v \in V(G)$. Then $S = A \cup (\cup_{v \in V(G)} S_v)$. Let $ab \in E(G) \subset E(G \circ H)$.

Since S is a vertex cover in $G \circ H$, it follows that $a \in A$ or $b \in A$. Hence, A is a vertex cover in G, showing that (i) holds. Let $v \in A \cap N_G(V(G) \setminus A)$ and let $pq \in E(H^v) \subset E(G \circ H)$. Again, because S is a vertex cover in $G \circ H$, $p \in S_v$ or $q \in S_v$. This implies that S_v is a vertex cover in H^v. Now, let $t \in V(H^v) \setminus S_v$. Since S is a super dominating set in $G \circ H$, there exists $x \in C$ such that $N_{G \circ H}(x) \cap [V(G \circ H) \setminus C] = \{t\}$. Since $v \in N_G(V(G) \setminus A$, $x \neq v$. This implies that $x \in S_v$ and $N_{H^v}(x) \cap [V(H^v) \setminus S_v] = \{t\}$. Thus, S_v is a super dominating set in H^v. This shows that (ii) holds. Suppose now that $v \in A \setminus N_G(V(G) \setminus A)$. Clearly, S_v is a vertex cover in H^v. If S_v is a super dominating set in H^v, then (iii) holds. So suppose S_v is not a super dominating set in H^v. Then there exists $q_v \in V(H^v) \setminus S_v$ such that for all $z \in S_v$, we have $N_H(z) \cap [V(H^v) \setminus S_v] \neq \{q_v\}$. Following a previous argument (see proof of Theorem 4), it can be shown that q_v is an isolated vertex in H^v and $S_v = V(H) \setminus \{q_v\}$. This shows that (iii) holds. Finally, let $v \notin A$ and let $s \in V(H^v)$. Since C is a vertex cover in $G \circ H$, $v \notin A$, and $vs \in E(G \circ H)$, we must have $s \in S_v$. As v was arbitrarily chosen, we have $S_v = V(H^v)$, showing that (iv) holds.

For the converse, suppose that S has the given form and satisties conditions (i), (ii), (iii), and (iv). Let $xy \in E(G \circ H)$. If $x, y \in V(G)$, then $x \in A$ or $y \in A$ because of (i). Suppose at most one of x and y is in $V(G)$. We may assume that $x \in V(G)$. Then $y \in V(H^x)$. If $x \in A$, then xy is incident to $x \in C$. If $x \notin A$, then $S_x = V(H^x)$ by (iv). It follows that $y \in S_v$. Hence, xy is incident to $y \in C$. Suppose now that $x, y \in V(H^v)$ for some $v \in V(G)$. If $v \notin A$, then $S_v = V(H^v)$. Hence, $x, y \in S_v \subset C$. Suppose that $v \in A \cap N_G(V(G) \setminus A)$. By (ii), S_v is a super vertex cover in H^v. This implies that $x \in S_v$ or $y \in S_v$. If $v \in A \setminus N_G(V(G) \setminus A)$, then $S_v = V(H^v) \setminus \{q_v\}$ for some isolated vertex q_v in H^v. Since $xy \in E(G \circ H)$, $x \neq q_v$ and $y \neq q_v$. Hence, $x, y \in S_v \subset C$. Therefore, C is a vertex cover in $G \circ H$. Next, let $p \in V(G \circ H) \setminus C$ and let $v \in V(G)$ such that $p \in V(v + H^v)$. If $p = v$, then $p \notin A$. By (iv), $S_v = V(H^v)$. Pick any $q \in S_v$. Then $N_{G \circ H}(q) \cap [V(G \circ H) \setminus C] = \{p\}$. Suppose $p \in V(H^v)$. Then $p \in V(H^v) \setminus S_v$. This implies that $v \in A$ (otherwise $S_v = V(H^v)$ by (iv), a contradiction). If $v \in A \cap N_G(V(G) \setminus A)$, then S_v is a super dominating set in H^v by (ii). Hence, there exists $d \in S_v \subset C$ such that $N_{G \circ H}(d) \cap [V(G \circ H) \setminus C] = N_{H^v}(d) \cap [V(H^v) \setminus S_v] = \{p\}$. If $v \in A \setminus N_G(V(G) \setminus A)$, then $S_v = V(H^v) \setminus \{q_v\}$ for some isolated vertex q_v in H^v. It follows that $p = q_v$. Clearly, $N_{G \circ H}(v) \cap [V(G \circ H) \setminus C] = \{p\}$. Therefore, C is a super dominating set in $G \circ H$. Accordingly, C is a super vertex cover in $G \circ H$. □

Corollary 6. *Let G and H be non-trivial connected graphs of orders m and n, respectively. Then*

$$\beta_s(G \circ H) = (\beta_s(H) - n + 1)\beta(G) + mn.$$

Proof. Let A be a β-set in G, S_v a β_s-set in H^v for each $v \in A$, and $S_w = V(H^v)$ for each $v \in V(G) \setminus A$. Then $S = A \cup (\cup_{v \in V(G)} S_v)$ is a super vertex cover of $G \circ H$ by Theorem 5. It follows that

$$\begin{aligned}
\beta_s(G \circ H) &\leq |S| \\
&= |A| + \sum_{v \in A} |S_v| + \sum_{v \in V(G) \setminus A} |S_v|
\end{aligned}$$

$$
\begin{aligned}
&= \beta(G) + \beta(G)\beta_s(H) + n(m - \beta(G)) \\
&= (\beta_s(H) - n + 1)\beta(G) + mn.
\end{aligned}
$$

On the other hand, let C_0 be a β_s-set in $G \circ H$. Then $C_0 = A_0 \cup (\cup_{v \in V(G)} S'_v)$ and satisfies properties (i), (ii), (iii), and (iv) of Theorem 5. Hence, A_0 is a vertex cover of G by (i) and $S'_v = V(H^v)$ for each $v \in V(G) \setminus A$ by (iv). Note that since H is a (non-trivial) connected graph, S'_v is a super vertex cover in H^v for each $v \in V(G)$ by (ii) and (iii). Hence,

$$
\begin{aligned}
\beta_s(G \circ H) &= |S_0| \\
&= |A_0| + \sum_{v \in A_0} |S'_v| + \sum_{v \in V(G) \setminus A_0} |S'_v| \\
&\geq |A_0| + \sum_{v \in A_0} \beta_s(H) + \sum_{v \in V(G) \setminus A_0} n \\
&= |A_0| + |A_0|\beta_s(H) + (m - |A_0|)n \\
&= (\beta_s(H) - n + 1)|A_0| + mn \\
&\geq (\beta_s(H) - n + 1)\beta(G) + mn.
\end{aligned}
$$

This establishes the desired equality. □

4. Conclusion

The variant super vertex cover of the standard vertex cover had been introduced and initially investigated in this study. For a non-empty graph G, its super vertex cover number is at least equal to the maximum of the super domination number and the vertex cover number of the graph and at most equal to $|V(G)| - 1$. It was shown that the difference of the super vertex cover number and the vertex cover number can be made arbitrarily large. In this study, the super vertex cover numbers of some graphs, including the join and the corona of two graphs, had been obtained. Further study or investigation of the newly defined parameter is recommended. In particular, it may be interesting to determine the value of the parameter for graphs resulting from other binary operations. Moreover, while the vertex cover problem is NP-complete, it remains open to show whether or not the super vertex cover problem is NP-complete.

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